Day 25

Perpetual Options

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Set-up

Single risky asset with price S and constant volatility σ, single Brownian motion, constant risk-free rate

Dividend paid by risky asset in time period $\mathrm{d}t$ is $qS_t\,\mathrm{d}t$ for constant q ("dividend yield")

Total return is

$$\frac{\mathrm{d}S + qS\,\mathrm{d}t}{S} = \frac{\mathrm{d}S}{S} + q\,\mathrm{d}t$$

Total expected return under RNP is risk-free rate, so

$$\frac{\mathrm{d}S}{S} = (r - q)\,\mathrm{d}t + \sigma\,\mathrm{d}B^*$$

for a risk-neutral Brownian motion B^*

Perpetual Call

Perpetual call option with strike K

Why exercise? To capture the dividend. But the asset price and dividend must be high enough before it is optimal to do so.

An example of a strategy is to pick a number x and exercise the first time S_t gets up to x. The optimal strategy will be of this type.

The problem of finding the optimal exercise time is in the class of problems often called optimal stopping.

Exercise Boundary

We will first calculate the value if we exercise the first time S_t gets up to x for an arbitrary $x > S_0$.

Let $\tau = \inf\{t \mid S_t \geq x\}$. This is called the hitting time of x.

By the time-homogeneity of S, the value at any $t < \tau$ depends only on S_t. Call it $f(S_t)$.

More formally,

$$f(s) = \mathsf{E}^*[\mathrm{e}^{-r\tau}((x - K) \mid S_0 = s] = \mathsf{E}^*[\mathrm{e}^{-r(\tau-t)}(x - K) \mid S_t = s]$$

Fundamental ODE

The fundamental ODE is

$$\frac{\text{drift}^* \text{ of } f}{f} = r$$

which is

$$(r - q)Sf' + \frac{1}{2}\sigma^2 S^2 f'' = rf$$

Trying a power solution $f(S) = S^\gamma$, we see that f satisfies the ODE if and only if

$$(r - q)\gamma + \frac{1}{2}\sigma^2\gamma(\gamma - 1) = r$$

The quadratic formula shows that there are two real roots of this equation. One is negative and the other is greater than 1.

General Solution and Boundary Conditions

Let $\gamma =$ absolute value of negative root, and $\beta =$ positive root. The general solution of the ODE is

$$aS^{-\gamma} + bS^{\beta}$$

for constants a and b that must be determined by boundary conditions.

The value f of the call exercised at the hitting time of x satisfies $f(0) = 0$ and $f(x) = x - K$. The condition $f(0) = 0$ implies $a = 0$, and the condition $f(x) = x - K$ implies $b = (x - K)x^{-\beta}$.

The value of the call is

$$f(S_t) = (x - K)\left(\frac{S_t}{x}\right)^{\beta}$$

Optimal Stopping

To optimize, maximize $(x-K)\left(\frac{S_t}{x}\right)^{\beta}$ over x. The factor S_t^{β} is a positive constant and is irrelevant for determining the optimum, so we can maximize

$$(x-K)x^{-\beta} = x^{1-\beta} - Kx^{-\beta}$$

The FOC is

$$(1-\beta)x^{-\beta} + \beta K x^{-\beta-1} = 0$$

Equivalently,

$$(1-\beta)x + \beta K = 0$$

So,

$$x^{*} = \frac{\beta}{\beta-1}K$$

Perpetual Put

Recall that the general solution of the ODe is $f(s) = as^{-\gamma} + bs^{\beta}$.

For a put, we exercise the first time S_t drops to a boundary x. The boundary conditions for a put are $f(\infty) = 0$, and $f(x) = K - x$. The condition $f(\infty) = 0$ implies $b = 0$. The condition $f(x) = K - x$ implies $a = (K - x)x^{\gamma}$.

So, the put value is

$$f(S_t) = (K - x)\left(\frac{x}{S_t}\right)^{\gamma}$$

The FOC for maximizing over x is

$$\gamma K x^{\gamma-1} - (1+\gamma)x^{\gamma} = 0$$

Maximizing over x yields $x^* = \gamma K/(1+\gamma)$.