Investigation of Minimum Length of Supersonic 2D Nozzle Design Using the Method of Characteristics

Kerry Sun[1]
California Polytechnic State University, San Luis Obispo, CA, 93407

Method of Characteristic was implemented to investigate the minimum length of supersonic 2D nozzle design. The lengths and area ratios of a minimum length nozzle designed for M=3.2 flow are found for various numbers of characteristic lines. For M=3.2 flow at the exit and using 25 lines, the area ratio was found to be 5.1457 and the minimum length was found to be 10.8234 (for a total throat height of 1). Error, based on isentropic flow equations was found to be only 0.4840%. With more lines used, the smaller error is obtained. Additionally, flow Mach numbers are found and depicted for the flow inside the nozzle. Method of Characteristic is found to be very useful to design minimum length nozzle.

Nomenclature

A	=	cross-sectional area
A_t	=	cross-sectional throat area
C	=	characteristic line
a	=	speed of sound
D	=	Denominator
K	=	constant
M	=	Mach number
N	=	Numerator
V	=	total velocity vector
a	=	speed of sound
du	=	acceleration of the flow in the x-direction
dv	=	acceleration of the flow in the y-direction
$\left(\frac{dy}{dx}\right)_{char}$	=	slope of the characteristic line
u	=	x component of velocity
v	=	y component of velocity
$\Delta\theta_w$	=	change in wall angle
x	=	horizontal direction
y	=	vertical direction
Φ	=	velocity potential function
Θ	=	flow deflection angle
θ	=	angle of velocity with respect to the horizontal
θ_w	=	wall angle
$\theta_{w,max}$	=	maximum wall angle
μ	=	Mach angle
υ	=	Prandtl-Meyer angle

Subscripts

+	=	left
-	=	right
char	=	characteristic

[1] Undergraduate, Aerospace Engineering Department, 1 Grand Avenue San Luis Obispo California 93407

I. Introduction

THIS paper details an investigation into the method of characteristics as used to design a minimum length supersonic nozzle. In the field of aerospace engineering, producing supersonic flows in an isentropic manner is very important. Particularly, nozzles on both air breathing and rocket engines tend to be converging-diverging nozzles design to produce supersonic flow at a particular Mach number. While simple conical nozzles are cheap and easy to produce, they do not produce the nearly isentropic flows seen in bell nozzles produced with the Method of Characteristics. As a result, there are significant losses in stagnation pressure as shocks form inside the nozzle flow. In addition to the inefficiency of non-isentropic flows, the shock waves can also cause severe damage to the nozzle.

Therefore, it is necessary to develop a method by which to produce contours that result in isentropic (or nearly isentropic) flow for a given design condition at the nozzle exit. One such method is the Method of Characteristics. This is a finite difference technique which can be used to numerically solve the partial differential equations of compressible flow.

II. Procedure

The supersonic nozzle is to be designed to a Mach number of 3.2 with a nozzle throat of 1 units. Since the two-dimensional nozzle is symmetric, the nozzle throat will be set at coordinate (0.5,0); however, this will be taken as right after the nozzle throat, or very close to it, for computational purposes. The nozzle is computed in the x-y plane, with symmetry about the x-axis. For a minimum length nozzle, an expansion fan is assumed at the throat, with 7, 25, and 100 expansion lines. These expansion lines are to originate from the coordinate (0.5,0).

First, characteristic lines are found in the x-y space, which are defined as the directions in which the flow variables of pressure, density, temperature, axial velocity, and radial velocity are continuous, but their derivatives with respect to the x- and y-directions are indeterminate. The partial differential conservation equations are then combined to obtain the compatibility equations that hold true only along the characteristic lines. These are then solved along the characteristic lines in a sequential manner to map the entire flowfield.

To begin mapping the flowfield within the supersonic nozzle, expansion lines are created originating at the wall of the nozzle throat. The increment of θ is determined for each expansion line, with respect to the number of divisions and maximum wall angle, as determined by the design Mach number. Using this, the constants K_+ and K_- for the positive and negative characteristic lines can be determined. The Prandtl-Meyer angle can be found as well, along with the Mach number. Using isentropic flow relations, other values such as temperature, pressure, and density can be found with the Mach number as well. Along the centerline of the nozzle, which is taken as x = 0, the flow is reflected. Figure 1 below shows the order at which the points in the flowfield are found.

To find the coordinates of each point, the slopes of the characteristic lines are used and intersected. For example, to find point 10, the positive slope of point 9 and negative slope of point 3 are used, assuming that points 1 through 9 have already been found. To find the wall coordinates, the average of the wall angles between the current point and the point to be found are used to determine the slope. For example, to find coordinate 15 of the wall, the wall angle of point 8 and point 15 are averaged and intersected with the positive characteristic line from point 14.

With this, the whole flowfield can be mapped out until the last expansion line reflects off the centerline.

III. Analysis

Equation 1 below is known to be the governing differential equation for a two-dimensional irrotational flow. Here, the equation has been rearranged and solved for $\partial u/\partial x$.

$$\frac{\partial u}{\partial x} = \frac{\frac{2uv}{a^2}\frac{\partial u}{\partial y} - \left(1 - \frac{v^2}{a^2}\right)\frac{\partial v}{\partial y}}{\left(1 - \frac{u^2}{a^2}\right)} \tag{1}$$

As an example of the process of applying finite difference techniques to such grids, Eq. 2 below shows how values at discrete points are related to each other using the partial differential equations, such as Eq. 1.

$$u_{i+1,j} = u_{i,j} + \left(\frac{\partial u}{\partial x}\right)_{i,j} \Delta x \tag{2}$$

In general the method of characteristics will proceed as follows. These characteristic lines will be found within the xy plane. Then the partial differential equations of conservation will be combined in such a way that ordinary differential equations can be obtained that are valid along, and only along, these characteristic lines. Finally, the compatibility equations will be solved along the characteristic lines. This will start at some known initial conditions and proceed until discrete points across the entire flow field have been solved.

Figure 1 below can be used to demonstrate the relationships between characteristic lines for an unsteady one-dimensional flow. It should be noted that at points b, c, and d all share a constant value for $\partial u/\partial x$. However, at points b and e, this derivative value is indeterminate, as in Eq. 1.

