Portfolio Approximation

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Notices

- The methods described here are patent pending.
- This work is my own; it does not necessarily reflect any views held or any approaches used by Convexity Capital Management LP.

Outline

The Problem

- Find simple, accurate approximations of portfolio values across a range of scenarios (no model assumed)
- Consider only exponential-affine instruments

    Bonds
    Interest rate swaps
    Credit default swaps
    Equities (as streams of dividends)

- State of the art: duration and higher-order relatives
- But these only work well for scenarios with small changes
- I seek simple approximations accurate for both small and large changes

Do We Care?

- An (optimal) n-factor approximation yields an (optimal) n-instrument approximating portfolio
- An approximation that works only for small changes $\rightarrow$ accurate portfolio replication only for small moves
- An approximation that works even for large changes $\rightarrow$ accurate portfolio replication even in large moves

History of the Problem

- Tables of discount / growth factors: Trenchant (1558)
- Continuous compounding: Bernoulli (1690)
- Duration (Macaulay (1938), rediscovered and reinterpreted by Hicks (1939) and Samuelson (1945))
- Immunization (Redington (1952), higher-order Taylor series terms appear here, seemingly for the first time)
- Model-based approximation (Vasicek (1977), Cox *et al.* (1979), Cox *et al.* (1985), Heath *et al.* (1992))

A Quartet of New Approximations

- No model is assumed; I seek to minimize worst-case approximation error or average squared approximation error across a range of scenarios.
- Both small and large changes are handled.

	Worst-case Error	**Avg. Squared Error**
Portfolio	Numerically-optimal	Interpolatory
Nested	Mathieu	Oblate

Relations to Recent Literature

- Conceptually, worst-case approximation error is a focus when robustness of replication is a concern
  - ⋆ Hansen & Sargent (1995), Hansen *et al.* (1999), Anderson *et al.* (2003), Hansen *et al.* (2006), Hansen & Sargent (2008), through Hansen & Sargent (2012)
  - ⋆ Chamberlain (2000) and Chamberlain (2001)
- Technically, the average squared approximation error analysis involves classical extensions of Perron-Frobenius theory
  - ⋆ Hansen & Scheinkman (2009), Hansen (2012), and Borovička *et al.* (2014)
  - ⋆ Ross (2014)

Further Potential Applications

- I do not assume any model
- However, the approximations I develop can be applied directly to standard affine models and some variants (Duffie *et al.* (2000), Piazzesi (2010), Campbell *et al.* (2013), Greenwood & Vayanos (2014))
- My approximations may also be useful in other contexts (Campbell *et al.* (2014), Barndorff-Nielsen & Shephard (2001))

The Key Initial Step

- All problems of this type can be reduced to approximating one integral operator.
- The exponential product operator P_c takes a signed measure $h(x)$ (a cashflow stream) to a function of y (a scenario profile):

$$P_c[h](y) = \int_{-1}^{1} \exp(cxy) dh(x). \tag{1}$$

- $x \in [-1, 1]$ is the (normalized) scenario shift times time, while $y \in [-1, 1]$ is a fraction of the scenario change.
- c represents the product of half the range of scenario changes and half the range of "move times time."

What Are the Domain and Range?

- P_c maps the Hilbert space $L^2[-1,1]$ to itself.
- This domain and range work for average squared approximation error and smooth cashflow streams.
- P_c also maps signed measures of bounded variation (with the total-variation norm) to continuous functions (with the sup norm).
- This domain and range are necessary to handle worst-case approximation error and general cashflow streams.

Rank-n Approximations

- Seek approximations of the type

$$\exp(cxy) \sim \sum_{i=1}^{n} f_i(x)\, g_i(y) \tag{2}$$

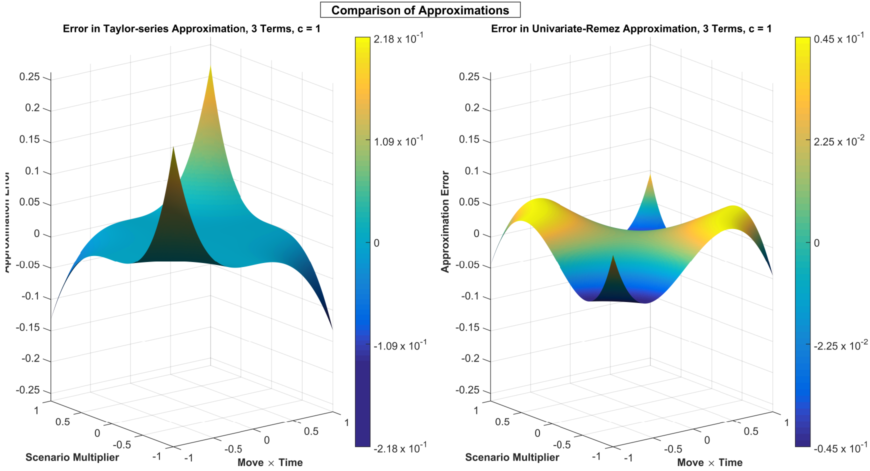
- What are the optimal functions f_i and g_i (optimal dimension reduction)?
- This is a problem in linear n-widths
- I provide methods for average squared error and *worst-case error*

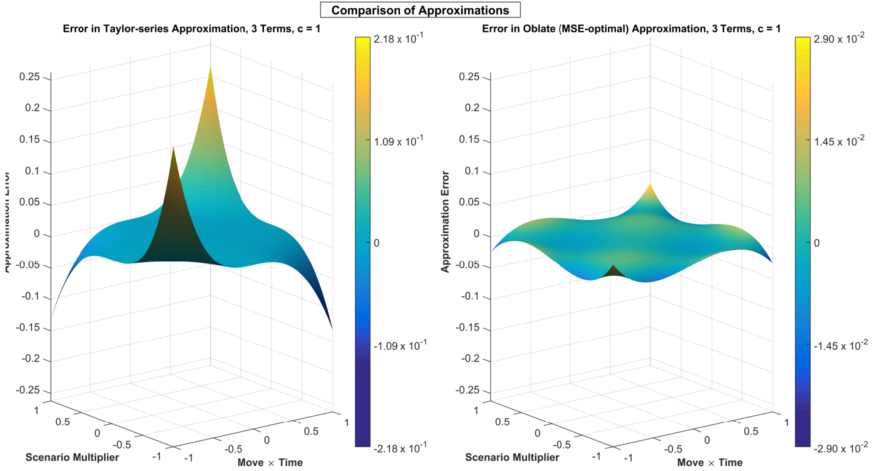
Some Suboptimal Approximations

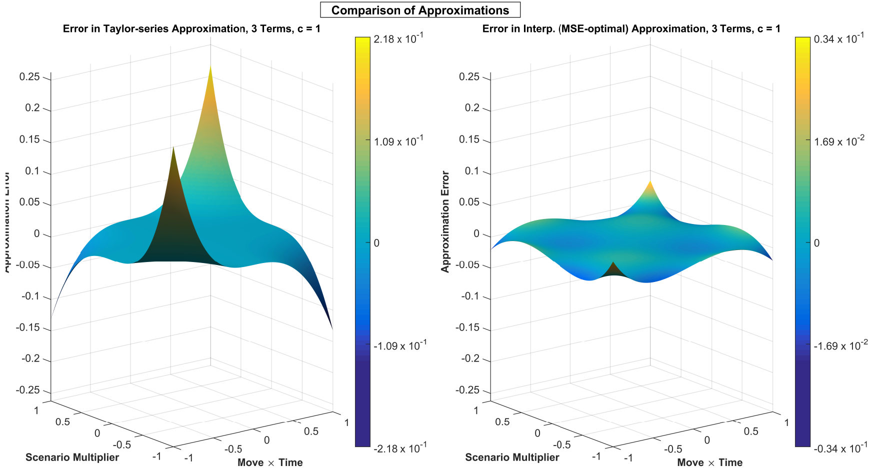
- Taylor series: $\exp(cxy) \sim \sum_{i=1}^{n} \frac{c^{i-1}}{(i-1)!} x^{i-1} y^{i-1}$.

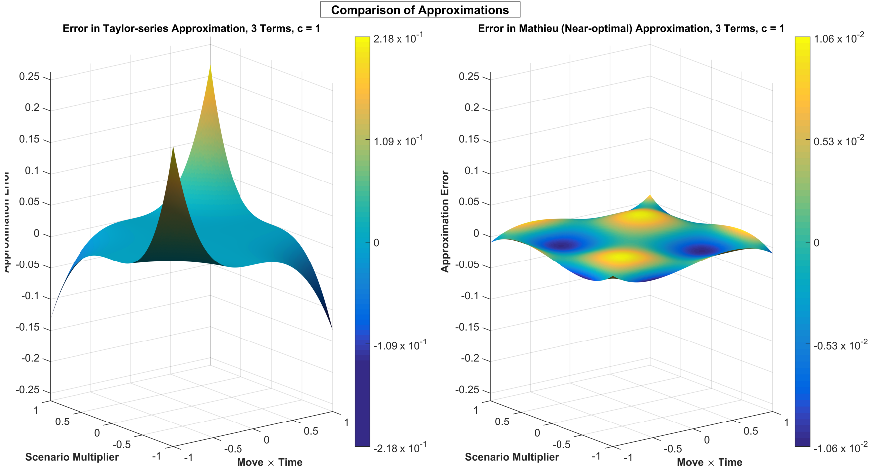
  Worst-case error on the order of $1/n!$.
- Univariate Remez: $\exp(cxy) \sim \sum_{i=1}^{n} a_i x^{i-1} y^{i-1}$.

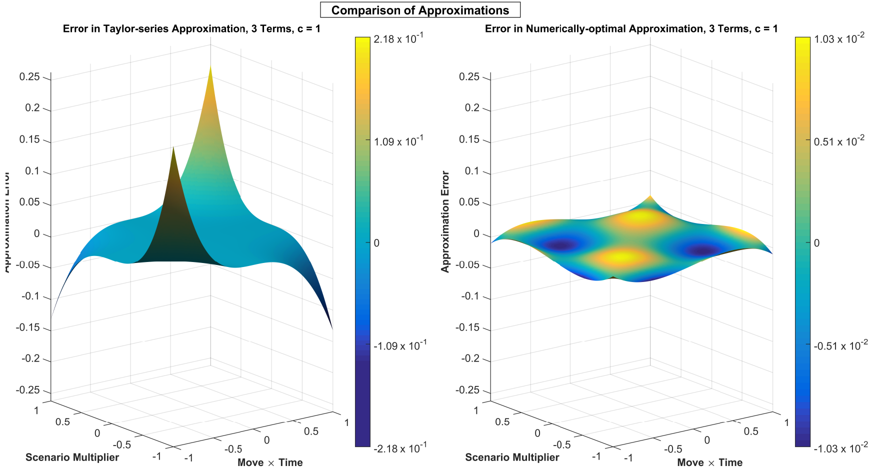
  Comes from the applying the Remez method to $\exp(cu)$ as a function of $u \in [-1, 1]$.
  Worst-case error on the order of $1/\left(2^{n-1} n!\right)$.
- How far are these from optimal? Very far indeed!

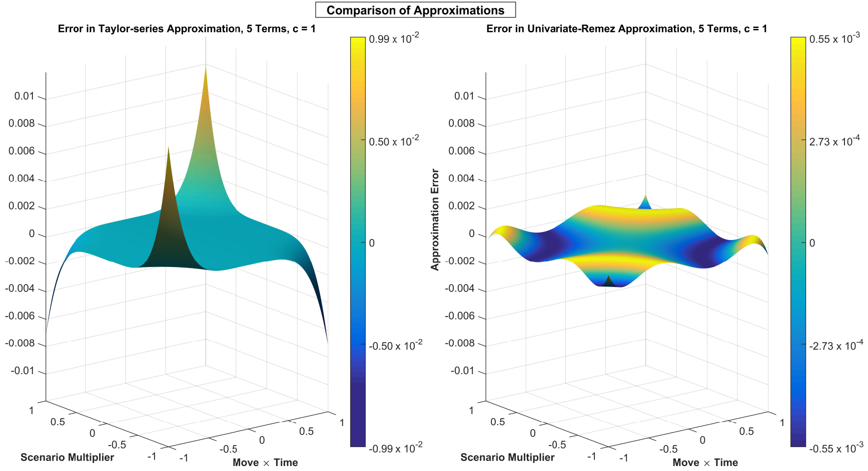

Comparison of Approximations
Error in Taylor-series Approximation, 3 Terms, c = 1
Approximation Error
Scenario Multiplier
Move × Time
2.18 x 10^-1
1.09 x 10^-1
0
-1.09 x 10^-1
-2.18 x 10^-1
Error in Univariate-Remez Approximation, 3 Terms, c = 1
Approximation Error
Scenario Multiplier
Move × Time
0.45 x 10^-1
2.25 x 10^-2
0
-2.25 x 10^-2
-0.45 x 10^-1

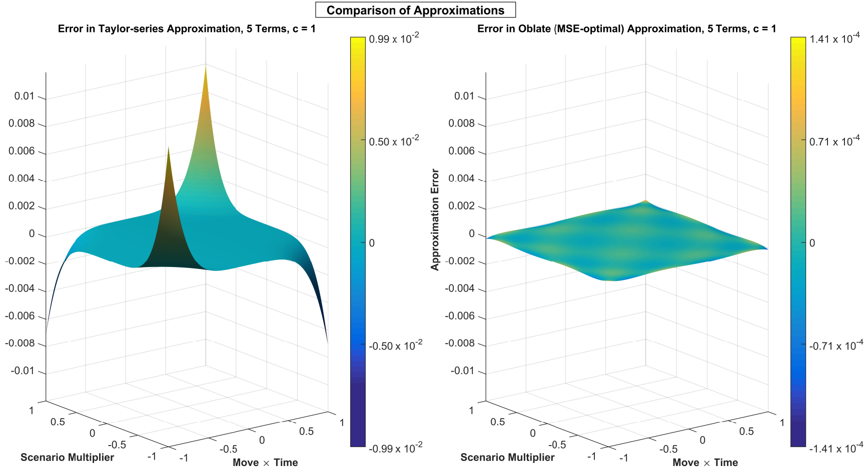

Comparison of Approximations
Error in Taylor-series Approximation, 3 Terms, c = 1
Error in Oblate (MSE-optimal) Approximation, 3 Terms, c = 1
Approximation Error
Scenario Multiplier
Move × Time
2.18 x 10^-1
1.09 x 10^-1
0
-1.09 x 10^-1
-2.18 x 10^-1
2.90 x 10^-2
1.45 x 10^-2
-1.45 x 10^-2
-2.90 x 10^-2

