The Determination of the Default Restoring Beam in Difmap

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1 Introduction

Observations of the sky with an interferometer sample the Fourier transform of the sky at discrete points. This Fourier transform plane is known as the UV plane, named after the U and V coordinates that are used to label its axes. After accumulating many samples at different locations in the UV plane, an image may be formed by gridding these samples onto a regular two-dimensional array and taking the inverse Fourier transform. The result is an image of the sky that has been convolved with the PSF of the interferometer. The PSF, known as the dirty beam, is the inverse Fourier transform of the sampling weights in the UV plane. The dirty beam, tends to be very irregular due to incomplete sampling of the UV plane, so the initial image of the sky can be hard to interpret, with each bright feature in the image being accompanied by side-lobes.

To obtain an image that is easier to analyze than the dirty image, a number of deconvolution schemes have been developed. These aim of re-convolving the image with a much simpler beam that doesnt have side-lobes, such as an elliptical Gaussian, but with similar average resolution to the irregular synthesized beam.

The simplest deconvolution scheme is known as CLEAN. This decomposes the dirty image into a set of delta functions, each one representing a separate bright point-like feature in the image, plus a residual image of the background noise. The delta functions, known as CLEAN components, are then convolved with the chosen elliptical Gaussian beam, known as the restoring beam, and the resulting images are added back to the residual image of the noise. The final result is known as the restored or clean image.

This document describes one way to choose the dimensions and orientation of an elliptical Gaussian restoring beam that best matches the resolution of the dirty beam. An obvious way to do this would be to perform a least-squares fit of an elliptical Gaussian to an image of the dirty beam. However this is not as straight forward as it might seem, because in general the dirty beam does not look much like a Gaussian, except very close to the center.

The method used in Difmap, which was inherited from the earlier Caltech VLBI programs, takes note of the fact that the center of the dirty beam is the only part of the synthesized beam that tends to look somewhat Gaussian. What it does is calculate the elliptical Gaussian that has the same curvature as the dirty beam at the center of the image. Image plane derivatives are trivial to calculate in the Fourier plane, so in practice this is done while calculating the weights used to grid the observed data in the UV plane.

2 Characterizing the clean beam

Let $f(\alpha, \beta)$ be an elliptical Gaussian beam with a peak value of unity, where α is a coordinate along the major axis, whose length is the Gaussian standard deviation $\sigma_{\rm maj}$, and β is the coordinate along the minor axis, whose length is the Gaussian standard deviation, $\sigma_{\rm min}$.

In the (α, β) coordinate system, the two-dimensional elliptical gaussian PSF can be written:

$$f(\alpha, \beta) = \exp(-0.5(\alpha^2/\sigma^2_{\rm maj} + \beta^2/\sigma^2_{\rm min})) \tag{1}$$

On the sky the major axis of the beam is rotated anticlockwise east of north by an angle θ. Within Difmap, however, images of the sky are stored with coordinates of (x, y), where x goes from west to east. In this coordinate system the beam is rotated clockwise by the angle θ.

Given a sky coordinate, (x, y), the corresponding position along the major and minor axes of the elliptical Gaussian are given by rotating the x and y clockwise by the angle θ, as follows.

$$\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

$$\beta = x\cos\theta - y\sin\theta \tag{3}$$

$$\alpha = x\sin\theta + y\cos\theta \tag{4}$$

If we substitute these equations for α and β into equation 1, we obtain the following equation for the beam in the (x, y) coordinate system.

$$s(x, y) = \exp(-0.5(x\sin\theta + y\cos\theta)^2/\sigma^2_{\rm maj} - 0.5(x\cos\theta - y\sin\theta)^2/\sigma^2_{\rm min}) \tag{5}$$

3 The curvature of the Gaussian at the origin

The second derivative of $s(x, y)$ with respect to x is is as follows.

$$\begin{aligned}\frac{d^2s}{dx^2} = ((-y\cos\theta\sin\theta - x\sin\theta^2)/\sigma^2_{\rm maj} + (y\cos\theta\sin\theta - x\cos\theta^2)/\sigma^2_{\rm min})^2 s(x, y) \\ + (-\sin\theta^2/\sigma^2_{\rm maj} - \cos\theta^2/\sigma^2_{\rm min})s(x, y)\end{aligned} \tag{6}$$

