Day 19

Continuous-Time Securities Markets

Kerry Back
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Rice University
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Securities Market Model

Money market account has price R with $\mathrm{d}R/R = r\,\mathrm{d}t$. n locally risky assets with dividend-reinvested prices S_i. $\mu =$ vector of n stochastic processes μ_i. $\sigma = n \times k$ matrix of stochastic processes, $\Sigma = \sigma\sigma'$. $B =$ vector of k independent Brownian motions. $k \geq n$.

$$\mathrm{d}S/S \stackrel{\text{def}}{=} \begin{pmatrix} \mathrm{d}S_{1t}/S_{1t} \\ \vdots \\ \mathrm{d}S_{nt}/S_{nt} \end{pmatrix} = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t$$

This means that, for $i = 1, \ldots, n$,

$$\frac{\mathrm{d}S_{it}}{S_{it}} = \mu_{it}\,\mathrm{d}t + \sum_{j=1}^{k} \sigma_{ijt}\,\mathrm{d}B_{jt}$$

Intertemporal Budget Constraint

Let ϕ_i denote the amount of the consumption good invested in risky asset i. Let $W =$ wealth, $C =$ consumption, $Y =$ labor income. The intertemporal budget constraint is

$$\mathrm{d}W = (Y - C)\,\mathrm{d}t + \theta'\,\mathrm{d}S + (W - \theta' S)r\,\mathrm{d}t$$

where $\theta = (\theta_1, \ldots, \theta_n)'$ denotes share holdings. Setting $\phi_i = \theta_i S_i$, we obtain

$$\mathrm{d}W = (Y - C)\,\mathrm{d}t + \phi'\,(\mathrm{d}S/S) + (W - \phi'\iota)r\,\mathrm{d}t$$

Equivalently,

$$\mathrm{d}W = (Y - C)\,\mathrm{d}t + rW\,\mathrm{d}t + \phi'(\mathrm{d}S/S - r\iota)\,\mathrm{d}t$$

Equivalently,

$$\mathrm{d}W = (Y - C)\,\mathrm{d}t + rW\,\mathrm{d}t + \phi'(\mu - r\iota)\,\mathrm{d}t + \phi'\sigma\,\mathrm{d}B$$

Assuming $W > 0$, we can define $\pi = \phi / W$ and write the intertemporal budget constraint as

$$\mathrm{d}W = (Y - C)\,\mathrm{d}t + rW\,\mathrm{d}t + W\pi'(\mu - r\iota)\,\mathrm{d}t + W\pi'\sigma\,\mathrm{d}B$$

Equivalently,

$$\frac{\mathrm{d}W}{W} = \frac{Y - C}{W}\,\mathrm{d}t + r\,\mathrm{d}t + \pi'(\mu - r\iota)\,\mathrm{d}t + \pi'\sigma\,\mathrm{d}B$$

If $Y = C = 0$, the wealth process is said to be self financing.

Stochastic Discount Factor Processes

Define a stochastic process M to be an SDF process if

- $M_0 = 1$
- $M_t > 0$ for all t with probability 1
- MR is a local martingale, where R denotes the price of the money market account,
- MS_i is a local martingale, for $i = 1, \ldots, n$, where the S_i are the dividend-reinvested asset prices.

'Local martingale' means zero drift (no $\mathrm{d}t$ part). We can show that if M is an SDF process and W is a self-financing wealth process, then MW is a local martingale.

Dynamics of SDF Processes

We can show the following: A stochastic process $M > 0$ with $M_0 = 1$ is an SDF process if and only if

$$\frac{\mathrm{d}M}{M} = -r\,\mathrm{d}t - \lambda'\,\mathrm{d}B$$

for a stochastic process λ satisfying

$$\sigma\lambda = \mu - r\iota$$

Notice that

$$\begin{aligned}(\mathrm{d}S/S)\left(\frac{\mathrm{d}M}{M}\right) &= -(\sigma\,\mathrm{d}B)(\lambda'\,\mathrm{d}B) \\ &= -(\sigma\,\mathrm{d}B)(\mathrm{d}B)'\lambda = -\sigma\lambda\,\mathrm{d}t = -(\mu - r\iota)\,\mathrm{d}t\end{aligned}$$

So,

$$(\mu - r\iota)\,\mathrm{d}t = -(\mathrm{d}S/S)\left(\frac{\mathrm{d}M}{M}\right)$$

Factor Pricing and Prices of Risk

As in a single-period model, an SDF process is a pricing factor. We read the equation

$$(\mu - r\iota)\,\mathrm{d}t \;=\; -(\mathrm{d}S/S)\left(\frac{\mathrm{d}M}{M}\right)$$

as saying the risk premium of each asset is minus its covariance with the SDF process.

