Robustness of the Mean-Variance Model with Truncated Probability Distributions

Author(s): Steven D. Hanson and George W. Ladd

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# Robustness of the Mean-Variance Model with Truncated Probability Distributions


## Steven D. Hanson and George W. Ladd



The known sufficient conditions for the mean-variance framework to produce expected utility results are violated in the presence of truncated probability distributions. A theoretical simulation is conducted to examine the ability of the linear mean-variance model to approximate expected utility results when the income distribution is truncated by the use of commodity option contracts. The mean-variance model is shown to produce solutions that are close approximations to the expected utility model results under the assumptions of constant absolute risk aversion and normally distributed prices. However, some inconsistency was found between the comparative static results of the two models.



*Key words:* futures, mean-variance, options, risk management.


The mean-variance (MV) framework has been a popular method for ordering choices into efficient and inefficient sets since its development by Markowitz. The problem of portfolio selection is reduced to choosing from a set of alternatives that provide minimum variance for given levels of expected return (i.e., the MV efficient set). The MV framework has been used to study a wide range of economic decisions made under risk such as the allocation of fixed assets in the presence of uncertain production processes (Freund), the demand for money (Tobin), and corporate financial decisions (Rubenstein). For a review of applications of mean-variance analysis in agricultural economics, see Robison and Brake; and Musser, Mapp, and Barry. Much of the previous research on risk management with futures contracts has employed the MV framework (e.g., see Anderson and Danthine, Kahl). Recently, commodity options contracts have become available as a risk management tool. Wolf has incorporated options contracts into an MV framework.


Steven D. Hanson is an assistant professor of agricultural economics at Michigan State University; George W. Ladd is a professor of economics and Charles F. Curtiss Distinguished Professor in Agriculture, Iowa State University.



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The inclusion of options in a decision maker's (DM's) portfolio results in truncated probability distributions that violate the standard assumptions used to justify the use of an MV representation of expected utility. Indeed, a variety of risky decision problems can be characterized by truncated probability distributions: bankruptcy, starvation, insurance schemes, limited liability, and price support and deficiency payment programs. This study examines the approximating ability of the standard linear MV model when the income probability distribution is truncated by the use of commodity options; it examines the robustness of the linear MV model to a violation of MV assumptions.

### The Mean-Variance Model

The MV framework focuses on the first two moments of the underlying probability distribution. Its popularity can be traced to the tractable theoretical results it produces and to its computational convenience. Unfortunately, its straightforward nature results from some restrictive assumptions that are violated when options are included in a portfolio.

Specifying an expected utility function in terms of the first two moments of the underlying attribute's distribution has been shown to be consistent with the expected utility hypothesis only if at least one of the following sufficient conditions is met: (*a*) the DM's utility function is quadratic (Tobin), (*b*) the DM has a concave utility function, and the random attribute is normally distributed (Samuelson), (*c*) the random attribute is a monotonic linear function of a single random variable (Meyer); sometimes referred to as the location-scale condition. Because these are only sufficient conditions, the MV framework may be consistent with the expected utility (EU) hypothesis under other conditions.

These sufficient conditions are often argued to be either violated or unacceptable. Baron shows that quadratic utility is a necessary and sufficient condition for MV representation of expected utility to be valid when the random variables are not restricted. Unfortunately, quadratic utility usually is viewed as an unacceptable preference function because it implies increasing absolute risk aversion and produces negative marginal utility as income rises above some given level.

Some have justified the use of the MV framework by assuming that condition (*b*) is satisfied (e.g., Anderson and Danthine). Others have criticized this assumption because few random variables take on values from negative to positive infinity and are symmetrically distributed, as is implied by the normality assumption.

Many utility functions are concave functions of income and can be used with a normally distributed income variable to yield an MV representation of the expected utility function. Only the negative exponential utility function, $U(y) = \alpha - \exp(-Ay)$, where $\alpha > 0$, in conjunction with a normally distributed attribute, $y$, will result in an expected utility function identical to the "standard" linear MV model, which is typically specified as[1]

$$(1) \qquad \underset{x}{\text{Max}}\ \hat{U} = \mu_y - (A/2)\,\sigma_y^2,$$

where $\mu_y$ is expected value of end-of-period income, $A$ is the Arrow-Pratt measure of absolute risk aversion, $\sigma_y^2$ is variance of end-of-period income, and $x$ is a vector of choice variables. The negative exponential utility function implies that the DM exhibits constant absolute risk aversion (CARA) and increasing relative risk aversion. The literature on risk preference characteristics is somewhat inconclusive, although there is some evidence that decision makers exhibit decreasing absolute risk aversion (King and Robison). Regardless, most studies have been willing to accept the assumption of CARA to obtain the desirable properties of the standard MV model, although not all studies employing MV have used the negative exponential function.

