”Kernel Learning”
Homework 2

December 11, 2025

Exercice 1. Non-expansiveness of the Gaussian kernel

Consider the Gaussian kernel $K : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ such that for all pair of points x, x' in $\mathbb{R}^p$,

$$K(x, x') = e^{-\frac{\alpha}{2}\|x-x'\|^2},$$

where $\|.\|$ is the Euclidean norm on $\mathbb{R}^p$. Call $\mathcal{H}$ the RKHS of K and consider its RKHS mapping $\varphi : \mathbb{R}^p \to \mathcal{H}$ such that $K(x, x') = \langle \varphi(x), \varphi(x') \rangle_{\mathcal{H}}$ for all x, x' in $\mathbb{R}^p$. Show that

$$\|\varphi(x) - \varphi(x')\|_{\mathcal{H}} \leq \sqrt{\alpha}\|x - x'\|.$$

The mapping is called non-expansive whenever $\alpha \leq 1$.

Exercice 2. B_n-splines

The convolution between two functions $f, g : \mathbb{R} \to \mathbb{R}$ is defined by:

$$f \star g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du,$$

when this integral exists.

Let now the function:

$$I(x) = \begin{cases} 1 & \text{si } -1 \leq x \leq 1, \\ 0 & \text{si } x < -1 \text{ ou } x > 1, \end{cases}$$

and $B_n = I^{\star n}$ for $n \in \mathbb{N}_*$ (that is, the function I convolved n times with itself: $B_1 = I, B_2 = I \star I, B_3 = I \star I \star I$, etc...).

Is the function $k(x,y) = B_n(x-y)$ a positive definite kernel over $\mathbb{R} \times \mathbb{R}$? If yes, describe the corresponding reproducing kernel Hilbert space.

Exercice 3. Sobolev spaces

Let $\mathcal{H} = \{f : [0,1] \to \mathbb{R}\,,\ \text{absolutely continuous}, f' \in L^2([0,1]), f(0) = 0\}$, endowed with the bilinear form

$$\forall f, g \in \mathcal{H}\,, \quad \langle f, g \rangle_{\mathcal{H}} = \int_0^1 \left(f(u)g(u) + f'(u)g'(u)\right) du\,.$$

Show that $\mathcal{H}$ is an RKHS, and compute its reproducing kernel.

Exercice 4. Gaussian RKHS

For any $\sigma > 0$, let K_σ be the normalized Gaussian kernel on $\mathbb{R}^d$:

$$\forall x, y \in \mathbb{R}^d \quad K_\sigma(x,y) = \frac{1}{\left(\sqrt{2\pi}\sigma\right)^d} \exp\left(-\frac{\| x - y \|^2}{2\sigma^2}\right)\,,$$

and let $\mathcal{H}_\sigma$ be its reproducing kernel Hilbert space (RKHS).

1. Recall a proof of the positive definiteness of K.

2. For any $0 < \sigma < \tau$, show that

$$\mathcal{H}_\tau \subset \mathcal{H}_\sigma \subset L_2(\mathbb{R}^d)\,,$$

3. For any $0 < \sigma < \tau$ and $f \in \mathcal{H}_\tau$, show that

$$\| f \|_{\mathcal{H}_\tau} \geq \| f \|_{\mathcal{H}_\sigma} \geq \| f \|_{L_2(\mathbb{R}^d)}\,,$$

and that

$$0 \leq \| f \|^2_{\mathcal{H}_\sigma} - \| f \|^2_{L_2(\mathbb{R}^d)} \leq \frac{\sigma^2}{\tau^2}\left(\| f \|^2_{\mathcal{H}_\tau} - \| f \|^2_{L_2(\mathbb{R}^d)}\right)\,.$$

4. For any $\tau > 0$ and $f \in \mathcal{H}_\tau$, show that

$$\lim_{\sigma \to 0} \| f \|_{\mathcal{H}_\sigma} = \| f \|_{L_2(\mathbb{R}^d)}\,.$$