Similarly, the second derivative of $s(x, y)$ with respect to y is as follows.

$$\begin{aligned}\frac{d^2s}{dy^2} = ((y\cos\theta^2 - x\cos\theta\sin\theta)/\sigma^2_{\rm maj} + (y\sin\theta^2 + x\cos\theta\sin\theta)/\sigma^2_{\rm min})^2 s(x, y) \\ + (-\cos\theta^2/\sigma^2_{\rm maj} - \sin\theta^2/\sigma^2_{\rm min})s(x, y)\end{aligned} \tag{7}$$

Finally, the second derivative of $s(x, y)$ with respect to $dxdy$ is as follows.

$$\frac{ds}{dxdy} = ((-y\cos\theta\sin\theta - x\sin\theta^2)/\sigma_{\text{maj}}^2 + (y\cos\theta\sin\theta - x\cos\theta^2)/\sigma_{\text{min}}^2)\times$$
$$((-x\cos\theta\sin\theta - y\cos\theta^2)/\sigma_{\text{maj}}^2 + (x\cos\theta\sin\theta - y\sin\theta^2)/\sigma_{\text{min}}^2)s(x,y)$$
$$+ (-\cos\theta\sin\theta/\sigma_{\text{maj}}^2 + \cos\theta\sin\theta/\sigma_{\text{min}}^2)s(x,y) \quad (8)$$

If we evaluate each of the above derivatives at the origin, $(x = 0, y = 0)$, then we obtain the following for the derivatives of the Gaussian at the origin of the image plane.

$$c_{xx} = d^2s(0,0)/dx^2 = -\sin\theta^2/\sigma_{\text{maj}}^2 - \cos\theta^2/\sigma_{\text{min}}^2 \quad (9)$$
$$c_{yy} = d^2s(0,0)/dy^2 = -\cos\theta^2/\sigma_{\text{maj}}^2 - \sin\theta^2/\sigma_{\text{min}}^2 \quad (10)$$
$$c_{xy} = d^2s(0,0)/dxdy = 0.5\sin 2\theta(1/\sigma_{\text{min}}^2 - 1/\sigma_{\text{maj}}^2) \quad (11)$$

Note that the two curvatures (equations 9 and 10) are negative, as expected for the curvature of a peak.

4 Obtaining Gaussian parameters from the second derivatives

As a first step towards determining the Gaussian parameters, σ_{maj}, σ_{min} and θ, we combine equations 9,10 and 11 as follows.

$$c_{xx} + c_{yy} = -1/\sigma_{\text{maj}}^2 - 1/\sigma_{\text{min}}^2 \quad (12)$$
$$c_{xx} - c_{yy} = -\cos 2\theta(1/\sigma_{\text{min}}^2 - 1/\sigma_{\text{maj}}^2) \quad (13)$$
$$(2c_{xy})^2 + (c_{xx} - c_{yy})^2 = (1/\sigma_{\text{min}}^2 - 1/\sigma_{\text{maj}}^2)^2 \quad (14)$$

In the above equations, note that we are aiming to obtain $(1/\sigma_{\text{min}}^2 - 1/\sigma_{\text{maj}}^2)$, which is positive, whereas $(1/\sigma_{\text{maj}}^2 - 1/\sigma_{\text{min}}^2)$ should be negative. This becomes important below where we would otherwise lose a negative by square-rooting the square of this term.

Solving the above equations for the Gaussian parameters, we obtain the following.

$$\theta = 0.5\,\text{atan2}(2c_{xy}, c_{yy} - c_{xx}) \quad (15)$$
$$\sigma_{\text{maj}} = \sqrt{\frac{2}{-\left(\sqrt{4c_{xy}^2 + (c_{xx} - c_{yy})^2} + c_{xx} + c_{yy}\right)}} \quad (16)$$
$$\sigma_{\text{min}} = \sqrt{\frac{2}{\sqrt{4c_{xy}^2 + (c_{xx} - c_{yy})^2} - (c_{xx} + c_{yy})}} \quad (17)$$

5 Obtaining derivatives via the Fourier plane

We can obtain c_{xx}, c_{yy} and c_{xy} at the origin by performing a simple operation to the Fourier transform of the beam.

