The Moment Generating Function of the Truncated Multi-normal Distribution

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Summary

In this paper the moment generating function (m.g.f.) of the truncated n-dimensional normal distribution is obtained. From the m.g.f., formulae for $E(X_i)$ and $E(X_i X_j)$ are derived, and are used to investigate certain special cases. Some applications of these results to statistical genetics are also discussed.

1. Introduction

The problem of finding the means, variances and covariances of a standardized n-dimensional normal distribution (here abbreviated to standard n-normal) truncated in $p \leqslant n$ coordinates was solved by Birnbaum and Meyer (1953). The solutions were obtained by direct integration and the general results left in a somewhat difficult form for explicit evaluation.

It is the purpose of this paper to present a different method of solving the same problem. Since the moment generating function (m.g.f.) approach is used, the required moments are obtained by differentiation rather than integration. General formulae for computing $E(X_i)$ and $E(X_i X_j)$ $(i,j = 1,2,\ldots,n)$ are given as well as explicit formulae for the same moments for the special case $n = 3$. Two examples are used to illustrate the methods of evaluating the general formulae.

2. Notation

It is convenient in the following development to let ϕ represent the frequency function of an arbitrary number of standardized normal variates. Thus, if X_s $(s = 1,2,\ldots,n)$ are n such variates with correlation matrix $\mathbf{R}$ (assumed positive definite), we have

$$\phi_n(x_1, x_2, \ldots, x_n; \mathbf{R}) = \phi_n(x_s; \mathbf{R}) = (2\pi)^{-n/2}|\mathbf{R}|^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}\mathbf{x}'\mathbf{R}^{-1}\mathbf{x}\}, \qquad (1)$$

where $\mathbf{x}$ is the column vector of the X_s. This distribution for $X_q = b_q$ and $X_q = b_q$, $X_r = b_r$ may be written

$$\phi_n(x_s, x_q = b_q; \mathbf{R}) = \phi(b_q)\,\phi_{n-1}(y_s; \mathbf{R}_q) \quad (s \neq q),$$

$$\phi_n(x_s, x_q = b_q, x_r = b_r; \mathbf{R}) = \phi(b_q, b_r; \rho_{qr})\,\phi_{n-2}(z_s; \mathbf{R}_{qr}) \quad (s \neq q \neq r),$$

where $\mathbf{R}_q$ and $\mathbf{R}_{qr}$ are the matrices of first- and second-order partial correlation coefficients of X_s for $s \neq q$, and for $s \neq q$, $s \neq r$ respectively, and

$$Y_s = (X_s - \rho_{qs} b_q)/\sqrt{(1-\rho_{qs}^2)},$$

$$Z_s = (X_s - \beta_{sq.r} b_q - \beta_{sr.q} b_r)/\sqrt{\{(1-\rho_{sq}^2)(1-\rho_{sr.q}^2)\}}.$$

In the above formulae $\beta_{sq.r}$ and $\beta_{sr.q}$ are the partial regression coefficients of X_s on X_q and X_r respectively and $\rho_{sr.q}$ is the partial correlation coefficient between X_s and X_r for fixed X_q.

Now, if the operator

$$\int_{b_1}^{\infty} \dots \int_{b_n}^{\infty} (\quad) \, dx_1 \dots dx_n$$

is abbreviated to

$$\overset{(n)}{\int_{b_s}^{\infty}} (\quad) \, dx_s,$$

and if we let

$$\Phi_n(b_s; \mathbf{R}) = \overset{(n)}{\int_{b_s}^{\infty}} \phi_n(x_s; \mathbf{R}) \, dx_s,$$

then it follows from the above formulae that

$$\overset{(n-1)}{\int_{b_s}^{\infty}} \phi_n(x_s, x_q = b_q; \mathbf{R}) \, dx_s = \phi(b_q) \Phi_{n-1}(B_{qs}; \mathbf{R}_q) \quad (s \neq q)$$

and

$$\overset{(n-2)}{\int_{b_s}^{\infty}} \phi_n(x_s, x_q = b_q, x_r = b_r; \mathbf{R}) \, dx_s = \phi(b_q, b_r; \rho_{qr}) \Phi_{n-2}(B_{rs}^q; \mathbf{R}_{qr}) \quad (s \neq q \neq r),$$

where

$$B_{qs} = (b_s - \rho_{qs} b_q)/\sqrt{(1-\rho_{qs}^2)},$$

$$B_{rs}^q = (b_s - \beta_{sq.r} b_q - \beta_{sr.q} b_r)/\sqrt{\{(1-\rho_{sq}^2)(1-\rho_{sr.q}^2)\}}.$$