Notice that

$$-(\mathrm{d}S/S)\left(\frac{\mathrm{d}M}{M}\right) \;=\; -(\mathrm{d}S/S)(-\lambda'\,\mathrm{d}B) \;=\; \sum_{j=1}^{k}\lambda_j(\mathrm{d}S/S)(\mathrm{d}B_j)$$

We call λ the vector of 'prices of risk.' Each λ_j is the 'price' for the covariance with B_j. If there is only a single risky asset, then $\lambda = (\mu - r)/\sigma$, which is the Sharpe ratio of the risky asset.

Projections of SDF Processes

One solution λ of the equation $\sigma\lambda = \mu - r\iota$ is

$$\lambda_p \stackrel{\text{def}}{=} \sigma'(\sigma\sigma')^{-1}(\mu - r\iota) = \sigma'\Sigma^{-1}(\mu - r\iota)$$

For this solution,

$$\begin{aligned} \lambda_p' \, \mathrm{d}B &= (\mu - r\iota)'\Sigma^{-1}\sigma \, \mathrm{d}B \\ &= \pi'\sigma \, \mathrm{d}B \end{aligned}$$

for $\pi = \Sigma^{-1}(\mu - r\iota)$ (the log-optimal portfolio). Thus, it is spanned by the assets.

Every solution λ of the equation $\sigma\lambda = \mu - r\iota$ is of the form

$$\lambda = \lambda_p + \zeta$$

where ζ is orthogonal to the assets in the sense that $\sigma\zeta = 0$.

Valuation

If MW is a martingale, for $u > t$,

$$M_t W_t = \mathsf{E}_t[M_u W_u] \quad \Leftrightarrow \quad W_t = \mathsf{E}_t\left[\frac{M_u}{M_t} W_u\right]$$

Under another martingale assumption, if (C, π, W) satisfy the intertemporal budget constraint with $Y = 0$, then

$$W_t = \mathsf{E}_t\left[\int_t^u \frac{M_\tau}{M_t} C_\tau \,\mathrm{d}\tau + \frac{M_u}{M_t} W_u\right]$$

In particular, for an asset with price process P and dividend process D,

$$P_t = \mathsf{E}_t\left[\int_t^u \frac{M_\tau}{M_t} D_\tau \,\mathrm{d}\tau + \frac{M_u}{M_t} P_u\right]$$

Euler Equation

The Euler equation is a necessary and sufficient condition for optimality for the investor with time-additive utility

$$\mathsf{E}\int_0^\infty \mathrm{e}^{-\delta t} u(C_t)\,\mathrm{d}t$$

The Euler equation is that the MRS

$$\frac{\mathrm{e}^{-\delta t} u'(C_t)}{u'(C_0)}$$

is an SDF process.

CRRA Representative Investor

Applying the Euler equation for a CRRA representative investor, we have

$$M_t = \mathrm{e}^{-\delta t}\left(\frac{C_t}{C_0}\right)^{-\rho}$$

("Assuming there is no bubble in the price of the market portfolio") the price is

$$P_t = \mathsf{E}_t \int_t^\infty \mathrm{e}^{-\delta(\tau - t)}\left(\frac{C_\tau}{C_t}\right)^{-\rho} C_\tau \,\mathrm{d}\tau$$

So, the price-dividend ratio is

$$\frac{P_t}{C_t} = \int_t^\infty \mathrm{e}^{-\delta(\tau - t)}\mathsf{E}_t\left[\left(\frac{C_\tau}{C_t}\right)^{1-\rho}\right]\mathrm{d}\tau$$