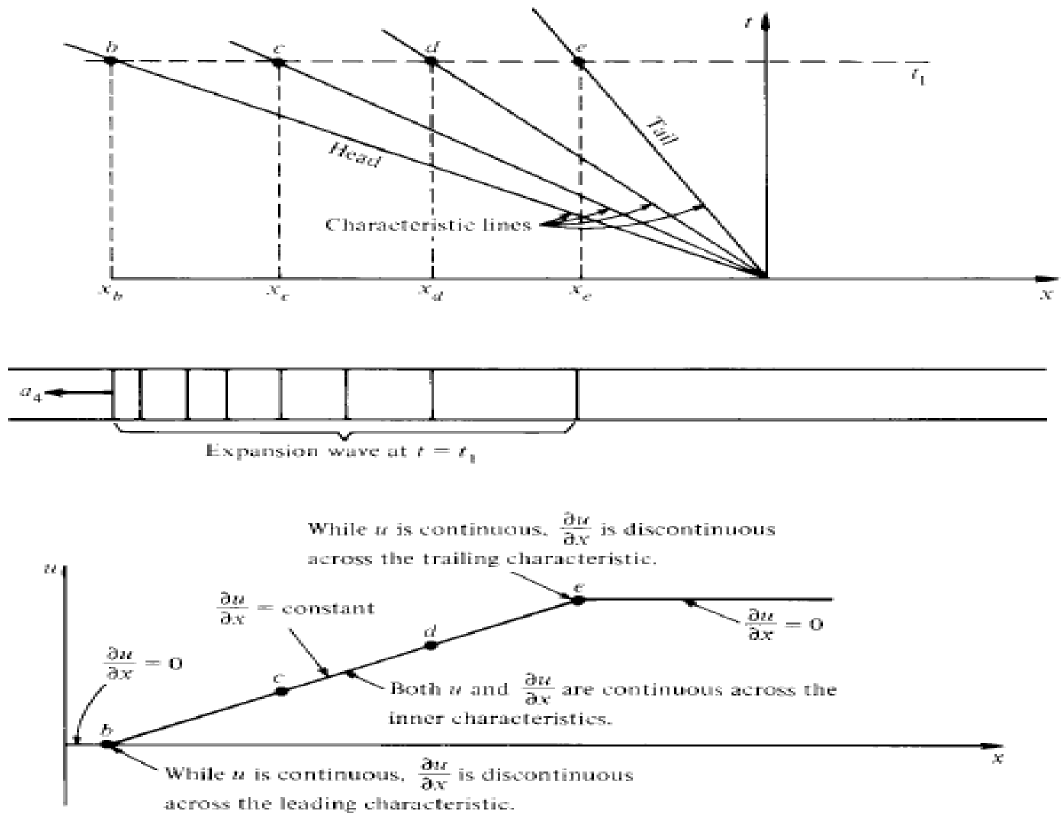

Figure 1. Relationships between characteristic lines in an unsteady one-dimensional flow.[2]

To begin the process of the Method of Characteristics described above, steady, adiabatic, two-dimensional, irrotational supersonic flow is considered. The nonlinear governing differential equation for this flow is as follows:

$$\left(1-\frac{\Phi_x^2}{a^2}\right)\Phi_{xx}+\left(1-\frac{\Phi_y^2}{a^2}\right)\Phi_{yy}-\frac{2\Phi_x\Phi_y}{a^2}\Phi_{xy}=0 \tag{3}$$

Here, Φ represents the full-velocity potential where the components and derivatives are defined as follows:

$$\Phi_x = u \quad \Phi_y = v \quad \boldsymbol{V} = u\mathbf{i}+\mathrm{v}\mathbf{j} \tag{4}$$

And because the velocity potential is function of both x and y, total derivatives can be found as follows.

$$d\Phi_x = \frac{\partial\Phi_x}{\partial x}dx+\frac{\partial\Phi_x}{\partial y}dy=\Phi_{xx}dx+\Phi_{xy}dy \tag{5}$$

$$d\Phi_y = \frac{\partial\Phi_y}{\partial x}dx+\frac{\partial\Phi_y}{\partial y}dy=\Phi_{xy}dx+\Phi_{yy}dy \tag{6}$$

These equations can be represented as a system of simultaneous linear algebraic equations. With the use of Cramer's Rule, the solution for Φ_{xy} is found as follows:

$$\Phi_{xy} = \frac{\begin{vmatrix} 1-\frac{u^2}{a^2} & 0 & 1-\frac{v^2}{a^2} \\ dx & du & 0 \\ 0 & dv & dy \end{vmatrix}}{\begin{vmatrix} 1-\frac{u^2}{a^2} & -\frac{2uv}{a^2} & 1-\frac{v^2}{a^2} \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}} = \frac{N}{D} \tag{7}$$

The particular geometry of these characteristic lines can be defined as follows based on the statements above and setting the numerator, N, and denominator, D, to be equal to zero.

$$\left(1-\frac{u^2}{a^2}\right)(dy)^2 + \frac{2uv}{a^2}dxdy + \left(1-\frac{v^2}{a^2}\right)(dx)^2 = 0 \tag{8}$$

Which can be rearranged as:

$$\left(1-\frac{u^2}{a^2}\right)\left(\frac{dy}{dx}\right)_{char}^2 + \frac{2uv}{a^2}\left(\frac{dy}{dx}\right)_{char} + \left(1-\frac{v^2}{a^2}\right) = 0 \tag{9}$$

Solving for $\left(\frac{dy}{dx}\right)_{char}$,

$$\left(\frac{dy}{dx}\right)_{char} = \frac{-\frac{uv}{a^2} \pm \sqrt{\left[\frac{(u^2+v^2)}{a^2}\right]-1}}{\left[1-\frac{u^2}{a^2}\right]} \tag{10}$$

Equation 10 has defined the slopes of characteristic lines in the xy-plane. This can be simplified into terms of Mach number as follows.

$$\frac{(u^2+v^2)}{a^2} - 1 = \frac{V^2}{a^2} - 1 = M^2 - 1 \tag{11}$$

It should be noted that Eq. 10 will result in two characteristic slopes for M>1 flow, meaning the equation is a hyperbolic partial differential equation. If M=1, one solution is generated and the equation is parabolic. If M<1 then all characteristic line slopes are imaginary and the equation is elliptic. This is expected as no Mach lines exist in subsonic flow.

