Day 1

Utility Functions, Risk Aversion, and Intro to Portfolio Choice

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1) A company can invest K to generate a cash flow of $\tilde{x}$ in one year. Under what circumstances should it make the investment?

2) Under what circumstances can we expect stock A to earn a higher return than stock B?

Risk Aversion

- Expected utility $\mathsf{E}[u(\tilde{w})]$
  - Utility function u is unique up to monotone affine transform: $f(w) = a + bu(w)$ for $b > 0$.
- Risk aversion: $\mathsf{E}[\tilde{\varepsilon}] = 0 \Rightarrow \mathsf{E}[u(w + \tilde{\varepsilon})] < \mathsf{E}[u(w)]$.
  - Equivalent to concavity (Jensen's inequality)
  - Equivalent to decreasing marginal utility: $u'' \leq 0$.
- Certainty equivalent: a constant x is the certainty equivalent of a random $\tilde{w}$ if $u(x) = \mathsf{E}[u(\tilde{w})]$. Risk aversion implies $x < \mathsf{E}[\tilde{w}]$.
- Absolute risk aversion: $\alpha(w) = -u''(w)/u'(w)$
- Risk tolerance: $\tau(w) = 1/\alpha(w)$

Second Order Risk Aversion

- The amount that an expected-utility investor would pay to avoid a small gamble is approximately proportional to the variance of the gamble with proportionality coefficient equal to one-half of absolute risk aversion.
- Here is a more precise statement of the result. Fix w. Let $\tilde{\varepsilon}$ be a bounded zero-mean random variable with unit variance. For $\sigma > 0$, define $\pi(\sigma)$ by

$$u(w - \pi(\sigma)) = \mathsf{E}[u(w + \sigma\tilde{\varepsilon})]\,.$$

- So, $w - \pi(\sigma)$ is the certainty equivalent of $w + \sigma\tilde{\varepsilon}$, and the variance of $w + \sigma\tilde{\varepsilon}$ is σ^2.
- We will show that

$$\lim_{\sigma \to 0} \frac{\pi(\sigma)}{\sigma^2} = \frac{1}{2}\alpha(w)$$

- Thus, for small σ,

$$\pi(\sigma) \approx \frac{1}{2}\alpha(w)\sigma^2$$

Proof

Take an exact Taylor series approximation:

$$\pi(\sigma) = \pi(0) + \pi'(0)\sigma + \frac{1}{2}\pi''(x_\sigma)\sigma^2$$

for $0 < x_\sigma < \sigma$ Clearly, $\pi(0) = 0$. Differentiate both sides of

$$u(w - \pi(\sigma)) = \mathrm{E}[u(w + \sigma\tilde{\varepsilon})] \qquad (\star)$$

and evaluate at $\sigma = 0$ to obtain

$$-u'(w)\pi'(0) = \mathrm{E}[u'(w)\tilde{\varepsilon}] = u'(w)\mathrm{E}[\tilde{\varepsilon}] = 0$$

Hence,

$$\frac{\pi(\sigma)}{\sigma^2} = \frac{1}{2}\pi''(x_\sigma) \to \frac{1}{2}\pi''(0)$$

Differentiate both sides of $(\star)$ twice and evaluate at $\sigma = 0$ to obtain $\pi''(0) = \alpha(w)$.

Relative Risk Aversion

- Relative risk aversion: $\rho(w) = w\alpha(w)$.
- As before, let Let $\tilde{\varepsilon}$ be a bounded zero-mean random variable with unit variance.
- Now, define $\pi(\sigma)$ by

$$u(w - \pi(\sigma)w) = \mathrm{E}[u(w + \sigma\tilde{\varepsilon}w)]$$

- So now π is the fraction of wealth you would pay to avoid a gamble that is proportional to wealth.
- If σ is small, then

$$\pi(\sigma) \approx \frac{1}{2}\rho(w)\sigma^2 \, .$$

- Proof: From the previous result, $\pi(\sigma)w \approx \alpha(w)\sigma^2 w^2/2$.

