Day 14

Rational Expectations Equilibria

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Informational Efficiency

An important issue, dating back to Hayek, is the extent to which market prices reflect and convey the information of market participants.

In finance, following Fama, prices are said to be

- weak-form efficient if prices contain all of the information that is in past prices,
- semi-strong-form efficient if prices contain all public information, and
- strong-form efficient if prices contain all public and private information

We will look at price formation under asymmetric information to study this. Two types of models: competitive (price-taking) and strategic (game-theoretic).

CARA-Normal

Suppose there is a risk-free asset and a single risky asset. Suppose the aggregate supply of the risky asset is 1 share.

Suppose the end-of-period value $\tilde{x}$ of the risky asset is normally distributed with mean μ and variance σ^2.

Suppose there are H investors and each investor h has CARA utility with absolute risk aversion α_h and risk tolerance $\tau_h = 1/\alpha_h$.

Review of Equilibrium

Let p denote the price of the risky asset, and set $\widetilde{R} = \tilde{x}/p$.

The demand for the risky asset by investor h is (from Chapter 2)

$$\phi_h = \frac{\mathsf{E}[\widetilde{R}] - R_f}{\alpha_h \operatorname{var}(\widetilde{R})}$$

The mean of $\widetilde{R}$ is μ/p, and the variance of $\widetilde{R}$ is σ^2/p^2, and $\phi = \theta p$ (dollars = shares times price), so

$$\theta_h p = \frac{\mu/p - R_f}{\alpha_h \sigma^2/p^2} \quad \Rightarrow \quad \theta_h = \tau_h \frac{\mu - R_f p}{\sigma^2}$$

Equating aggregate demand to supply yields

$$\tau \frac{\mu - R_f p}{\sigma^2} = 1 \quad \Rightarrow \quad p = \frac{\mu - \alpha\sigma^2}{R_f}$$

where $\tau = \sum_h \tau_h$ and $\alpha = 1/\tau$.

Asymmetric Information

Assume some investors observe a signal $\tilde{s} = \tilde{x} + \tilde{\varepsilon}$ before trade. Let $I \subset \{1, \ldots, H\}$ denote the informed investors. Let U denote the other investors, who do not observe the signal.

Let μ_I denote the expectation of $\tilde{x}$ for the informed traders (which depends on $\tilde{s}$) and let μ_U denote the expectation for the uninformed traders (which maybe depends on $\tilde{s}$).

Write $\nu = 1/\sigma^2$ (called a precision). Let ν_I denote the precision (reciprocal of variance) for the informed traders and let ν_U denote the precision for the uninformed traders.

Assume both types of investors still regard $\tilde{x}$ as normally distributed. We can calculate μ_I and ν_I using standard statistical arguments. But μ_U and ν_U depend on how much uninformed investors learn from prices.

Assume R_f is given exogenously, so information comes only from p.

Equilibrium

Informed investors' demands are $\theta_h = \tau_h \nu_I(\mu_I - R_f p)$.

Uninformed investors' demands are $\theta_h = \tau_h \nu_U(\mu_U - R_f p)$.

Market clearing is

$$\sum_{h \in I} \tau_h \nu_I(\mu_I - R_f p) + \sum_{h \in U} \tau_h \nu_U(\mu_U - R_f p) = 1$$

Set $\tau_I = \sum_{h \in I} \tau_h$ and $\tau_U = \sum_{h \in U} \tau_h$. Market clearing is

$$\tau_I \nu_I(\mu_I - R_f p) + \tau_U \nu_U(\mu_U - R_f p) = 1$$

Equilibrium is

$$p = \frac{\tau_I \nu_I \mu_I + \tau_U \nu_U \mu_U - 1}{(\tau_I \nu_I + \tau_U \nu_U) R_f} = \frac{1}{R_f} \left(\frac{\tau_I \nu_I \mu_I + \tau_U \nu_U \mu_U}{\tau_I \nu_I + \tau_U \nu_U} - \frac{1}{\tau_I \nu_I + \tau_U \nu_U} \right)$$

Posterior Mean

Suppose the signal is truth plus noise: $\tilde{s} = \tilde{x} + \tilde{e}$, where $\tilde{e}$ is normally distributed, zero mean, and uncorrelated with $\tilde{x}$.

To calculate μ_I and ν_I, start with a projection:

$$\tilde{x} - \mu \; = \beta(\tilde{s} - \mu_s) + \tilde{\varepsilon}$$

where $\tilde{\varepsilon}$ is uncorrelated with $\tilde{s}$ and mean zero.

We have $\mu_s = \mu$. This implies

$$\mu_I := \mathsf{E}[\tilde{x} \mid \tilde{s}] \; = \; (1 - \beta)\mu + \beta\tilde{s}\,.$$

In words: the posterior mean is a weighted average of the prior mean and the signal.

Also, $\beta = \mathrm{cov}(\tilde{x}, \tilde{s})/\sigma_s^2 = \sigma^2/\sigma_s^2 = \sigma^2/(\sigma^2 + \sigma_e^2)$.

Posterior Variance

From the projection equation,

$$\sigma^2 = \beta^2 \sigma_s^2 + \sigma_\varepsilon^2$$

The conditional variance of $\tilde{x}$ for the informed investors is the variance of $\tilde{\varepsilon}$, which is

$$\sigma_\varepsilon^2 = \sigma^2 - \beta^2 \sigma_s^2 = \sigma^2(1 - \beta)$$

Notice that the conditional mean depends on $\tilde{s}$, but the conditional variance does not.