Equation 10 can be rewritten with component velocities, u and v, in terms of the flow angle θ using the following relations:

$$u = Vcos(\theta),\ v = Vsin(\theta),\ \sin(\mu) = \frac{1}{M} \tag{12}$$

Therefore Eq. 10 becomes:

$$\left(\frac{dy}{dx}\right)_{char} = \frac{-\frac{\cos(\theta)\sin(\theta)}{\sin^2(\mu)} \pm \sqrt{\left[\frac{\cos^2(\theta)+\sin^2(\theta)}{\sin^2(\mu)}\right]-1}}{1-\frac{\cos^2(\theta)}{\sin^2(\mu)}} \tag{13}$$

Noticing the trigonometric relationship,

$$\sqrt{\frac{\cos^2(\theta)+\sin^2(\theta)}{\sin^2(\mu)} - 1} = \sqrt{\frac{1}{\sin^2(\mu)} - 1} = \sqrt{\csc^2(\mu) - 1} = \sqrt{\cot^2(\mu)} = \frac{1}{\tan(\mu)} \tag{14}$$

Eq. 13 then becomes:

$$\left(\frac{dy}{dx}\right)_{char} = \frac{-\frac{\cos(\theta)\sin(\theta)}{\sin^2(\mu)} \pm \frac{1}{\tan(\mu)}}{1-\frac{\cos^2(\theta)}{\sin^2(\mu)}} \tag{15}$$

When simplified through trigonometric manipulation, Eq. 15 becomes:

$$\left(\frac{dy}{dx}\right)_{char} = \tan(\theta \mp \mu) \tag{16}$$

The above derivations considered the consequences of setting the denominator, D, to be equal to zero in Eq. 7. In order to make the derivatives finite but indeterminate, as previously discussed, the numerator, N, is also set to be equal to zero. The following equations analyze those consequences.

$$\left(1-\frac{u^2}{a^2}\right)dudy+\left(1-\frac{v^2}{a^2}\right)dxdv=0 \tag{17}$$

Rearranging and combining Eq. 17 with Eq. 10,

$$\frac{dv}{du}=-\frac{\left(1-\frac{u^2}{a^2}\right)}{\left(1-\frac{v^2}{a^2}\right)}\left[\frac{-\frac{uv}{a^2}\pm\sqrt{\frac{(u^2+v^2)}{a^2}-1}}{\left(1-\frac{u^2}{a^2}\right)}\right] \tag{18}$$

This simplifies to:

$$\frac{dv}{du}=\frac{\frac{uv}{a^2}\pm\sqrt{\frac{(u^2+v^2)}{a^2}-1}}{\left(1-\frac{v^2}{a^2}\right)} \tag{19}$$

The relationships in Eq. 12 are used to simply further to:

$$\frac{d(Vsin(\theta))}{d(Vcos(\theta))}=\frac{M^2cos(\theta)\sin(\theta)\mp\sqrt{M^2-1}}{1-M^2\sin^2(\theta)} \tag{20}$$

which finally results in the following.

$$d(\theta)=\mp\sqrt{M^2-1}\frac{dV}{V} \tag{21}$$

This equation applies only along characteristic lines. Noticing the order of the $\mp$ signs, the positive value clearly applies along C+ and the negative along C-. Equation 21 is realized to be identical to the Prandtl-Meyer flow equation. Integrating, the following relationships are developed.

$$\theta\pm\nu(M)=K_{\mp} \tag{22}$$

In Equation 22, the K- and K+ values are constants of integration which apply along the C- and C+ lines, respectively. These represent the compatibility equations along the characteristic lines.

Now that the slopes of characteristic lines and the compatibility equations along those lines have been established, the method can be applied to a flow problem. Figure 2 below shows how a point, point 3, can be found using from the intersection of a C+ and C- line coming from two previous points, points 1 and 2. If the positions and slopes of the characteristic lines are known at points 1 and 2, their intersection can be found easily. Then the compatibility equations can be used to calculate the flow parameters, θ and μ, at the new point. Once these are known the Mach number can be found, and based on that Mach number, parameters like temperature and pressure ratios can be found.

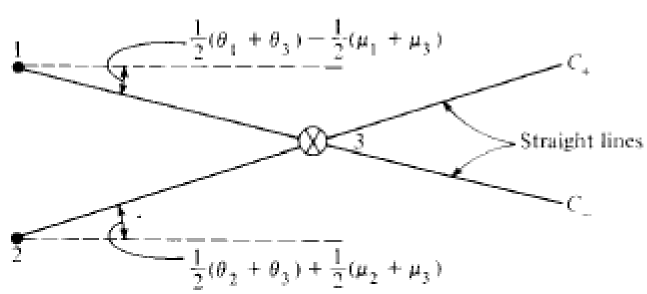

Figure 2. Determination of points in a flow field using the Method of Characterstics.[2]

The following equations are a manipulation of the characteristic slope equations, Eq. 16, and the compatibility equations, Eq. 22. In the following format, flow values at points are readily calculated form those at previous points.

$$\theta_3 = \frac{1}{2}[(K_-)_1 + (K_+)_2] \tag{23}$$

$$\nu_3 = \frac{1}{2}[(K_-)_1 - (K_+)_2] \tag{24}$$

Finally, this entire procedure is applied to the problem of defining the contour of a minimum length converging-diverging nozzle. Figure 3 below shows the various sections of a converging-diverging nozzle. For the purpose of this investigation, only the diverging section will be considered. That section consists of an expansion region and a straightening region. The expansion region allows the flow to gradually expand over a long distance before the straightening region forces the flow into the desired flow direction. The two sections are separated by an inflection point in the contour of the diverging section wall. Because practical factors of size and weight limit the length of a nozzle, the expansion zone is often collapsed to a single point.

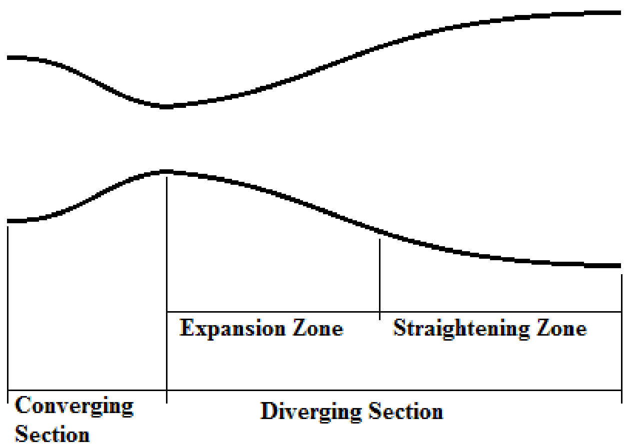

Figure 3. Different regions of a converging-diverging nozzle.

Figure 4 below shows an example of the expansion region being collapsed to a single point. At this point, right at the throat, the flow encounters a sudden change in flow angle and an expansion fan is created. This expansion fan is used as the initially known condition for the Method of Characteristics. The equations developed of course only are valid along characteristic lines, which are known to be Mach lines. An expansion fan is essentially an infinite number of Mach lines emanating from a single point at all angles. As a result, the slopes of characteristic lines starting at this throat point can be arbitrarily selected, because in any direction the line will be a Mach line. For simplicity only one half of the nozzle design will be considered. This is allowable based on simple symmetry. Additionally, the number of characteristic lines initially created is arbitrary. More lines will improve the accuracy of the solution but can have a high cost in computing time.

