Analysis Notes

Kerry Tarrant

January 17, 2021

Contents

Chapter 1

Measure Theory

1.1 Preliminaries

Note: Understand the analogy between measuring "volume" and the concept of this section.

Note: Every open set in $\mathbb{R}$ is a countable union of disjoint open intervals. Every open set in $\mathbb{R}^d$, $d \geq 2$, is almost the disjoint union of closed cubes. Almost means that only the boundaries of the cubes can overlap.

A **point** $x \in \mathbb{R}^d$ consists of a d-tuple of real numbers

$$x = (x_1, x_2, \cdots, x_d), \qquad x_i \in \mathbb{R}, \text{ for } i = 1, \cdots, d.$$

The **norm** of x is denoted by $|x|$ and is defined to be the standard Euclidean norm given by

$$|x| = (x_1^2 + \cdots + x_d^2)^{1/2}.$$

The **distance between two points** x and y is then simply $|x - y|$.

The **complement** of a set E in $\mathbb{R}^d$ is denoted by E^c and defined by

$$E^c = \{x \in \mathbb{R}^d : x \notin E\}.$$

The **distance between two sets** E and F is defined by

$$d(E, F) = \inf |x - y|,$$

where the infimum is taken over all $x \in e$ and $y \in F$.

The **open ball** in $\mathbb{R}^d$ centered at x and of radius r is defined by

$$B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r.$$

A subset $E \subset \mathbb{R}^d$ is **open** if for every $x \in E$ there exists $r > 0$ with $B_r(x) \subset E$. A set is **closed** if its complement is open.

A point $x \in \mathbb{R}^d$ is a **limit point** of the set E if for every $r > 0$, the ball $B_r(x)$ contains points of E.

An **isolated point** of E is a point $x \in E$ such that there exists an $r > 0$ where $B_r(x) \cap E$ is equal to $\{x\}$.

A closed set E is **perfect** if E does not have any isolated points.

A (closed) **rectangle** R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d],$$

where $a_j \leq b_j$ are real numbers, $j = 1, 2, \ldots, d$.

The **volume** of the rectangle R is denoted by $|R|$, and is defined to be

$$|R| = (b_1 - a_1) \cdots (b_d - a_d).$$

A union of rectangles is said to be **almost disjoint** if the interiors are disjoint.

Lemma 1.1.1. *If a rectangle is the almost disjoint union of finitely many other rectangles, say $R = \cup_{k=1}^{N} R_k$, then*

$$|R| = \textstyle\sum_{k=1}^{N} |R_k|.$$