Day 2

More on Portfolio Choice

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Review of Portfolio Choice Problem

- Choose $\theta_1, \ldots, \theta_n$ to

$$\max \ \mathrm{E}\left[u\left(\sum_{i=1}^n \theta_i \tilde{x}_i\right)\right] \quad \text{subject to} \quad \sum_{i=1}^n p_i \theta_i = w\,.$$

- Choose $\phi_1, \ldots, \phi_n$ to

$$\max \ \mathrm{E}\left[u\left(\sum_{i=1}^n \phi_i \widetilde{R}_i\right)\right] \quad \text{subject to} \quad \sum_{i=1}^n \phi_i = w\,.$$

- Choose $\pi_1, \ldots, \pi_n$ to

$$\max \ \mathrm{E}\left[u\left(w\sum_{i=1}^n \pi_i \widetilde{R}_i\right)\right] \quad \text{subject to} \quad \sum_{i=1}^n \pi_i = 1\,.$$

Can define $\hat{u}(a) = u(wa)$ so objective is $\max \mathrm{E}[\hat{u}(\sum \pi_i \widetilde{R}_i)]$.

First-Order Condition

- If $p_i > 0$ and $p_j > 0$,

$$\mathrm{E}\left[u'\left(\sum_{i=1}^{n}\theta_i\tilde{x}_i\right)(\widetilde{R}_i - \widetilde{R}_j)\right] = 0$$

- In words: marginal utility at the optimal wealth is orthogonal to excess returns.
  - A return is the payoff of a unit-cost portfolio.
  - An excess return is the payoff of a zero-cost portfolio (for example, a difference of returns).
- Why? Investing a little less in asset j and a little more in asset i (or the reverse) cannot increase expected utility at the optimum.

A Single Risky Asset: Go Long if Risk Premium is Positive

- Assume a risk-free asset with return R_f and a single risky asset with return $\widetilde{R}$. Set $\mu = \mathsf{E}[\widetilde{R}]$ and $\sigma^2 = \mathrm{var}(\widetilde{R})$.
- Let $\phi =$ amount invested in risky asset, so $w_0 - \phi$ is invested in risk-free asset.
- Date–1 wealth is

$$\tilde{w} = (w_0 - \phi)R_f + \phi\widetilde{R} = w_0 R_f + \phi(\widetilde{R} - R_f)$$

- Claim: $\mu > R_f \Rightarrow \phi^* > 0$ (by symmetry, $\mu < R_f \Rightarrow \phi^* < 0$).

Proof that $\phi > 0$ when $\mu > R_f$

- We want to compare the utility of an investment $\phi > 0$ to the utility of $\phi = 0$. That is, we want to compare $\mathrm{E}[u(w_0 R_f + \phi(\widetilde{R} - R_f)]$ to $u(w_0 R_f)$.
- Given ϕ, expected wealth is $\overline{w} = w_0 R_f + \phi(\mu - R_f)$. Define $\tilde{\varepsilon} = \tilde{w} - \overline{w}$.
- The investor will pay π to avoid the gamble $\tilde{\varepsilon}$ where

$$u(\overline{w} - \pi) = \mathrm{E}[u(\overline{w} + \tilde{\varepsilon})] .$$

- The variance of $\tilde{\varepsilon}$ is $\phi^2\sigma^2$, so by second-order risk aversion,

$$\pi \approx \frac{1}{2}\alpha(\overline{w})\phi^2\sigma^2 < (\mu - R_f)\phi$$

  when $\phi > 0$ and small, so

$$\mathrm{E}[u(\tilde{w})] = u(\overline{w} - \pi) > u(\overline{w} - (\mu - R_f)\phi) = u(w_0 R_f)$$

DARA Implies Risky Asset is a Normal Good

- Normal good: demand rises when income (wealth) rises. Inferior: demand falls when income (wealth) rises.
- A single risky asset with $\mu > R_f$ is a normal good if the investor has decreasing absolute risk aversion.
- Proof: The FOC is

$$\mathsf{E}[u'(\tilde{w})(\widetilde{R} - R_f)] = 0$$

Differentiate it:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}w_0}\mathsf{E}[u'(w_0 R_f + \phi(\widetilde{R} - R_f))(\widetilde{R} - R_f)] \\ &= \mathsf{E}[u''(\tilde{w})\{R_f + \phi'(w_0)(\widetilde{R} - R_f)\}(\widetilde{R} - R_f)] \end{aligned}$$

Rearrange as

$$\phi'(w_0) = -\frac{R_f \mathsf{E}[u''(\tilde{w})(\widetilde{R} - R_f)]}{\mathsf{E}[u''(\tilde{w})(\widetilde{R} - R_f)^2]}\,.$$

Can show: DARA $\Rightarrow \phi' > 0$.

Multiple Risky Assets

- $\tilde{\mathbf{R}} =$ n–vector of risky asset returns
- $\mu =$ n–vector of expected returns
- $\phi =$ n–vector of investments in consumption good units
- $\pi = (1/w_0)\phi$
- $\iota =$ n–vector of 1's
- $\Sigma =$ $n \times n$ covariance matrix, $\Sigma_{ij} = \text{cov}(\tilde{R}_i, \tilde{R}_j)$

$$\Sigma = \mathsf{E}[(\tilde{R} - \mu)(\tilde{R} - \mu)']$$

- date–1 wealth $\tilde{w} = w_0 R_f + \phi'(\tilde{\mathbf{R}} - R_f \iota)$
- expected wealth $\overline{w} = w_0 R_f + \phi'(\mu - R_f \iota)$
- variance of wealth $= \phi' \Sigma \phi$. Proof:

$$\mathsf{E}[(\tilde{w} - \overline{w})^2] = \mathsf{E}[\{\phi'(\tilde{\mathbf{R}} - \mu)\}^2] = \mathsf{E}[\phi'(\tilde{R} - \mu)(\tilde{R} - \mu)'\phi] = \phi' \Sigma \phi$$

Eliminate Redundant Assets

- Each portfolio variance $\phi'\Sigma\phi \geq 0$ so a covariance matrix is positive semidefinite.
- A positive semidefinite matrix is nonsingular if and only if it is positive definite.
- Positive definiteness means all portfolio variances are positive, equivalently, there is no risk-free portfolio of risky assets.
- If there is a risk-free portfolio of risky assets, one risky asset is a linear combination of the others and can be eliminated. Deleting it does not change investment opportunities.
- By eliminating redundant assets, we can assume Σ is nonsingular and positive definite.