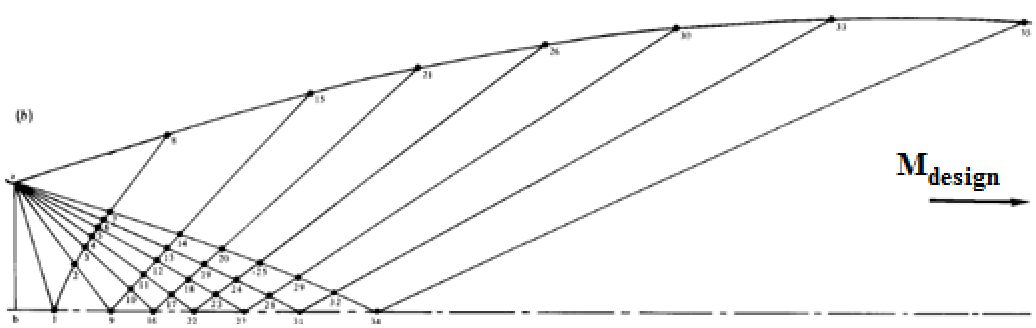

Figure 4. Illustration of the application of the Method of Characteristics to a minimum length nozzle.[2]

The slopes are known at the throat point but the initial values of θ and μ still need to be determined. In Figure 4, the final point on the nozzle will have flow properties based on the design Mach number. The following considers the characteristic line from this point to the centerline of the nozzle and to the throat wall point. Figure 5 below is used to illustrate the determination of initial points at the throat in terms of design conditions required at the exit.

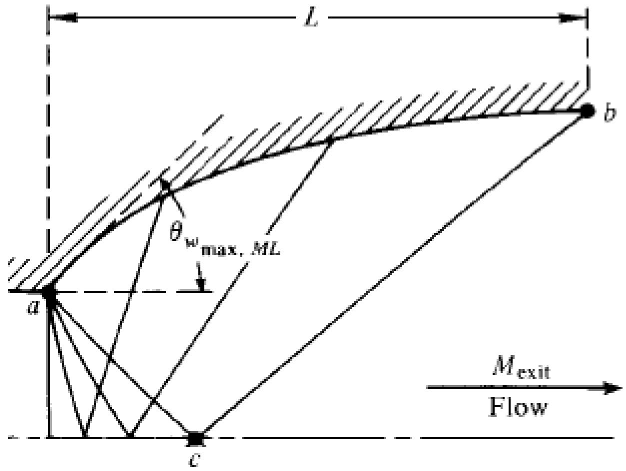

Figure 5. Determination of initial throat conditions.[2]

Along the characteristic line c-b, knowing that the flow deflection, θ_b, will be zero at the exit and all flow along the centerline has a flow deflection, θ_c, of zero:

$$\theta_c - \nu_c = (K_+)_c = \theta_b - \nu_b \tag{23}$$

$$\nu_c = \nu_b = \upsilon(M_{design}) \tag{24}$$

Now considering the characteristic line a-c,

$$\theta_a + \nu_a = (K_-)_a = \theta_c + \nu_c = \nu_{M_{design}} \tag{25}$$

Additionally the flow at the throat, point a, is known to be chocked so M=1 and, by definition, $\upsilon(M = 1) = \theta_{wall,max}$. As a result, Eq. 25 becomes:

$$\theta_{wall,max} + \theta_{wall,max} = \nu_{M_{design}} \tag{26}$$

So now it is known that:

$$\theta_{wall,max} = \frac{\nu_{M_{design}}}{2} \tag{27}$$

Now that all conditions are known at the throat, characteristic lines of arbitrary slope, as previously described, can be drawn and reflected off the centerline due to symmetry. A grid of points based on the known slopes and compatibility equations is calculated, one at a time, until the entire flow field is established. All that remains to be done is to determine when the right running C+ lines terminate into a wall point. Once this is done, the contour of the desired isentropic nozzle will have been found.

At intersections of C+ lines and the wall of the nozzle, it is known that the flow deflection angle θ will be equal to zero. This is because if the angle of the wall is smaller than the flow deflection angle, the characteristic line will reflect back a compression wave and if the wall angle is larger than the flow angle, an entirely new expansion fan will be generated. Because the nozzle is desired to be isentropic, compression shocks and new expansion fans are not wanted. Therefore the wall angle will be set exactly equal to the flow angle of the previous node. The position of the wall point is found from the intersection of the C+ line coming from that node and the previous wall point. The previous wall point will have a slope that is an average of the flow deflection angle at the previous wall point and the wall point that is being calculated, as follows:

$$\left(\frac{dy}{dx}\right)_{previous\ wall\ point} = \tan\left(\frac{\theta_{previous\ wall\ point}+\theta_{current\ wall\ point}}{2}\right) \tag{28}$$

This allows for the locations of all the wall points to be calculated. All that remains to be done is to calculate the area ratio of the flow for comparison to the theoretical value. The area ratio of the designed nozzle contour will be the y-coordinate of the final wall point divided by the y-coordinate of the throat. This is compared to the theoretical area ratio based on isentropic flow, as follows:

$$\frac{A}{A^*} = \left(\frac{\gamma+1}{2}\right)^{\frac{-(\gamma+1)}{2(\gamma-1)}} \frac{1}{M}\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{(\gamma+1)}{2(\gamma-1)}} \tag{29}$$

IV. Results and Discussion

The objective of this investigation was to develop minimum length supersonic nozzles using the Method of Characteristics. A Matlab script was developed to do this and results are evaluated for a design Mach number of M=3.2 using different numbers of initial lines. Figure 6 shows results of the wall contour and the intersections of characteristic lines for n=7 total characteristic lines and a design Mach of 3.2. Similarly, Figure 7 shows results of the wall contour and the intersections of characteristic lines for n=100 total characteristic lines and a design Mach of 3.2.

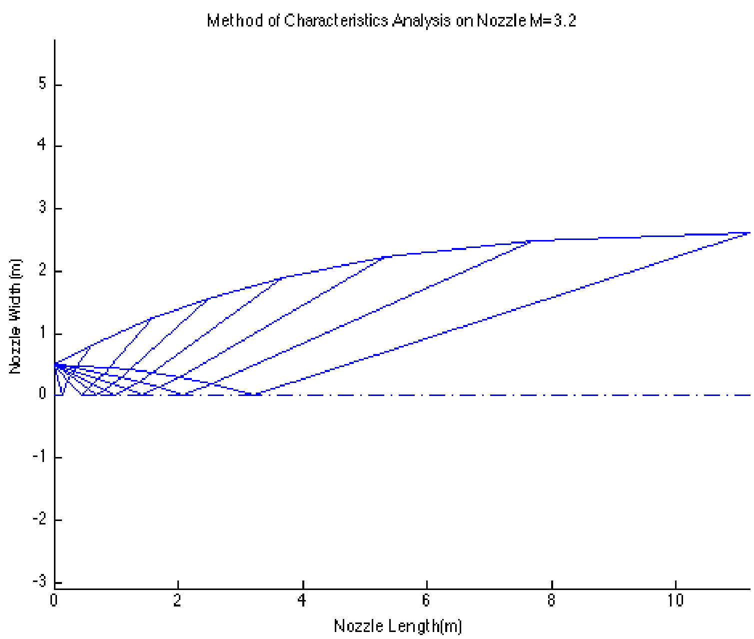

Figure 6. Results of M=3.2 nozzle contour with 7 characteristic lines.

