

Analysis Notes

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Chapter 1

Measure Theory

1.1 Preliminaries

Note: Understand the analogy between measuring "volume" and the concept of this section.

Note: Every open set in \mathbb{R} is a countable union of disjoint open intervals. Every open set in \mathbb{R}^d , $d \geq 2$, is almost the disjoint union of closed cubes. Almost means that only the boundaries of the cubes can overlap.

A **point** $x \in \mathbb{R}^d$ consists of a d -tuple of real numbers

$$x = (x_1, x_2, \dots, x_d), \quad x_i \in \mathbb{R}, \text{ for } i = 1, \dots, d.$$

The **norm** of x is denoted by $|x|$ and is defined to be the standard Euclidean norm given by

$$|x| = (x_1^2 + \dots + x_d^2)^{1/2}.$$

The **distance between two points** x and y is then simply $|x - y|$.

The **complement** of a set E in \mathbb{R}^d is denoted by E^c and defined by

$$E^c = \{x \in \mathbb{R}^d : x \notin E\}.$$

The **distance between two sets** E and F is defined by

$$d(E, F) = \inf |x - y|,$$

where the infimum is taken over all $x \in E$ and $y \in F$.

The **open ball** in \mathbb{R}^d centered at x and of radius r is defined by

$$B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r\}.$$

A subset $E \subset \mathbb{R}^d$ is **open** if for every $x \in E$ there exists $r > 0$ with $B_r(x) \subset E$. A set is **closed** if its complement is open.

A point $x \in \mathbb{R}^d$ is a **limit point** of the set E if for every $r > 0$, the ball $B_r(x)$ contains points of E .

An **isolated point** of E is a point $x \in E$ such that there exists an $r > 0$ where $B_r(x) \cap E$ is equal to $\{x\}$.

A closed set E is **perfect** if E does not have any isolated points.

A (closed) **rectangle** R in \mathbb{R}^d is given by the product of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d],$$

where $a_j \leq b_j$ are real numbers, $j = 1, 2, \dots, d$.

The **volume** of the rectangle R is denoted by $|R|$, and is defined to be

$$|R| = (b_1 - a_1) \cdots (b_d - a_d).$$

A union of rectangles is said to be **almost disjoint** if the interiors are disjoint.

Lemma 1.1.1. *If a rectangle is the almost disjoint union of finitely many other rectangles, say $R = \cup_{k=1}^N R_k$, then*

$$|R| = \sum_{k=1}^N |R_k|.$$

Lemma 1.1.2. *If R, R_1, \dots, R_N are rectangles, and $R \subset \cup_{k=1}^N R_k$, then*

$$|R| \leq \sum_{k=1}^N |R_k|.$$

Theorem 1.1.3. *Every open subset \mathcal{O} of \mathbb{R} can be written uniquely as a countable union of disjoint open intervals.*

Theorem 1.1.4. *Every open subset \mathcal{O} of \mathbb{R}^d , $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.*

Construction of the **Cantor set**:

1. Subintervals, starting at $[0, 1]$, delete a third of the remaining closed intervals. This iterative process is performed countable many times. Example:

- (0) $C_0 = [0, 1]$
- (1) $C_1 = [0, 1/3] \cup [2/3, 1]$
- (2) $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$
- \vdots
- (n) C_n
- \vdots

2. This procedure yields a sequence of nested compact sets with

$$C_0 \supset C_1 \supset \cdots \supset C_k \supset C_{k+1} \supset \cdots.$$

3. The **Cantor set** is defined by the intersection of all C_k 's:

$$\mathcal{C} = \bigcap_{k=0}^{\infty} C_k.$$

The Cantor set has a length of zero.

C_k is a disjoint union of 2^k intervals of length 3^{-k} , making the total length of C_k equal to $(2/3)^k$.

1.2 The Exterior Measure

If E is *any* subset of \mathbb{R}^d , the exterior measure of E is

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|,$$

where the infimum is taken over all countable coverings $E \subset \cup_{j=1}^{\infty} Q_j$ by closed cubes.

Properties of the exterior measure:

Observation 1 (**Monotonicity**): If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$.

Observation 2 (**Countable sub-additivity**):

If $E = \cup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.

Observation 3 If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf m_*(\mathcal{O})$, where the infimum is taken over all open sets \mathcal{O} containing E .

Observation 4 If $E = E_1 \cup E_2$, and $d(E_1, E_2) > 0$, then $m_*(E) = m_*(E_1) + m_*(E_2)$.

Observation 5 If a set E is the countable union of almost disjoint cubes $E = \cup_{j=1}^{\infty} Q_j$, then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j|.$$

1.3 Measurable Sets and the Lebesgue Measure