Diversification

- Portfolio variance is

$$\pi'\Sigma\pi = \sum_{i=1}^{n} \pi_i^2 \operatorname{var}(\widetilde{R}_i) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \pi_i\pi_j \operatorname{cov}(\widetilde{R}_i, \widetilde{R}_j)$$

- We can generally make $\sum_{i=1}^{n} \pi_i^2 \operatorname{var}(\widetilde{R}_i)$ small by diversifying, if there are many assets.
- Suppose for example that the risky assets are uncorrelated and have the same variance σ^2 ($\Sigma = \sigma^2 I$). Then

$$\pi'\Sigma\pi = \sigma^2 \sum_{i=1}^{n} \pi_i^2$$

Among portfolios fully invested in risky assets (π_i sum to 1), this variance is minimized at $\pi_i = 1/n$ and

$$\pi'\Sigma\pi = \sigma^2 \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 = \frac{\sigma^2}{n} \to 0 \quad \text{as } n \to \infty$$

CARA-Normal with Single Risky Asset

Assume CARA utility $\mathsf{E}[-\mathrm{e}^{-\alpha\tilde{w}}]$. Assume $\widetilde{R} \sim$ normal (μ, σ). Then $\tilde{w}$ is normally distributed.

Recall: If $\tilde{x}$ is normally distributed with mean μ_x and std dev σ_x, then

$$\mathsf{E}[\mathrm{e}^{\tilde{x}}] = \mathrm{e}^{\mu_x + \sigma_x^2/2}$$

Given an investment ϕ in the risky asset, $-\alpha\tilde{w}$ is normal with mean $-\alpha w_0 R_f - \alpha\phi(\mu - R_f)$ and std dev $\alpha\phi\sigma$. Hence,

$$\mathsf{E}[-\mathrm{e}^{-\alpha\tilde{w}}] = -\mathrm{e}^{-\alpha[w_0 R_f + \phi(\mu - R_f) - \alpha\phi^2\sigma^2/2]}$$

Thus,

$$w_0 R_f + \phi(\mu - R_f) - \alpha\phi^2\sigma^2/2$$

is the certainty equivalent (mean minus one-half risk aversion times variance).

CARA-Normal cont.

Optimal portfolio maximizes the certainty equivalent. Therefore, the optimum is

$$\phi^* = \frac{\mu - R_f}{\alpha\sigma^2}$$

The optimal fraction of wealth to invest is

$$\pi^* = \frac{\mu - R_f}{(\alpha w_0)\sigma^2}$$

Usually assume αw_0 (relative risk aversion at w_0) is between 2 and 10.

CARA-Normal with Multiple Risky Assets

- Certainty equivalent is mean minus one-half risk aversion times variance:

$$w_0 R_f + \phi'(\mu - R_f \iota) - \frac{1}{2}\alpha\phi'\Sigma\phi$$

- FOC is

$$\mu - R_f \iota - \alpha\Sigma\phi = 0\,.$$

- Optimum is

$$\phi^* = \frac{1}{\alpha}\Sigma^{-1}(\mu - R_f \iota)$$

  Note no wealth effects.

- Similar form to single risky asset case. Optimum investment in each asset depends on its covariances with other assets unless Σ is diagonal.

LRT Utility and Wealth Expansion Paths

- CARA utility (even without normal returns) implies investments ϕ_i in risky assets are independent of initial wealth (absence of wealth effects).
- CRRA utility implies portfolio weights π_i are independent of initial wealth.
- More generally, linear risk tolerance ($\tau(w) = A + Bw$) implies that the investment in each risky asset i is

$$\phi_i = \xi_i A + \xi_i B R_f w_0$$

  where ξ_i does not depend on A or w_0.
  - CARA is the case $B = 0$, so investments are independent of w_0 and inversely proportional to absolute risk aversion.
  - CRRA is the case $A = 0$, so

$$\pi_i = \frac{\phi_i}{w_0} = \xi_i B R_f \,.$$

    Remember $B = 1/\rho$, so weights are inversely proportional to relative risk aversion.

Proof for Log Utility

- End-of-period wealth is (where $n =$ number of risky assets):

$$\tilde{w} = w_0 R_f + \sum_{i=1}^{n} \phi_i(\tilde{R}_i - R_f) = w_0 \left[R_f + \sum_{i=1}^{n} \pi_i(\tilde{R}_i - R_f) \right] .$$

- Therefore, expected utility is

$$\mathrm{E}[\log(\tilde{w})] \;=\; \log w_0 + \mathrm{E}\left[\log\left(R_f + \sum_{i=1}^{n} \pi_i(\tilde{R}_i - R_f) \right) \right] .$$

- The optimal π_i's maximize

$$\mathrm{E}\left[\log\left(R_f + \sum_{i=1}^{n} \pi_i(\tilde{R}_i - R_f) \right) \right] ,$$

which does not depend on w_0.