Day 16

Dynamic Models

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Returns

Dates $t = 0, 1, 2, \ldots$. No tildes anymore for random things. Information grows over time as random variables are observed.

$D_{it} =$ dividend of asset i at date t. Ex-dividend price $P_{it} > 0$.

Return from t to $t+1$ is

$$R_{i,t+1} := \frac{P_{i,t+1} + D_{i,t+1}}{P_{it}}$$

Risk-free return from t to $t+1$ is $R_{f,t+1}$.

SDFs

Denote SDF at date 0 for pricing payoffs that occur at date t by M_t.

This means that the date–0 price of X_t paid at t is $\mathsf{E}[M_t X_t]$. And, $M_0 = 1$.

With no uncertainty, or with risk neutrality,

$$M_t = \frac{1}{R_{f,1}} \times \frac{1}{R_{f,2}} \times \cdots \times \frac{1}{R_{f,t}}$$

Pricing at Later Dates

What is the value at date s of a payoff X_t at $t > s$?

It must be

$$\mathsf{E}_s\left[\frac{M_t}{M_s}X_t\right]$$

where E_s denotes expectation conditional on date s information. Call this X_s and multiply by M_s to get

$$M_s X_s = \mathsf{E}_s[M_t X_t]$$

Thus, SDF $\times$ Value = martingale (Value = value of asset that pays no dividends, or dividends are reinvested.)

No Uncertainty or Risk Neutrality

$$M_s = \frac{1}{R_{f,1}} \times \cdots \times \frac{1}{R_{f,s}}$$
$$M_t = \frac{1}{R_{f,1}} \times \cdots \times \frac{1}{R_{f,s}} \times \frac{1}{R_{f,s+1}} \times \cdots \times \frac{1}{R_{f,t}}$$

So,

$$\frac{M_t}{M_s} = \frac{1}{R_{f,s+1}} \times \cdots \times \frac{1}{R_{f,t}}$$

One-Period SDF

The SDF at t for pricing payoffs at $t+1$ is

$$\frac{M_{t+1}}{M_t}$$

With no uncertainty or risk neutrality,

$$\frac{M_{t+1}}{M_t} = \frac{1}{R_{f,t+1}}$$

In general, \$1 at t invested in asset i grows to $R_{i,t+1}$ at $t+1$, so

$$1 = \mathsf{E}_t\left[\frac{M_{t+1}}{M_t} R_{i,t+1}\right]$$

Compounding One-Period SDFs

In general,

$$M_t = \frac{M_t}{M_0} = \frac{M_1}{M_0} \times \frac{M_2}{M_1} \times \cdots \times \frac{M_t}{M_{t-1}}$$

and the SDF at s for pricing payoffs at $t > s$ is

$$\frac{M_t}{M_s} = \frac{M_{s+1}}{M_s} \times \frac{M_{s+2}}{M_{s+1}} \times \cdots \times \frac{M_t}{M_{t-1}}$$

Dynamic Factor Model

From

$$1 = \mathrm{E}_t\left[\frac{M_{t+1}}{M_t} R_{i,t+1}\right]$$

we get

$$1 = \frac{\mathrm{E}_t[R_{i,t+1}]}{R_{f,t+1}} + \mathrm{cov}_t\left(\frac{M_{t+1}}{M_t}, R_{i,t+1}\right)$$

So

$$\mathrm{E}_t[R_{i,t+1}] - R_{f,t+1} = -R_{f,t+1}\,\mathrm{cov}_t\left(\frac{M_{t+1}}{M_t}, R_{i,t+1}\right)$$

Portfolio Choice

Stack returns into an n–vector R_{t+1}. One may be risk-free (return $= R_{f,t+1}$).

Investor chooses consumption C_t and a portfolio $\pi_t \in \mathbb{R}^n$. $\iota'\pi_t = 1$. Labor income Y_t.

Suppose investor seeks to maximize

$$\sum_{t=0}^{\infty} \delta^t u(C_t)$$

Wealth (actually financial wealth) W satisfies the intertemporal budget constraint

$$W_{t+1} = (W_t - C_t)\pi_t' R_{t+1} + Y_{t+1}$$

Euler Equation

A necessary condition for consumption/investment optimality is that, for all dates t and assets i,

$$\mathsf{E}_t\left[\frac{\delta u'(C_{t+1})}{u'(C_t)} R_{i,t+1}\right] = 1$$

This is called the Euler equation. It is derived by the same logic as in a single-period model.