3. General Results

Let W_s $(s = 1, 2, \dots, n)$ have the standard n-normal distribution with correlation matrix $\mathbf{R}$ and let W_s be truncated at a_s so that

$$\alpha = \text{prob}(W_1 > a_1, W_2 > a_2, \dots, W_n > a_n) = \Phi_n(a_s; \mathbf{R}).$$

The joint m.g.f. of the truncated population $W_1 > a_1, W_2 > a_2, \dots, W_n > a_n$ is

$$m(t_s) = m = \alpha^{-1} \overset{(n)}{\int_{a_s}^{\infty}} e^{\mathbf{t}'\mathbf{w}} \phi_n(w_s; \mathbf{R}) \, dw_s$$

$$= \alpha^{-1} (2\pi)^{-n/2} |\mathbf{R}|^{-\frac{1}{2}} \overset{(n)}{\int_{a_s}^{\infty}} \exp\left[-\tfrac{1}{2}\{\mathbf{w}'\mathbf{R}^{-1}\mathbf{w} - 2\mathbf{t}'\mathbf{w}\}\right] dw_s,$$

where $\mathbf{t}$ is the column vector of the t_s $(s = 1, 2, \dots, n)$. Now the identity

$$-\tfrac{1}{2}\{\mathbf{w}'\mathbf{R}^{-1}\mathbf{w} - 2\mathbf{t}'\mathbf{w}\} \equiv \tfrac{1}{2}\mathbf{t}'\mathbf{R}\mathbf{t} - \tfrac{1}{2}(\mathbf{w} - \boldsymbol{\zeta})'\mathbf{R}^{-1}(\mathbf{w} - \boldsymbol{\zeta})$$

is easily verified by noticing that $\boldsymbol{\zeta} = \mathbf{Rt}$ and then expanding the right-hand side. It can then be shown that the above integral for m may be written

$$m = \alpha^{-1}(2\pi)^{-n/2}|\mathbf{R}|^{-\frac{1}{2}} e^T \overset{(n)}{\int_{a_s}^{\infty}} \exp\{-\tfrac{1}{2}(\mathbf{w}-\boldsymbol{\zeta})' \mathbf{R}^{-1}(\mathbf{w}-\boldsymbol{\zeta})\}\, dw_s,$$

where $T = \frac{1}{2}\mathbf{t}'\mathbf{Rt}$, $\mathbf{t}$ and $\mathbf{w}-\boldsymbol{\zeta}$ are column vectors of t_s and $W_s - \boldsymbol{\zeta}_s$, and $\boldsymbol{\zeta}_s = \Sigma \rho_{sv} t_v$. By the change of variables $X_s = W_s - \boldsymbol{\zeta}_s$, we obtain immediately

$$\alpha m = e^T \Phi_n(b_s; \mathbf{R}) \quad (b_s = a_s - \zeta_s). \tag{2}$$

With the results of section 2, equation (2) may be readily differentiated, first with respect to t_i and then with respect to t_j. It can be verified that, when the derivatives are calculated with all $t_s = 0$,

$$\alpha \frac{\partial m}{\partial t_i} = \alpha E(X_i) = \sum_{q=1}^{n} \rho_{iq}\, \phi(a_q)\, \Phi_{n-1}(A_{qs}; \mathbf{R}_q), \tag{3}$$

$$\alpha \frac{\partial^2 m}{\partial t_j\, \partial t_i} = \alpha E(X_i X_j) = \rho_{ij}\,\alpha + \sum_{q=1}^{n} \rho_{qi}\, \rho_{qj}\, a_q\, \phi(a_q)\, \Phi_{n-1}(A_{qs}; \mathbf{R}_q)$$