If the Fourier transform of $s(x, y)$ is denoted $S(u, v)$, then the image plane derivatives needed in the previous section can be obtained as follows.

$$d^2s/dx^2 <=> -4\pi^2 u^2 S(u, v) \tag{18}$$
$$d^2s/dy^2 <=> -4\pi^2 v^2 S(u, v) \tag{19}$$
$$ds/dxdy <=> -4\pi^2 uvS(u, v) \tag{20}$$

The value at the origin of an inverse Fourier transform is the integral or sum of the values in the Fourier transform, so we can calculate the desired derivatives in the image plane as follows.

$$c_{xx} = -4\pi^2 \sum u^2 S(u, v) \tag{21}$$
$$c_{yy} = -4\pi^2 \sum v^2 S(u, v) \tag{22}$$
$$c_{xy} = -4\pi^2 \sum uvS(u, v) \tag{23}$$

To simplify the subsequent equations, we define the following values.

$$s_{uu} = \sum u^2 S(u, v) \tag{24}$$
$$s_{vv} = \sum v^2 S(u, v) \tag{25}$$
$$s_{uv} = \sum uvS(u, v) \tag{26}$$

In terms of these values, the derivatives can now be written as follows.

$$c_{xx} = -4\pi^2 s_{uu} \tag{27}$$
$$c_{yy} = -4\pi^2 s_{vv} \tag{28}$$
$$c_{xy} = -4\pi^2 s_{uv} \tag{29}$$

6 Computing the Gaussian parameters via the Fourier plane

If we insert the derivatives of equations 27, 28 and 29 into equations 15, 17 and 16, we obtain the following equations for the Gaussian parameters.

$$\theta = -0.5\,\mathrm{atan2}(2s_{uv}, s_{uu} - s_{vv}) \tag{30}$$
$$\sigma_{\mathrm{min}} = 1/\sqrt{2}/\pi/\sqrt{\sqrt{4s_{uv}^2 + (s_{uu} - s_{vv})^2} + (s_{uu} + s_{vv})} \tag{31}$$
$$\sigma_{\mathrm{maj}} = 1/\sqrt{2}/\pi/\sqrt{-(\sqrt{4s_{uv}^2 + (s_{uu} - s_{vv})^2} - (s_{uu} + s_{vv}))} \tag{32}$$

In difmap, the dimensions of the Gaussian beam are specified as Full Widths at Half Maximum (FWHM). We can obtain those by scaling the standard deviations given above by $\sqrt{8\ln 2}$.

$$b_{\mathrm{min}} = \sqrt{4\ln 2}/\pi/\sqrt{s_{uu} + s_{vv} + \sqrt{4s_{uv}^2 + (s_{uu} - s_{vv})^2}} \tag{33}$$

$$b_{\mathrm{maj}} = \sqrt{4\ln 2}/\pi/\sqrt{s_{uu} + s_{vv} - \sqrt{4s_{uv}^2 + (s_{uu} - s_{vv})^2}} \tag{34}$$

7 The algorithm historically used in Difmap

In Difmap versions up to 2.5b (25 Nov 2018)), the algorithm difmap used was inherited verbatim from the VLBI programs, which used the following equations:

$$k = 0.7 \tag{35}$$

$$\theta = -0.5\,\mathrm{atan2}(2s_{uv}, s_{uu} - s_{vv}) \tag{36}$$

$$b_{\mathrm{min}} = k/\sqrt{2}/\sqrt{s_{uu} + s_{vv} + \sqrt{4s_{uv}^2 + (s_{uu} - s_{vv})^2}} \tag{37}$$

$$b_{\mathrm{maj}} = k/\sqrt{2}/\sqrt{s_{uu} + s_{vv} - \sqrt{4s_{uv}^2 + (s_{uu} - s_{vv})^2}} \tag{38}$$

From the correct equations of the previous section, k ought to have had the following value.

$$k_{\mathrm{true}} = \sqrt{8\ln 2}/\pi \tag{39}$$

$$= 0.7495625 \tag{40}$$

This is 6.6% larger than the value used in Difmap, such that the default Gaussian beam used to restore images in Difmap was been 6.6% too small.

This was corrected in Difmap version 2.5c (20 May 2019).