Day 10

Equilibrium

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BUSI 521–ECON 505
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Spring 2022

Competitive Equilibrium

n assets, possibly including a risk-free asset. Date-1 values $\tilde{x}_i$ are exogenously given. Investors have endowments of shares $\bar{\theta}_h \in \mathbb{R}^n$ at date 0 and endowments $\tilde{y}_h$ of the date–1 consumption good. Choose portfolios $\theta_h \in \mathbb{R}^n$ subject to budget constraint

$$p'\theta_h \;\leq\; p'\bar{\theta}_h$$

Maximize expected utility of date–1 wealth $\tilde{y}_h + \sum_{i=1}^n \theta_{hi}\tilde{x}_i$.
Equilibrium is $(p, \theta_1^*, \ldots, \theta_H^*)$ such that θ_h^* is optimal for each investor h given p, and markets clear:

$$\sum_{h=1}^H \theta_h^* \;=\; \sum_{h=1}^H \bar{\theta}_h\,.$$

Existence??

Optimality??

Equilibrium risk premia??

Arrow-Debreu Model

k states. Assets are Arrow securities: pay 1 in single state and 0 otherwise. Denote price vector by $q \in \mathbb{R}^k$. Endowment of date–1 consumption good is $y_h \in \mathbb{R}^k$. Portfolio θ_h determines date–1 wealth as $w_h = \theta_h + y_h$. Budget constraint is

$$q'\theta_h \leq q'\bar{\theta}_h \quad \Leftrightarrow \quad q'w_h \leq q'(\bar{\theta}_h + y_h)$$

Market clearing is

$$\sum_{h=1}^{H} \theta_h = \sum_{h=1}^{H} \bar{\theta}_h \quad \Leftrightarrow \quad \sum_{h=1}^{H} w_h = \sum_{h=1}^{h} (\bar{\theta}_h + y_h)$$

Existence: standard result

Optimality: standard welfare theorems

When is a competitive equilibrium in a securities market equivalent to an equilibrium in an Arrow-Debreu model?

Answer: if the securities market is complete.

Let $X = n \times k$ matrix of asset payoffs. Suppose rank of X is k. Then for each $w \in \mathbb{R}^k$, there is some $\theta \in \mathbb{R}^n$ such that $X'\theta = w$. Furthermore, securities prices p and state prices q correspond as $Xq = p$.

So, equilibria in complete markets are Pareto optimal.

Pareto Optimum

Let $\tilde{w}_m$ denote end-of-period market wealth.

An allocation is $\tilde{w}_1, \ldots, \tilde{w}_H$ such that $\sum_h \tilde{w}_h = \tilde{w}_m$.

A Pareto optimum is an allocation such that any other allocation that makes at least one person better off also makes at least one person worse off.

A Pareto optimum solves a social planner's problem: for some weights $\lambda_1, \ldots, \lambda_H$,

$$\max \quad \sum_{h=1}^{H} \lambda_h \mathrm{E}[u_h(\tilde{w}_h)] \quad \text{subject to} \quad \sum_{h=1}^{H} \tilde{w}_h = \tilde{w}_m \,.$$

Social Planner's Problem

The resource constraint is a separate constraint for each state. And, expected utility is additive across states. So, we can solve the maximization problem state-by-state.

What does this mean? Consider the problem $\max a + b$ subject to $a \leq 3$ and $b \leq 5$. We can solve this as separate problems: $\max a$ s.t. $a \leq 3$ and $\max b$ s.t. $b \leq 5$.

In each state of the world ω, the social planner solves

$$\max \quad \sum_{h=1}^{H} \lambda_h u_h(\tilde{w}_h(\omega)) \quad \text{subject to} \quad \sum_{h=1}^{H} \tilde{w}_h(\omega) = \tilde{w}_m(\omega)\,.$$

Let $\tilde{\eta}(\omega)$ denote the Lagrange multiplier in state ω. Then, for all h,

$$\lambda_h u_h'(\tilde{w}_h(\omega)) = \tilde{\eta}(\omega)\,.$$

Sharing Rules

At a Pareto optimum, there is equality of MRS's: for all individuals h and ℓ and states i and j,

$$\frac{u_h'(\tilde{w}_h(\omega_i))}{u_h'(\tilde{w}_h(\omega_j))} = \frac{u_\ell'(\tilde{w}_\ell(\omega_i))}{u_\ell'(\tilde{w}_\ell(\omega_j))}$$

