Day 5

Mean Variance Analysis

Kerry Back
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Rice University
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Notation

- n risky assets with returns $\widetilde{R}_i$. $\widetilde{\mathbf{R}} = (\widetilde{R}_1 \cdots \widetilde{R}_n)'$
- $\mu =$ vector of expected returns. At least two of the assets have different expected returns.
- $\Sigma =$ covariance matrix. Assume no redundant assets, so Σ is positive definite.
- $\iota =$ n–vector of 1's.
- $\pi \in \mathbb{R}^n$ is a portfolio (of risky assets). If the portfolio is fully invested in risky assets, then $\iota'\pi = 1$. Otherwise, $1 - \iota'\pi$ is the fraction of wealth invested in the risk-free asset.

Portfolio Mean and Standard Deviation

Consider two assets with expected returns μ_i, standard deviations σ_i, and correlation ρ. Consider a portfolio (π_1, π_2) with $\pi_1 + \pi_2 = 1$. The portfolio return is

$$\pi' \widetilde{\mathbf{R}} = \pi_1 \tilde{R}_1 + \pi_2 \tilde{R}_2$$

The portfolio expected return is

$$\pi' \mu = \pi_1 \mu_1 + \pi_2 \mu_2$$

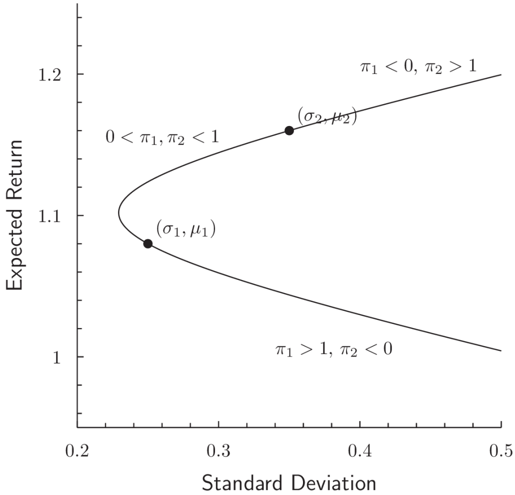
Write the covariance matrix as

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

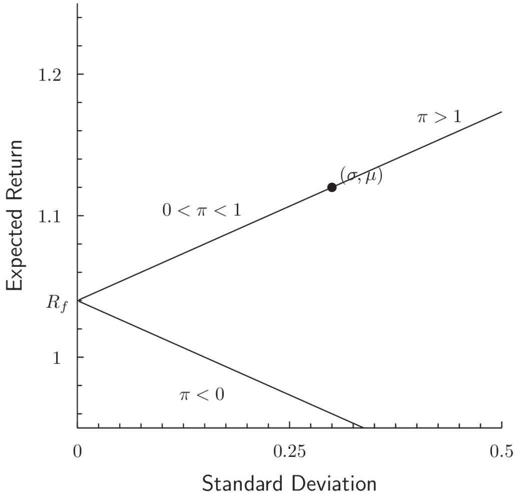
The portfolio variance is

$$\pi' \Sigma \pi = \pi_1^2 \sigma_1^2 + \pi_2^2 \sigma_2^2 + 2\pi_1\pi_2\rho\sigma_1\sigma_2$$

Portfolios of Two Risky Assets



Portfolios of a Risky and Risk-Free Asset



Global Minimum Variance Portfolio

The portfolio of risky assets with minimum variance is called the Global Minimum Variance (GMV) portfolio.

It solves the optimization problem

$$\min \quad \frac{1}{2}\pi'\Sigma\pi \quad \text{subject to} \quad \iota'\pi = 1$$

The Lagrangean for this problem is

$$\frac{1}{2}\pi'\Sigma\pi - \gamma(\iota'\pi - 1)$$

The FOC is

$$\Sigma\pi = \gamma\iota \quad \Leftrightarrow \quad \pi = \gamma\Sigma^{-1}\iota$$

Impose the constraint $\iota'\pi = 1$ and solve for γ to obtain

$$\pi = \frac{1}{\iota'\Sigma^{-1}\iota}\Sigma^{-1}\iota$$

In other words, take the vector $\Sigma^{-1}\iota$ and divide by the sum of its elements, so the rescaled vector sums to 1.