Meyer and Robison have used condition (*c*) to justify the MV framework. Meyer has shown that the MV and EU frameworks are consistent regardless of the distribution of the random variable whenever a random outcome variable is a monotonic linear function of a single underlying random variable. Current work by Nelson is attempting to generalize Meyer's results to a class of models containing multivariate risks which are elliptically symmetric.

Some risky decision-making environments violate the known sufficient conditions for consistency. Newbery and Stiglitz recognize that risk is not symmetric for some decisions such as insurance schemes, and MV analysis may be seriously misleading for those decisions. One implication is that the MV framework may produce significant errors when the income distribution is truncated in the tails.

Much of the debate involving MV and EU models has focused on the approximating ability of the MV model. Levy and Markowitz show that the MV model provides reasonable approximations to EU maximization when the opportunity set contains a low probability of extreme outcomes. Tsiang shows that an MV framework approximates the EU maximization model when certain restrictions are placed on the skewness of the income distribution. Meyer and Rache show that, with "few enough" observations, there is no statistically significant difference between MV and EU rankings because the random alternatives are estimated imprecisely. Porter shows that MV sets of randomly constructed stock portfolios are consistent with EU models, except for portfolios having small means and variances. Tew and Reid explore the performance of the MV framework using firm specific, nontradable investments for a representative farm in Georgia. Their results indicate the MV framework performs well in selecting investments that maximize expected utility. Robison and Barry argue for the use of the MV model on the basis of its ability to produce useful analytical results when more general models become too complex. Thus, there is some support, both empirical and deductive, for use of the MV framework even when its sufficient conditions are

[1] Robison and Barry formulate a linear MV model by claiming that $A$ is the slope of a linear tangent at equilibrium to the isoutility line and its MV-efficient set. The assumptions of constant, increasing, and decreasing absolute risk aversion are then introduced through slope changes.

violated. In what follows, we show that conditions ($b$) and ($c$) are violated when the income distribution is truncated by including options in a portfolio. Then we examine the robustness of the linear MV model in the presence of truncated probability distributions for a DM with constant absolute risk aversion.

## Income Probability Distribution with Commodity Options

To derive the distribution of income when options are available, assume that the DM has the amount $Q$ of a single asset that may be sold at a specified future date in the cash market for some unknown spot price, $P$, or in the futures market for a certain price, $P_f$, or the DM may purchase the option to sell the output for a certain price, $P_e$. The output $Q$ may be divided among the three markets in any way. The DM may sell a total amount in the cash, futures, and options markets that is greater than $Q$, may go long in the futures market, or write put options instead of selling them. There are assumed to be no transaction costs, basis uncertainty, or margin calls.

The only uncertainty faced by the DM is the output price $P$, and the DM can reduce uncertainty by marketing output in the futures or the options market. The DM's choice variables are the amounts marketed in the futures and options markets. The model is single period in nature and assumes that the initial positions are maintained until the end of the period at which time all cash, futures, and options transactions will take place. The quantity of output to be sold, the futures contract selling price, the exercise price, and the put option premium are known, exogenous variables to the DM. The DM's end-of-period income can be represented as equation (2):[2]

$$y(P) = PQ + x(P_f - P) + Iz(P_e - P) - zk, \tag{2}$$

where $y(P)$ is total end-of-period income, $P$ is random end-of-period spot price, $P_f$ is localized futures price for end-of-period delivery, $Q$ is certain end-of-period output level, $I$ is an indicator variable (= 1 if $P < P_e$, or = 0 if $P \geq P_e$), $P_e$ is localized exercise price of a put option, $k$ is put option premium per bushel, $x$ is the number of bushels of futures contracts, $z$ is the number of bushels of put option contracts.