$$+ \sum_{q=1}^{n} \left\{ \rho_{qi} \sum_{r \neq q} \phi(a_q, a_r; \rho_{qr})\, \Phi_{n-2}(A^q_{rs}; \mathbf{R}_{qr})\, (\rho_{rj} - \rho_{qr}\, \rho_{qj}) \right\}, \tag{4}$$

where
$$A_{qs} = (a_s - \rho_{sq}\, a_q)/\sqrt{\{1-\rho^2_{sq}\}},$$

$$A^q_{rs} = (a_s - \beta_{sq.r}\, a_q - \beta_{sr.q}\, a_r)/\sqrt{\{(1-\rho^2_{sq})(1-\rho^2_{sr.q})\}}$$

and $s \neq q$ in Φ_{n-1} and $s \neq q \neq r$ in Φ_{n-2}.

The expressions (3) and (4) necessitate the evaluation of such integrals as $\Phi_n(a_s; \mathbf{R})$. These integrals have been tabulated for $n = 1$ and $n = 2$ (Pearson, 1931), $n = 2$ (Owen, 1956). For $n \geqslant 3$, see Plackett (1954) and Steck (1958).

Some special cases for $E(X_i)$ and $E(X_i X_j)$ arise when certain $a_s = -\infty$. In these instances, the appropriate modifications to (3) and (4) may be obtained by noticing that:

(a) all terms involving ϕ, where ϕ is a function of any $a_s = -\infty$, are zero since $\phi = 0$;

(b) by definition, if $a_s = -\infty$, then $A_{qs} = A^q_{rs} = -\infty$. Hence all integrals involving A_{qs} or A^q_{rs} have their dimension reduced by one for each negatively infinite parameter;

(c) obviously if $a_s = -\infty$, $\Phi(A_{qs}) = \Phi(A^q_{rs}) = 1$.

4. SPECIAL CASES IN TWO AND THREE DIMENSIONS

In order to illustrate the use of expressions (3) and (4), $E(X_1)$, $E(X_1^2)$ and $E(X_1 X_2)$ will be evaluated for the special case $n = 3$. The expression for $E(X_1)$ is obtained by setting $i = 1$ in (3), and $E(X_1^2)$ and $E(X_1 X_2)$ are obtained by setting $i = j = 1$ and $i = 1, j = 2$ in (4) respectively.

The results are

$$\alpha E(X_1) = \phi(a_1)\,\Phi(A_{12}, A_{13};\ \rho_{23.1}) + \rho_{12}\,\phi(a_2)\,\Phi(A_{21}, A_{23};\ \rho_{13.2}) + \rho_{13}\,\phi(a_3)\,\Phi(A_{31}, A_{32};\ \rho_{12.3}),$$

$$\begin{aligned}\alpha E(X_1^2) = \alpha &+ a_1\,\phi(a_1)\,\Phi(A_{12}, A_{13};\ \rho_{23.1}) + \rho_{12}^2\,a_2\,\phi(a_2)\,\Phi(A_{21}, A_{23};\ \rho_{13.2})\\ &+ \rho_{13}^2\,a_3\,\phi(a_3)\,\Phi(A_{31}, A_{32};\ \rho_{12.3}) + \rho_{12}(1-\rho_{12}^2)\,\phi(a_1, a_2;\ \rho_{12})\,\Phi(A_{13}^2)\\ &+ \rho_{13}(1-\rho_{13}^2)\,\phi(a_1, a_3;\ \rho_{13})\,\Phi(A_{12}^3)\\ &+ \phi(a_2, a_3;\ \rho_{23})\,\{\Phi(A_{31}^2)\,\rho_{12}(\rho_{13}-\rho_{12}\,\rho_{23})\\ &+ \Phi(A_{21}^3)\,\rho_{13}(\rho_{12}-\rho_{23}\,\rho_{13})\},\end{aligned}$$