If market wealth is higher in state i than in state j, then at any Pareto optimum (assuming strict risk aversion) all investors have higher wealth in state i than in state j:

$$\begin{aligned}
\widetilde{w}_h(\omega_i) > \widetilde{w}_h(\omega_j) &\Rightarrow \frac{u_h'(\widetilde{w}_h(\omega_i))}{u_h'(\widetilde{w}_h(\omega_j))} < 1 \\
&\Rightarrow \frac{u_\ell'(\widetilde{w}_\ell(\omega_i))}{u_\ell'(\widetilde{w}_\ell(\omega_j))} < 1 \\
&\Rightarrow \widetilde{w}_\ell(\omega_i) > \widetilde{w}_\ell(\omega_j)\,.
\end{aligned}$$

This implies each investor's wealth is a function of market wealth. The function is called a sharing rule.

Example

Suppose there are two risk-averse investors and two possible states of the world, with $\widetilde{w}_m$ being the same in both states, say, $\widetilde{w}_m = 6$, and with the two states being equally likely. Can the allocation

$$\widetilde{w}_1 = \begin{cases} 2 & \text{in state 1} \\ 4 & \text{in state 2} \end{cases}$$

$$\widetilde{w}_2 = \begin{cases} 4 & \text{in state 1} \\ 2 & \text{in state 2} \end{cases}$$

be Pareto optimal?

Sharing Rules with Linear Risk Tolerance

Assume $\tau_h(w) = A_h + Bw$ with same cautiousness parameter $B \geq 0$ for all individuals. Note $B > 0$ implies DARA. Then, either

- Everyone has CARA utility: $-\mathrm{e}^{-\alpha_h w}$, or
- Everyone has shifted log utility: $\log(w - \zeta_h)$, or
- Everyone has shifted power utility with $\rho > 0$:

$$\frac{1}{1-\rho}(w - \zeta_h)^{1-\rho}$$

In this case, Pareto optimal sharing rules are affine: $\tilde{w}_h = a_h + b_h \tilde{w}_m$.

Pareto optimal sharing rules with LRT utility and same cautiousness parameter are affine ($\tilde{w}_h = a_h + b_h \tilde{w}_m$) with

- $\sum_{h=1}^{H} a_h = 0$, and
- $\sum_{h=1}^{H} b_h = 1$.

With CARA utility, $b_h = \tau_h / \sum_{j=1}^{H} \tau_j$.

With shifted log ($\rho = 1$) or shifted power,

$$b_h = \frac{\lambda_h^{1/\rho}}{\sum_{j=1}^{H} \lambda_j^{1/\rho}}$$

where the λ's are the weights in the social planning problem.

Proof for CARA Utility

Social planner's problem is (in each state of the world)

$$\max \quad \sum_{h=1}^{H} \lambda_h e^{-\alpha_h w_h} \quad \text{subject to} \quad \sum_{h=1}^{H} w_h = w$$

where w denotes the value of $\tilde{w}_m$ in the given state.

FOC is: $(\forall\, h)$ $\lambda_h \alpha_h e^{-\alpha_h w_h} = \eta$ where η is the Lagrange multiplier (in the given state). Set $\tau = \sum_h \tau_h$. We have

$$w_h = -\tau_h \log \eta + \tau_h \log(\lambda_h \alpha_h)$$

$$\Rightarrow \quad w = -\tau \log \eta + \sum_{\ell=1}^{H} \tau_\ell \log(\lambda_\ell \alpha_\ell)$$

$$\Rightarrow \quad -\log \eta = \frac{1}{\tau} w - \frac{1}{\tau} \sum_{\ell=1}^{H} \tau_\ell \log(\lambda_\ell \alpha_\ell)$$

$$\Rightarrow \quad w_h = \frac{\tau_h}{\tau} w - \frac{\tau_h}{\tau} \sum_{\ell=1}^{H} \tau_\ell \log(\lambda_\ell \alpha_\ell) + \tau_h \log(\lambda_h \alpha_h)$$

Competitive Equilibria with LRT Utility

- Assume there is a risk-free asset. Assume all investors have linear risk tolerance $\tau_h(w) = A_h + Bw$ with the same cautiousness parameter B. Assume there are no $\tilde{y}_h$'s.
- The set of equilibrium prices does not depend on the distribution of wealth across investors.
  - Called Gorman aggregation
  - Due to wealth expansion paths being parallel (Chapter 2)
- Any Pareto optimal allocation can be implemented in the securities market.
  - Due to affine sharing rules, we only need the risk-free asset and market portfolio
  - Example of two-fund separation (Chapter 2)
- Any competitive equilibrium is Pareto optimal.