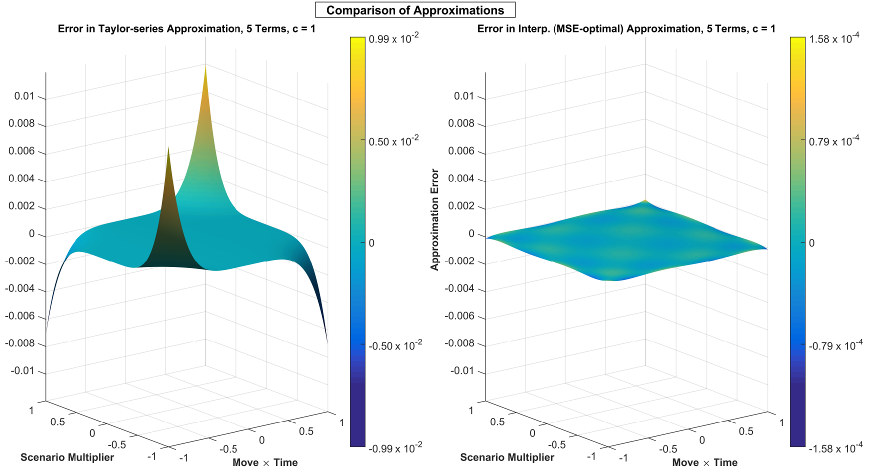

Comparison of Approximations
Error in Taylor-series Approximation, 3 Terms, c = 1
Approximation Error
Scenario Multiplier
Move × Time
2.18 x 10^-1
1.09 x 10^-1
0
-1.09 x 10^-1
-2.18 x 10^-1
Error in Interp. (MSE-optimal) Approximation, 3 Terms, c = 1
Approximation Error
Scenario Multiplier
Move × Time
0.34 x 10^-1
1.69 x 10^-2
0
-1.69 x 10^-2
-0.34 x 10^-1

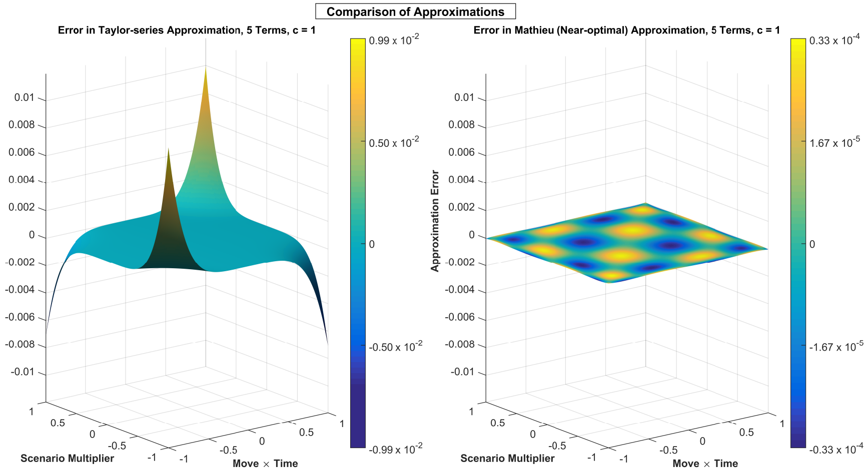

Comparison of Approximations
Error in Taylor-series Approximation, 3 Terms, c = 1
Error in Mathieu (Near-optimal) Approximation, 3 Terms, c = 1
Approximation Error
Scenario Multiplier
Move × Time
2.18 x 10^-1
1.09 x 10^-1
0
-1.09 x 10^-1
-2.18 x 10^-1
1.06 x 10^-2
0.53 x 10^-2
-0.53 x 10^-2
-1.06 x 10^-2

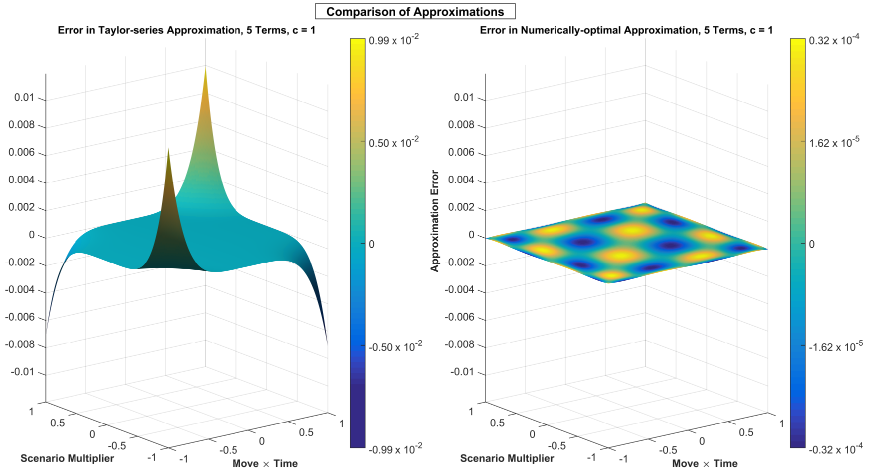

Comparison of Approximations
Error in Taylor-series Approximation, 3 Terms, c = 1
Error in Numerically-optimal Approximation, 3 Terms, c = 1
Approximation Error
Scenario Multiplier
Move × Time
2.18 x 10^-1
1.09 x 10^-1
0
-1.09 x 10^-1
-2.18 x 10^-1
1.03 x 10^-2
0.51 x 10^-2
-0.51 x 10^-2
-1.03 x 10^-2

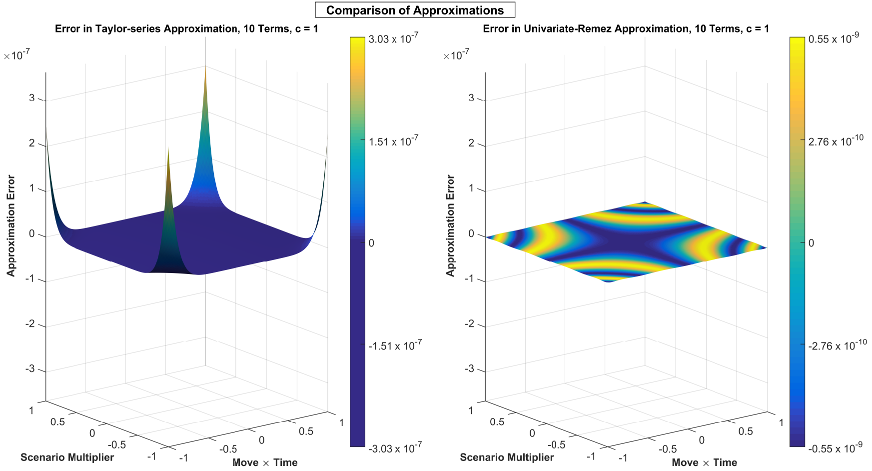

Comparison of Approximations
Error in Taylor-series Approximation, 5 Terms, c = 1
0.99 x 10^-2
0.50 x 10^-2
0
-0.50 x 10^-2
-0.99 x 10^-2
0.01
0.008
0.006
0.004
0.002
0
-0.002
-0.004
-0.006
-0.008
-0.01
Scenario Multiplier
Move × Time
Error in Univariate-Remez Approximation, 5 Terms, c = 1
Approximation Error
0.55 x 10^-3
2.73 x 10^-4
0
-2.73 x 10^-4
-0.55 x 10^-3
Scenario Multiplier
Move × Time

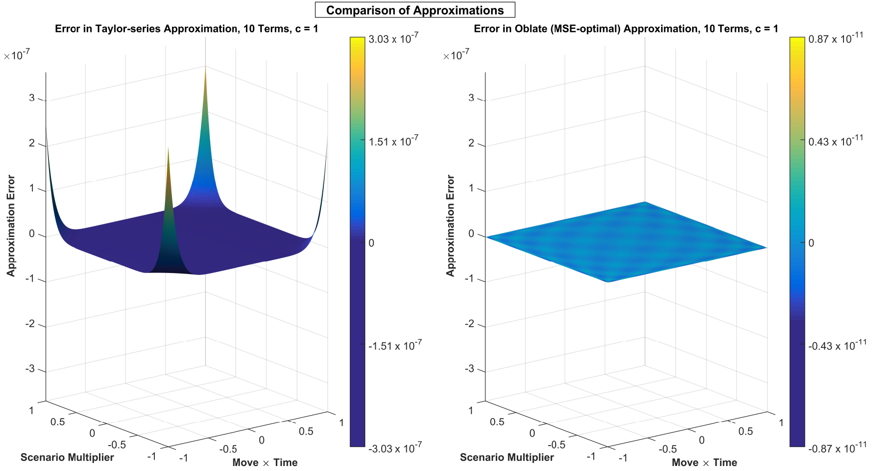

Comparison of Approximations
Error in Taylor-series Approximation, 5 Terms, c = 1
Error in Oblate (MSE-optimal) Approximation, 5 Terms, c = 1
Approximation Error
Scenario Multiplier
Move × Time
0.99 x 10^-2
0.50 x 10^-2
0
-0.50 x 10^-2
-0.99 x 10^-2
1.41 x 10^-4
0.71 x 10^-4
-0.71 x 10^-4
-1.41 x 10^-4

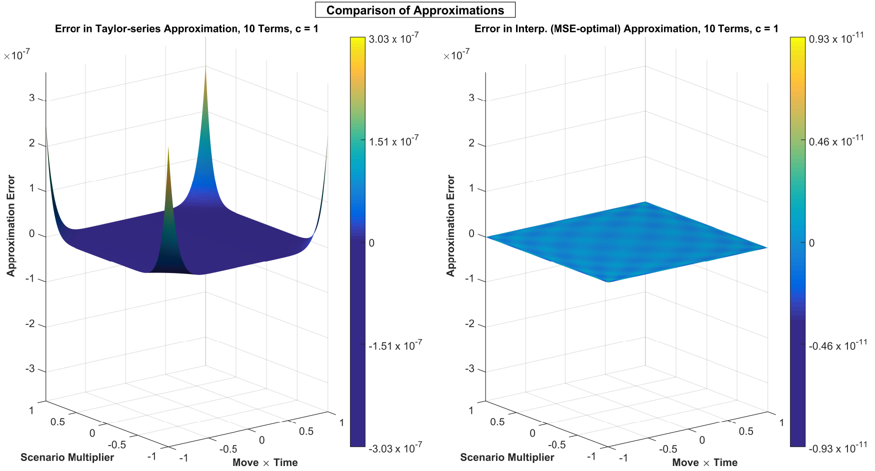

Comparison of Approximations
Error in Taylor-series Approximation, 5 Terms, c = 1
0.99 x 10^-2
0.50 x 10^-2
0
-0.50 x 10^-2
-0.99 x 10^-2
Scenario Multiplier
Move × Time
Error in Interp. (MSE-optimal) Approximation, 5 Terms, c = 1
Approximation Error
1.58 x 10^-4
0.79 x 10^-4
0
-0.79 x 10^-4
-1.58 x 10^-4
Scenario Multiplier
Move × Time

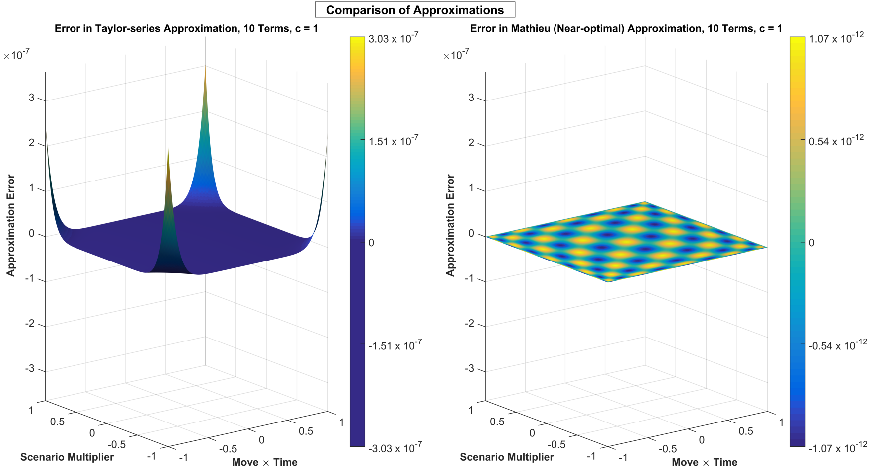

Comparison of Approximations
Error in Taylor-series Approximation, 5 Terms, c = 1
0.99 x 10^-2
0.50 x 10^-2
0
-0.50 x 10^-2
-0.99 x 10^-2
Scenario Multiplier
Move × Time
Error in Mathieu (Near-optimal) Approximation, 5 Terms, c = 1
Approximation Error
0.33 x 10^-4
1.67 x 10^-5
-1.67 x 10^-5
-0.33 x 10^-4
Scenario Multiplier
Move × Time

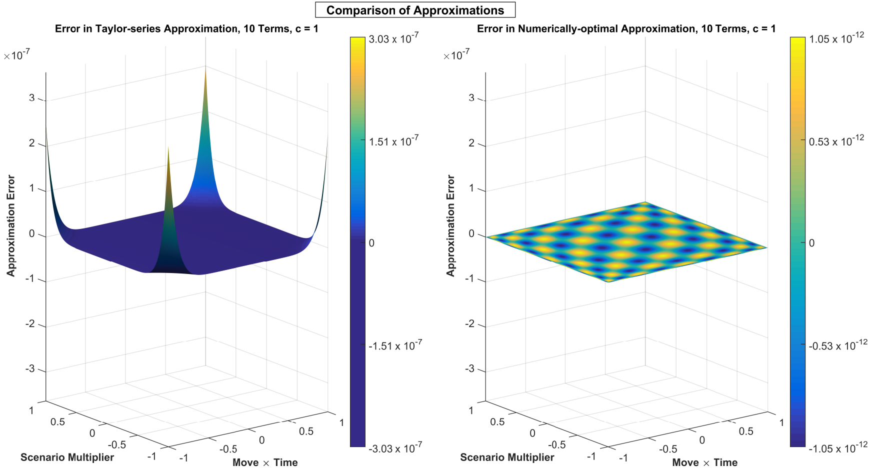

Comparison of Approximations
Error in Taylor-series Approximation, 5 Terms, c = 1
Error in Numerically-optimal Approximation, 5 Terms, c = 1
0.01
0.008
0.006
0.004
0.002
0
-0.002
-0.004
-0.006
-0.008
-0.01
Approximation Error
Scenario Multiplier
Move × Time
0.99 x 10-2
0.50 x 10-2
-0.50 x 10-2
-0.99 x 10-2
0.32 x 10-4
1.62 x 10-5
-1.62 x 10-5
-0.32 x 10-4