$$\begin{aligned}\alpha E(X_1\,X_2) = \alpha\rho_{12} &+ \rho_{12}\,a_1\,\phi(a_1)\,\Phi(A_{12}, A_{13};\ \rho_{23.1})\\ &+ \rho_{12}\,a_2\,\phi(a_2)\,\Phi(A_{21}, A_{23};\ \rho_{13.2})\\ &+ \rho_{13}\,\rho_{23}\,a_3\,\phi(a_3)\,\Phi(A_{31}, A_{32};\ \rho_{12.3}) + (1-\rho_{12}^2)\,\phi(a_1, a_2;\ \rho_{12})\,\Phi(A_{23}^1)\\ &+ \rho_{13}(1-\rho_{23}^2)\,\phi(a_2, a_3;\ \rho_{23})\,\Phi(A_{21}^3)\\ &+ \phi(a_1, a_3;\ \rho_{13})\,\{(\rho_{23}-\rho_{13}\,\rho_{12})\,\Phi(A_{32}^1) + \rho_{13}(\rho_{12}-\rho_{13}\,\rho_{23})\,\Phi(A_{12}^3)\},\end{aligned}$$

where A_{qs} and A_{rs}^q are as defined in the previous section. If now

$$a_3 = A_{13}^2 = A_{23}^1 = A_{13} = A_{23} = -\infty,$$

the first and second moments of the truncated standard bi-normal distribution are obtained. These formulae agree with those presented by Weiler (1959).

5. Applications

Example 1. Young and Weiler (1961) have considered the case of the selection of animals (or plants) by the method of independent culling levels, using two bi-normally distributed characters W_1 and W_2, with frequency function

$$N(\mu_1, \mu_2, P_{11}, P_{22}, \rho_p).$$

This technique involves the simultaneous truncation of W_1 and W_2 at p_1 and p_2 in such a manner that $\text{prob}(W_1 > p_1, W_2 > p_2) = \alpha$. From their formulae for the first moments of the truncated bivariate distribution, it is possible to compute the phenotypic advance due to selection. However, it is also of interest to calculate the total genetic gains.

In order to make further progress with the latter problem, we assume the usual genetic models $W_1 = G_1 + E_1$ and $W_2 = G_2 + E_2$, where the G_i and E_i are the additive genetic and environmental contributions to phenotype respectively. The components G_i and E_i $(i = 1, 2)$ of the models are assumed to be independently and normally distributed. In this notation, the genetic value of an animal, relative to the population, may be defined as

$$G = \gamma_1\{G_1 - E(G_1)\} + \gamma_2\{G_2 - E(G_2)\},$$

where γ_1 and γ_2 are the economic weights for W_1 and W_2. Now let the variance of G be σ_G^2, let $X_1 = (W_1-\mu_1)/\sqrt{P_{11}}$, $X_2 = (W_2-\mu_2)/\sqrt{P_{22}}$ and $X_3 = G/\sigma_G$. Then X_i $(i = 1,2,3)$ are assumed to have a standard tri-normal distribution with correlation coefficients,

$$\rho_{12} = \rho_p, \rho_{13} = \frac{\gamma_1 V(G_1)+\gamma_2 C(G_1, G_2)}{\sigma_G\sqrt{P_{11}}}, \quad \rho_{23} = \frac{\gamma_1 C(G_1, G_2)+\gamma_2 V(G_2)}{\sigma_G\sqrt{P_{22}}}.$$

Here ρ_p is the phenotypic correlation between W_1 and W_2 and V and C denote variance and covariance. Since the truncation points of X_1, X_2 and X_3 are given by

$$a_1 = (p_1-\mu_1)/\sqrt{P_{11}}, \quad a_2 = (p_2-\mu_2)/\sqrt{P_{22}} \quad \text{and} \quad a_3 = -\infty \text{ respectively,}$$

it is possible to deduce from the formula for $E(X_1)$ in section 4 (by symmetry) that

$$\alpha E(X_3) = \rho_{13}\,\phi(a_1)\,\Phi(A_{12})+\rho_{23}\,\phi(a_2)\,\Phi(A_{21}).$$

From the definition of X_3 it is clear that $E(G) = \sigma_G E(X_3)$. It can be shown that this result is algebraically equivalent to the results obtained by Young and Weiler (1961) by a method analogous to linear interpolation. With the aid of the formula for $E(X_1^2)$, it can also be verified that

$$\alpha E(X_3^2) = 1+\rho_{23}^2\, a_2\,\phi(a_2)\,\Phi(A_{21})+\rho_{13}^2\, a_1\,\phi(a_1)\,\Phi(A_{12}) \\ +\{2\rho_{23}\,\rho_{13}-\rho_{12}(\rho_{23}^2+\rho_{13}^2)\}\,\phi(a_1, a_2;\ \rho_{12}).$$