Expression (2) is derived from two distinct income functions:

$$y(P) = PQ + x(P_f - P) + z(P_e - P) - zk \quad \text{if } P < P_e, \tag{3}$$

$$y(P) = PQ + x(P_f - P) - zK \quad \text{if } P \geq P_e \tag{4}$$

by introducing the indicator variable $I$. It is clear from these expressions that sufficient condition (c) for adequacy of the standard MV framework is violated in the presence of options. For example, if $Q - x > 0$ and $Q - x - z < 0$, then $\partial y/\partial P < 0$ for all $P < P_e$ and $\partial y/\partial P \geq 0$ for all $P \geq P_e$. Income is not a monotonic transformation of price; it may be a negative or a positive transformation, which violates condition ($c$).[3]

The DM is assumed to believe that output price is normally distributed with a known mean $\mu_P$ and variance $\sigma_P^2$. Because $P$ is normally distributed and $y(P)$ is a linear function of $P$, if $P < P_e$ the unconditional mean and variance of income are [from (3)]

$$\mu_{y1} = (Q - x - z)\mu_P + xP_f + z(P_e - k),$$

$$\sigma_{y1}^2 = (Q - x - z)^2\sigma_P^2.$$

If $P \geq P_e$, $y(P)$ is normally distributed with unconditional mean and variance $\mu_{y2}$ and $\sigma_{y2}^2$ [from (4)],

$$\mu_{y2} = (Q - x)\mu_P + xP_f - zk,$$

$$\sigma_{y2}^2 = (Q - x)^2\sigma_P^2.$$

Define $y(P_e)$ as the value of $y(P)$ from (3) or from (4) with $P = P_e$. The income density function corresponding to equation (2) can be shown to be[4]

$$w(y) = I_1 \exp[-(y - \mu_{y1})^2/2\sigma_{y1}^2]/\sigma_{y1}\sqrt{2\pi} + I_2 \exp[-(y - \mu_{y2})^2/2\sigma_{y2}^2]/\sigma_{y2}\sqrt{2\pi}, \tag{5}$$

where

$I_1 = 1$ for $y < y(P_e)$ if $(Q - x - z) > 0$,
or for $y \geq y(P_e)$ if $(Q - x - z) \leq 0$,
$= 0$ otherwise;

[2] Commodity options currently traded on the organized exchanges are options on commodity futures contracts. The income function in equation (2) treats the options as options on the spot commodity. The assumption of a certain basis level results in equivalence between the two types of options. That is, if the basis is known with certainty, then an option on a futures contract is equivalent to an option on the underlying spot commodity.

[3] Sufficient condition ($c$) is violated when options are included in a portfolio. However, as Meyer points out, it may be difficult to reject the null hypothesis that sufficient condition ($c$) holds when using empirical data.

[4] The derivation is presented by Hanson.

$$I_2 = 1 \text{ for } y \geq y(P_e) \quad \text{if } (Q - x) \geq 0,$$
$$\text{or for } y < y(P_e,) \quad \text{if } (Q - x) < 0,$$
$$= 0 \text{ otherwise.}$$

The first term of (5) corresponds to equation (3); the second, to equation (4). This distribution is the sum of two truncated normal distributions, for which the mean, variance, and bounds of integration depend on $\mu_P$, $\sigma^2_P$, $P_f$, $P_e$, $k$, $Q$ and the DM's choices of $x$ and $z$. The distribution represented by equation (5) can take on a variety of forms dependent on the DM's positions in the futures and options markets. It is clear from (5) that sufficient condition (*b*) is violated because the income distribution is not normal when the DM takes a position in the options market. Thus, sufficient conditions (*b*) and (*c*) for the standard MV model to produce results that are consistent with expected utility maximization are violated when options contracts are available, and an alternative framework is needed.

## Expected Utility Maximization

If the DM is assumed to maximize expected utility and faces the income function and conditions indicated by equation (2), then the DM chooses futures and options positions to satisfy equation (6):

$$\text{(6)} \qquad \underset{x,z}{\text{MAX}}\, \overline{U} = \int_{-\infty}^{\infty} U(y)w(y)dy,$$

where $\overline{U}$ is the DM's expected utility given the choice of $x$ and $z$; $U(y)$ is the DM's utility function, assumed to exhibit CARA; and $w(y)$ is the income distribution specified by equation (5). Solving equation (6) yields the optimal futures and options positions, $x^*$ and $z^*$. Because equation (6) has no explicit algebraic solutions when the DM uses options, solution techniques involving numerical integration and optimization are required.