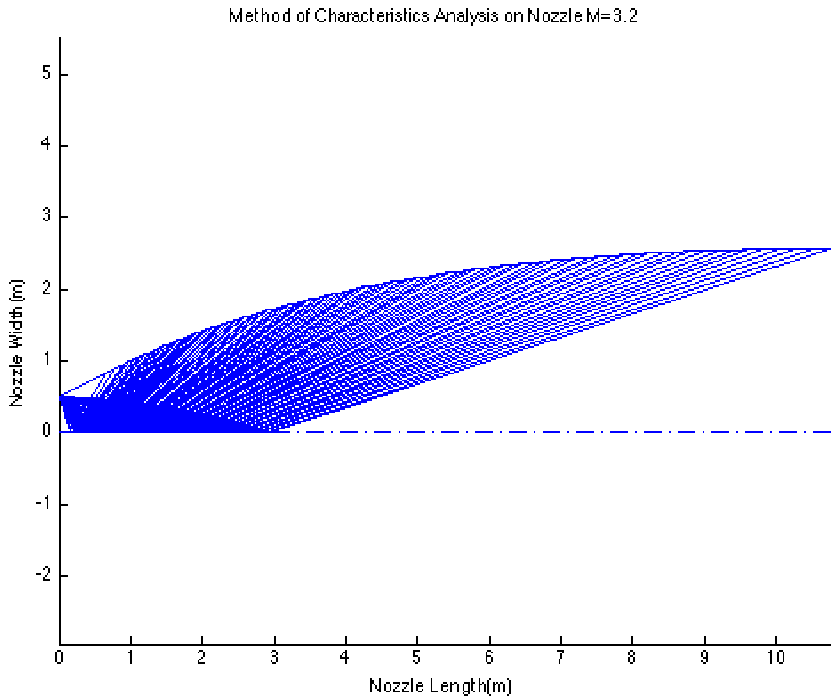

Figure 7. Results of M=3.2 nozzle contour with 100 characteristic lines.

Figure 8 below shows the Mach numbers calculated for all the interior flow points, using interpolated shading. As expected, the flow at the throat has Mach number M=1 and the final wall point has Mach number M=3.2, the design value.

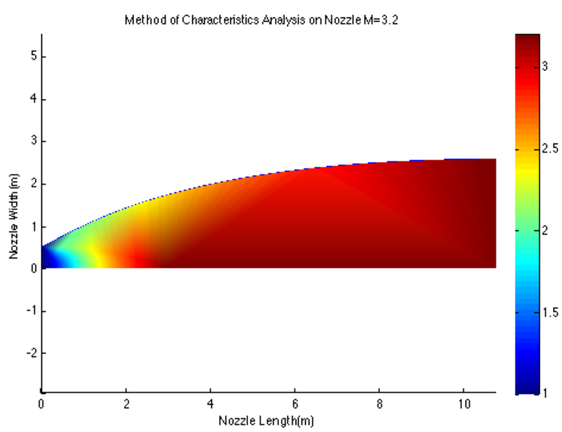

Figure 8. Results of Mach number in the nozzle flow.

Table 1 below shows the values calculated at each point in the flow field, as a numerical demonstration of the derived equations. For simplicity, n=7 lines are used for the table.

Table 1. Flow and Characteristic Line Properties for design M=3.2 and n=7 lines.

Point Number	K- = θ+υ	K+ = θ-υ	Θ=½(K- + K+)	υ=½(K- - K+)	M	μ
1	0.75	0	0.375	0.375	1.04	74.1
2	6.75	0	3.375	3.375	1.19	57.2
3	12.75	0	6.375	6.375	1.31	49.8
4	18.75	0	9.375	9.375	1.41	45.2
5	24.75	0	12.375	12.375	1.52	41.1
6	30.75	0	15.375	15.375	1.62	38.1
7	36.75	0	18.375	18.375	1.72	35.6
8	36.75	0	18.375	18.375	1.72	35.6
9	6.75	-6.75	0	6.75	1.32	49.3
10	12.75	-6.75	3	9.75	1.43	44.4
11	18.75	-6.75	6	12.75	1.53	40.8
12	24.75	-6.75	9	15.75	1.63	37.8
13	30.75	-6.75	12	18.75	1.73	35.3
14	36.75	-6.75	15	21.75	1.84	32.9
15	36.75	-6.75	15	21.75	1.84	32.9
16	12.75	-12.75	0	24.75	1.53	40.8
17	18.75	-12.75	3	24.75	1.63	37.8
18	24.75	-12.75	6	18.75	1.73	35.3
19	30.75	-12.75	9	21.75	1.84	32.0
20	36.75	-12.75	12	24.75	1.94	31.0
21	36.75	-12.75	12	24.75	1.94	31.0
22	18.75	-12.75	0	18.75	1.73	35.3
23	24.75	-12.75	3	21.75	1.84	32.9
24	30.75	-12.75	6	24.75	1.94	31.0
25	36.75	-12.75	9	27.75	2.05	29.2
26	36.75	-12.75	9	27.75	2.05	29.2
27	24.75	-24.75	0	24.75	1.94	31.0
28	30.75	-24.75	3	27.75	2.05	29.2
29	36.75	-24.75	6	30.75	2.16	27.6
30	36.75	-24.75	6	30.75	2.16	27.6
31	30.75	-30.75	0	30.75	2.16	27.6
32	36.75	-30.75	3	33.75	2.28	26.0
33	36.75	-30.75	3	33.75	2.28	26.0
34	36.75	36.75	0	36.75	2.4	24.6
35	36.75	36.75	0	36.75	2.4	24.6

The previous figures and table valid the method and code. After this, a study of the number of initial characteristic lines was performed for the same design Mach of M=3.2. Figure 9 shows the area ratios for various numbers of initial characteristic lines. For comparison, the theoretical value as calculated with isentropic flow equations is shown in green. The results indicate that an acceptably accurate answer can be found with only about n=20 initial characteristic lines. After about n=40 lines, the results converge so close to the theoretical answer that additional lines would be a waste of computing time.