Therefore, the new variance of G, $\sigma_{G'}^2$, is

$$\sigma_{G'}^2 = \sigma_G^2[E(X_3^2)-\{E(X_3)\}^2].$$

Now, if a sample of N animals is taken from the truncated population $W_1 > b_1$, $W_2 > b_2$, then $\bar{G} = \Sigma G/N$. Although G cannot be measured directly, by virtue of the Central Limit Theorem we have for N sufficiently large

$$\text{prob}\{E(G)-t_\beta\,\sigma_{G'}/\sqrt{N} < \bar{G} < E(G)+t_\beta\,\sigma_{G'}/\sqrt{N}\} \simeq 1-\beta,$$

where t_β is the standard normal deviate corresponding to the 100β per cent., two-tailed probability level. Thus, although in practice the required parameters for calculating $E(G)$ and $\sigma_{G'}^2$ have to be estimated, it is possible to obtain some idea of the interval in which $\bar{G}$ is expected to lie with given probability.

Example 2. As a final illustration of these methods, consider the n variables $W_s = Y_s+Z_s$ $(s = 1,2,\ldots,n)$, where the Y_s and Z_s are normally and independently distributed with zero expectations. Now, if all $W_s < a_s\{V(W_s)\}^{\frac{1}{2}}$ are discarded, it may be of interest to investigate the changes in the means, variances and covariances of the $2n$ variables W_s and Y_s.

In order to proceed with the problem, it is convenient to let

$$W_1/[V(W_1)]^{\frac{1}{2}} = X_1, \ldots, W_n/[V(W_n)]^{\frac{1}{2}} = X_n, Y_1/[V(Y_1)]^{\frac{1}{2}} = X_{n+1}, \ldots, Y_n/[V(Y_n)]^{\frac{1}{2}} = X_{2n}$$

and let $\mathbf{R}$ be the $2n\times 2n$ correlation matrix of X_s $(s = 1,2,\ldots,2n)$. Then it is possible to write $\mathbf{R}$ as the partitioned matrix

$$\mathbf{R} = \begin{bmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{M} & \mathbf{N} \end{bmatrix},$$

where $\mathbf{K}, \mathbf{L}, \mathbf{M}$ and $\mathbf{N}$ are $n \times n$. We thus have for $s, t = 1, 2, \ldots, n$

$$\mathbf{K} = [k_{st}] = \left[\frac{C(W_s W_t)}{\{V(W_s)\,V(W_t)\}^{\frac{1}{2}}}\right], \quad \mathbf{L} = [l_{st}] = \left[\frac{C(Y_s Y_t)}{\{V(W_s)\,V(Y_t)\}^{\frac{1}{2}}}\right],$$

$$\mathbf{M} = [m_{st}] = \left[\frac{C(Y_s Y_t)}{\{V(Y_s)\,V(W_t)\}^{\frac{1}{2}}}\right], \quad \mathbf{N} = [n_{st}] = \left[\frac{C(Y_s Y_t)}{\{V(Y_s)\,V(Y_t)\}^{\frac{1}{2}}}\right].$$

With the above notation and the rules for special cases, it is now possible to write down formulae for $E(X_i)$ and $E(X_i X_j)$, remembering $a_s = -\infty$ $(n < s \leqslant 2n)$. For $i,j = 1, 2, \ldots, 2n$, we have that

$$\alpha E(X_i) = \sum_{q=1}^{n} \rho_{iq}\,\phi(a_q)\,\Phi_{n-1}(A_{qs};\, \mathbf{K}_q),$$

$$\alpha E(X_i X_j) = \rho_{ij}\,\alpha + \sum_{q=1}^{n} \rho_{qi}\,\rho_{qj}\,a_q\,\phi(a_q)\,\Phi_{n-1}(A_{qs};\, \mathbf{K}_q)$$
$$+ \sum_{q=1}^{n}\left\{\rho_{qi} \sum_{\substack{r \neq q \\ \leqslant n}} \phi(a_q, a_r;\, \rho_{qr})\,\Phi_{n-2}(A^q_{rs};\, \mathbf{K}_{qr})\,(\rho_{rj} - \rho_{qr}\,\rho_{qj})\right\},$$

where $s \neq q$ in Φ_{n-1}, $s \neq q \neq r$ in Φ_{n-2} and $s \leqslant n$ in all cases.

