Day 11

Representative Investors

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Equilibrium with Date–0 Consumption

Assume there is no labor income. Investors $h = 1, \ldots, H$ have endowments of date–0 consumption $\bar{c}_{h0}$ and asset shares $\bar{\theta}_h$. Assets $i = 1, \ldots, n$ have payoffs $\tilde{x}_i$.

Take date–0 consumption to be the numeraire (price=1). An equilibrium is a price vector $p \in \mathbb{R}^n$ for assets, a date–0 consumption allocation $(c_{10}, \ldots, c_{H0})$ and asset allocations $(\theta_1, \ldots, \theta_H)$ such that

- date–0 consumption c_{h0} and portfolio θ_h are optimal for investor h, for all h
- the date–0 consumption market: $\sum_h c_{h0} = \sum_h \bar{c}_{h0}$
- the asset markets clear: $\sum_h \theta_h = \sum_h \bar{\theta}_h$

Representative Investor

There is a representative investor if each asset price vector p that is part of a securities market equilibrium is also part of a securities market equilibrium in the economy in which there is only the representative investor, and the representative investor's endowments are $\bar{c}_0 := \sum_h \bar{c}_{h0}$ and $\bar{\theta} := \sum_h \bar{\theta}_{h0}$.

By the FOC in the representative investor economy, the representative investor's MRS is an SDF.

Plan for Today

Assume there is a representative investor with CRRA utility. Derive

- formula for market return
- formula for risk-free rate
- formula for log equity premium (assuming also lognormal consumption growth)
- variation of the CAPM

Then discuss:

- There is a representative investor if the first welfare theorem holds (complete markets or LRT utility with same cautiousness parameter)
- With LRT utility for all investors and same cautiousness parameter, the representative investor has the same utility function.

Representative Investor with CRRA Utility

Assume there is a representative investor with utility function

$$(c_0, c_1) \mapsto u(c_0) + \delta u(c_1)$$

where

$$u(c) = \frac{1}{1-\rho} c^{1-\rho}$$

Let c_0 denote aggregate consumption at date 0, and let $\tilde{c}_1$ denote aggregate consumption at date 1.

Then

$$\delta \left(\frac{\tilde{c}_1}{c_0} \right)^{-\rho}$$

is an SDF.

Market Return

Assume $\tilde{c}_1$ is spanned by the assets. Its cost is

$$\mathsf{E}[\tilde{m}\tilde{c}_1] = \mathsf{E}\left[\frac{\delta\tilde{c}_1^{-\rho}}{c_0^{-\rho}}\tilde{c}_1\right] = c_0\mathsf{E}\left[\frac{\delta\tilde{c}_1^{-\rho}}{c_0^{-\rho}}\cdot\frac{\tilde{c}_1}{c_0}\right] = \delta c_0\mathsf{E}\left[\left(\frac{\tilde{c}_1}{c_0}\right)^{1-\rho}\right]$$

The market return is

$$\widetilde{R}_m := \frac{\tilde{c}_1}{\mathsf{E}[\tilde{m}\tilde{c}_1]} = \frac{1}{\delta\mathsf{E}\left[\left(\frac{\tilde{c}_1}{c_0}\right)^{1-\rho}\right]}\cdot\frac{\tilde{c}_1}{c_0} := \frac{1}{\nu_1}\cdot\frac{\tilde{c}_1}{c_0}$$

Risk-Free Return

The risk-free return is

$$R_f = \frac{1}{\mathsf{E}[\tilde{m}]} = \frac{1}{\delta \mathsf{E}[(\tilde{c}_1/c_0)^{-\rho}]} := \frac{1}{\nu_0}$$

Log Equity Premium

$$\frac{\widetilde{R}_m}{R_f} = \frac{\nu_0}{\nu_1} \cdot \frac{\tilde{c}_1}{c_0}$$