CARA and CRRA Utility

- A utility function has constant absolute risk aversion (CARA) if $\alpha(w)$ is constant (same for all w).
- Every CARA utility function with absolute risk aversion α is a monotone affine transform of

$$u(w) = -\mathrm{e}^{-\alpha w}$$

- A utility function has constant relative risk aversion (CRRA) if $\rho(w)$ is constant (same for all w).
- Every CRRA utility function with relative risk aversion $\rho = 1$ is a monotone affine transform of

$$u(w) = \log w$$

- Every CRRA utility function with relative risk aversion $\rho \neq 1$ is a monotone affine transform of

$$u(w) = \frac{1}{1-\rho} w^{1-\rho}$$

Proofs

CARA is negative exponential:

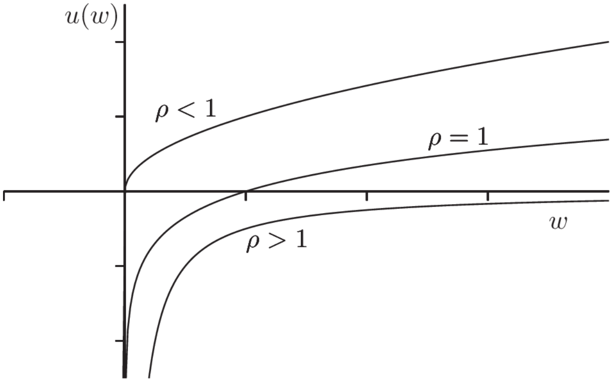
$\alpha(w) = -\mathrm{d}\log u'(w)/\mathrm{d}w$, so CARA implies

$$\begin{aligned} \log u'(w) = \log u'(0) - \alpha w &\Rightarrow u'(w) = u'(0)\mathrm{e}^{-\alpha w} \\ &\Rightarrow u(w) = u(0) + u'(0)\int_0^w \mathrm{e}^{-\alpha x}\,\mathrm{d}x \\ &\Rightarrow u(w) = u(0) - \frac{u'(0)}{\alpha}\left(\mathrm{e}^{-\alpha w} - 1\right) \end{aligned}$$

CRRA utilities:

Use $\rho(w) = -\mathrm{d}\log u'(w)/\mathrm{d}\log w$.

CRRA Utility Functions



CARA Utility and Normal Gambles

- What would someone with CARA utility pay to avoid a normally-distributed zero-mean gamble with variance σ^2?
- Solve, for $\tilde{\varepsilon}$ a standard normal,

$$u(w-\pi) = \mathrm{E}\left[u(w+\sigma\tilde{\varepsilon})\right]$$

$$\Leftrightarrow \qquad -e^{-\alpha(w-\pi)} = \mathrm{E}\left[-e^{-\alpha(w+\sigma\tilde{\varepsilon})}\right]$$

- Use the following fact: if $\tilde{x}$ is normal (μ, σ), then

$$\mathrm{E}\left[e^{\tilde{x}}\right] = e^{\mu+\sigma^2/2}$$

- Solve

$$e^{-\alpha(w-\pi)} = e^{-\alpha w+\alpha^2\sigma^2/2}$$

- Solution: $\pi = \alpha\sigma^2/2$.

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Linear Risk Tolerance

- CARA risk tolerance $= 1/\alpha$
- CRRA absolute risk aversion $= \rho/w$. Risk tolerance $= w/\rho$
- Linear risk tolerance: $\tau(w) = A + Bw$.
  - For CARA, $A = 1/\alpha$, $B = 0$
  - For CRRA, $A = 0$, $B = 1/\rho$
  - In general, B is called the cautiousness parameter.

LRT Utility Functions with $B > 0$

- CARA
- CRRA
- Shifted CRRA
  - Shifted log: $u(w) = \log(w - \zeta)$
  - Shifted power: $u(w) = \frac{1}{1-\rho}(w - \zeta)^{1-\rho}$

Quadratic Utility

$$u(w) = -\frac{1}{2}(\zeta - w)^2$$

- Quadratic utility is monotone increasing for $w < \zeta$ (ζ is bliss point).
- Risk tolerance: $\tau(w) = \zeta - w$, so $A = \zeta$ and $B = -1$.
- Implies mean-variance preferences:

$$\mathrm{E}[u(\tilde{w})] \;\sim\; \zeta\overline{w} - \frac{1}{2}\overline{w}^2 - \frac{1}{2}\mathrm{var}(\widetilde{w})$$

Decreasing Absolute Risk Aversion

- Recall that an investor will pay approximately $\alpha(w)\sigma^2/2$ to avoid a small gamble with variance σ^2.
- Suppose an investor's wealth increases. Should she be willing to pay more or less to avoid a given gamble?
- Paying less seems reasonable. This means $\alpha(w)$ is a decreasing function of w (DARA utility).
- If an investor has LRT utility with $B > 0$, does she have DARA utility?
- If an investor has quadratic utility, does she have DARA utility?