## Optimal Hedging in a Mean-Variance Framework

Expression (5) shows that the assumptions underlying the linear MV model are violated when options are held by the DM. The linear MV model may, however, produce acceptable quantitative results or useful insights into the impacts of options on optimal market positions. To investigate this possibility, this section extends the linear MV model to include options.

Three different MV models are used to study the optimal market positions under various situations. Model 1 presents the standard result for a DM who uses only the cash and futures markets and corresponds to the use of a security which does not truncate the income distribution.[5] Model 2 includes the cash and options market and corresponds to the use of a security which truncates the income distribution. Model 3 includes the cash, futures and options markets and corresponds to the use of a security which truncates the income distribution and a security that does not truncate the income distribution. Model 1 contains no truncation problem but models 2 and 3 do.[6] Each model is solved using (1), and implications of the solutions are discussed. As will be seen later, despite the violation of the sufficient conditions for MV consistency with EU, the model does provide useful insights into the numerically generated EU results.

These models use the following additional definitions: $T$ is localized value of the option at the end of the period = max $[0, P_e - P]$; $\mu_T$ is expected value of the option; $\sigma^2_T$ is variance of the value of the option; $\sigma_{P,T}$ is covariance between the value of the option and the spot price.

### *Model 1: Cash and Futures with Certain Output (CF)*

Model 1 examines the optimal hedge in a futures market for a DM having a fixed end-of-period output level, The end-of-period income function is

$$\text{(7)} \qquad y = PQ + x(P_f - P).$$

Solving (1) subject to (7) yields (8) as the optimal futures position

$$\text{(8)} \qquad \hat{x} = Q + \frac{(P_f - \mu_p)}{A\sigma^2_p}.$$

It is useful to discuss the results in each MV model in terms of the hedging and speculative components. The first term in the solution (8) is the hedging component; it is simply the futures position that minimizes the variance of end-of-period income. The second term is the spec-

[5] These results have been reported in previous work: for example, see Anderson and Danthine.

[6] Models 2 and 3 are derived in Hanson.

ulative component; it is the futures position that results if the DM does not have any cash position and obtains his entire end-of-period income from the futures market. If the futures market is considered unbiased (i.e., $P_f = \mu_p$), then the speculative component disappears, and the optimal futures position is the traditional hedge where the futures position is equal and opposite the cash position. However, if the DM believes the futures market is biased, then the DM believes that, on average, profits can be made by speculating in the futures market. The futures position is altered by the amount of believed bias adjusted by the level of risk aversion and price variability. Increases in risk aversion lead to smaller speculative positions. If a DM is infinitely risk averse, the speculative component is zero, and the futures position equals the traditional hedge.

### *Model 2: Cash and Options with Certain Output (CO)*

Model 2 examines the options market position. End-of-period income consists of revenue from the cash and options markets less the cost of purchasing options. The income function can be written as

$$(9) \qquad y = PQ + zT - zk,$$

where both $P$ and $T$ are random variables. Choosing $z$ to maximize (1) subject to (9) yields

$$(10) \qquad \hat{z} = -\frac{Q\sigma_{p,T}}{\sigma_T^2} + \frac{(\mu_T - k)}{A\sigma_T^2}.$$

The first and second terms represent hedging and speculative components analogous to model 1. It can be shown that the absolute value of the ratio $\sigma_{P,T}/\sigma_T^2$ will always be greater than one. This implies that in the absence of any speculative position, the DM will always overhedge in the options market. The result occurs because the overhedge position in the options market minimizes the variance of end-of-period income. The speculative component will modify the optimal market position in a manner analogous to model 1.

### *Model 3: Cash, Futures, and Options with Certain Output (CFO)*

Model 3 considers a DM who is able to use futures and options contracts. Income now includes revenue from the cash, futures, and options markets, and the cost of purchasing options, and can be expressed as

$$(11) \qquad y = PQ + x(P_f - P) + zT - zk.$$

Maximizing (1) subject to (11) yields the optimal market positions in the futures and options markets:

$$(12) \qquad \hat{x} = Q - \frac{(P_f - \mu_p)\sigma_T^2 + (\mu_T - k)\sigma_{p,T}}{A(\sigma_{p,T}^2 - \sigma_T^2\sigma_p^2)},$$

$$\hat{z} = -\frac{(P_f - \mu_p)\sigma_{p,\mathrm{T}}^2 + (\mu_T - k)\sigma_p^2}{A(\sigma_{p,T}^2 - \sigma_T^2\sigma_p^2)}.$$

The optimal futures market position consists of a hedging and speculative component, and the optimal options market position simply consists of a speculative component. In the absence of any speculative positions, the optimal market positions become the traditional hedge in the futures market and no position in the options market. The DM is acting to minimize variance of the end-of-period income, which can be completely eliminated by taking the traditional hedge in the futures market. If either the futures market or options market is perceived to be biased, then the speculative components adjust the futures and options positions accordingly.

