Day 6

Mean Variance Analysis with a Risk-Free Asset

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Mean-Variance Frontier with a Risk-Free Asset

Now, we add a risk-free asset. We continue to let $\pi \in \mathbb{R}^n$ denote the portfolio of risky assets. We no longer require $\iota'\pi = 1$. The weight on the risk-free asset is $1 - \iota'\pi$. This can be negative (borrowing).

A portfolio's expected return is

$$(1 - \iota'\pi)R_f + \mu'\pi \;=\; R_f + (\mu - R_f\iota)'\pi$$

A frontier portfolio solves the following for some μ_{targ}:

$$\min \quad \frac{1}{2}\pi'\Sigma\pi \quad \text{subject to} \quad R_f + (\mu - R_f\iota)'\pi = \mu_{\text{targ}}$$

FOC is

$$\Sigma\pi - \delta(\mu - R_f\iota) = 0 \quad \Leftrightarrow \quad \pi = \delta\Sigma^{-1}(\mu - R_f\iota)$$

So, all frontier portfolios are scalar multiples of the vector $\Sigma^{-1}(\mu - R_f\iota)$.

In other words, the frontier portfolios form a line through the origin and the vector $\Sigma^{-1}(\mu - R_f\iota)$.

Tangency Portfolio

We can probably divide the vector $\Sigma^{-1}(\mu - R_f\iota)$ by the sum of its elements to form a portfolio of purely risky assets (satisfying $\iota'\pi = 1$).

We can do that as long as the sum is nonzero. That is, we need

$$\iota'\Sigma^{-1}(\mu - R_f\iota) \neq 0$$

This expression is $B - R_f C$. It is nonzero if and only if $B/C \neq R_f$. The term B/C is the expected return of the GMV portfolio:

$$\mu'\pi_{\text{gmv}} = \mu'\left(\frac{1}{\iota'\Sigma^{-1}\iota}\Sigma^{-1}\iota\right) = \frac{1}{\iota'\Sigma^{-1}\iota}\mu'\Sigma^{-1}\iota = \frac{B}{C}$$

So, when the expected return of the GMV portfolio is different from R_f, we can define

$$\pi_{\text{tang}} = \frac{1}{\iota'\Sigma^{-1}(\mu - R_f\iota)}\Sigma^{-1}(\mu - R_f\iota)$$

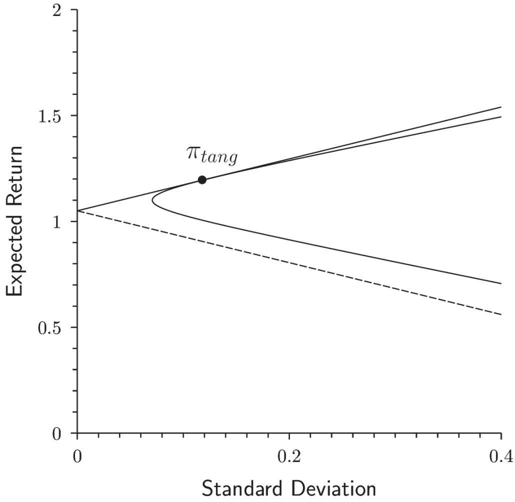
We call this the tangency portfolio because it is on two frontiers: the frontier including the risk-free asset and the frontier of only risky assets.

How do we know it is on the frontier of only risky assets?

