Analysis Notes

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Contents

Chapter 1

Measure Theory

1.1 Preliminaries

Note: Understand the analogy between measuring "volume" and the concept of this section.

Note: Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals. Every open set in $\mathbb{R}^d$, $d \geq 2$, is almost the disjoint union of closed cubes. Almost means that only the boundaries of the cubes can overlap.

A **point** $x \in \mathbb{R}^d$ consists of a d-tuple of real numbers

$$x = (x_1, x_2, \cdots, x_d), \qquad x_i \in \mathbb{R}, \text{ for } i = 1, \cdots, d.$$

The **norm** of x is denoted by $|x|$ and is defined to be the standard Euclidean norm given by

$$|x| = (x_1^2 + \cdots + x_d^2)^{1/2}.$$

The **distance between two points** x and y is then simply $|x - y|$.

The **complement** of a set E in $\mathbb{R}^d$ is denoted by E^c and defined by

$$E^c = \{x \in \mathbb{R}^d : x \notin E\}.$$

The **distance between two sets** E and F is defined by

$$d(E, F) = \inf |x - y|,$$

where the infimum is taken over all $x \in e$ and $y \in F$.

The **open ball** in $\mathbb{R}^d$ centered at x and of radius r is defined by

$$B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r.$$

A subset $E \subset \mathbb{R}^d$ is **open** if for every $x \in E$ there exists $r > 0$ with $B_r(x) \subset E$. A set is **closed** if its complement is open.

A point $x \in \mathbb{R}^d$ is a **limit point** of the set E if for every $r > 0$, the ball $B_r(x)$ contains points of E.

An **isolated point** of E is a point $x \in E$ such that there exists an $r > 0$ where $B_r(x) \cap E$ is equal to $\{x\}$.

A closed set E is **perfect** if E does not have any isolated points.

A (closed) **rectangle** R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d],$$

where $a_j \leq b_j$ are real numbers, $j = 1, 2, \ldots, d$.

The **volume** of the rectangle R is denoted by $|R|$, and is defined to be

$$|R| = (b_1 - a_1) \cdots (b_d - a_d).$$

A union of rectangles is said to be **almost disjoint** if the interiors are disjoint.

Lemma 1.1.1. *If a rectangle is the almost disjoint union of finitely many other rectangles, say $R = \cup_{k=1}^{N} R_k$, then*

$$|R| = \textstyle\sum_{k=1}^{N} |R_k|.$$

Lemma 1.1.2. *If $R, R_1, \ldots, R_N$ are rectangles, and $R \subset \cup_{k=1}^{N} R_k$, then*

$$|R| \leq \textstyle\sum_{k=1}^{N} |R_k|.$$

Theorem 1.1.3. *Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.*

Theorem 1.1.4. *Every open subset $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.*

Construction of the **Cantor set**:

1. Subintervals, starting at $[0, 1]$, delete a third of the remaining closed intervals. This iterative process is performed countable many times. Example:
   (0) $C_0 = [0, 1]$
   (1) $C_1 = [0, 1/3] \cup [2/3, 1]$
   (2) $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$
   $\vdots$
   (n) C_n
   $\vdots$
2. This procedure yields a sequence of nested compact sets with
$$C_0 \supset C_1 \supset \cdots \supset C_k \supset C_{k+1} \supset \cdots.$$
3. The **Cantor set** is defined by the intersection of all C_k's:
$$\mathcal{C} = \cap_{k=0}^{\infty} C_k.$$

The Cantor set a length of zero.

C_k is a disjoint union of 2^k intervals of length 3^{-k}, making the total length of C_k equal to $(2/3)^k$.

1.2 The Exterior Measure

If E is *any* subset of $\mathbb{R}^d$, the exterior measure of E is

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|,$$

where the infimum is taken over all countable coverings $E \subset \cup_{j=1}^{\infty} Q_j$ by closed cubes.

Properties of the exterior measure:

Observation 1 (**Monotonicity**): If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$.

Observation 2 (**Countable sub-additivity**):
If $E = \cup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.

Observation 3 If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf m_*(\mathcal{O})$, where the infimum is taken over all open sets $\mathcal{O}$ containing E.

Observation 4 If $E = E_1 \cup E_2$, and $d(E_1, E_2) > 0$, then $m_*(E) = m_*(E_1) + m_*(E_2)$.

Observation 5 If a set E is the countable union of almost disjoint cubes $E = \cup_{j=1}^{\infty} Q_j$, then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j|.$$

1.3 Measurable Sets and the Lebesgue Measure

A subset E of $\mathbb{R}^d$ is **Lebesgue measurable**, or simply **measurable**, if for any $\epsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and

$$m_*(\mathcal{O} - E) \leq \epsilon.$$

If E is measurable, we define its **Lebesgue measure** (or **measure**) $m(E)$ by

$$m(E) = m_*(E).$$

Properties of Lebesgue measure:

Property 1: Every open set in $\mathbb{R}^d$ is measurable.

Property 2: If $m_*(E) = 0$, then E is measurable. In particular, if F is a subset of a exterior measure 0, then F is measurable.

Property 3: A countable union of measurable sets is measurable.

Property 4: Closed sets are measurable.

Property 5: The complement of a measurable set is measurable.

Property 6: A countable intersection of measurable sets is measurable.