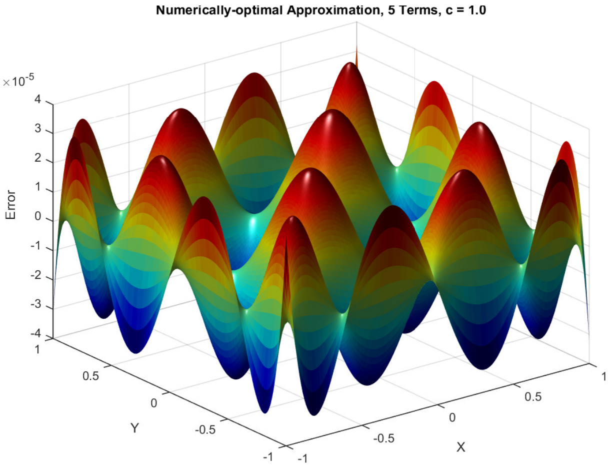

Comparison of Approximations
Error in Taylor-series Approximation, 10 Terms, c = 1
×10^-7
Approximation Error
Scenario Multiplier
Move × Time
3.03 x 10^-7
1.51 x 10^-7
0
-1.51 x 10^-7
-3.03 x 10^-7
Error in Univariate-Remez Approximation, 10 Terms, c = 1
×10^-7
Approximation Error
Scenario Multiplier
Move × Time
0.55 x 10^-9
2.76 x 10^-10
0
-2.76 x 10^-10
-0.55 x 10^-9

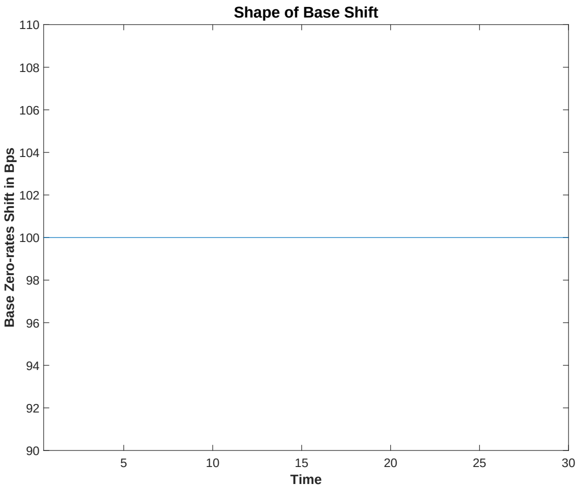

Comparison of Approximations
Error in Taylor-series Approximation, 10 Terms, c = 1
×10^-7
Approximation Error
Scenario Multiplier
Move × Time
3.03 x 10^-7
1.51 x 10^-7
0
-1.51 x 10^-7
-3.03 x 10^-7
Error in Oblate (MSE-optimal) Approximation, 10 Terms, c = 1
0.87 x 10^-11
0.43 x 10^-11
-0.43 x 10^-11
-0.87 x 10^-11

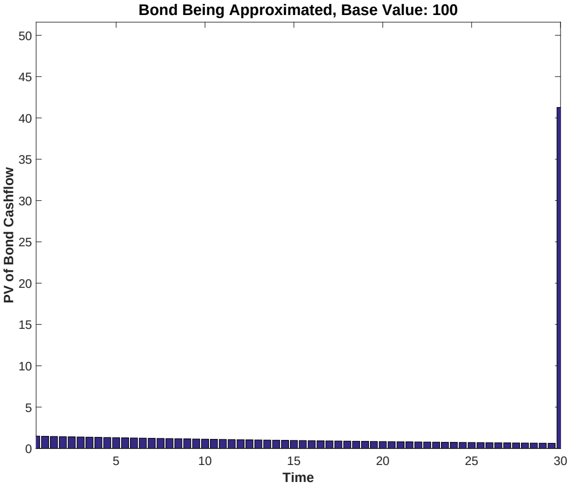

Comparison of Approximations
Error in Taylor-series Approximation, 10 Terms, c = 1
Error in Interp. (MSE-optimal) Approximation, 10 Terms, c = 1
Approximation Error
Scenario Multiplier
Move × Time
×10^-7
3.03 x 10^-7
1.51 x 10^-7
0
-1.51 x 10^-7
-3.03 x 10^-7
0.93 x 10^-11
0.46 x 10^-11
-0.46 x 10^-11
-0.93 x 10^-11

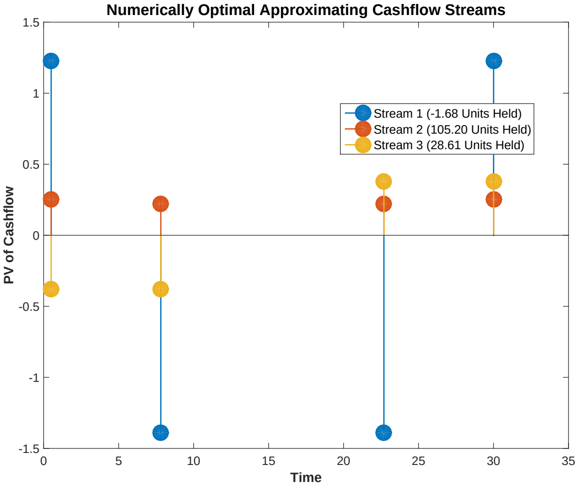

Comparison of Approximations
Error in Taylor-series Approximation, 10 Terms, c = 1
×10^-7
Approximation Error
Scenario Multiplier
Move × Time
3.03 x 10^-7
1.51 x 10^-7
0
-1.51 x 10^-7
-3.03 x 10^-7
Error in Mathieu (Near-optimal) Approximation, 10 Terms, c = 1
1.07 x 10^-12
0.54 x 10^-12
-0.54 x 10^-12
-1.07 x 10^-12

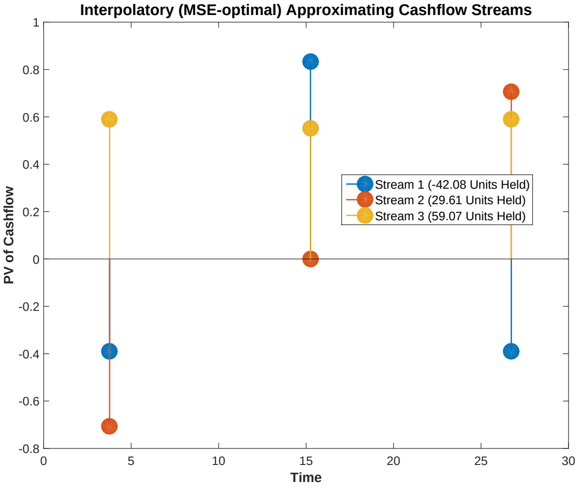

Comparison of Approximations
Error in Taylor-series Approximation, 10 Terms, c = 1
Error in Numerically-optimal Approximation, 10 Terms, c = 1
Approximation Error
Scenario Multiplier
Move × Time
×10^-7
3.03 x 10^-7
1.51 x 10^-7
0
-1.51 x 10^-7
-3.03 x 10^-7
1.05 x 10^-12
0.53 x 10^-12
-0.53 x 10^-12
-1.05 x 10^-12

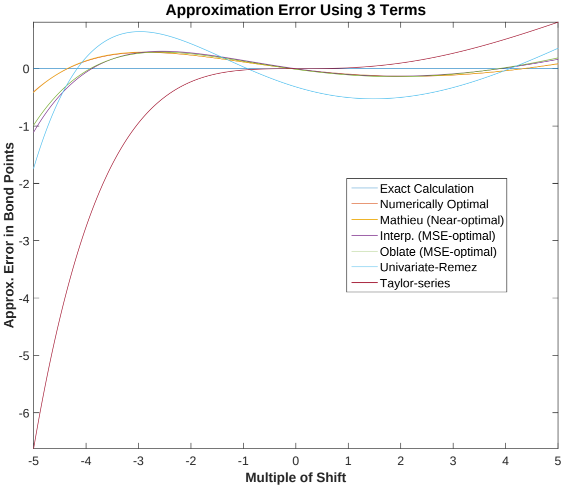

Numerically-optimal Approximation, 5 Terms, c = 1.0
×10^-5
4
3
2
1
0
-1
-2
-3
-4
Error
1
0.5
0
-0.5
-1
Y
-1
-0.5
0
0.5
1
X

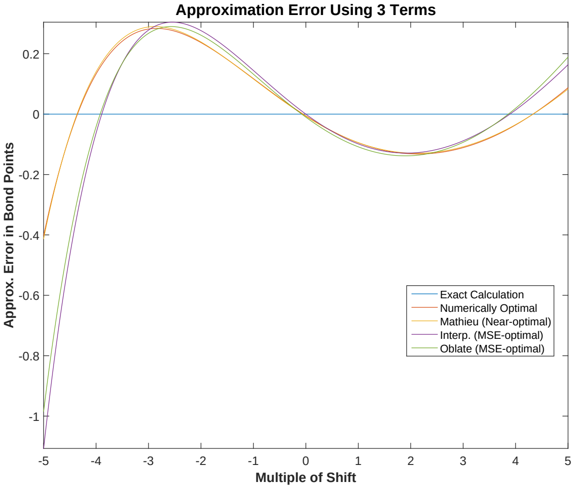

Shape of Base Shift
110
108
106
104
102
100
98
96
94
92
90
Base Zero-rates Shift in Bps
5
10
15
20
25
30
Time

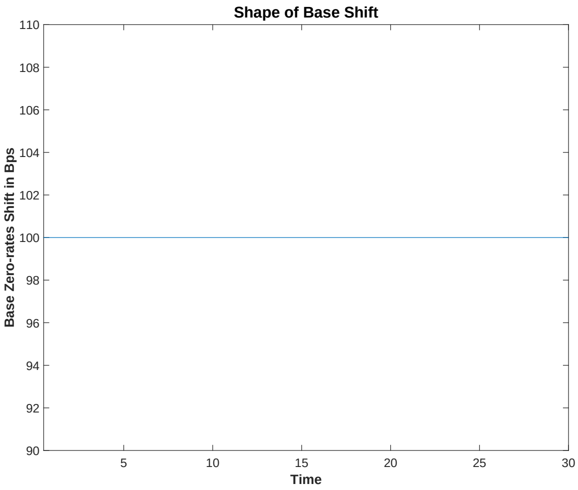

Bond Being Approximated, Base Value: 100
PV of Bond Cashflow
Time
0
5
10
15
20
25
30
35
40
45
50
5
10
15
20
25
30

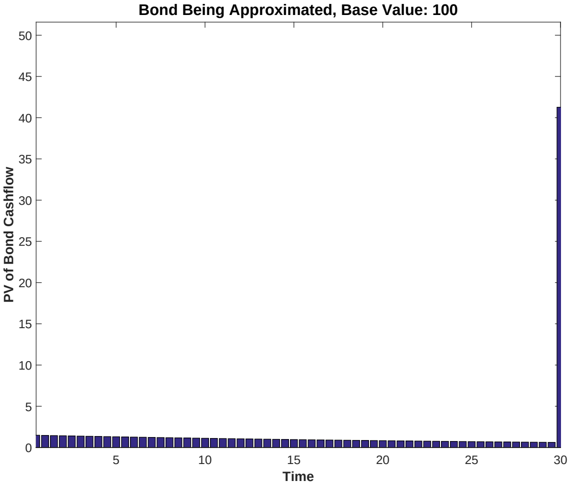

Numerically Optimal Approximating Cashflow Streams
PV of Cashflow
Time
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30
35
Stream 1 (-1.68 Units Held)
Stream 2 (105.20 Units Held)
Stream 3 (28.61 Units Held)

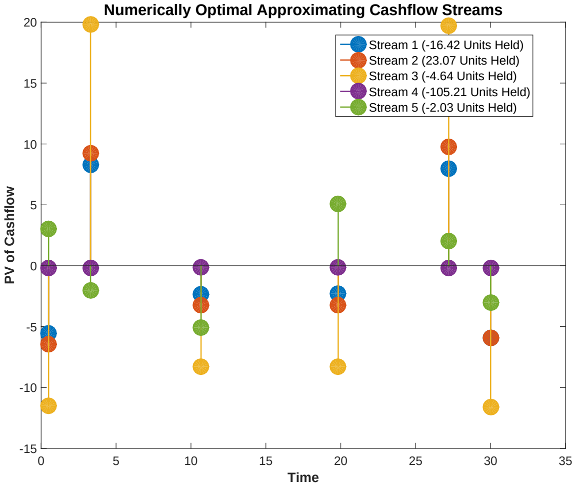

Interpolatory (MSE-optimal) Approximating Cashflow Streams
PV of Cashflow
Time
Stream 1 (-42.08 Units Held)
Stream 2 (29.61 Units Held)
Stream 3 (59.07 Units Held)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
5
10
15
20
25
30

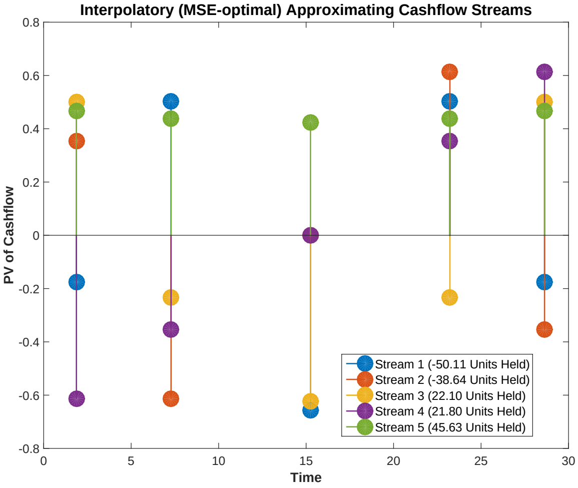

Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
0
-1
-2
-3
-4
-5
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)
Univariate-Remez
Taylor-series