## Opportunity Cost of MV Solution

The opportunity cost to the DM of using the MV market position instead of the EU position is the amount of additional income that must be given to the DM to provide the same level of expected utility as would be provided by using the EU market position. This money value, which we denote by $V$, is obtained by solving equation (13) for $V$.

$$(13) \qquad EU(y(\hat{x}, \hat{z}) + V) = EU(y(x^*, z^*)),$$

where $\hat{x}$ and $\hat{z}$ are the MV market positions and $x^*$ and $z^*$ are the EU market positions.

Three dimensions of the MV and EU solutions are compared: (*a*) value of $V$, (*b*) correlation between $x^*$ and $\hat{x}$, and (*c*) correlation between $z^*$ and $\hat{z}$. For the DM studied here, the value of $V$ is the only relevant measure of goodness of fit of MV to EU solutions. Large variations in options and futures positions are economically important only if they affect $E(U)$. It is possible that wide differences in the values of the decision variables between the MV and EU frameworks do not make much difference in the level of the DM's expected utility.

## Experimental Design

Equations (5) and (6) were used to derive optimal futures and options positions of a representative midwestern soybean producer who exhibits constant absolute risk aversion (CARA) subject to the preceding assumptions. These EU solutions were compared with corresponding MV solutions from (1). We could solve (6) for historical values of the market factors. But because commodity options have existed for a short time, this would provide a small sample of results. Consequently, we elected to solve (6) for a simulated set of market factors. To efficiently extract information from a given set of market factors, the factors were organized in a modified factorial design.

Values of the market factors were selected to represent conditions in midwestern soybean markets. The exogenous factors of the model were specified in terms of the levels of four "market factors," amount of output, and the level of risk aversion exhibited by the producer. The four market factors are: $\sigma_p^2$, the variance of the soybean cash price; $B_f = P_f - \mu_p$, the perceived expected bias in the futures market; $B_o = E[P_e - P|P < P_e] - K$, the perceived expected bias in the options market; and $E_m = P_e - \mu_p$, the difference between the exercise price and the expected soybean cash price. Setting the exogenous factors of the model—$\sigma_p^2$, $\mu_p$, $P_e$, $P_f$, and $k$—in relation to these market factors ensured that various combinations of variable values in the design did not result in unrealistic specifications of market behavior.

The variance of average deflated November spot prices received by Iowa soybean producers during 1976–86 was \$1.08. We assigned values of $\sigma_p^2$ ranging from \$0.50 to \$1.50. From studies of pricing in futures markets (Gray; Just and Rausser; Conorella and Pollard; Telser; Cootner 1977a, b) and in options markets (Tucker, Whaley, Jordan et al., Luft and Fielitz), we conclude that, if any bias does exist in either futures or options markets, it is small and short-lived. The extreme values of each bias were set at ±\$.04. The extreme values of the difference between the exercise price and the expected spot price were set at ±\$0.10.

The assigned values of the market factors and of $Q$ and $A$ were used in a central composite experimental design (Khuri and Cornell). A factorial design is one in which all values of a given factor are combined with all levels of every other factor in the data set. A central composite design consists of a full factorial design for two levels of variable values called the upper and lower factorial levels and additional selected combinations of variable values around the two-level factorial design called the center and axial points. The intent of the central composite design is to capture much of the information of a full factorial design that uses more than two levels of each variable without the data requirements of a full three-level, or higher, factorial design. Table 1 presents the levels of the factors used in the design. The levels in this table were used in 77 different combinations to form a central composite design of $n = 77$ points.

To solve (6) we then applied MINTDF numerical integration (Kaylen and Preckel) and GQOPT/PC (Quandt and Goldfeld) numerical optimization to each combination of variable values to find the optimal futures and options positions, $x^*$ and $z^*$.