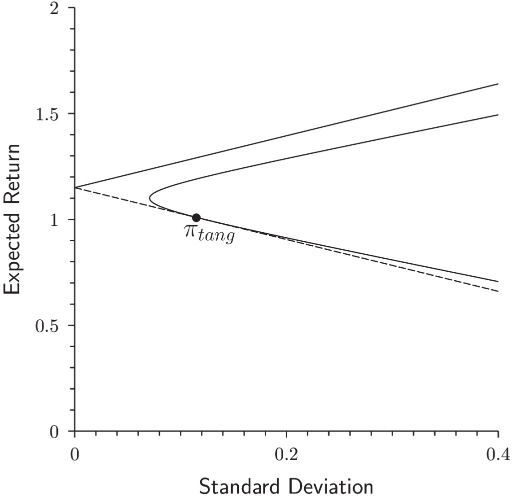
1. It is a portfolio constructed from the two vectors $\Sigma^{-1}\mu$ and $\Sigma^{-1}\iota$.
2. Anything that solves a less constrained optimization problem (not requiring $\iota'\pi = 1$) and satisfies the constraints of a more constrained problem (satisfies $\iota'\pi = 1$ anyway) must solve the more constrained problem too.

Thus, the two frontiers (in std dev/mean space) must be tangent at this point.

Mean-Variance Frontier: $B/C > R_f$



Mean-Variance Frontier: $B/C < R_f$



What if $B/C = R_f$?

If $B/C = R_f$, all frontier portfolios have 100% invested in the risk-free asset and then go long and short in proportion to the positive and negative elements of $\Sigma^{-1}(\mu - R_f\iota)$.

The cone and hyperbola never touch.

Two Fund Spanning with a Risk-Free Asset

All frontier portfolios lie on the line through the origin and the vector $\Sigma^{-1}(\mu - R_f\iota)$ in $\mathbb{R}^n$.

Any vector on the line is a portfolio, because we are not requiring $\iota'\pi = 1$.

The origin represents 100% in the risk-free asset.

Any two portfolios on the line span the frontier in the sense that any frontier portfolio is a combination λ and $(1 - \lambda$ of the portfolios.

Maximum Sharpe Ratio

What is the risk premium of the portfolio $\Sigma^{-1}(\mu - R_f\iota)$?

What is the variance of the return of the portfolio $\Sigma^{-1}(\mu - R_f\iota)$?

What is its Sharpe ratio (risk premium divided by standard deviation)?

Return Proportional to $\tilde{m}_p$

Consider the SDF in the span of the assets, denoted by $\tilde{m}_p$.

This is the payoff of some portfolio (that's what it means to be in the span of the assets).

If we divide by its cost, the re-scaled payoff will have a cost of 1 and hence be a return. Call this return $\tilde{R}_p$. From Exercise 3.3, if there is a risk-free asset,

$$\tilde{m}_p = \frac{1}{R_f} + \left(\iota - \frac{1}{R_f}\right)\Sigma^{-1}(\tilde{\mathbf{R}} - \mu)\,.$$

This implies that $\widetilde{R}_p$ is an inefficient frontier portfolio.

HJ Bound with a Risk-Free Asset

For any SDF $\tilde{m}$ and any return $\widetilde{R}$,

$$\frac{\text{stdev}(\tilde{m})}{\mathsf{E}[\tilde{m}]} \geq \frac{\text{stdev}(\tilde{m}_p)}{\mathsf{E}[\tilde{m}_p]} \geq \frac{|\mathsf{E}[\tilde{R}] - R_f|}{\text{stdev}(\tilde{R})}$$

The right-hand side is the absolute value of the Sharpe ratio of $\widetilde{R}$.

The second inequality is an equality for the maximum Sharpe ratio. In fact,

$$\frac{\text{stdev}(\tilde{m}_p)}{\mathsf{E}[\tilde{m}_p]} = \frac{|\mathsf{E}[\tilde{R}_p] - R_f|}{\text{stdev}(\tilde{R}_p)}$$

The FOC with a Risk-Free Asset

The FOC for the frontier with a risk-free asset is

$$\Sigma\pi = \delta(\mu - R_f\iota)$$

We solved it for π:

$$\pi = \delta\Sigma^{-1}(\mu - R_f\iota)$$

We can instead solve it for the vector of risk premia:

$$\mu - R_f\iota = \frac{1}{\delta}\Sigma\pi$$

What is $\Sigma\pi$?