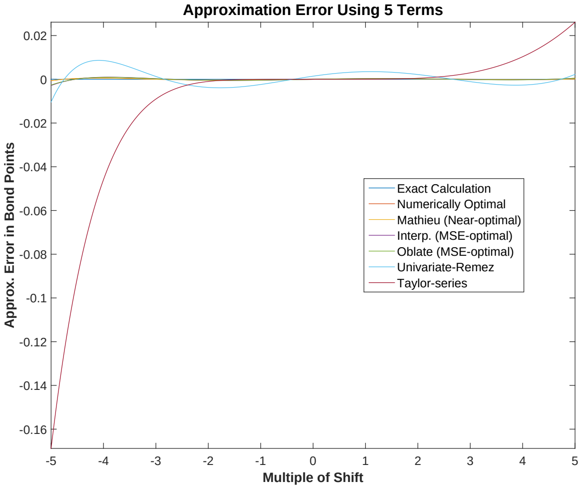

Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)

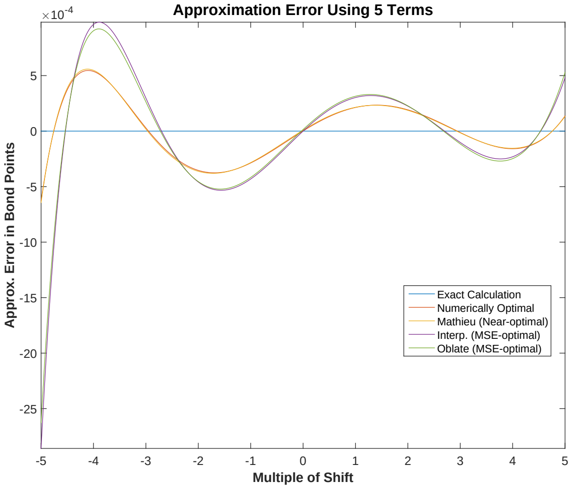

Shape of Base Shift
Base Zero-rates Shift in Bps
110
108
106
104
102
100
98
96
94
92
90
5
10
15
20
25
30
Time


Bond Being Approximated, Base Value: 100
PV of Bond Cashflow
Time
0
5
10
15
20
25
30
35
40
45
50
5
10
15
20
25
30


Numerically Optimal Approximating Cashflow Streams
Stream 1 (-16.42 Units Held)
Stream 2 (23.07 Units Held)
Stream 3 (-4.64 Units Held)
Stream 4 (-105.21 Units Held)
Stream 5 (-2.03 Units Held)
PV of Cashflow
Time
20
15
10
5
0
-5
-10
-15
0
5
10
15
20
25
30
35


Interpolatory (MSE-optimal) Approximating Cashflow Streams
PV of Cashflow
Time
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
5
10
15
20
25
30
Stream 1 (-50.11 Units Held)
Stream 2 (-38.64 Units Held)
Stream 3 (22.10 Units Held)
Stream 4 (21.80 Units Held)
Stream 5 (45.63 Units Held)


Approximation Error Using 5 Terms
Approx. Error in Bond Points
Multiple of Shift
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
-0.12
-0.14
-0.16
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)
Univariate-Remez
Taylor-series


Approximation Error Using 5 Terms
×10^-4
5
0
-5
-10
-15
-20
-25
Approx. Error in Bond Points
-5
-4
-3
-2
-1
0
1
2
3
4
5
Multiple of Shift
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)

n-Widths

- *n*-width theory measures how well *n*-dimensional sets can approximate infinite-dimensional sets (Kolmogorov (1936), Pinkus (1985))
- Prior to my work, applications were not financial: splines, signal processing, statistics
- I consider sets of scenario profiles generated by cashflow streams with gross present value of one
- Use the theory of *n*-widths to study optimal approximations to these sets

Total Positivity

- Totally positive functions have variation-diminishing properties when used to weight averages
- Applications in approximation problems, differential equations, statistics, and mechanism design
- Theory of n-widths for totally positive kernels provides important building blocks (reviewed in Pinkus (1985))
- Examples: $\exp(cxy)$ (the exponential product kernel) on $\mathbb{R}^2$, $\frac{1}{x+y}$ on $[0, \infty) \times (0, \infty)$, and $\exp\left(-(x-y)^2\right)$ on $\mathbb{R}^2$

Chebyshev Systems

- $n + 1$ functions form a Chebyshev system if any nontrivial linear combination of the functions has $\leq n$ zeros
- Must always be functions of one variable, effectively: Mairhuber (1956)
- The approximation from a Chebyshev system that minimizes worst-case error is unique and its approximation error equioscillates
- Examples: polynomials, exponentials, trigonometric functions, eigenfunctions of some differential operators
- The Remez method works for these systems and shows how to compute best approximations

A Topological Result and A Lower Bound

- The Borsuk Antipodality Theorem (Borsuk (1933))
- Classic n-width result (Tikhomirov (1960)) uses this to lower-bound approximation errors
- I show that any list of $n+1$ numbers $\rho_i \in [-1, 1]$ provides the lower bound on worst-case approximation error of

$$\min_{\boldsymbol{a}: \sum_{i=1}^{n+1} |a_i| = 1} \max_{y \in [-1,1]} \left| \sum_{i=1}^{n+1} a_i \exp\left(c \rho_i y\right) \right| \qquad (3)$$

for the exponential product operator $\exp(cxy)$, $x, y \in [-1, 1]$.

Finding the Largest Lower Bound

- I develop a nontrivial variant of the Remez method to solve (3)
- Then I use this variant as a building block in a fixed-point procedure that, empirically, finds the points ρ that give the largest lower bound of this type

Sample Output: Fixed Point Procedure I

» [wRhoVector, wCoeffs, wAmatrix, wNodes, wErrorsAtNodes, wDiscrepancy, wNumIter] = worstRhoForExp(3, 1 , 1e-15 , 100 , true)

wRhoVector =

-1 0 1

wCoeffs =

-0.989586650561472 0.000000000000001
0.143938393177520

wAmatrix =

0.596243191512492 -1.669693283914562
0.596243191512493 -0.461226118145786
0.000000000000000 0.461226118145786

Sample Output: Fixed Point Procedure II

-0.323775931284113 -0.231239110157009
-0.323775931284113
wNodes =
-1.000000000000000 0.000000000000000
1.000000000000000
wErrorsAtNodes =
-0.345745818123157 0.345745818123157
-0.345745818123156
wDiscrepancy =
0
wNumIter =
1

Sample Output: Computing the Lower Bound I

```
» [ bCoeffs , bNodes , bErrorsAtNodes , bDiscrepancy ,
bNumIter] = borsukLowerBound(wRhoVector , 1 , 1e-15 , 100)
bCoeffs =
-0.220114957312587 0.559770085374825
-0.220114957312587
bNodes =
-1 0 1
bErrorsAtNodes =
-0.119540170749650 0.119540170749650
-0.119540170749650
bDiscrepancy =
0
```

Sample Output: Computing the Lower Bound II

bNumIter =
0

Example of Numerical Results for the Lower Bound

	c	n	Lower Bound
	0.900	1	1.02651672570817 5
	0.900	2	0.09769410018048 4
	0.900	3	0.00750061556494 3
	0.900	4	0.00042429071143 4
	0.900	5	0.00001914082408 5
	0.900	6	0.00000071872854 8
	0.900	7	0.00000002312007 5
	0.900	8	0.00000000065057 9
	0.900	9	0.00000000001627 0
	0.900	10	0.00000000000036 6
	0.900	11	0.00000000000000 7
	0.900	12	0.00000000000000 0

Numerically-optimal Approximation: Motivation

- Speculate: the optimal rank-n linear approximation may also be optimal in approximating a rank-$(n+1)$ operator
- Imagine a game: market plays a grid with $n+1$ values of x and $n+1$ values of y
- What is the best rank-n response?
- Hope that the Bernstein, Gel'fand, Kolmogorov, and linear n-widths are identical
- Apply the results of Micchelli & Pinkus (1979)
- Upper bound on approximation error using interval analysis (computationally intensive)

Numerically-optimal Approximation: Construction

To construct $P^*_{c,n}(x,y)$ I find very particular:

- $\rho^\dagger$, a $(n+1)\times 1$ vector (from fixed-point procedure)
- a $(n+1)\times n$ matrix V^*

Then

$$P^*_{c,n}(x,y) = \exp\left(cx\rho^\dagger\right)^T V^* {V^*}^T \exp\left(c\rho^\dagger y\right) \quad (4)$$

$$= \sum_{i=1}^{n} f_i(x)\, f_i(y)\,, \quad (5)$$

where $f_i(x) = \sum_{j=1}^{n+1} V^*_{ji} \exp\left(cx\rho^\dagger_j\right)$.

Numerically-optimal Approximation: Interpretation

- The numerically-optimal n-dim'l. approximating space is spanned by n simple financial instruments
- Each of these instruments is a cashflow stream
- All n of these streams have their cashflows on a common set of $n+1$ times
- The approximating portfolio is linear in the cashflow stream

Results of Interval Analysis

	c	n	Lower Bound	Numerically-optimal
	0.900	1	1.026516725708175	1.026516725708176
	0.900	2	0.097694100180484	0.097694100180486
	0.900	3	0.007500615564943	0.007500615564952
	0.900	4	0.000424290711434	0.000424290711437
	0.900	5	0.000019140824085	0.000019140824093
	0.900	6	0.000000718728548	0.000000718728562
	0.900	7	0.000000023120075	0.000000023120094
	0.900	8	0.000000000650579	0.000000000650595
	0.900	9	0.000000000016270	0.000000000016291
	0.900	10	0.000000000000366	0.000000000000377
	0.900	11	0.000000000000007	0.000000000000012
	0.900	12	0.000000000000000	0.000000000000007

Interpolatory Approximation: Motivation

- Optimal approximations are generally not unique
- Eigenfunction expansions can be truncated to obtain optimal approximations for average squared error
- The eigenfunctions of P_c are the oblates (oblate spheroidal wave functions of order zero)
- Melkman & Micchelli (1978) showed that totally positive kernels also have interpolatory approximations that are optimal for average squared error
- Interpolate at the n zeros of the $(n+1)^{\text{st}}$ eigenfunction

Interpolatory Approximation: Construction

$$P_{c,n}^{\text{Interp}}(x, y) = \begin{pmatrix} \exp(cx\eta_1) \\ \vdots \\ \exp(cx\eta_n) \end{pmatrix}^T U^T \Lambda^{-1} U \begin{pmatrix} \exp(c\eta_1 y) \\ \vdots \\ \exp(c\eta_n y) \end{pmatrix}$$

where the η_i are the zeros of the $(n+1)^{\text{st}}$ oblate (it has index n, since indexing starts with zero for the oblates) and

$$\begin{pmatrix} \exp(c\eta_1\eta_1) & \cdots & \exp(c\eta_1\eta_n) \\ \vdots & \ddots & \vdots \\ \exp(c\eta_n\eta_1) & \cdots & \exp(c\eta_n\eta_n) \end{pmatrix} = U^T \Lambda U$$

Interpolatory Approximation: Interpretation

- The rank-n interpolatory approximation uses a portfolio of a fixed set of n approximating cashflow streams
- All n approximating cashflow streams have cashflows at the same n times
- The approximating portfolio is linear in the target cashflow stream
- Different cashflow streams have distinct approximating holdings of the same n approximating cashflow streams

Mathieu Approximation: Motivation

- In one-variable problems, there is a weighting that gives excellent polynomial approximations (in worst-case error)
- These are Chebyshev polynomial approximations (Trefethen (2013))
- Will a similar result hold for my two-variable problem?
- I show and exploit: Mathieu differential operator commutes with exponential cosine product operator

The Mathieu Operator

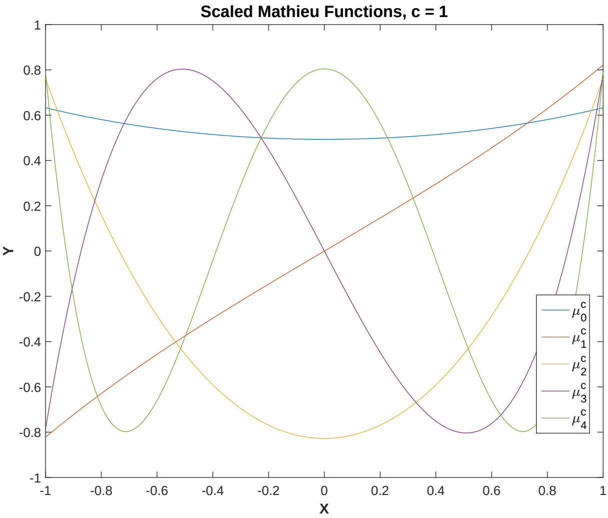
$$M_c[f](t) \equiv -\frac{d^2 f}{dt^2}(t) - \frac{c^2}{2}\cos(2t) f(t), \tag{6}$$

though its standard form is written in terms of $q = -\frac{c^2}{4}$.