## Comparison of MV and EU Results

The MV results from (1) and the EU results from (6) were found for each combination of variables in the central composite design subject to the income functions in (7), (9), and (11). Thus, the MV solution could be directly compared to the EU solution for each model. The general behavioral characteristics of the EU results correspond to the economic behavior described by the MV model in the previous section. As the MV suggested, the EU maximizer hedges to reduce income variability and speculates according to any believed bias in the commodity markets adjusted by risk aversion and income variability.

The MV and EU results were identical for the CF model because the necessary and sufficient condition is met, which causes the linear MV model to produce the EU maximizing solution. As expected, the CO and CFO results for the MV model were different from those for the EU model because of the violation of sufficient conditions for equivalence between the MV and EU models. However, visual examination of the results of the MV and EU models suggests that the MV model does approximate the EU results. The twelve tabular pages of MV and EU solutions are not presented here but are available from the authors.

**Table 1. Observation Levels in the Central Composite Design**

| Levels | Factors | | | | | |
|---|---|---|---|---|---|---|
| | $E_m$ | $\sigma^2$ | $B_f$ | $B_0$ | $Q$ | $A$ |
| | ($) | | | | (bu.) | |
| Upper axial | 0.10 | 1.50 | 0.04 | 0.04 | 25,000 | 0.00045 |
| Upper factorial | 0.05 | 1.25 | 0.02 | 0.02 | 20,000 | 0.00035 |
| Center point | 0.00 | 1.00 | 0.00 | 0.00 | 15,000 | 0.00025 |
| Lower factorial | −0.05 | 0.75 | −0.02 | −0.02 | 10,000 | 0.00015 |
| Lower axial | −0.10 | 0.50 | −0.04 | −0.04 | 5,000 | 0.00005 |

*Cash and Options*

Results for the CO model are summarized in tables 2 and 3 and in the regression

$$z^* = \underset{(313)}{1{,}602} + \underset{(0.015)}{0.874}\ \hat{z} + e,$$

where $r^2 = 0.978$, the standard deviation of $e$ = 847 bushels, and the values in parentheses are the standard errors of the estimated coefficients. The standard deviation of the errors amounted to only 17% of the smallest value of $Q$ included in the experiment. An $F$-test rejected the null hypothesis that the intercept equalled zero and the slope equalled one at the 1% level. All values of $\hat{z}$ and $z^*$ were positive.

Table 2 contains the frequency distribution of the relative absolute differences (RADs) between the EU and MV market positions: i.e., the absolute differences divided by 5,000 bushels, the size of a standard futures or options contract. The majority of the observations (57%) have a RAD of less than 15%, while the remaining observations are spread uniformly up to a RAD of 70%. The largest error in the MV solutions was 3,366 bushels, which amounted to 14.6% of the EU solution. Table 3 contains the frequency distribution of the opportunity costs of the MV market positions. The mean is around $55 and the median is about $13. The distribution is skewed with a range of nearly zero to $350. The largest concentration of observations (64%) is in the $0–$20 range. Of the seventy-seven values of $V$, sixty-eight are less than 1¢ per bushel of $Q$. The largest value of $V$ amounts of $342, or slightly less than 2¢ per bushel.

**Table 2. Distribution of Relative Absolute Differences Between EU and MV Options Positions**

| Relative Absolute Difference[a] | Frequency | | |
|---|---|---|---|
| | CO | CFO | |
| | Options | Futures | Options |
| (%) | | | |
| 0–5 | 8 | 47 | 41 |
| 5–10 | 14 | 15 | 11 |
| 10–15 | 22 | 9 | 10 |
| 15–20 | 7 | 4 | 6 |
| 20–25 | 4 | 2 | 5 |
| 25–30 | 4 | 0 | 2 |
| 30–35 | 0 | 0 | 2 |
| 35–40 | 3 | 0 | 0 |
| 40–45 | 2 | 0 | 0 |
| 45–50 | 0 | 0 | 0 |
| 50–55 | 4 | 0 | 0 |
| 55–60 | 4 | 0 | 0 |
| 60–65 | 3 | 0 | 0 |
| 65–70 | 2 | 0 | 0 |
| 70 and up | 0 | 0 | 0 |

[a] $100|z^* - \hat{z}|/5{,}000$ and $100|x^* - \hat{x}|/5{,}000$. These are absolute differences expressed as percentages of number of bushels in a futures or options contract.