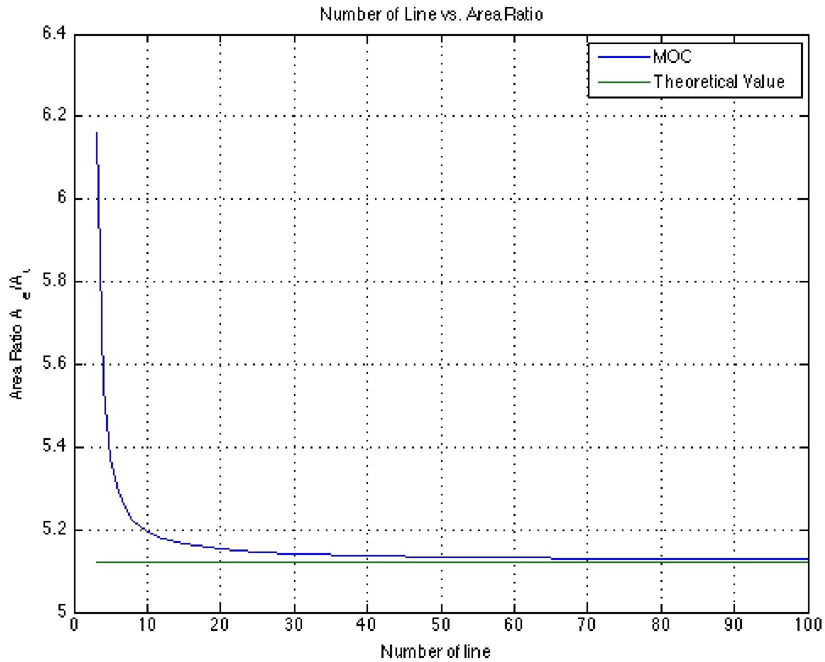

Figure 9. Calculated area ratios vs number of initial lines.

Figure 10 below shows the percentage of error in the results compared to the theoretical value. Error is extremely low with n=40 or more lines are used. However, results also showed that even very small numbers of lines can produce accurate solutions as long as at least one characteristic line intersects the center line very close to the throat. As previously discussed, the initial slopes can be set arbitrarily for an expansion fan. Making one of those slopes very close to vertical places a node close to the throat. In reality, a Mach line is perfectly vertical at the throat. Placing a characteristic (Mach) line as close as possible to the throat reduces error and can allow for accurate solutions with even small numbers of characteristic lines.

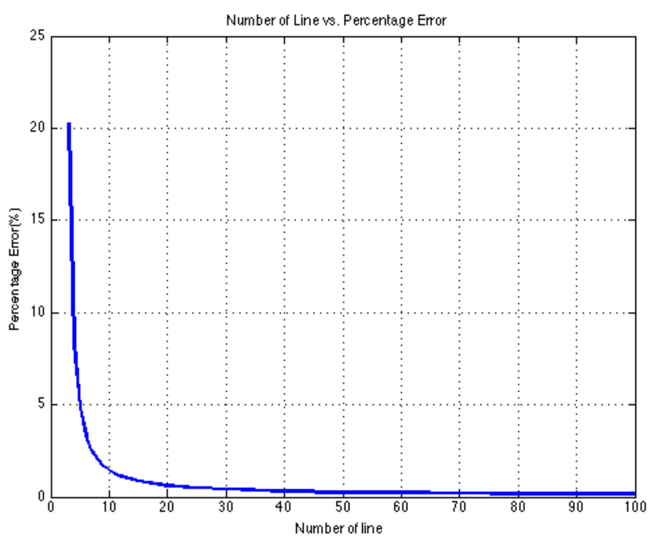

Figure 10. Error in calculated area ratios vs number of initial lines.

Figure 11 below shows the length of the contours being produced with this method. As previously discussed, the answer quickly converges to a value after about n=40 lines. After this, additional lines and computing time does not yield much change in the minimum length. For the design condition of M=3.2 at the exit, area ratios and lengths are shown in Table 2.

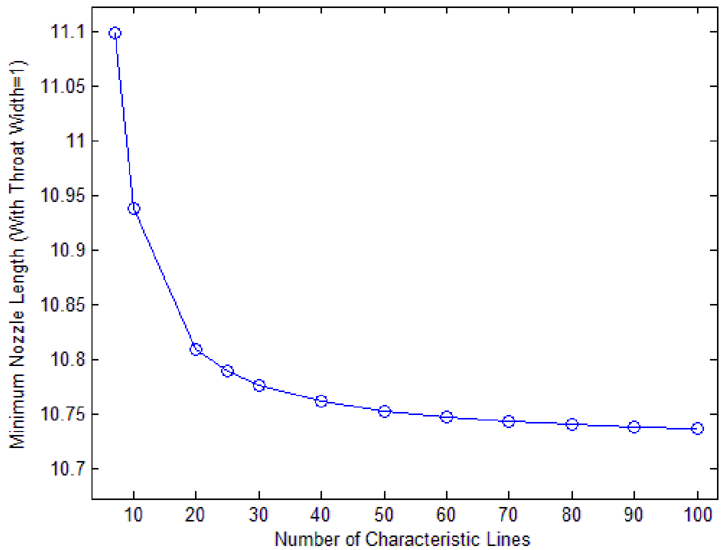

Figure 11. Minimum length of nozzle (M=3.2) vs. number of initial lines.

Table 2. Area ratios and lengths for M=3.2 nozzle.

	n=7	n=25	n=100
Area Ratio	5.2498	5.1457	5.1288
Minimum Length (for throat of total height 1)	1,1925	10.8234	10.7489
Error % (compare to 5.1210)	2.5175	0.4840	0.1506

V. Conclusions

The objective of this investigation was to develop the equations of the Method of Characteristics and produce nozzle contours for a given design Mach number of M=3.2. The method was found to be very accurate with an area ratio error of 0.4045% with n=30 characteristic lines. The nozzle contours developed accurately represent those that could be manufactured and used to produce supersonic flow at the given Mach number and are very close to isentropic. It is recommended that at least 30 lines be used for calculations, as this amount provides a very accurate answer with little computing time required.