- The eigenfunctions of this differential operator are the Mathieu functions
- They are special functions of classical mathematical physics

Mathieu Approximation: Construction

- Expand Mathieu functions in series of Chebyshev polynomials (minor variation on traditional Fourier series)
- Result is two symmetric tridiagonal eigenproblems for series coefficients (eigenvectors)
- Solve these by truncation (as in Zhang & Jin (1996))
- Use eigenfunction identity to get the eigenvalues for the weighted exponential product operator (new)


Scaled Mathieu Functions, c = 1
Y
X
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
μ_0^c
μ_1^c
μ_2^c
μ_3^c
μ_4^c

Oblate Approximation: Motivation

- Eigenfunction expansions can be truncated to give average-squared-error optimal approximations
- What are the eigenfunctions of the exponential product operators?
- Use a commuting differential operator, the oblate spheroidal wave operator of order zero (Zayed (2007) notes this briefly)
- The eigenfunctions are the oblate spheroidal wave functions of order zero (oblates)

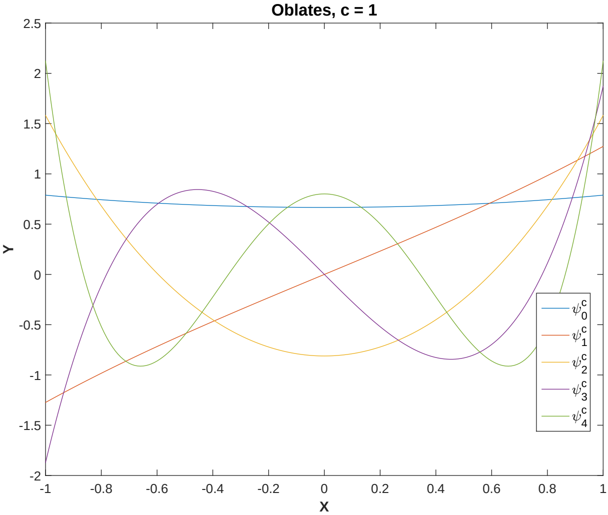
The Oblate Spheroidal Wave Operator

$$L_c[f](x) \equiv -\frac{d}{dx}\left(\left(1-x^2\right)\frac{df}{dx}(x)\right) - c^2 x^2 f(x) \tag{7}$$

- The eigenfunctions of this differential operator are the oblates (full name: oblate spheroidal wave functions of order zero)
- They are special functions of classical mathematical physics

Oblate Approximation: Construction

- Expand the oblates in series of Legendre polynomials (standard approach)
- Result is two symmetric tridiagonal eigenproblems for series coefficients (eigenvectors)
- Solve these by truncation (as in Hodge (1970))
- Use eigenfunction identity to get the eigenvalues for the exponential product operator (as in prolate case, though new here)

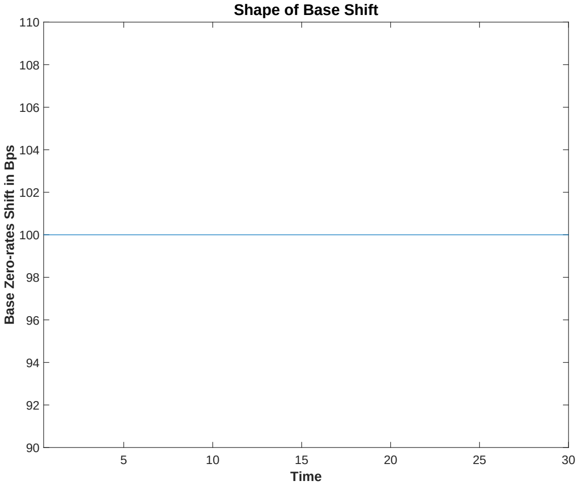

Oblates, c = 1
Y
X
ψ_0^c
ψ_1^c
ψ_2^c
ψ_3^c
ψ_4^c
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

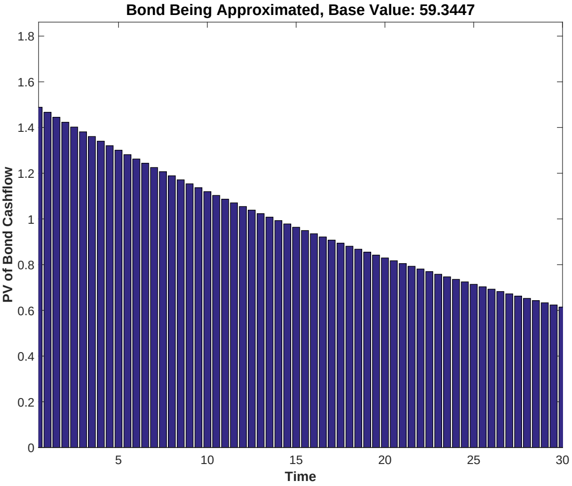
Comparisons of Approximations: Worst-case Error

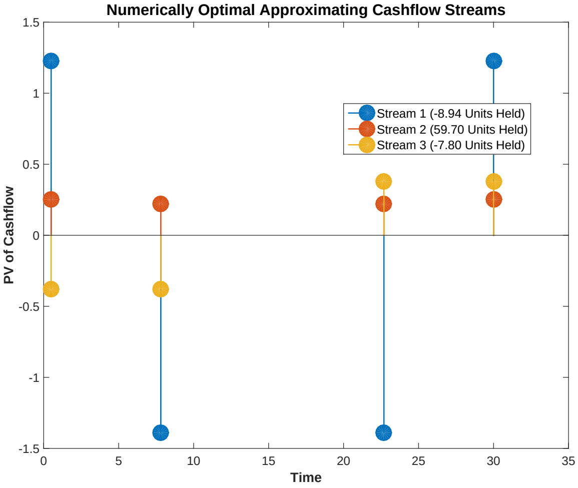
c	n	Num.-optimal	Mathieu	Taylor-series
0.900	1	1.026516725708176	1.142627510037619	1.459603111156950
0.900	2	0.097694100180486	0.123445382124493	0.559603111156950
0.900	3	0.007500615564952	0.008533145878606	0.154603111156950
0.900	4	0.000424290711437	0.000468349741963	0.033103111156950
0.900	5	0.000019140824093	0.000020725649064	0.005765611156950
0.900	6	0.000000718728562	0.000000768395225	0.000844861156950
0.900	7	0.000000023120094	0.000000024492435	0.000106748656950
0.900	8	0.000000000650595	0.000000000684441	0.000011848478378
0.900	9	0.000000000016291	0.000000000017024	0.000001172208289
0.900	10	0.000000000000377	0.000000000000381	0.000000104581280
0.900	11	0.000000000000012	0.000000000000008	0.000000008494849
0.900	12	0.000000000000007	0.000000000000000	0.000000000633232

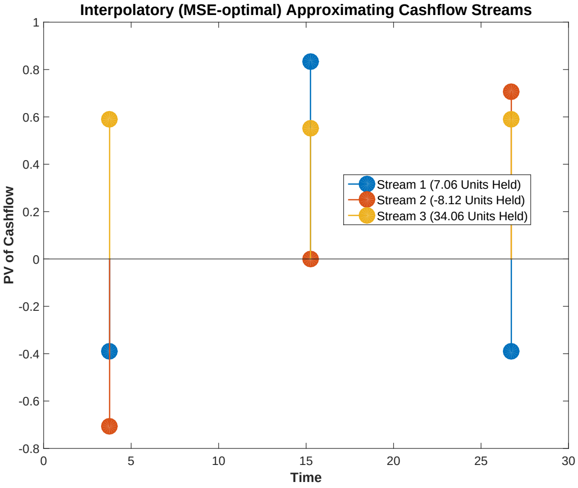
Comparisons of Approximations: RMSE

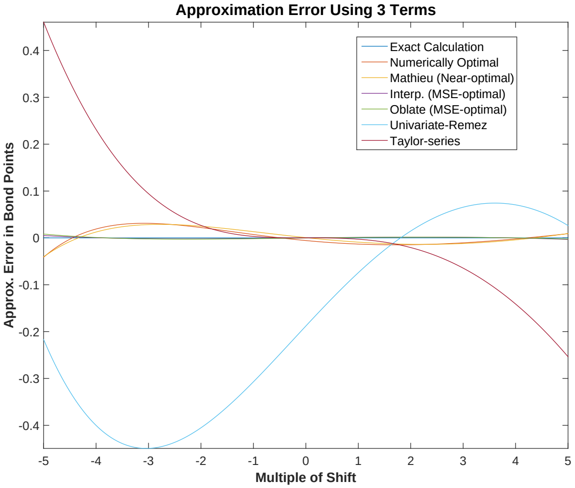
c	n	Interpolatory	Oblate	Taylor-series
0.900	1	0.630046304453232	0.630046304453232	0.630046304453232
0.900	2	0.072215718962917	0.072215718962917	0.167688553451906
0.900	3	0.005560981010366	0.005560981010366	0.035576959630715
0.900	4	0.000317559230476	0.000317559230476	0.006186015811949
0.900	5	0.000014430879283	0.000014430879283	0.000907144882218
0.900	6	0.000000544878914	0.000000544878914	0.000114798404740
0.900	7	0.000000017602704	0.000000017602704	0.000012764853380
0.900	8	0.000000000497000	0.000000000497000	0.000001265127833
0.900	9	0.000000000012463	0.000000000012463	0.000000113062188
0.900	10	0.000000000000281	0.000000000000281	0.000000009198080
0.900	11	0.000000000000006	0.000000000000006	0.000000000686633
0.900	12	0.000000000000002	0.000000000000001	0.000000000047350

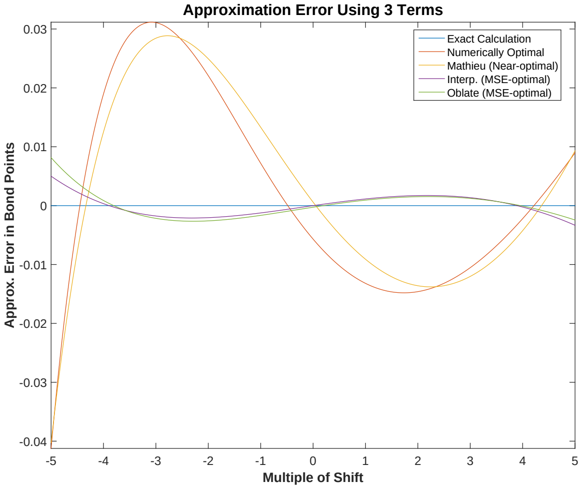

Shape of Base Shift
Base Zero-rates Shift in Bps
110
108
106
104
102
100
98
96
94
92
90
5
10
15
20
25
30
Time

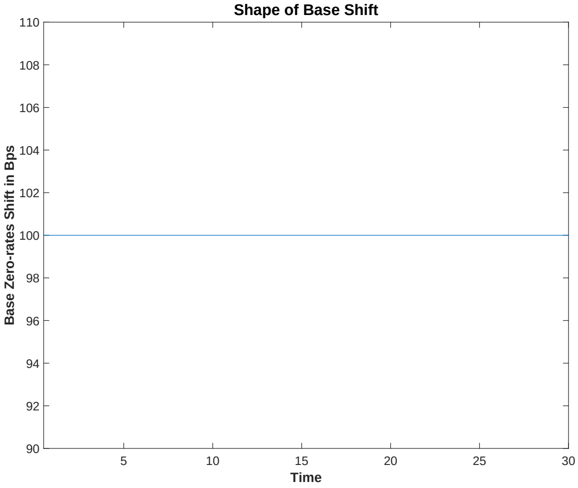

Bond Being Approximated, Base Value: 59.3447
PV of Bond Cashflow
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
5
10
15
20
25
30
Time

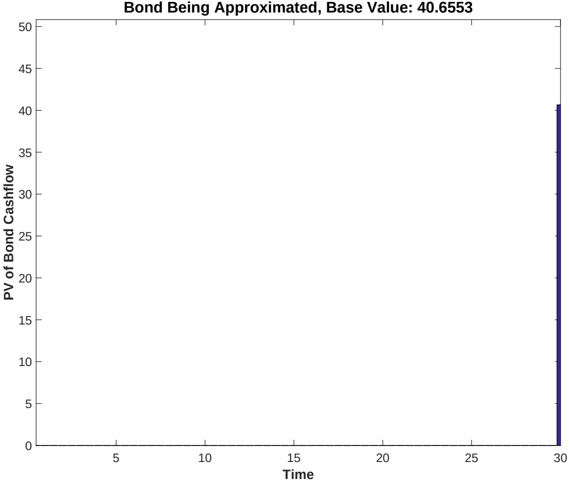

Numerically Optimal Approximating Cashflow Streams
PV of Cashflow
Time
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30
35
Stream 1 (-8.94 Units Held)
Stream 2 (59.70 Units Held)
Stream 3 (-7.80 Units Held)

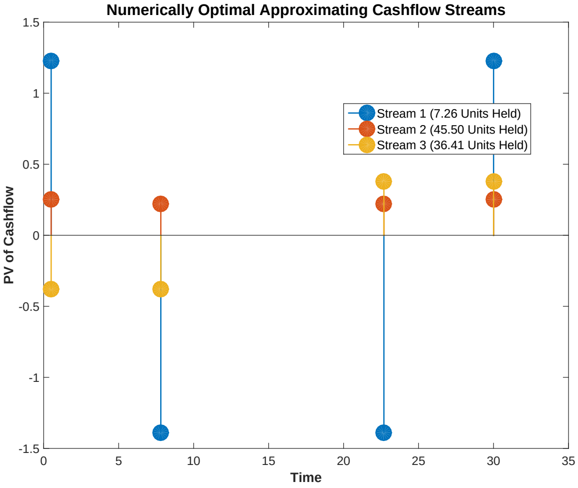

Interpolatory (MSE-optimal) Approximating Cashflow Streams
PV of Cashflow
Time
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
5
10
15
20
25
30
Stream 1 (7.06 Units Held)
Stream 2 (-8.12 Units Held)
Stream 3 (34.06 Units Held)

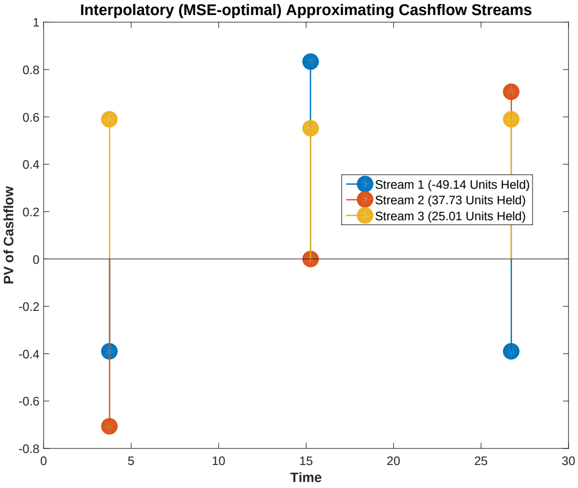

Approximation Error Using 3 Terms
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)
Univariate-Remez
Taylor-series
Approx. Error in Bond Points
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-5
-4
-3
-2
-1
0
1
2
3
4
5
Multiple of Shift

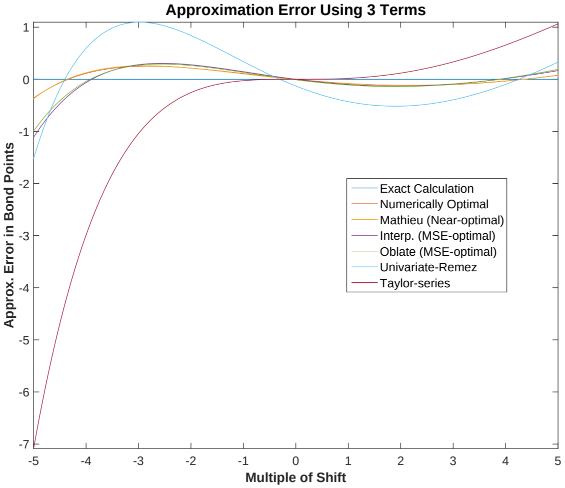

Approximation Error Using 3 Terms
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)
Approx. Error in Bond Points
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-5
-4
-3
-2
-1
0
1
2
3
4
5
Multiple of Shift