As a particular illustration let W_1 and W_2 be two phenotypic characters (as in Example 1), then Y_1 and Y_2 represent the additive genetic contributions and Z_1 and Z_2 the environmental contributions to phenotype respectively. In this instance

$$\mathbf{R} = \left[\begin{array}{cc:cc} 1 & \rho_p & h_1 & h_1\,\rho_g \\ \rho_p & 1 & h_2\,\rho_g & h_2 \\ \hdashline h_1 & h_2\,\rho_g & 1 & \rho_g \\ h_1\,\rho_g & h_2 & \rho_g & 1 \end{array}\right],$$

where ρ_p and ρ_g are the phenotypic and genetic correlations between the two characters and $h_1 = \{V(Y_1)/V(W_1)\}^{\frac{1}{2}}$ and $h_2 = \{V(Y_2)/V(W_2)\}^{\frac{1}{2}}$. In this case we have, for $i,j = 1, 2, 3, 4$, that

$$\alpha E(X_i) = \rho_{i1}\,\phi(a_1)\,\Phi(A_{12}) + \rho_{i2}\,\phi(a_2)\,\Phi(A_{21}),$$

$$\alpha E(X_i X_j) = \rho_{ij}\,\alpha + \rho_{1i}\,\rho_{1j}\,a_1\,\phi(a_1)\,\Phi(A_{12}) + \rho_{2i}\,\rho_{2j}\,a_2\,\phi(a_2)\,\Phi(A_{21})$$
$$+ \phi(a_1, a_2;\, \rho_{12})\{\rho_{1i}(\rho_{2j} - \rho_{12}\,\rho_{1j}) + \rho_{2i}(\rho_{1j} - \rho_{12}\,\rho_{2j})\}.$$

Therefore, it is clear from the last results that, by evaluating $E(X_i)$ and $E(X_i X_j)$ for appropriate i,j, it is possible to study the effects of phenotypic truncation on heritability, h_i^2, and genetic correlation, ρ_g. Moreover, the work required to accomplish this for two characters is relatively small and can be completed with the aid of existing tables for the bivariate normal distribution.

6. Extensions

The methods of section 3 may be used to investigate certain additional problems related to the truncation of multi-normally distributed variates. For instance, the evaluation when all $t_s = 0$ of $\partial^3 m/\partial t_i^3$ and $\partial^4 m/\partial t_i^4$ would provide the third and fourth moments of the marginal distributions of the X_i. Moreover, a further generalization is achieved by considering the X_s $(s = 1, 2, ..., n)$ as doubly truncated so that $\text{prob}(a_1 \leqslant X_1 \leqslant c_1, ..., a_n \leqslant X_n \leqslant c_n) = \alpha$. In this case the required m.g.f. is

$$\alpha m = e^T \int_{b_s}^{d_s}{}^{(n)} \phi(x_s; \mathbf{R})\, dx_s,$$

where $$b_s = a_s - \sum_{v=1}^{n} \rho_{sv} t_v \quad \text{and} \quad d_s = c_s - \sum_{v=1}^{n} \rho_{sv} t_v.$$

For example, the m.g.f. for the bi-normal distribution under double truncation is

$$\alpha m = e^T \int_{b_1}^{d_1} \int_{b_2}^{d_2} \phi(x_1, x_2; \rho_{12})\, dx_1\, dx_2$$

which may be written with advantage

$$\alpha m = e^T\{\Phi(b_1, b_2; \rho_{12}) + \Phi(d_1, d_2; \rho_{12}) - \Phi(d_1, b_2; \rho_{12}) - \Phi(b_1, d_2; \rho_{12})\}$$

and hence it is clear that

$$\alpha m = \alpha_1 m_1 + \alpha_2 m_2 - \alpha_3 m_3 - \alpha_4 m_4,$$

where the subscripts 1, 2, 3 and 4 refer to the bi-normal distribution truncated at $(a_1, a_2), (c_1, c_2), (c_1, a_2)$ and (a_1, c_2) respectively. The first and second moments may now be obtained in an obvious way from the formulae of Weiler (1959).

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