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Notation

- Single consumption good at each of two dates 0 and 1
- Date–0 wealth w (in units of consumption good)
- Assets
  - Assets $i = 1, \dots, n$
  - Date–0 prices p_i (in units of consumption good)
  - Date–1 payoffs $\tilde{x}_i$ (in units of consumption good)
- Returns
  - Returns $\widetilde{R}_i = \tilde{x}_i / p_i$ (assuming $p_i > 0$)
  - Rates of return $(\tilde{x} - p_i)/p_i = \widetilde{R}_i - 1$
  - If there is a risk-free asset ($\tilde{x}$ constant) then return is R_f
- Portfolios
  - $\theta_i =$ number of shares held in portfolio
  - $\phi_i = \theta_i p_i =$ units of consumption good invested
  - $\pi_i = \theta_i p_i / w =$ fraction of wealth invested

Portfolio Choice Problem

- Choose $\theta_1, \ldots, \theta_n$ to

$$\max \ \mathsf{E}\left[u\left(\sum_{i=1}^{n} \theta_i \tilde{x}_i\right)\right] \quad \text{subject to} \quad \sum_{i=1}^{n} p_i \theta_i = w\,.$$

- Choose $\phi_1, \ldots, \phi_n$ to

$$\max \ \mathsf{E}\left[u\left(\sum_{i=1}^{n} \phi_i \widetilde{R}_i\right)\right] \quad \text{subject to} \quad \sum_{i=1}^{n} \phi_i = w\,.$$

- Choose $\pi_1, \ldots, \pi_n$ to

$$\max \ \mathsf{E}\left[u\left(w\sum_{i=1}^{n} \pi_i \widetilde{R}_i\right)\right] \quad \text{subject to} \quad \sum_{i=1}^{n} \pi_i = 1\,.$$

Can define $\hat{u}(a) = u(wa)$ so objective is $\max \mathsf{E}[\hat{u}(\sum \pi_i \widetilde{R}_i)]$.

Comments

- Short sales are allowed ($\theta_i < 0$)
- There are no margin requirements.
  - In the U.S. stock market, an investor with \$100 cash can only buy \$200 of stock (borrowing \$100).
  - In our formulation, there are no limits on borrowing, except that $\sum \theta_i \tilde{x}_i$ must be in the domain of $u(\cdot)$—for example, positive if $u = \log$.
  - In real markets, collateral (margin) also has to be posted against short sales, but we do not require that in our formulation.
- Here, we take date–0 consumption and investment as given and optimize over the portfolio. We can also optimize over date–0 consumption and investment.
- We can sometimes allow for other non-portfolio income $\tilde{y}$ at date–1 (for example, labor income).

First-Order Condition

- Lagrangean:

$$\mathsf{E}\left[u\left(\sum_{i=1}^{n}\theta_i\tilde{x}_i\right)\right]-\lambda\left(\sum_{i=1}^{n}p_i\theta_i-w\right)$$

- Assume interior optimum and assume can interchange differentiation and expectation to obtain

$$(\forall\, i)\quad \mathsf{E}\left[u'\left(\sum_{i=1}^{n}\theta_i\tilde{x}_i\right)\tilde{x}_i\right]=\lambda p_i$$

- If $p_i > 0$,

$$\mathsf{E}\left[u'\left(\sum_{i=1}^{n}\theta_i\tilde{x}_i\right)\widetilde{R}_i\right]=\lambda$$

First-Order Condition cont.

- If $p_i > 0$ and $p_j > 0$,

$$\mathrm{E}\left[u'\left(\sum_{i=1}^{n}\theta_i\tilde{x}_i\right)(\widetilde{R}_i - \widetilde{R}_j)\right] = 0$$

- In words: marginal utility at the optimal wealth is orthogonal to excess returns.
  - A return is the payoff of a unit-cost portfolio.
  - An excess return is the payoff of a zero-cost portfolio (for example, a difference of returns).
- Why? Investing a little less in asset j and a little more in asset i (or the reverse) cannot increase expected utility at the optimum.