Mean-Variance Frontier of Risky Assets

We continue to look at only risky assets so we continue to require portfolio weights to sum to 1 ($\iota'\pi = 1$).

A frontier portfolio is a portfolio that achieves a target expected return with minimum risk. It solves an optimization problem

$$\min \quad \frac{1}{2}\pi'\Sigma\pi \quad \text{subject to} \quad \mu'\pi = \mu_{\text{targ}} \quad \text{and} \quad \iota'\pi = 1$$

where μ_{targ} denotes the given target expected return.

By varying the target expected return, we trace out the frontier.

Solving for Frontier Portfolios

The Lagrangean for the optimization problem is

$$\frac{1}{2}\pi'\Sigma\pi - \delta(\mu'\pi - \mu_{\text{targ}}) - \gamma(\iota'\pi - 1)$$

The FOC is

$$\Sigma\pi - \delta\mu - \gamma\iota = 0 .$$

The solution is

$$\pi = \delta\Sigma^{-1}\mu + \gamma\Sigma^{-1}\iota$$

This means that π is a linear combination of the two vectors $\Sigma^{-1}\mu$ and $\Sigma^{-1}\iota$. Use constraints to solve for δ and γ.

More Notation

Denote the GMV portfolio by π_{gmv}. It is $\Sigma^{-1}\iota$ rescaled to sum to 1:

$$\pi_{\text{gmv}} = \frac{1}{\iota\Sigma^{-1}\iota}\Sigma^{-1}\iota$$

Let's also rescale the vector $\Sigma^{-1}\mu$ to sum to 1 and call it π_μ:

$$\pi_\mu = \frac{1}{\iota\Sigma^{-1}\mu}\Sigma^{-1}\mu$$

To simplify, define $A = \mu'\Sigma^{-1}\mu$, $B = \mu'\Sigma^{-1}\iota$, and $C = \iota'\Sigma^{-1}\iota$. Then,

$$\pi_{\text{gmv}} = \frac{1}{C}\Sigma^{-1}\iota$$

$$\pi_\mu = \frac{1}{B}\Sigma^{-1}\mu$$

Mean-Variance Frontier Again

We saw that a frontier portfolio is

$$\pi = \delta\Sigma^{-1}\mu + \gamma\Sigma^{-1}\iota$$

for some δ and γ. We can write this as

$$\begin{aligned}\pi &= \delta B\frac{1}{B}\Sigma^{-1}\mu + \gamma C\frac{1}{C}\Sigma^{-1}\mu \\ &= \delta B\pi_{\mu} + \gamma C\pi_{\text{gmv}}\end{aligned}$$

The constraint $\iota'\pi = 1$ implies

$$\delta B + \gamma C = 1$$

The constraint $\mu'\pi = \mu_{\text{targ}}$ implies

$$\delta BA + \gamma CB = \mu_{\text{targ}}$$

Can solve these two equations in two unknowns for δ and γ.

Solution

Set $\lambda = \delta B$. The constraint $\iota'\pi = 1$ implies $\gamma C = 1 - \lambda$, so the frontier portfolio is $\pi = \lambda\pi_\mu + (1 - \lambda)\pi_{\text{gmv}}$. This is the general description of frontier portfolios.

To find the particular frontier portfolio meeting the target return constraint, we can calculate

$$\mu'\pi = \lambda\mu'\pi_\mu + (1 - \lambda)\mu'\pi_{\text{gmv}} \;=\; \lambda\frac{A}{B} + (1 - \lambda)\frac{B}{C}$$

and set equal to μ_{targ} to obtain

$$\lambda \;=\; \frac{\mu_{\text{targ}} - B/C}{A/B - B/C} \;=\; \frac{BC\mu_{\text{targ}} - B^2}{AC - B^2}$$

Two Fund Spanning

The characterization $\pi = \lambda\pi_\mu + (1-\lambda)\pi_{\text{gmv}}$ means that π lies on the line through π_μ and π_{gmv} in $\mathbb{R}^n$. Every frontier portfolio is a combination of π_μ and π_{gmv}. We say that these two portfolios span the frontier.

We can consider the portfolios to be funds – like mutual funds. If you want a frontier portfolio, you can just invest in these two funds. We call this two-fund spanning.

Any other two points on the line also span the line. So, any two frontier portfolios can serve as the funds.