The Euler equation is equivalent to:

$$M_t := \frac{\delta^t u'(C_t)}{u'(C_0)}$$

is an SDF process. The one-period SDFs are

$$\frac{M_{t+1}}{M_t} = \frac{\delta u'(C_{t+1})}{u'(C_t)}$$

Representative Investor and SDF Process

Let C denote aggregate consumption. Assume there is a representative investor with CRRA utility and risk aversion ρ. Then, the one-period SDF is

$$\frac{M_{t+1}}{M_t} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho}$$

The SDF process is

$$M_t = \delta^t \left(\frac{C_t}{C_0} \right)^{-\rho}$$

Market Price-Dividend Ratio

Define the market portfolio as the claim to future consumption. Consumption is the dividend of the portfolio. Assume consumption growth C_{t+1}/C_t is iid lognormal.

The ex-dividend date–t price of the market portfolio is

$$P_t := \mathsf{E}_t \sum_{u=t+1}^{\infty} \frac{M_u}{M_t} C_u = \mathsf{E}_t \sum_{u=t+1}^{\infty} \delta^{u-t} \left(\frac{C_u}{C_t}\right)^{-\rho} C_u$$

So, the price-dividend ratio is

$$\begin{aligned} \frac{P_t}{C_t} &= \mathsf{E}_t \sum_{u=t+1}^{\infty} \delta^{u-t} \left(\frac{C_u}{C_t}\right)^{1-\rho} \\ &= \mathsf{E} \sum_{u=1}^{\infty} \delta^{u} \left(\frac{C_u}{C_0}\right)^{1-\rho} \end{aligned}$$

Assume $\log C_{t+1} = \log C_t + \mu + \sigma \varepsilon_{t+1}$ for iid standard normals ε.

Then

$$\log C_u = \log C_0 + u\mu + \sigma \sum_{n=1}^{u} \varepsilon_n$$

Hence,

$$\begin{aligned} \mathsf{E}\left[\left(\frac{C_u}{C_0}\right)^{1-\rho}\right] &= \mathsf{E}\left[\exp\left((1-\rho)u\mu + (1-\rho)\sigma \sum_{n=1}^{u} \varepsilon_n\right)\right] \\ &= \exp\left((1-\rho)u\mu + \frac{1}{2}(1-\rho)^2 u\sigma^2\right) \end{aligned}$$

So, the price-dividend ratio is

$$\sum_{u=1}^{\infty} \delta^u \exp\left((1-\rho)u\mu + \frac{1}{2}(1-\rho)^2 u\sigma^2\right) = \frac{\nu_1}{1-\nu_1}$$

where

$$\nu_1 = \delta \mathrm{E}\left[\left(\frac{C_1}{C_0}\right)^{1-\rho}\right] = \delta \mathrm{e}^{(1-\rho)\mu+(1-\rho)^2\sigma^2/2}$$

provided $\nu_1 < 1$.

This is the same ν_1 we saw in Chapter 7. Everything else—risk-free return, expected market return, log equity premium, equity premium puzzle—is exactly the same as in Chapter 7.

Risk-Neutral Probability

Consider an arbitrary finite (possibly large) horizon T. Define a probability measure Q by $Q(A) = \mathsf{E}[M_T 1_A]$ for each event A that can be distinguished by date T (at date T, you know whether A happened or not).

Define E^* as expectation with respect to Q. Then for all assets i and dates t,

$$\mathsf{E}_t^*[R_{i,t+1}] = R_{f,t+1}$$

and the price at t of a payoff X_{t+1} at date $t+1$ is

$$\frac{\mathsf{E}_t^*[X_{t+1}]}{R_{f,t+1}}$$

Martingales

For any $s \leq t$, the price at s of a payoff X_{t+1} at $t+1$ is

$$E^*_s\left[\frac{X_{t+1}}{R_{f,s+1} \times \cdots \times R_{f,t+1}}\right]$$

By switching measures, we change the discounting factor from M_{t+1}/M_s to this reciprocal of the product of risk-free returns. More generally, we switch the SDF process M to the process

$$Y_t := \frac{1}{R_{f,1} \times R_{f,2} \times \cdots \times R_{f,t}}$$

We have $M\times$ Value is a martingale and also $Y\times$ Value is a Q–martingale, where Value is the value of any portfolio with dividends reinvested.

Testing Conditional Models

Suppose we have a model for an SDF. Call the model value $\hat{M}$. We want to test whether

$$(\forall\ t, i) \qquad E_t[\hat{M}_{t+1} R_{i,t+1}] = 1 \qquad (\star)$$

Let I_t be any variable observed at t. Multiply both sides by I_t and rearrange as:

$$(\forall\ t, i) \qquad E_t[I_t \hat{M}_{t+1} R_{i,t+1}] - I_t = 0$$

Take the expectation and use the law of iterated expectations to obtain

$$(\forall\ t, i) \qquad E[I_t(\hat{M}_{t+1} R_{i,t+1} - 1)] = 0 \qquad (\star\star)$$

The conditional model $(\star)$ implies the unconditional moment condition $(\star\star)$ for every instrument I. If we reject the unconditional moment conditions, then we reject the model.