Posterior Precision

The variance of $\tilde{\varepsilon}$ is

$$\sigma^2(1-\beta) = \sigma^2\left(1 - \frac{\sigma^2}{\sigma^2+\sigma_e^2}\right)$$
$$= \frac{\sigma^2\sigma_e^2}{\sigma^2+\sigma_e^2}$$

So, the informed precision is

$$\nu_I = \frac{1}{\sigma^2} + \frac{1}{\sigma_e^2}$$

In words: the posterior precision is the sum of the prior precision and the precision of the signal noise term.

Model I: Naive Uninformed Investors

Suppose uninformed investors learn nothing from the price, so $\mu_U = \mu$ and $\nu_U = \nu$ (prior mean and precision). Then, the equilibrium is

$$p = \frac{1}{R_f}\left(\frac{\tau_I \nu_I[(1-\beta)\mu + \beta\tilde{s}] + \tau_U \nu \mu}{\tau_I \nu_I + \tau_U \nu} - \frac{1}{\tau_I \nu_I + \tau_U \nu}\right)$$

The naive investors are irrational, because they should realize that a higher price means a higher $\tilde{s}$. They should extract information from the price.

Model II: Fully Revealing Rational Expectations Equilibrium

Suppose uninformed investors are fully rational. Conjecture that they learn $\tilde{s}$ fully from the price, so they have the same posterior mean and precision as do the informed investors.

Substituting a common mean and precision into the demand functions and solving for equilibrium, we obtain

$$p = \frac{1}{R_f}\left((1-\beta)\mu + \beta\tilde{s} - \frac{1}{\tau\nu_I}\right)$$

The uninformed can indeed learn $\tilde{s}$ from this price, so our conjecture is established.

Grossman-Stiglitz paradox: who would pay to collect information if they can get it free from the price?

Model III: Grossman-Stiglitz

Suppose the supply of the asset is a random variable $\tilde{z}$. Randomness could be due to random demand from other traders, sometimes called liquidity traders, because they may be trading due to cash needs or cash surpluses (liquidity shocks).

Suppose $\tilde{z}$ is normally distributed and independent of all other random variables in the model.

We solve by 'guess and verify.' We conjecture that the equilibrium price is $p = a_0 + a_1\tilde{s} + a_2\tilde{z}$ with $a_1 \neq 0$. We need to compute equilibrium coefficients a_0, a_1, and a_2.

Uninformed Investors

From the price, uninformed investors can calculate

$$\begin{aligned}\frac{p - a_0 - a_2\mu_z}{a_1} &= \tilde{s} + \frac{a_2}{a_1}(\tilde{z} - \mu_z) \\ &= \tilde{s} + b(\tilde{z} - \mu_z) \\ &= \tilde{x} + \tilde{e} + b(\tilde{z} - \mu_z)\,.\end{aligned}$$

where we define $b = a_2/a_1$.

So, the price provides a truth-plus-noise signal for uninformed investors.

The noise for uninformed investors is $\tilde{e} + b(\tilde{z} - \mu_z)$, which is ‘noisier’ than the informed investors’ signal.

Uninformed Demands

For uninformed investors, the conditional (posterior) mean of $\tilde{x}$ is

$$(1-\beta_U)\mu + \beta_U[\tilde{s} + b(\tilde{z}-\mu_z)]$$

where

$$\beta_U = \frac{\sigma^2}{\sigma^2 + \sigma_e^2 + b^2\sigma_z^2}\,.$$

The posterior precision is

$$\nu_U \stackrel{\text{def}}{=} \nu + \frac{1}{\sigma_e^2 + b^2\sigma_z^2}$$

Aggregate uninformed demands are

$$\tau_U\nu_U\Big[(1-\beta_U)\mu + \beta_U[\tilde{s} + b(\tilde{z}-\mu_z)] - R_f p\Big]\,.$$

Market Clearing

Aggregate demand is

$$\tau_I \nu_I \left[(1-\beta)\mu + \beta \tilde{s} - R_f p \right] + \tau_U \nu_U \left[(1-\beta_U)\mu + \beta_U [\tilde{s} + b(\tilde{z} - \mu_z)] - R_f p \right]$$

Substituting $p = a_0 + a_1 \tilde{s} + a_2 \tilde{z}$, we see that aggregate demand is of the form $A + B\tilde{s} + C\tilde{z}$, so market-clearing is

$$A + B\tilde{s} + C\tilde{z} = \tilde{z}$$

where A, B, and C depend on the unknown coefficients a_0, a_1, and a_2 and on R_f. For this to hold in all states of the world, it must be that $A = 0$, $B = 0$, and $C = 1$.

Solution

Solve $A = 0$, $B = 0$ and $C = 1$ for a_0, a_1, and a_2.

In equilibrium,

$$b = -\frac{1}{\beta \tau_I \nu_I} = -\frac{(1-\beta)\sigma^2}{\beta \tau_I}$$

Recall that uninformed investors can calculate

$$\tilde{s} + b(\tilde{z} - \mu_z)$$

Their information is better (i.e., the variance of $b\tilde{z}$ is smaller) when

- the variance of $\tilde{z}$ is smaller
- informed investors are more risk tolerant
- β is larger (the informed investors have a better signal, so they face less residual risk), or
- σ^2 is smaller (again, the informed investors face less residual risk).