Day 21

Dynamic Programming

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BUSI 521–ECON 505
Rice University
Spring 2022

Optimal Capital Investment

A firm combines capital k and labor ℓ to produce output $q = k^\alpha \ell^\psi$ with $\alpha + \psi \leq 1$. The wage rate w is constant.

The firm is a monopolist facing a constant elasticity demand curve:

$$\frac{\mathrm{d}\log q}{\mathrm{d}\log p} = \gamma \quad \Rightarrow \quad p = aq^{-1/\gamma}$$

for a constant (soon to be stochastic process) a. We call $pq - w\ell$ the firm's operating cash flow.

Capital depreciates over time. Also, adjusting the capital stock is costly. To adjust by Δk costs $\Delta k + (1/2)(\Delta k)^2 b$. We call $-\Delta k - (1/2)(\Delta k)^2 b$ the firm's investment cash flow. Δk can be negative.

Total cash flow is operating cash flow plus financing cash flow.

We formulate the problem in continuous time. The firm chooses an investment rate I_t and capital K_t evolves as

$$\mathrm{d}K_t = -\delta K_t \,\mathrm{d}t + I_t \,\mathrm{d}t$$

Given the capital stock K at time t, the firm chooses the labor input L_t to maximize the operating cash flow $pq - w\ell$, i.e.,

$$L_t = \operatorname{argmax}_{\ell} \; a \left(K_t^{\alpha} \ell^{\psi}\right)^{1-1/\gamma} - w\ell$$

There is random time-varying demand: the constant a is a stochastic process A. The firm chooses an investment policy to maximize

$$\mathsf{E} \int_0^{\infty} M_t \left[\text{Operating Cash Flow} - I_t - (1/2) b I_t^2\right] \,\mathrm{d}t$$

where M is the SDF process.

Assume the risk-free rate is constant. By switching to the risk-neutral probablility, we can formulate the firm's objective as

$$\mathsf{E}^* \int_0^\infty \mathrm{e}^{-rt} \left[\text{Operating Cash Flow} - I_t - (1/2)bI_t^2\right] \,\mathrm{d}t$$

Assume A is a geometric Brownian motion (hence Markov) under the risk-neutral probability. The state variables are A_t and K_t.

The problem is time-stationary, so the firm's value function is of the form $J(A_t, K_t)$.

What sort of solution can we find?

We want to find the optimal investment rate I_t as a function of A_t and K_t.

We can show that the optimum satisfies

$$I_t^* = \frac{1}{b}\left[\frac{\partial J(A_t, K_t)}{\partial K_t} - 1\right]$$

Can we compute $\partial J / \partial K$?

Not analytically. But, we can show that J satisfies a partial differential equation (PDE), which can be solved numerically.

The PDE is the Bellman equation, called Hamilton-Jacobi-Bellman (HJB) equation in continuous time.

The HJB equation is

$$\max_{\text{controls}} \left\{ \text{current reward} - rJ + \text{drift of J} \right\} = 0$$

The optimal controls must achieve the max here.

Let $\pi(A_t, K_t)$ denote the operating cash flow maximized over L_t. Then the current reward is

$$\pi(A_t, K_t) - I_t - \frac{1}{2} b I_t^2$$

The control is I_t.

The drift of J is

$$\frac{\partial J}{\partial A} \text{drift of A} + \frac{\partial J}{\partial k}(-\delta K + I) + \frac{1}{2} \frac{\partial^2 J}{\partial A^2} (\mathrm{d}A)^2$$

dropping the $\mathrm{d}t$ from $(\mathrm{d}A)^2$.

Substituting into the HJB equation and dropping terms that do not depend on I, the maximization problem is

$$\max_I \left\{ -I - \frac{1}{2}bI^2 + \frac{\partial J}{\partial K}I \right\}$$

The solution is

$$I = \frac{\partial J/\partial K - 1}{b}$$

Write J_k for the partial derivative. So, the solution is $I = (J_k - 1)/b$.

At the max $I = (J_k - 1)/b$, the HJB equation tells us

$$\text{current reward} - rJ + \text{drift of J} = 0$$

This is a PDE in J.

Write J_a for $\partial J/\partial A$ and J_{aa} for the second partial. Then, suppressing the arguments (A, K), we have

$$\pi - \frac{J_k - 1}{b} - \frac{1}{2}b\left(\frac{J_k - 1}{b}\right)^2 - rJ \\ + J_a\text{drift of A} + \left(\frac{J_k - 1}{b} - \delta K\right)J_k + \frac{1}{2}J_{aa}(\mathrm{d}A)^2 = 0$$

This is an equation that must hold for $A > 0$ and $K > 0$, with boundary conditions at 0 and ∞.

HJB Equation

Discrete-time, no uncertainty, control c, state variable x, and state evolution equation $x' = f(x, c)$:

$$J(x) = \max_c \ u(c) + \mathrm{e}^{-\delta} J(x')$$

With uncertainty and expected utility

$$J(x) = \max_c \ u(c) + \mathrm{e}^{-\delta} \mathrm{E}[J(x') \mid x, c]$$

Equivalently,

$$0 = \max_c \ u(c) + \mathrm{E}[J(x') - J(x) \mid x, c] - (1 - \mathrm{e}^{-\delta}) \mathrm{E}[J(x') \mid x, c]$$

In continuous time,

$$0 = \max_c \ u(c) + \text{drift of J} - \delta J$$

Another Derivation

Let C control process, $X =$ state process, $u =$ utility function, $\delta =$ discount rate (r in capital investment problem), $\mathsf{E} =$ appropriate expectation. Problem is

$$\max_{C} \mathsf{E} \int_0^\infty \mathrm{e}^{-\delta t} u(C_t, X_t) \, \mathrm{d}t$$

where evolution of X depends on current X and C.

Let J denote value function. Suppose there is an optimal policy C^*, which induces the state process X^*, so

$$J(X_0) = \mathsf{E} \int_0^\infty \mathrm{e}^{-\delta t} u(C_t^*, X_t^*) \, \mathrm{d}t$$

Bellman's principle of optimality states that, for any s,

$$J(X_0) = \mathsf{E}\left[\int_0^s \mathrm{e}^{-\delta t} u(C_t^*, X_t^*)\,\mathrm{d}t + \max_C \; \mathsf{E}_s \int_s^\infty \mathrm{e}^{-\delta t} u(C_t, X_t)\,\mathrm{d}t\right]$$

In other words, if a plan is optimal, then if we re-optimize at some later date, we cannot improve upon the original plan.

So,

$$J(X_0) = \mathsf{E}\left[\int_0^s \mathrm{e}^{-\delta t} u(C_t^*, X_t^*)\,\mathrm{d}t + \mathrm{e}^{-\delta s} J(X_s)\right]$$

So, the expectation is constant with respect to s. It is not very much extra work to show that, in fact,

$$\int_0^s \mathrm{e}^{-\delta t} u(C_t^*, X_t^*)\,\mathrm{d}t + \mathrm{e}^{-\delta s} J(X_s)$$

is a martingale.

A martingale has no drift. Calculate the drift and set to zero:

$$e^{-\delta s}u(C_s^*, X_s^*) + e^{-\delta s}\text{drift of J} - \delta e^{-\delta s}J = 0$$

Multiply by $e^{\delta s}$ to cancel the $e^{-\delta s}$ factor:

$$u(C_s^*, X_s^*) + \text{drift of J} - \delta J = 0$$

This is almost the HJB equation. The actual HJB equation states that the maximum drift is zero.

The maximum drift is zero, because if you could get a positive drift, you would contradict the principle of optimality (re-optimizing would improve upon the original plan).