**Table 3. Frequency Distribution of the Opportunity Cost of MV Market Positions**

| Value V | Frequency | |
|---|---|---|
| | CO Model | CFO Model |
| ($) | | |
| 0–2[a] | 13 | 41 |
| 2–4 | 13 | 10 |
| 4–6 | 4 | 6 |
| 6–8 | 4 | 8 |
| 8–10 | 1 | 5 |
| 10–12 | 3 | 0 |
| 12–14 | 2 | 4 |
| 14–16 | 1 | 1 |
| 16–18 | 2 | 2 |
| 18–20 | 6 | 0 |
| 20–100 | 12 | 0 |
| 100–200 | 8 | 0 |
| 200–300 | 6 | 0 |
| 300–350 | 2 | 0 |
| Mean | $54.66 | $3.80 |
| Median | $12.94 | $1.50 |
| Mode | $ 3.50 | $0.50 |

[a] Includes lower limit and excludes upper limit on range.

### Cash, Futures, and Options

Results for the CFO Model are summarized in tables 2 and 3 and in the simple regressions:

$$z^* = 51 + 1.68\,\hat{z} + e, \quad (23)\ (0.03)$$

where $r^2 = 0.978$, and the standard deviation of $e = 201$ bushels. An $F$-test rejected the hypothesis of a zero intercept and a unit slope at the 1% level.

$$x^* = -113 + 1.005\,\hat{x} + e, \quad (153)\ (0.010)$$

where $r^2 = 0.993$, and the standard deviation of $e = 412$ bushels. An $F$-test failed to reject the hypothesis of a zero intercept and a unit slope at the 40% level. The standard deviations of the errors in these two equations are less than 10% of the smallest value of $Q$ included in the experiment. Some values of $\hat{z}$ and $z^*$ are positive, and some negative. Every value of $z^*$ is less than 5,000 bushels (one options contract) and only eight exceed 2,500 bushels.

Table 2 shows that all of the RADs in the futures positions are less than 25% and 61% have a RAD of 5% or less. The RADs of the options positions are all less than 35%, and 53% of the observations have a RAD of less than 5%. The largest errors in $\hat{x}$ and $\hat{z}$ are 1,159 bushels and 1,628 bushels, which are 14% and 51% of the correct values $x^*$ and $z^*$. The futures position has a relatively low percentage error because the traditional hedge is in the futures market in MV and EU models. The entire error in the options position is related to the speculative component. The resulting speculative error in the futures position for this particular observation was 61%.

Table 3 shows the opportunity cost to the DM of using the MV market positions instead of the EU positions has a mean of \$3.80 and a median of \$1.50. The distribution is skewed with all observations falling in the range \$0–\$18. The largest concentration of observations (53%) is in the \$0–\$2 range. The largest value of $V$ is \$16.38, less than 0.2¢ per bushel of output.

Values of $V$ and $\hat{z}$ are positively correlated, as are values of $V$ and $A$. As the size of the options position in the expected utility solution increases, the error in the standard MV solution goes up, and the value of the EU decision over the MV solution also rises. As the options position increases, the resulting income distribution deviates further from the normal distribution assumed by the MV model. Consequently, the MV model results contain more error as the income distribution becomes "less normal." As the level of risk aversion rises, any deviation from the optimal solution is translated into a higher opportunity cost being attached to the MV solutions.

### Comparative Statics: MV versus EU

The previous results show that the MV model can be used to produce results that approximate the solutions of the EU model. The analytical results of the MV model are also useful in providing insights into the numerical results generated by the EU model. For example, in EU solutions to the CO model the optimal options position is always greater than the output level. Using the analytical result from (10), it is easy to show that the variance of end-of-period income is minimized by taking an options position that is greater than the output level.

One test of the MV model's analytical usefulness is its ability to produce comparative static results consistent with those produced by the EU model. The comparative static results can be generated for the MV model directly by using equations (10) and (12). In some cases, the MV model comparative static results cannot be signed in general; e.g., $\partial x^*/\partial\sigma^2$ and $\partial z^*/\partial\sigma^2$ in the CFO strategy. However, the comparative static results for both the MV and EU models can be generated numerically over local regions. Table 4 presents the local comparative results over the region considered in this study. The comparative static results of the MV model are not always consistent with the EU model. In the CO strategy, there is some inconsistency between signs of $\partial x^*/\partial\sigma^2$ and $\partial x^*/\partial A$ across the various combinations of exogenous variables. In the CFO strategy there is some inconsistency in the two models between $\partial x^*/\partial E_m$ and $\partial z^*/\partial E_m$. The remaining comparative static results are consistent between the two models over the region considered.