Appendix

1. Published Matlab Code:

```
close all
clc
clear all
format compact
tic;
%% Aero 401 Project 2
%% Name: Kerry Sun
% Method of Characteristics
% This code gives you the minimal length to reach the desired exit Mach
% Number, given the following:
% 1. Mach at exit
% 2. gamma
% 3. Numbers of characteristic line
% 4. Throat area
%% Given:
M_exit =3.2; % Mach exit number
gamma = 1.4; % gamma
N=7;        % numbers of characteristic line
a = [0 .5];  % Throat area (half of the nozzle)
%%
[M_exit, nu_exit, mu_exit] = flowprandtlmeyer(gamma,M_exit,'Mach');
theta_max = 0.5*nu_exit;
theta_initial = 0.375;%  or theta_max/N;
for i = 1:N
    line(i) =
struct('theta',[],'nu',[],'Mach',[],'Kplus',[],'Kminus',[],'mu',[],'slope_p',
[],'slope_m',[],'coords',[]);
end
%% Line 1, the initial expansion line
n = N;
% since this is from the wall, all K+ values are 0;
line(1).Kplus = zeros(1,n);
line(1).theta = linspace(theta_initial,theta_max,n);
for i = 1:n
    line(1).nu(i) = line(1).theta(i)-line(1).Kplus(i); % find nu values from
K+ line values

    [line(1).Mach(i) line(1).nu(i) line(1).mu(i)] =
flowprandtlmeyer(1.4,line(1).nu(i),'nu'); % Mach and Mach angle from Prandtl
Meyer angle
    line(1).Kminus(i) = line(1).theta(i) + line(1).nu(i); % Find K- values
end
i=n;
line(1).Kplus(i+1) = line(1).Kplus(n);line(1).nu(i+1) = line(1).nu(n);
line(1).Kminus(i+1) = line(1).Kminus(n);line(1).mu(i+1) = line(1).mu(n);
line(1).Mach(i+1) = line(1).Mach(n);line(1).theta(i+1) = line(1).theta(n);

%%


X = 1;
n = n-1;
while n >= 1
    X = X+1;
```

```
    line(X).Kminus(1) = line(X-1).Kminus(2); line(X).theta(1) = 0;
    line(X).Kplus(1) = line(X).theta(1)*2-line(X).Kminus(1);
    line(X).nu(1) = 0.5*(line(X).Kminus(1)-line(X).Kplus(1));
    [line(X).Mach(1) line(X).nu(1) line(X).mu(1)] =
flowprandtlmeyer(1.4,line(X).nu(1),'nu');
    if n>=2
        for i = 2:1:n
            line(X).Kminus(i) = line(X-1).Kminus(i+1);
            line(X).Kplus(i) = line(X).Kplus(i-1);
            line(X).theta(i) = 0.5*(line(X).Kplus(i)+line(X).Kminus(i));
            line(X).nu(i) = 0.5*(line(X).Kminus(i)-line(X).Kplus(i));
            [line(X).Mach(i) line(X).nu(i) line(X).mu(i)] =
flowprandtlmeyer(1.4,line(X).nu(i),'nu');
        end
    end
    line(X).theta(n+1) = line(X).theta(n);line(X).Kplus(n+1) =
line(X).Kplus(n);
    line(X).Kminus(n+1) = line(X).Kminus(n); line(X).nu(n+1) = line(X).nu(n);
    line(X).mu(n+1) = line(X).mu(n);line(X).Mach(n+1) = line(X).Mach(n);
    n = n-1;
end

%% Slopes
for i = 1:N % Line number
    for k = 1:length(line(i).theta)
        line(i).slope_p(k) = tand(line(i).theta(k)+line(i).mu(k));
        line(i).slope_m(k) = tand(line(i).theta(k)-line(i).mu(k));
    end
end

%% all the coordinates inside
line(1).coords(1,:) = intersect(line(1).slope_m(1),a, 0, [0,0]);
for i = 2:length(line(1).theta)-1
    line(1).coords(i,:) = intersect(line(1).slope_m(i),a, line(1).slope_p(i-
1), line(1).coords(i-1,:));
end
for i = 2:N
    line(i).coords(1,:) = intersect(line(i-1).slope_m(2),line(i-
1).coords(2,:),0,[0,0]);
    for k = 2:length(line(i).theta)-1
        line(i).coords(k,:) = intersect(line(i-1).slope_m(k+1),line(i-
1).coords(k+1,:),line(i).slope_p(k-1),line(i).coords(k-1,:));
    end
end

%% Wall point

line(1).coords(N+2-1,:) = intersect(tand(theta_max),a,line(1).slope_p(N+2-
1),line(1).coords(N+2-2,:));
for i = 1:N-1
    point1 = line(i).coords(N+2-i,:); slope1 = tand(line(i).theta(N+2-i)/2 +
line(i+1).theta(N+2-(i+1)-1)/2);
    point2 = line(i+1).coords(N+2-(i+1)-1,:); slope2 = line(i+1).slope_p(N+2-
(i+1)-1);
    line(i+1).coords(N+2-(i+1),:) = intersect(slope1,point1,slope2,point2);
end
```

```
%%
figure(1)
hold on
axis equal
xlabel('Nozzle Length(m)')
ylabel('Nozzle Width(m)')
title('Method of Characteristics Analysis on Nozzle M=3.2')
for i = 1:1:N
    plot([a(1) line(1).coords(i,1)],[a(2) line(1).coords(i,2)],'b-');
end

for i = 1:1:N
    if i == 1
        plot([a(1) line(i).coords(N+2-i,1)], [a(2) line(i).coords(N+2-
i,2)],'b-');
        %plot3([a(1) line(i).coords(N+2-i,1)], [a(2) line(i).coords(N+2-
i,2)],[M_exit M_exit],'b-');
    else
        for k = 1:1:length(line(i).theta)
        plot([line(i-1).coords(k+1,1) line(i).coords(k,1)],[line(i-
1).coords(k+1,2) line(i).coords(k,2)],'b-');
            %plot3([line(i-1).coords(j+1,1) line(i).coords(j,1)],[line(i-
1).coords(j+1,2) line(i).coords(j,2)],[M_exit M_exit],'b-');
        end
    end
    for k = 1:1:N+2-i-1
        plot([line(i).coords(k,1) line(i).coords(k+1,1)],[line(i).coords(k,2)
line(i).coords(k+1,2)],'b-');
        %plot3([line(i).coords(j,1)
line(i).coords(j+1,1)],[line(i).coords(j,2) line(i).coords(j+1,2)],[M_exit
M_exit],'b-');
    end
end
% plot center line
for i=1:1:N-1
    if i==1
     %   plot([0 line(1).coords(i,1)], [0 line(1).coords(i,2)],'b-.');
    %plot3([0 line(1).coords(i,1)], [0 line(1).coords(i,2)],[M_exit
M_exit],'b-.'); %(0,0) to first line
    else
        plot([line(i+1).coords(1,1) line(i).coords(1,1)],
[line(i+1).coords(1,2) line(i).coords(1,2)],'b-.');
    %plot3([line(i+1).coords(1,1) line(i).coords(1,1)],
[line(i+1).coords(1,2) line(i).coords(1,2)],[M_exit M_exit],'b-.');
    end
end

plot([line(N).coords(end,1) line(N).coords(1,1)], [0 line(N).coords(1,2)],'b-
.');
%plot3([line(N).coords(end,1) line(N).coords(1,1)], [0
line(N).coords(1,2)],[M_exit M_exit],'b-.');

%% Fill the plots with color
coords = [];
Mach = [];
for i = 1:N
```

```
    coords = [coords; line(i).coords];
    Mach = [Mach; line(i).Mach'];
end
coords(N+1,1)=0;
coords(N+1,2)=0;
coords(N+2,1)=a(1);
coords(N+2,2)=a(2);
coords(N+3,1)=line(N).coords(end,1);
coords(N+3,2)=0;
Mach(N+1)=1;
Mach(N+2)=1;
Mach(N+3)=M_exit;
tri=delaunay(coords(:,1),coords(:,2));
h=trisurf(tri,coords(:,1),coords(:,2),Mach);
shading interp
colorbar

%% Area ratio calculation and comparison, minimium length
Length=line(N).coords(end,1);
A_exit_Area_ratio=line(N).coords(end,2)/a(2);
[~,~,~,~,A_theo]=flowisentropic(gamma,M_exit,'mach');
error=abs(A_exit_Area_ratio-A_theo)/A_theo*100;
time=toc;
disp('Results:')
disp('                                          ')
fprintf('Number of line: %f \n',N)
fprintf('Time running the code: %f secs \n',time)
fprintf('Throat area length: %f meters \n',2*a(2))
fprintf('Minimum length of the nozzle: %f meters\n',Length);
fprintf('A exit Area ratio: %f \n',A_exit_Area_ratio);
fprintf('Percentage error: %f%%\n',error);
%%
Line=[3,4,5,6,7,8,9,10,11,12,15,20,25,30,40,50,60,70,80,90,100]';
A=[6.161599,5.546148,5.369534,5.291912,5.249879,5.224049,...
   5.206766,5.194473,5.185318,5.178252,5.164308,5.152240,5.145744,...
   5.141674,5.136836,5.134047,5.132228,5.130947,5.129994,5.129257,5.128671]';
err=[20.321235,8.302954,4.854093,3.338331,2.517520,2.013129,...
     1.675640,1.435585,1.256808,1.118832,0.846522,0.610876,0.484024,...
     0.404543,0.310070,0.255611,0.220094,0.195064,0.176458,0.162078,...
     0.150626]';
 figure(2)
 A_theo1=zeros(21,1);
 for i=1:length(A_theo1)
     A_theo1(i)=A_theo;
 end
 plot(Line,A,Line,A_theo1,'LineWidth',1)
 xlabel('Number of line')
 ylabel('Area Ratio A_e/A_t')
 title('Number of Line vs. Area Ratio')
 grid on
 legend('MOC','Theoretical Value')
 figure(3)
 plot(Line,err,'LineWidth',2)
 xlabel('Number of line')
 ylabel('Percentage Error(%)')
 title('Number of Line vs. Percentage Error')
 grid on
```

```
% Finds the intersection of two lines, given slope and point in x-y plane
% using Cramer's rule
function x4 = intersect(m1, x1, m2, x2)
    x3 = det([-m1*x1(1)+x1(2),1; -m2*x2(1)+x2(2),1])/det([-m1,1; -m2,1]);
    y3 = det([-m1,-m1*x1(1)+x1(2); -m2,-m2*x2(1)+x2(2)])/det([-m1,1; -m2,1]);

    x4 = [x3 y3];
end
```