So,

$$\frac{\mathsf{E}[\widetilde{R}_m]}{R_f} = \frac{\nu_0 \mathsf{E}[\tilde{c}_1/c_0]}{\nu_1} = \frac{\mathsf{E}[(\tilde{c}_1/c_0)^{-\rho}]\mathsf{E}[\tilde{c}_1/c_0]}{\mathsf{E}[(\tilde{c}_1/c_0)^{1-\rho}]} = \frac{\mathsf{E}[\tilde{c}_1]\mathsf{E}[\tilde{c}_1^{-\rho}]}{\mathsf{E}[\tilde{c}_1^{1-\rho}]}$$

Lognormal Consumption

Assume $\log \tilde{c}_1 - \log c_0 = \mu + \sigma\tilde{\varepsilon}$ for constants μ and σ and a standard normal $\tilde{\varepsilon}$.

$$\tilde{c}_1 = c_0 \mathrm{e}^{\mu+\sigma\tilde{\varepsilon}} \Rightarrow \mathsf{E}[\tilde{c}_1] = c_0 \mathrm{e}^{\mu+\sigma^2/2}$$

$$\tilde{c}_1^{-\rho} = c_0^{-\rho} \mathrm{e}^{-\rho\mu-\rho\sigma\tilde{\varepsilon}} \Rightarrow \mathsf{E}[\tilde{c}_1^{-\rho}] = c_0^{-\rho} \mathrm{e}^{-\rho\mu+\rho^2\sigma^2/2}$$

$$\tilde{c}_1^{1-\rho} = c_0^{1-\rho} \mathrm{e}^{(1-\rho)\mu+(1-\rho)\sigma\tilde{\varepsilon}} \Rightarrow \mathsf{E}[\tilde{c}_1^{1-\rho}] = c_0^{1-\rho} \mathrm{e}^{(1-\rho)\mu+(1-\rho)^2\sigma^2/2}$$

This implies

$$\frac{\mathsf{E}[\widetilde{R}_m]}{R_f} = \mathrm{e}^{\rho\sigma^2}$$

So,

$$\log \mathsf{E}[\widetilde{R}_m] - \log R_f = \rho\sigma^2$$

Equity Premium and Risk-Free Rate Puzzles

- To match this model to the historical equity premium, a risk aversion around 50 is required. Much too high.
- Using $\rho = 10$ and $\delta = 0.99$, the model implies a high risk-free rate (12.7%) and low equity premium ($\mathsf{E}[\widetilde{R}_m] - R_f = 1.4\%$).
- The historical (U.S.) numbers are around 1% for the real risk-free rate and 6% for the equity premium.

SDF and Market Return

The market return is

$$\tilde{R}_m = \frac{1}{\nu_1} \cdot \frac{\tilde{c}_1}{c_0}$$

and the SDF is

$$\tilde{m} = \delta \left(\frac{\tilde{c}_1}{c_0} \right)^{-\rho}$$

so the SDF is

$$\tilde{m} = \delta \nu^{-\rho} \tilde{R}_m^{-\rho}$$

CAPM Alternative

Risk premia of all assets are

$$\mathsf{E}[\widetilde{R}] - R_f = -R_f \,\mathrm{cov}(\widetilde{R}, \tilde{m}) = -\delta\nu^{-\rho} R_f \,\mathrm{cov}(\widetilde{R}, \widetilde{R}_m^{-\rho})$$

This implies

$$\mathsf{E}[\widetilde{R}] - R_f = \lambda \frac{\mathrm{cov}(\widetilde{R}, \widetilde{R}_m^{-\rho})}{\mathrm{var}(\widetilde{R}_m^{-\rho})}$$

for a λ that is the same for all assets. So, risk premia depend on betas with respect to $\widetilde{R}_m^{-\rho}$.

Social Planner's Problem

For each value w of market wealth, the social planner solves

$$\max \quad \sum_{h=1}^{H} \lambda_h u_h(w_h) \quad \text{subject to} \quad \sum_{h=1}^{H} w_h = w$$

Let $U(w)$ denote the maximum value. This is the social planner's utility function.

