# Analysis Notes

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# Chapter 1

# Measure Theory

## 1.1 Preliminaries

*Note*: Understand the analogy between measuring "volume" and the concept of this section.

*Note*: Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals. Every open set in $\mathbb{R}^d$, $d \geq 2$, is almost the disjoint union of closed cubes. Almost means that only the boundaries of the cubes can overlap.

A **point** $x \in \mathbb{R}^d$ consists of a $d$-tuple of real numbers

$$x = (x_1, x_2, \cdots, x_d), \qquad x_i \in \mathbb{R}, \text{ for } i = 1, \cdots, d.$$

The **norm** of $x$ is denoted by $|x|$ and is defined to be the standard Euclidean norm given by

$$|x| = (x_1^2 + \cdots + x_d^2)^{1/2}.$$

The **distance between two points** $x$ and $y$ is then simply $|x - y|$.

The **complement** of a set $E$ in $\mathbb{R}^d$ is denoted by $E^c$ and defined by

$$E^c = \{x \in \mathbb{R}^d : x \notin E\}.$$

The **distance between two sets** $E$ and $F$ is defined by

$$d(E, F) = \inf |x - y|,$$

where the infimum is taken over all $x \in e$ and $y \in F$.

The **open ball** in $\mathbb{R}^d$ centered at $x$ and of radius $r$ is defined by

$$B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r.$$

A subset $E \subset \mathbb{R}^d$ is **open** if for every $x \in E$ there exists $r > 0$ with $B_r(x) \subset E$. A set is **closed** if its complement is open.

A point $x \in \mathbb{R}^d$ is a **limit point** of the set $E$ if for every $r > 0$, the ball $B_r(x)$ contains points of $E$.

An **isolated point** of $E$ is a point $x \in E$ such that there exists an $r > 0$ where $B_r(x) \cap E$ is equal to $\{x\}$.

A closed set $E$ is **perfect** if $E$ does not have any isolated points.

A (closed) **rectangle** $R$ in $\mathbb{R}^d$ is given by the product of $d$ one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d],$$

where $a_j \leq b_j$ are real numbers, $j = 1, 2, \ldots, d$.

The **volume** of the rectangle $R$ is denoted by $|R|$, and is defined to be

$$|R| = (b_1 - a_1) \cdots (b_d - a_d).$$

A union of rectangles is said to be **almost disjoint** if the interiors are disjoint.

**Lemma 1.1.1.** *If a rectangle is the almost disjoint union of finitely many other rectangles, say $R = \cup_{k=1}^{N} R_k$, then*

$$|R| = \textstyle\sum_{k=1}^{N} |R_k|.$$

**Lemma 1.1.2.** *If $R, R_1, \ldots, R_N$ are rectangles, and $R \subset \cup_{k=1}^{N} R_k$, then*

$$|R| \leq \textstyle\sum_{k=1}^{N} |R_k|.$$

**Theorem 1.1.3.** *Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.*

**Theorem 1.1.4.** *Every open subset $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.*

Construction of the **Cantor set**:

1. Subintervals, starting at $[0, 1]$, delete a third of the remaining closed intervals. This iterative process is performed countable many times. Example:
   (0) $C_0 = [0, 1]$
   (1) $C_1 = [0, 1/3] \cup [2/3, 1]$
   (2) $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$
   $\vdots$
   (n) $C_n$
   $\vdots$

2. This procedure yields a sequence of nested compact sets with

$$C_0 \supset C_1 \supset \cdots \supset C_k \supset C_{k+1} \supset \cdots.$$

3. The **Cantor set** is defined by the intersection of all $C_k$'s:

$$\mathcal{C} = \cap_{k=0}^{\infty} C_k.$$

The Cantor set a length of zero.

$C_k$ is a disjoint union of $2^k$ intervals of length $3^{-k}$, making the total length of $C_k$ equal to $(2/3)^k$.

## 1.2 The Exterior Measure