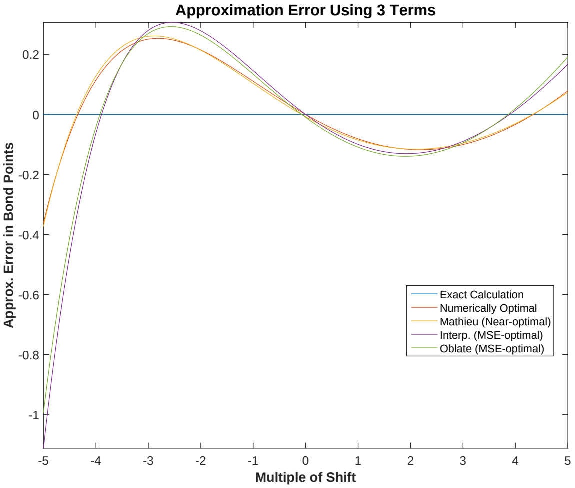

Shape of Base Shift
Base Zero-rates Shift in Bps
110
108
106
104
102
100
98
96
94
92
90
5
10
15
20
25
30
Time

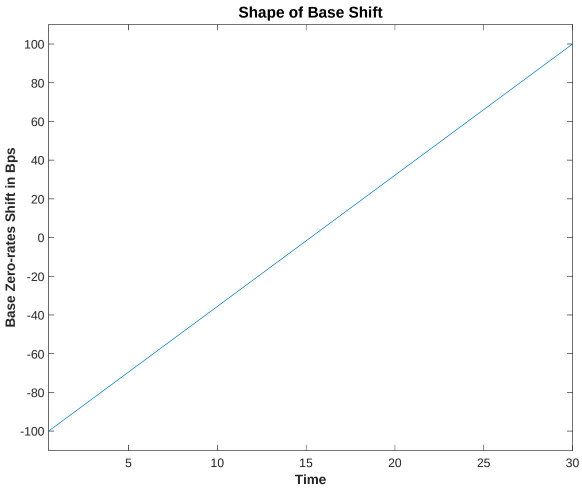

Bond Being Approximated, Base Value: 40.6553
PV of Bond Cashflow
Time
0
5
10
15
20
25
30
35
40
45
50
5
10
15
20
25
30

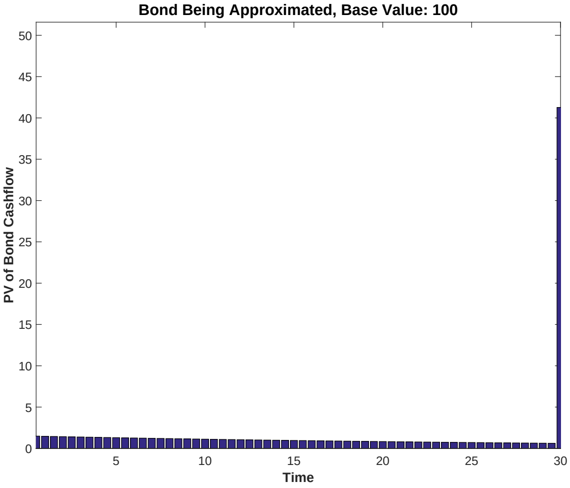

Numerically Optimal Approximating Cashflow Streams
PV of Cashflow
Time
Stream 1 (7.26 Units Held)
Stream 2 (45.50 Units Held)
Stream 3 (36.41 Units Held)
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
20
25
30
35

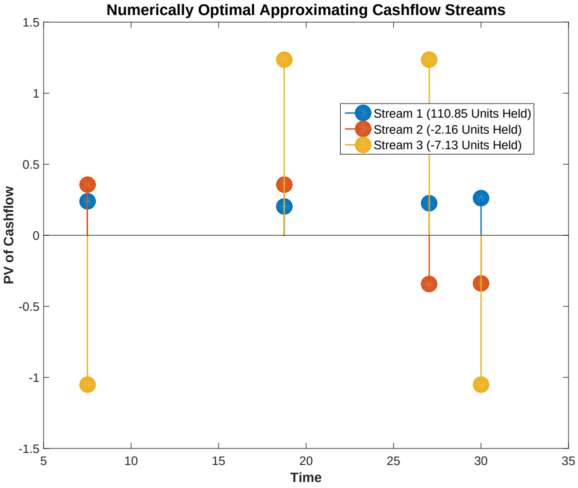

Interpolatory (MSE-optimal) Approximating Cashflow Streams
PV of Cashflow
Time
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
5
10
15
20
25
30
Stream 1 (-49.14 Units Held)
Stream 2 (37.73 Units Held)
Stream 3 (25.01 Units Held)

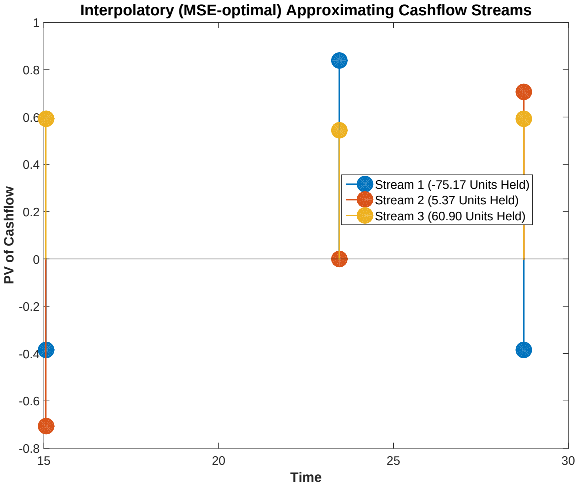

Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
1
0
-1
-2
-3
-4
-5
-6
-7
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)
Univariate-Remez
Taylor-series

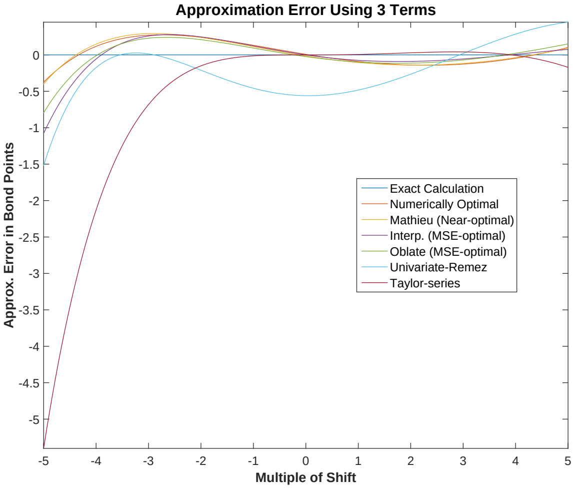

Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)

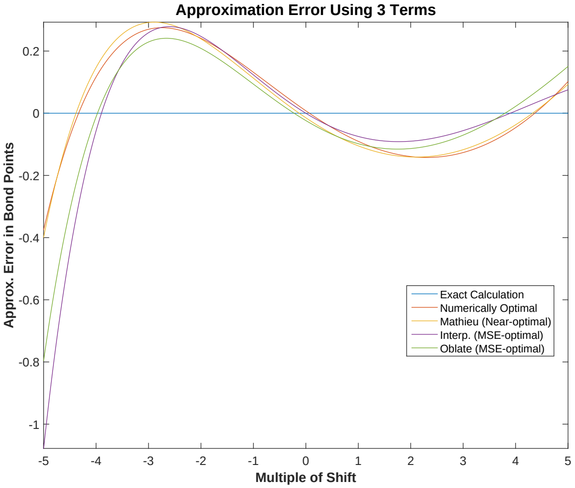

Shape of Base Shift
Base Zero-rates Shift in Bps
Time
100
80
60
40
20
0
-20
-40
-60
-80
-100
5
10
15
20
25
30

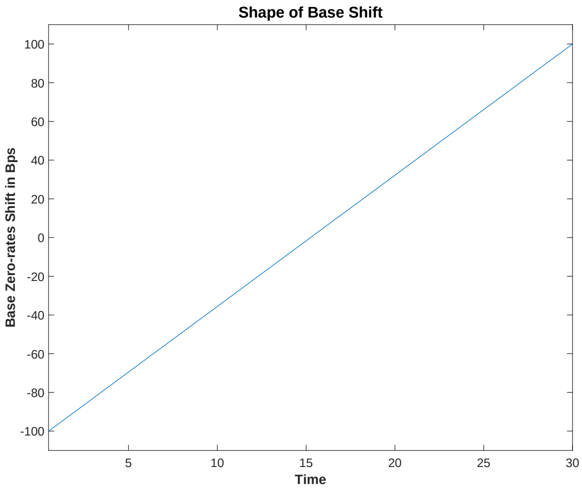

Bond Being Approximated, Base Value: 100
PV of Bond Cashflow
Time
0
5
10
15
20
25
30
35
40
45
50
5
10
15
20
25
30

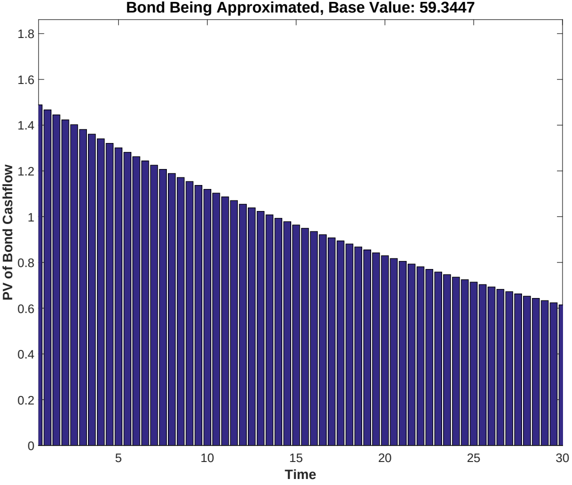

Numerically Optimal Approximating Cashflow Streams
1.5
1
0.5
0
-0.5
-1
-1.5
PV of Cashflow
5
10
15
20
25
30
35
Time
Stream 1 (110.85 Units Held)
Stream 2 (-2.16 Units Held)
Stream 3 (-7.13 Units Held)

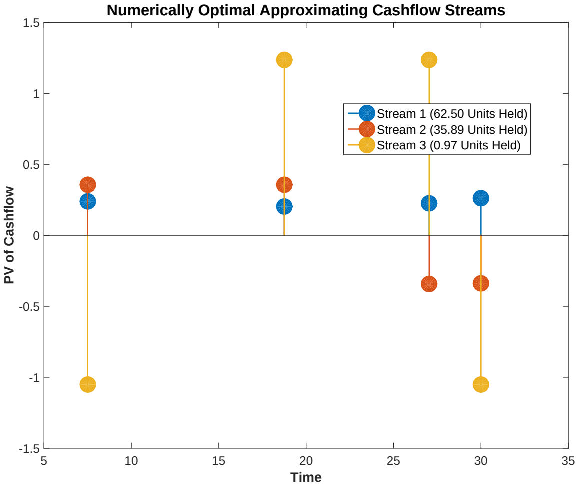

Interpolatory (MSE-optimal) Approximating Cashflow Streams
PV of Cashflow
Time
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
15
20
25
30
Stream 1 (-75.17 Units Held)
Stream 2 (5.37 Units Held)
Stream 3 (60.90 Units Held)

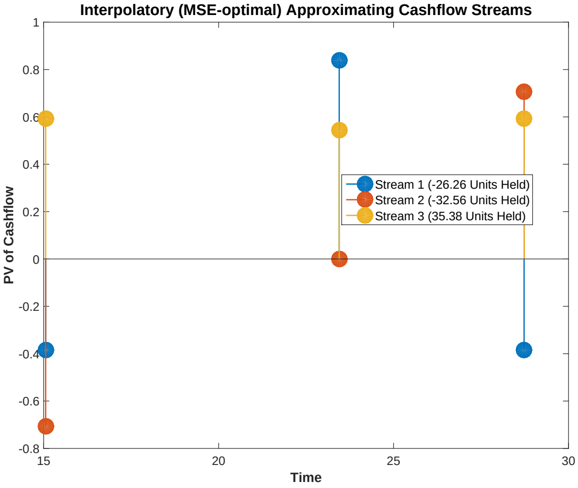

Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)
Univariate-Remez
Taylor-series

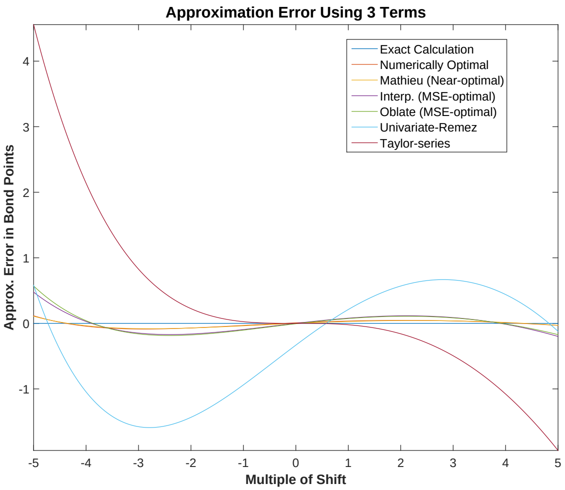

Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)

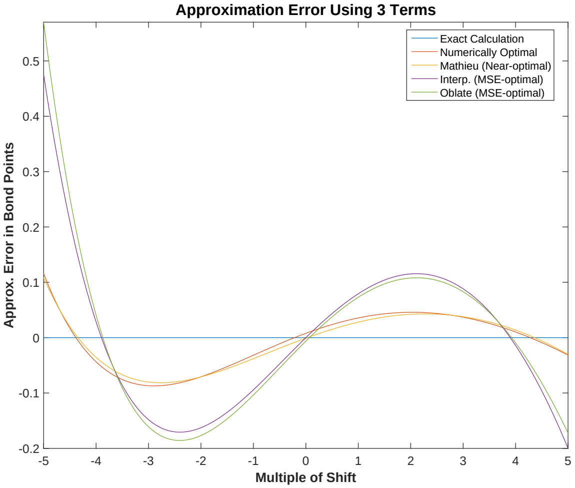

Shape of Base Shift
Base Zero-rates Shift in Bps
100
80
60
40
20
0
-20
-40
-60
-80
-100
5
10
15
20
25
30
Time