2. Sample Calculation

Given Designed Mach number: 3.2

$$\nu(3.2) = \sqrt{\frac{1.4+1}{1.4-1}\tan^{-1}\frac{1.4-1}{1.4+1}(3.2^2-1)} - \tan^{-1}\sqrt{3.2^2-1} = 53.47^\circ$$

$$\theta_{wall\ max} = \frac{\nu_M}{2} = \frac{53.47^\circ}{2} = 26.74^\circ$$

For line n = 7:

$$\Delta\theta = \frac{\theta_{wall\ max}}{7} = \frac{26.74^\circ}{7} = 3.82^\circ$$

For high accuracy, $\Delta\theta = 0.375^\circ$ was chosen.

$$\theta_a = \Delta\theta = 0.375^\circ$$

$$\theta_a + \nu_{a,after} = \theta_a + (\theta_a + \nu_{a,before}) = 2\theta_a + 0 = 2 \cdot 0.375^\circ = 0.75^\circ = (K_-)_a$$

$$(K_-)_1 = (K_-)_a = 0.75$$

$$(K_+)_1 = \theta_a - \nu_{a,after} = 0$$

$$\theta_1 = \frac{(K_-)_1 + (K_+)_1}{2} = 0.375^\circ$$

$$\nu_1 = \frac{(K_-)_1 - (K_+)_1}{2} = 0.375^\circ$$

With prandtl meyer function:

$$M_1 = 1.04, \mu_1 = 74.1$$

To find intersection point, the following steps are done:

$$\theta_1 + \nu_1 = (K_-)_1 = 0.75$$

$$\theta_2 - \nu_2 = (K_+)_2 = -9.5367$$

$$\theta_3 + \nu_3 = (K_+)_3 = (K_+)_2$$

$$\theta_3 - \nu_3 = (K_-)_3 = (K_-)_1$$

$$\theta_3 = \frac{(K_-)_1 + (K_+)_2}{2} = -4.3933$$

$$\nu_3 = \frac{(K_-)_1 - (K_+)_2}{2} = 0.6165$$

$$\left(\frac{A}{A^*}\right)_{theo} = \left(\frac{\gamma+1}{2}\right)^{\frac{-(\gamma+1)}{2(\gamma-1)}} \frac{1}{M}\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{(\gamma+1)}{2(\gamma-1)}} = \left(\frac{1.4+1}{2}\right)^{\frac{-(1.4+1)}{2(1.4-1)}} \frac{1}{3.2}\left(1+\frac{1.4-1}{2}M^2\right)^{\frac{(1.4+1)}{2(1.4-1)}} = 5.1210$$

$$\left(\frac{A}{A^*}\right)_{Line=7} = 5.2498$$

$$error = \frac{\left(\frac{A}{A^*}\right)_{Line=7} - \left(\frac{A}{A^*}\right)_{theo}}{\left(\frac{A}{A^*}\right)_{theo}} = \frac{5.2498 - 5.1210}{5.1210} \cdot 100\% = 2.518\%$$

References

[1]Mehta, Piyush, *Rocket Final Project: Design of a Minimum Length 2D Supersonic Nozzle*, Cal Poly, 2013.
[2]Anderson, John David. "Modern Compressible Flow with Historical Perspective." 3rd Ed.