Day 24
Option Pricing

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Assumptions

single risky asset and single Brownian motion, constant risk-free rate

no dividends

geometric Browinan motion price process

$$\frac{dS}{S} = \mu\,dt + \sigma\,dB$$

for constants μ and σ

option that expires at some future date T

European = can only be exercised at T or American = can be exercised at any time prior to T

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$$\log S_T = \log S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma B_T$$

$$S_T = S_0 \mathrm{e}^{(\mu - \sigma^2/2)T + \sigma B_T}$$

SDF Process

$$\frac{\mathrm{d}M}{M} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B$$

$$\sigma\lambda = \mu - r \quad \Rightarrow \quad \lambda = \frac{\mu - r}{\sigma}$$

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$$M_T = \mathrm{e}^{-(r+\lambda^2/2)T - \lambda B_T}$$

European Call Option

Value of call at maturity is $\max(0, S_T - K) = (S_T - K)^+$

Let A denote the event $S_T > K$ and let 1_A denote its zero-one indicator.

The value of the call at maturity is

$$(S_T - K)1_A = S_T 1_A - K1_A$$

The value of the call at date 0 is

$$\mathrm{E}[M_T S_T 1_A] - K\mathrm{E}[M_T 1_A]$$

The event A is the event $\log S_T > \log K$, which is the event

$$\log S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma B_T > \log K$$

Let $\tilde{z}$ denote the standard normal $-B_T/\sqrt{T}$. The event A is the event

$$\log S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) T - \sigma\sqrt{T}\tilde{z} > \log K$$

This is equivalent to

$$\tilde{z} < \bar{z} \overset{\text{def}}{=} \frac{\log S_0 - \log K + (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}$$

Also,

$$S_T = S_0 \mathrm{e}^{(\mu-\sigma^2/2)T - \sigma\sqrt{T}\tilde{z}} \quad \text{and} \quad M_T = \mathrm{e}^{-(r+\lambda^2/2)T - \lambda\sqrt{T}\tilde{z}}$$

So, we're integrating exponentials of a standard normal over the region $(-\infty, \bar{z})$.

Black-Scholes Formula

Value of the call at date 0 is

$$S_0 \, \mathrm{N}(d_1) - \mathrm{e}^{-rT} K \, \mathrm{N}(d_2)$$

where N is the standard normal cdf and

$$d_1 = \frac{\log S_0 - \log K + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Risk-Neutral Probability

The asset price process is also

$$\frac{\mathrm{d}S}{S} = r\,\mathrm{d}t + \sigma\,\mathrm{d}B^*$$

where B^* is a Brownian motion under the risk-neutral probability.

The value of the call at date 0 is

$$\mathrm{e}^{-rT}\mathsf{E}^*[S_T 1_A] - \mathrm{e}^{-rT}K\mathsf{E}^*(1_A) = \mathrm{e}^{-rT}\mathsf{E}^*[S_T 1_A] - \mathrm{e}^{-rT}KQ(A)$$

Fundamental PDE

Let $f(t, S_t)$ denote the value at t of some future payoff (e.g., the value of $(S_T - K)1_A$).

It is convenient to use the risk-neutral probability. The RN expected rate of return is the risk-free rate:

$$\frac{\text{drift of } f}{f(t, S_t)} = r$$

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$$\begin{aligned} \mathrm{d}f &= f_t\,\mathrm{d}t + f_S\,\mathrm{d}S + \frac{1}{2} f_{SS} (\mathrm{d}S)^2 \\ &= f_t\,\mathrm{d}t + f_S (rS\,\mathrm{d}t + \sigma S\,\mathrm{d}B^*) + \frac{1}{2} f_{SS} (S^2 \sigma^2\,\mathrm{d}t) \end{aligned}$$

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So

$$\text{drift of } f = f_t + rSf_S + \frac{1}{2}\sigma^2 S^2 f_{SS}$$

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The Black-Scholes formula is

$$f(t, S_t) = S_t \, \mathrm{N}(d_1(t, S_t)) - \mathrm{e}^{-r(T-t)} K \, \mathrm{N}(d_2(t, S_t))$$

where

$$d_1(t, S_t) = \frac{\log S_t - \log K + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$
$$d_2(t, S_t) = d_1(t, S_t) - \sigma\sqrt{T - t}$$

We can differentiate and verify that the Black-Scholes formula satisfies the fundamental PDE. It also satisfies the boundary conditions $f(t, 0) = 0$, $f(t, \infty) = \infty$, and $\lim_{t\to T} f(t, S) = (S - K)^+$.

Americans and Perpetuals

For a finite-lived American option there is an optimal exercise boundary $s^*(t)$. Exercise at the first time that $S_t = s^*(t)$. Value satisfies the fundamental PDE in the "inaction region" and conditions at the boundary.

Have to find the boundary. Called a free boundary problem. Have to solve numerically.

For a time-stationary perpetual option, the value does not depend on t. Need to find a function $f(S_t)$. The fundamental PDE becomes an ODE. There is an analytic solution. Appplicable in many timing games – innovation races, etc. – and corporate investment problems — called real options.

Solution is derived from following. Pick a number s^* and let τ denote the hitting time of s^* (first time $S_t = s^*$). Value getting \$1 at τ. Value is $f(S_t) = \mathrm{E}^*[\mathrm{e}^{-r(\tau - t)} \mid S_t]$. Solve the ODE.