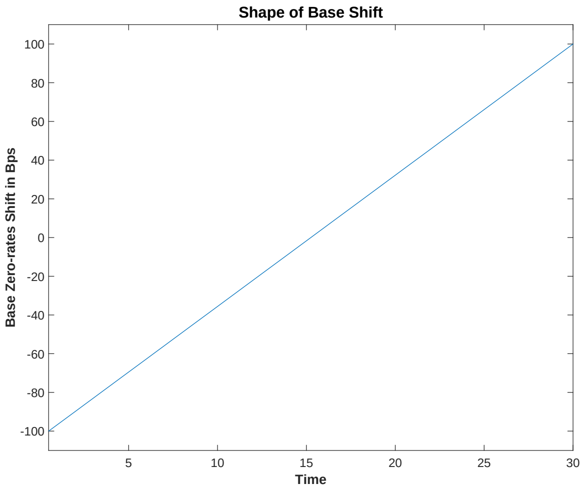

Bond Being Approximated, Base Value: 59.3447
PV of Bond Cashflow
Time
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
5
10
15
20
25
30

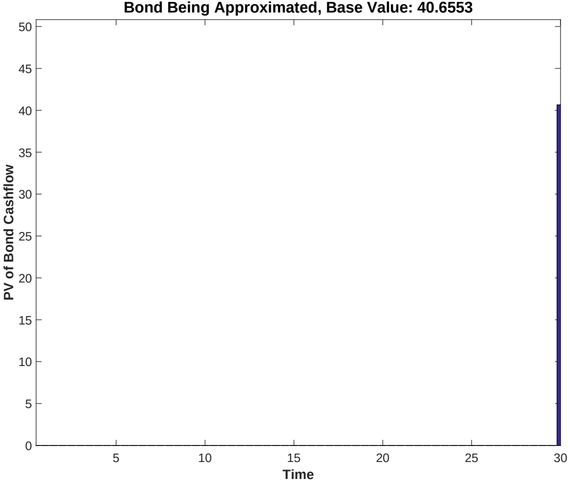

Numerically Optimal Approximating Cashflow Streams
Stream 1 (62.50 Units Held)
Stream 2 (35.89 Units Held)
Stream 3 (0.97 Units Held)
PV of Cashflow
Time
1.5
1
0.5
0
-0.5
-1
-1.5
5
10
15
20
25
30
35

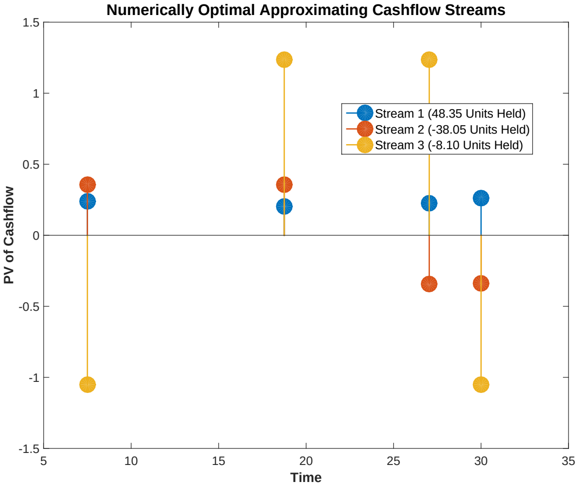

Interpolatory (MSE-optimal) Approximating Cashflow Streams
PV of Cashflow
Time
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
15
20
25
30
Stream 1 (-26.26 Units Held)
Stream 2 (-32.56 Units Held)
Stream 3 (35.38 Units Held)

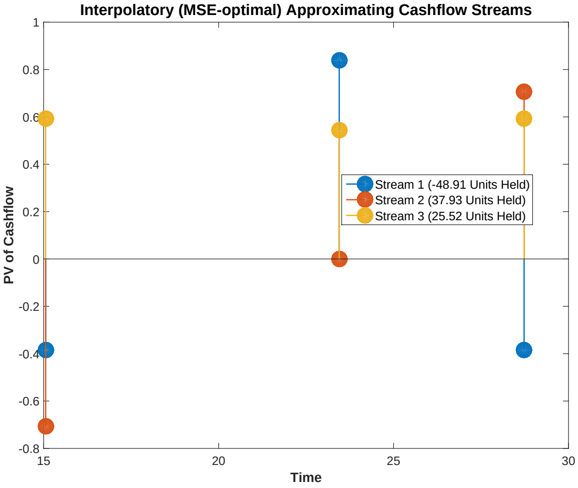

Approximation Error Using 3 Terms
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)
Univariate-Remez
Taylor-series
Approx. Error in Bond Points
Multiple of Shift
4
3
2
1
0
-1
-5
-4
-3
-2
-1
0
1
2
3
4
5

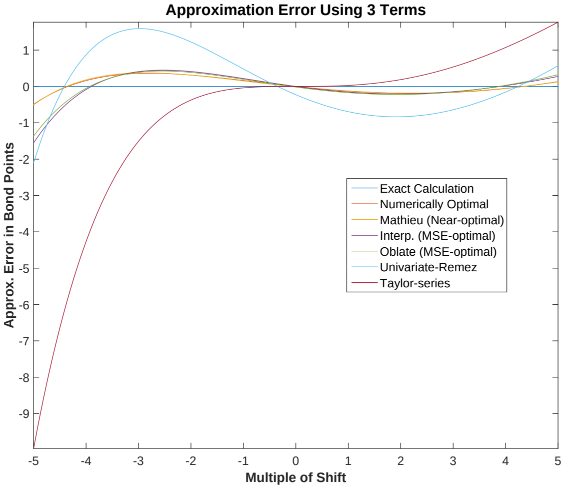

Approximation Error Using 3 Terms
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)
Approx. Error in Bond Points
Multiple of Shift
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-5
-4
-3
-2
-1
0
1
2
3
4
5

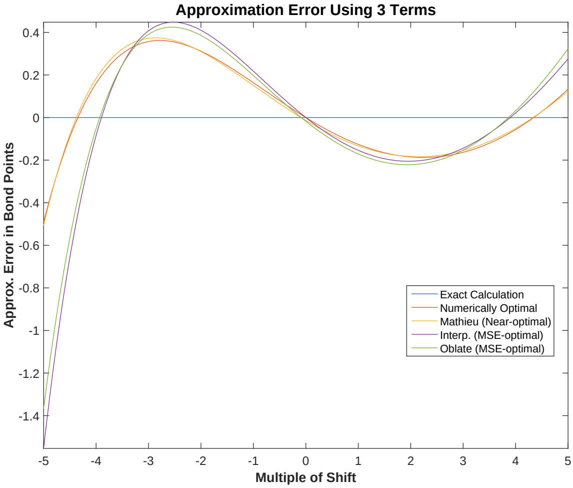

Shape of Base Shift
Base Zero-rates Shift in Bps
Time
100
80
60
40
20
0
-20
-40
-60
-80
-100
5
10
15
20
25
30

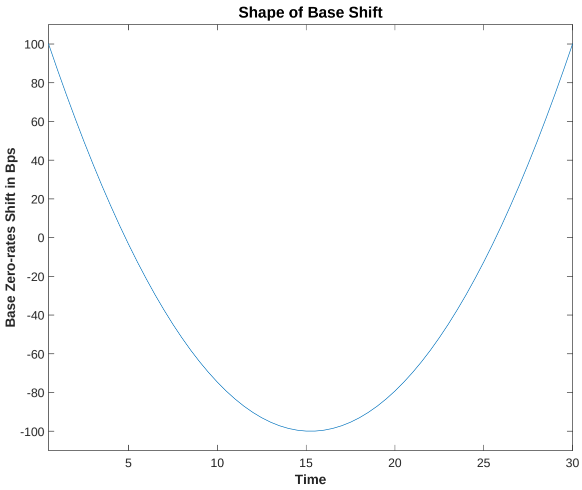

Bond Being Approximated, Base Value: 40.6553
PV of Bond Cashflow
Time
0
5
10
15
20
25
30
35
40
45
50
5
10
15
20
25
30

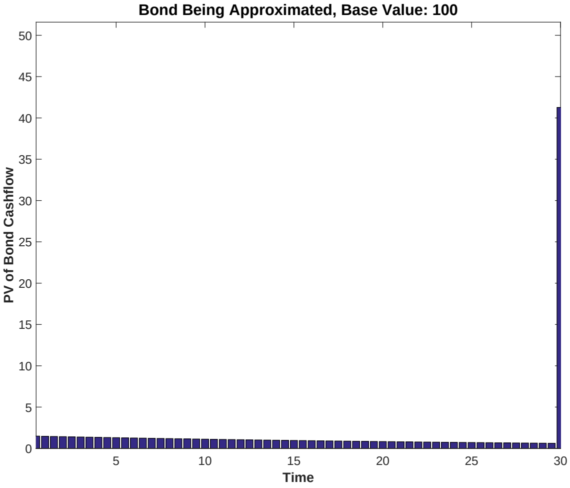

Numerically Optimal Approximating Cashflow Streams
1.5
1
0.5
0
-0.5
-1
-1.5
PV of Cashflow
5
10
15
20
25
30
35
Time
Stream 1 (48.35 Units Held)
Stream 2 (-38.05 Units Held)
Stream 3 (-8.10 Units Held)

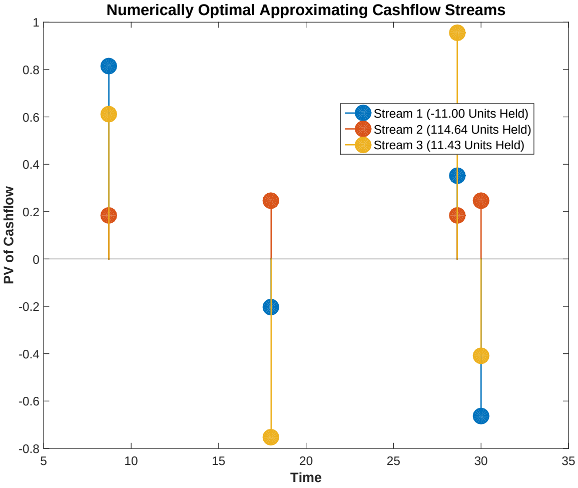

Interpolatory (MSE-optimal) Approximating Cashflow Streams
PV of Cashflow
Time
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
15
20
25
30
Stream 1 (-48.91 Units Held)
Stream 2 (37.93 Units Held)
Stream 3 (25.52 Units Held)

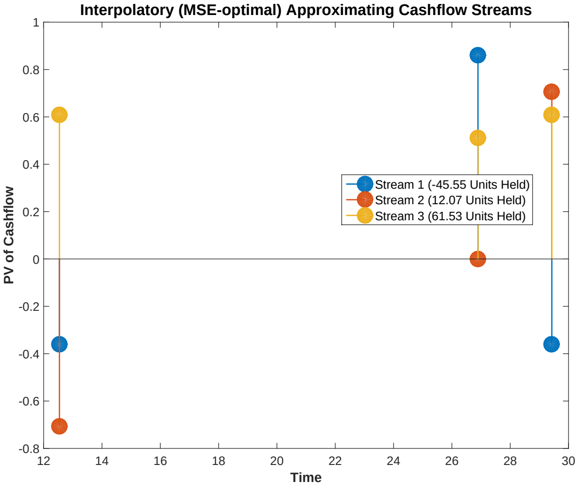

Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)
Univariate-Remez
Taylor-series

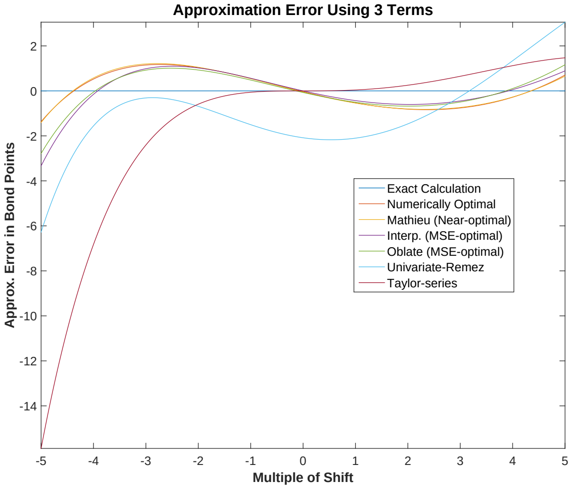

Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)

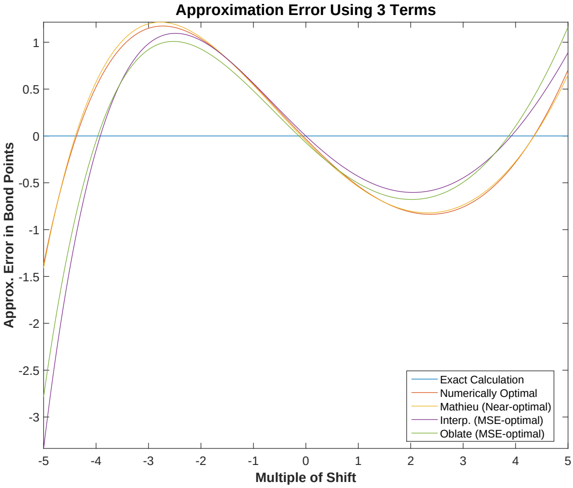

Shape of Base Shift
Base Zero-rates Shift in Bps
100
80
60
40
20
0
-20
-40
-60
-80
-100
5
10
15
20
25
30
Time

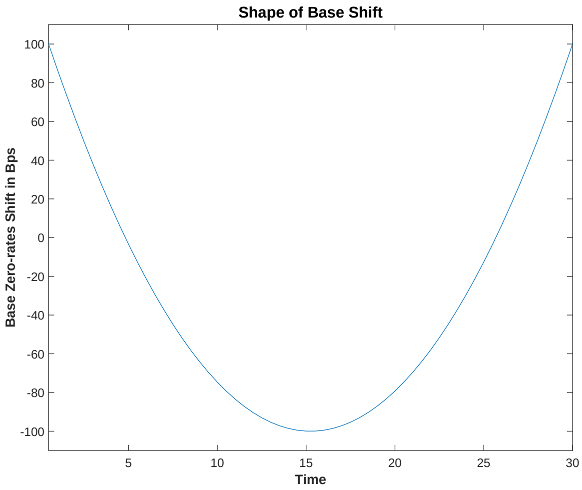

Bond Being Approximated, Base Value: 100
PV of Bond Cashflow
Time
0
5
10
15
20
25
30
35
40
45
50