Let η denote the Lagrange multiplier (which depends on market wealth w). Then, for all h,

$$\lambda_h u_h'(w_h) = \eta$$

Also, the social planner's marginal utility (the marginal value of market wealth) is equal to η. So, for all h, we have the envelope result:

$$U'(w) = \lambda_h u_h'(w_h)$$

Hence, the social planner's marginal utility is proportional to an SDF.

Social Planner's Problem with Date–0 Consumption

Suppose investor h has utility $u_h(c_{h0}) + \delta_h u_h(c_{h1})$. The social planner's problem is now separable in dates and in states. Given aggregate date–0 consumption c_{m0} and aggregate date–1 consumption c_{m1}, the social planner solves

$$U_0(c_{m0}) := \max \quad \sum_{h=1}^{H} \lambda_h u_h(c_{h0}) \quad \text{subject to} \quad \sum_{h=1}^{H} c_{h0} = c_{m0}$$

and

$$U_1(c_{m1}) := \max \quad \sum_{h=1}^{H} \lambda_h \delta_h u_h(c_{h1}) \quad \text{subject to} \quad \sum_{h=1}^{H} c_{h1} = c_{m1}$$

The envelope theorem tells us that, for all h,

$$U_0'(c_{m0}) = \lambda_h u_h'(c_{h0}) \quad \text{and} \quad U_1'(c_{m1}) = \lambda_h \delta_h u_h'(c_{h1})$$

So,

$$\frac{U_1'(c_{m1})}{U_0'(c_{m0})} = \frac{\delta_h u_h'(c_{h1})}{u_h'(c_{h0})} = \text{SDF}$$

Common Discount Factors

If all investors have the same discount factor δ, then we can pull δ outside the sum in the definition of U_1 and see that, as functions, $U_1 = \delta U_0$.

Writing $U = U_0$, an SDF is

$$\frac{\delta U'(\tilde{c}_{m1})}{U'(c_{m0})}$$

Linear Risk Tolerance

Suppose all investors have linear risk tolerance $\tau_h(c) = A_h + Bc$ with same cautiousness parameter $B \geq 0$. Then, the social planner's utility functions U_0 and U_1 have linear risk tolerance with the same cautiousness parameter.

Example: all investors have CRRA utility with risk aversion ρ and the same discount factor δ. Then, an SDF is

$$\frac{\delta U'(\tilde{c}_{m1})}{U'(c_{m0})}$$

where

$$U(c) = \frac{1}{1-\rho} c^{1-\rho}$$

So, the SDF is

$$\delta \left(\frac{\tilde{c}_{m1}}{c_{m0}} \right)^{-\rho}$$

Proof of LRT Social Planner in CARA Case

We solved the social planner's problem in the last class and found

$$w_h = \frac{\tau_h}{\tau} w - \frac{\tau_h}{\tau} \sum_{\ell=1}^{H} \tau_\ell \log(\lambda_\ell \alpha_\ell) + \tau_h \log(\lambda_h \alpha_h)$$

which we wrote as $w_h = a_h + b_h w$ with $b_h = \tau_h / \tau$ So,

$$U(w) = -\sum_{h=1}^{H} \lambda_h \mathrm{e}^{-\alpha_h (a_h + b_h w)} = -\sum_{h=1}^{H} \lambda_h \mathrm{e}^{-\alpha_h a_h} \mathrm{e}^{-\alpha_h b_h w}$$

Moreover,

$$\alpha_h b_h w = \frac{\alpha_h \tau_h w}{\tau} = \frac{w}{\tau} = \alpha w$$

So

$$U(w) = -\mathrm{e}^{-\alpha w} \sum_{h=1}^{H} \lambda_h \mathrm{e}^{-\alpha_h a_h}$$

which is a monotone affine transform of CARA utility.