IID Consumption Growth

Assume

$$\frac{\mathrm{d}C}{C} = \alpha\,\mathrm{d}t + \gamma'\,\mathrm{d}B$$

for constant α and γ (geometric Brownian motion). Then

$$\mathrm{d}\log C = \left(\alpha - \frac{1}{2}\gamma'\gamma\right)\mathrm{d}t + \gamma'\,\mathrm{d}B$$

This implies

$$C_\tau = C_t \mathrm{e}^{(\alpha-\gamma'\gamma/2)(\tau-t)+\gamma'(B_\tau-B_t)}$$

The exponent is normal with mean $(\tau - t)(\alpha - \gamma'\gamma/2)$ and variance $(\tau - t)\gamma'\gamma$. We can easily calculate

$$\mathrm{E}_t\left[\left(\frac{C_\tau}{C_t}\right)^{1-\rho}\right]$$

as $\mathrm{e}^{-\eta(\tau-t)}$ for a constant η and then, assuming $\eta > 0$, integrate to get

$$\frac{P_t}{C_t} = \frac{1}{\delta + \eta}$$

Risk-Neutral Probabilities in Continuous Time

Consider $T < \infty$. Let R denote the money market account price with $R_0 = 1$. Let M be an SDF process. Assume MR is a martingale so

$$\mathsf{E}[M_T R_T] = R_0 = 1$$

Define

$$\mathbb{Q}(A) = \mathsf{E}\left[M_T R_T 1_A\right]$$

for each event A that is distinguishable at date T, where $1_A = 1$ when the state of the world is in A and 0 otherwise.

It follows that $\mathbb{Q}$ is a probability (measure) and

$$\mathsf{E}^*[X_T] = \mathsf{E}\left[M_T R_T X_T\right]$$

for any random variable X_T depending on date–T information, where E^* denotes expectation with respect to $\mathbb{Q}$.

Risk-Neutral Valuation

Let W be such that MW is a martingale under the physical probability. Because we changed the probability using MR, a theorem in probability theory tells us that

$$\frac{MW}{MR}$$

is a $\mathbb{Q}$–martingale.

So, W/R is a $\mathbb{Q}$–martingale. Thus,

$$W_t = R_t \mathsf{E}_t^* \left[\frac{W_T}{R_T} \right] = \mathsf{E}_t^* \left[\exp\left(-\int_t^T r_u \, \mathrm{d}u \right) W_T \right].$$

In other words, asset values are expected discounted values, taking expectations with respect to the risk neutral probability and discounting at the instantaneous risk-free rate.

It follows that expected returns under the RNP equal the risk-free rate.

Girsanov's Theorem

Let M be an SDF process with

$$\frac{\mathrm{d}M}{M} = -r\,\mathrm{d}t - \lambda'\,\mathrm{d}B$$

Here, r and λ can be stochastic processes. Define the risk-neutral probability $\mathbb{Q}$ using the martingale MR.

The vector B is not a vector of Brownian motions under $\mathbb{Q}$

- Its drift is nonzero.
- But, we still have quadratic variation $(\mathrm{d}B)(\mathrm{d}B)' = I\,\mathrm{d}t$, so it is "close" to being a vector of Brownian motions.

Girsanov's theorem states that B^* defined by $B^*_0 = 0$ and

$$\mathrm{d}B^* = \mathrm{d}B + \lambda\,\mathrm{d}t$$

is a vector of independent Brownian motions under the risk-neutral probability $\mathbb{Q}$.

Asset Returns under a Risk-Neutral Probability

Recall that the vector of asset returns is

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$$

Define $\mathrm{d}B^* = \mathrm{d}B + \lambda\,\mathrm{d}t$. Substitute to obtain

$$\begin{aligned}\frac{\mathrm{d}S}{S} &= \mu\,\mathrm{d}t + \sigma\,(\mathrm{d}B^* - \lambda\,\mathrm{d}t) \\ &= (\mu - \sigma\lambda)\,\mathrm{d}t + \sigma\,\mathrm{d}B^* \\ &= r\iota\,\mathrm{d}t + \sigma\,\mathrm{d}B^*\end{aligned}$$