The standard linear MV model used in this analysis is a special MV model. In general, the form of the MV function will not be known. It may be that the EU choices would be in the MV efficient sets generated by some alternative nonlinear MV preference functions. In an alternative specification of the MV preference function, the differences between the two sets of results may be even smaller. Relaxing the assumptions such as CARA, normally distributed output price, no transactions costs, no basis un-

**Table 4. Numerical Comparative Static Results for the MV and EU Models**

| | CO | | | CFO | | | |
|---|---|---|---|---|---|---|---|
| | $\Delta$ Options | | | $\Delta$ Futures | | $\Delta$ Options | |
| $\Delta$ Factor[a] | MV | EU | Special Conditions | MV | EU | MV | EU |
| $\Delta E_m > 0$ | $< 0$ | $< 0$ | $B_f = B_0 > 0$ | $> 0$ | $\approx 0$ | $< 0$ | $< 0$ |
| | | | $B_f = B_0 < 0$ | $< 0$ | $\approx 0$ | $> 0$ | $> 0$ |
| | | | $B_f > 0,\ B_0 < 0$ | $< 0$ | $> 0$ | $> 0$ | $< 0$ |
| | | | $B_f < 0,\ B_0 > 0$ | $> 0$ | $< 0$ | $< 0$ | $> 0$ |
| $\Delta\sigma^2 > 0$ | $\lesseqgtr 0$ | $\lesseqgtr 0$ | $B_f = B_0 > 0$ | $> 0$ | $> 0$ | $< 0$ | $< 0$ |
| | | | $B_f = B_0 < 0$ | $< 0$ | $< 0$ | $> 0$ | $> 0$ |
| | | | $B_f > 0,\ B_0 < 0$ | $< 0$ | $< 0$ | $> 0$ | $> 0$ |
| | | | $B_f < 0,\ B_0 > 0$ | $> 0$ | $> 0$ | $< 0$ | $< 0$ |
| $\Delta Bf > 0$ | $= 0$ | $= 0$ | | $> 0$ | $> 0$ | $< 0$ | $< 0$ |
| $\Delta B_0 > 0$ | $> 0$ | $> 0$ | | $< 0$ | $< 0$ | $> 0$ | $> 0$ |
| $\Delta Q > 0$ | $> 1$ | $> 1$ | | $= 1$ | $= 1$ | $= 0$ | $= 0$ |
| Special Conditions | | | | | | | |
| $\Delta A > 0$ $B_0 > 0$ | $< 0$ | $< 0$ | $B_f = B_0 > 0$ | $> 0$ | $> 0$ | $< 0$ | $< 0$ |
| $B_0 < 0$ | $> 0$ | | $B_f = B_0 < 0$ | $< 0$ | $< 0$ | $> 0$ | $> 0$ |
| | | | $B_f > 0,\ B_0 < 0$ | $< 0$ | $< 0$ | $> 0$ | $> 0$ |
| | | | $B_f < 0,\ B_0 > 0$ | $> 0$ | $> 0$ | $< 0$ | $< 0$ |

[a] The symbol $\Delta$ is used to represent a change in the indicated variable.

certainty, and no margin calls may also change some of the results. The impact of these assumptions is left for further study.

## Conclusions and Implications

This study considered the approximating ability of the standard linear MV model for a DM with constant absolute risk aversion when the income distribution is truncated by the use of commodity options. The MV model was shown to provide analytical results useful in describing the general behavioral characteristics of the EU model. However, the comparative static results of the MV model were not always consistent with the comparative static results of the EU model. Thus, some care must be taken in interpreting the analytical results of the MV model when the sufficient conditions for expected utility equivalence are violated.

The MV model also produced empirical solutions that were close approximations of the EU model results. The opportunity costs of using the mean-variance solutions instead of the EU solutions were small. In the situations studied here, the payoff for performing the additional computations (the numerical integration and numerical optimization) instead of using the simpler MV solution is small. The MV model is robust to the particular violation of the assumptions that was studied here. These results will also be of interest to those concerned with modeling other decision-making problems under risk which are characterized by truncated probability distributions.

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