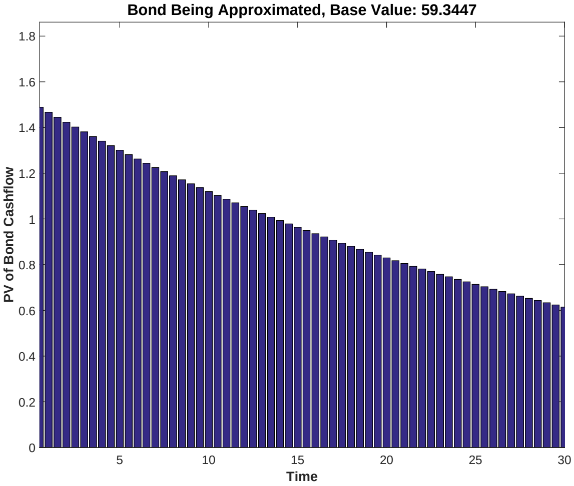

Numerically Optimal Approximating Cashflow Streams
Stream 1 (-11.00 Units Held)
Stream 2 (114.64 Units Held)
Stream 3 (11.43 Units Held)
PV of Cashflow
Time
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
5
10
15
20
25
30
35

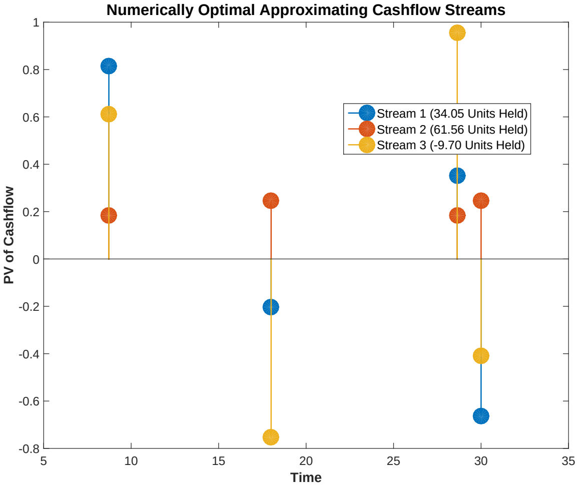

Interpolatory (MSE-optimal) Approximating Cashflow Streams
PV of Cashflow
Time
Stream 1 (-45.55 Units Held)
Stream 2 (12.07 Units Held)
Stream 3 (61.53 Units Held)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
12
14
16
18
20
22
24
26
28
30

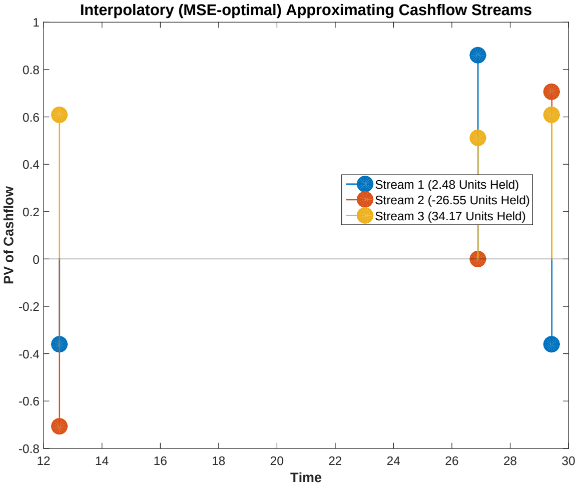

Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
2
0
-2
-4
-6
-8
-10
-12
-14
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)
Univariate-Remez
Taylor-series

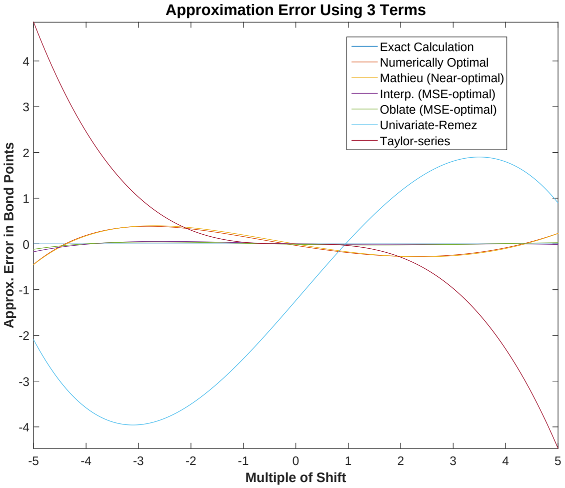

Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)

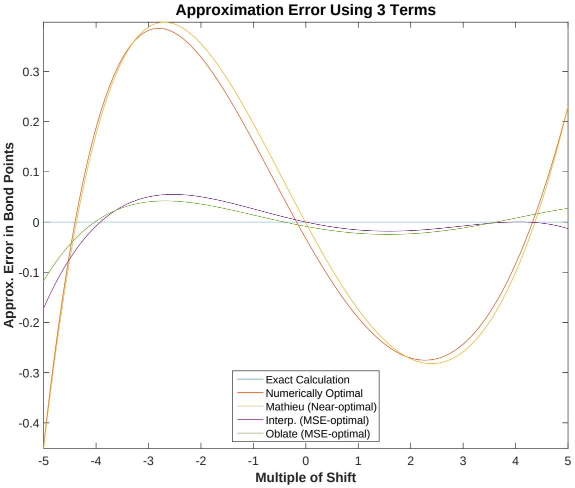

Shape of Base Shift
Base Zero-rates Shift in Bps
Time
100
80
60
40
20
0
-20
-40
-60
-80
-100
5
10
15
20
25
30

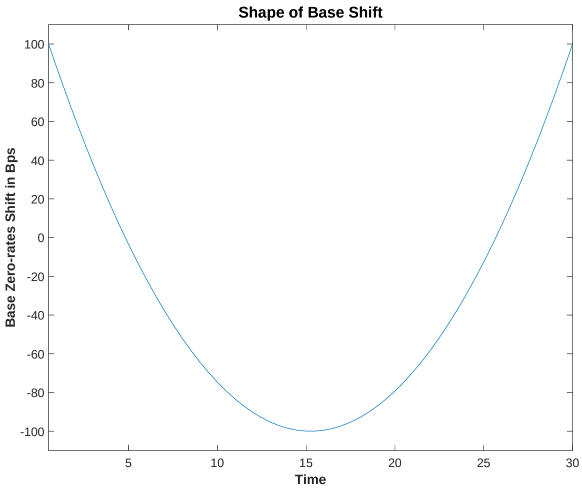

Bond Being Approximated, Base Value: 59.3447
PV of Bond Cashflow
Time
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
5
10
15
20
25
30

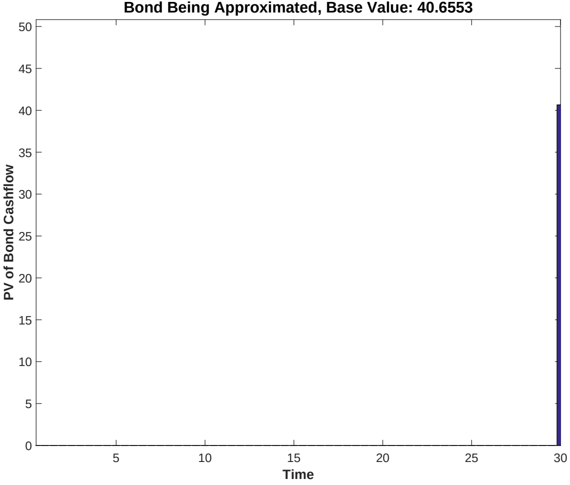

Numerically Optimal Approximating Cashflow Streams
PV of Cashflow
Time
Stream 1 (34.05 Units Held)
Stream 2 (61.56 Units Held)
Stream 3 (-9.70 Units Held)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
5
10
15
20
25
30
35

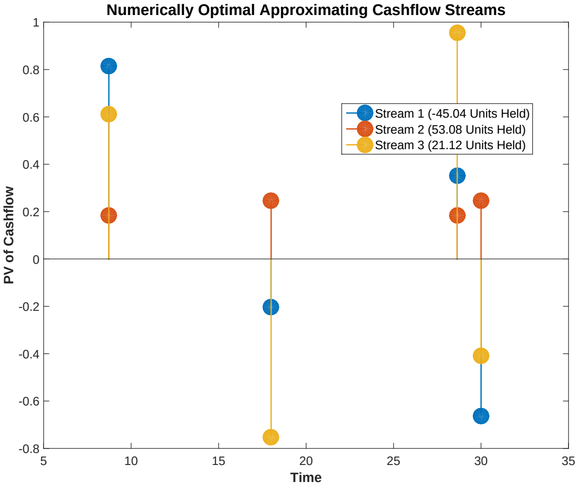

Interpolatory (MSE-optimal) Approximating Cashflow Streams
PV of Cashflow
Time
Stream 1 (2.48 Units Held)
Stream 2 (-26.55 Units Held)
Stream 3 (34.17 Units Held)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
12
14
16
18
20
22
24
26
28
30

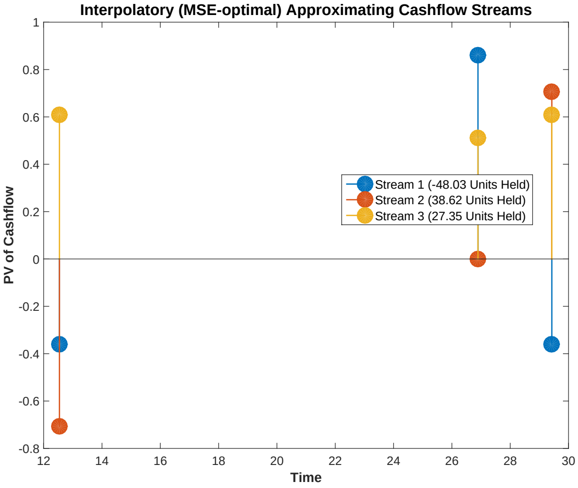

Approximation Error Using 3 Terms
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)
Univariate-Remez
Taylor-series
Approx. Error in Bond Points
Multiple of Shift
4
3
2
1
0
-1
-2
-3
-4
-5
-4
-3
-2
-1
0
1
2
3
4
5

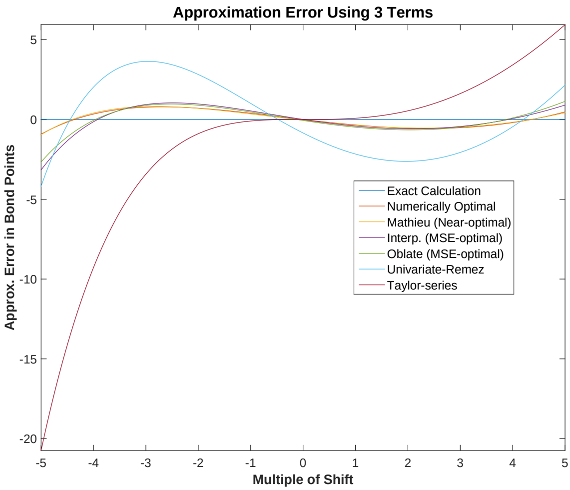

Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)

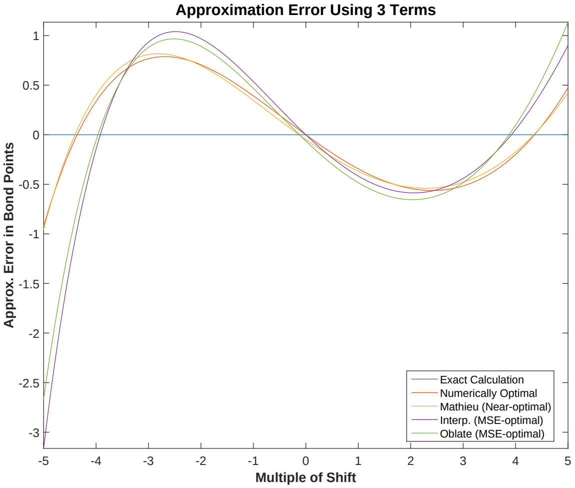

Shape of Base Shift
Base Zero-rates Shift in Bps
100
80
60
40
20
0
-20
-40
-60
-80
-100
5
10
15
20
25
30
Time

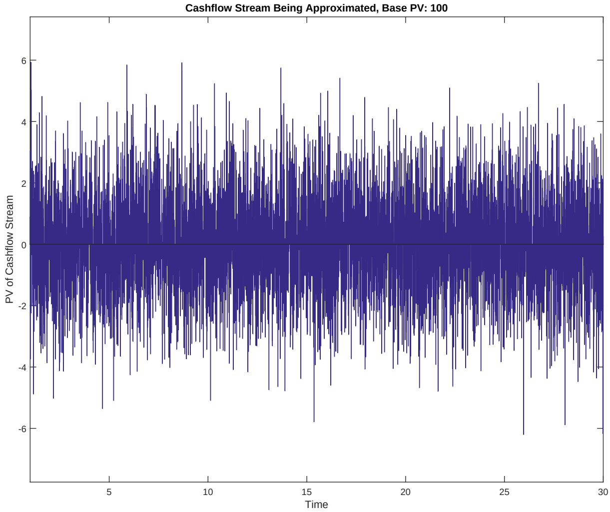

Bond Being Approximated, Base Value: 40.6553
PV of Bond Cashflow
Time
0
5
10
15
20
25
30
35
40
45
50
5
10
15
20
25
30


Numerically Optimal Approximating Cashflow Streams
PV of Cashflow
Time
Stream 1 (-45.04 Units Held)
Stream 2 (53.08 Units Held)
Stream 3 (21.12 Units Held)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
5
10
15
20
25
30
35


Interpolatory (MSE-optimal) Approximating Cashflow Streams
PV of Cashflow
Time
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
12
14
16
18
20
22
24
26
28
30
Stream 1 (-48.03 Units Held)
Stream 2 (38.62 Units Held)
Stream 3 (27.35 Units Held)


Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
5
0
-5
-10
-15
-20
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)
Univariate-Remez
Taylor-series


Approximation Error Using 3 Terms
Approx. Error in Bond Points
Multiple of Shift
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-5
-4
-3
-2
-1
0
1
2
3
4
5
Exact Calculation
Numerically Optimal
Mathieu (Near-optimal)
Interp. (MSE-optimal)
Oblate (MSE-optimal)


Cashflow Stream Being Approximated, Base PV: 100
PV of Cashflow Stream
Time
6
4
2
0
-2
-4
-6
5
10
15
20
25
30

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