Arbitrage Pricing

Foundation of Derivatives Pricing Theory

Finance 987

Section 1

Introduction

Learning Objectives

By the end of this lecture, you will be able to:

- Explain the concept of linear pricing and its implications
- Calculate state prices and risk-neutral probabilities
- Apply the fundamental pricing formula with different numeraires
- Understand how probability measures change under different numeraires
- Value derivatives using arbitrage-free pricing methods

Why Study Arbitrage Pricing?

Key Insight: Most of modern finance rests on one simple principle:
There are no free lunches in well-functioning markets

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This principle leads to:

- Black-Scholes formula
- Risk-neutral valuation
- Change of numeraire techniques
- Martingale pricing methods

Section 2

Linear Pricing

The Fundamental Concept

Linear Pricing is the cornerstone of asset pricing theory

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In finance:

- Price of portfolio = sum of (quantity × price) for each asset
- This applies to **contingent claims** (payoffs in different states)

No Arbitrage Implies Linear Pricing

Definition: An arbitrage opportunity is a trading strategy with:

- Non-negative cash flows in all states
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Key Result: If there are no arbitrage opportunities, then the pricing operator is linear:

$$P(aX + bY) = aP(X) + bP(Y)$$

State Prices

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$$P(X) = \sum_{j=1}^{J} \pi_j X_j$$

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where:

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- X_j = payoff in state j
- State prices are **strictly positive** (no arbitrage)

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Time Value of Money:

$$1 = \sum_{j=1}^{J} \pi_j e^{rt}$$

Section 3

Binomial Model

State Prices in the Binomial Model

Two states (up and down), two equations:

$$1 = \pi_u e^{rt} + \pi_d e^{rt}$$
$$S = \pi_u S_u + \pi_d S_d$$

State Prices in the Binomial Model

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$$
\begin{aligned}
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S &= \pi_u S_u + \pi_d S_d
\end{aligned}
$$

Solution:

$$\pi_u = \frac{S - e^{-rt} S_d}{S_u - S_d}, \quad \pi_d = \frac{e^{-rt} S_u - S}{S_u - S_d}$$

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No-arbitrage condition: $\pi_u, \pi_d > 0$ requires:

$$\frac{S_u}{S} > e^{rt} > \frac{S_d}{S}$$

Arrow Securities

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Any derivative can be decomposed as a portfolio of Arrow securities:

$$C = \pi_u C_u + \pi_d C_d$$

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Define: $p_u = \pi_u e^{rt}$ and $p_d = \pi_d e^{rt}$

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Derivative pricing formula:

$$C = e^{-rt}[p_u C_u + p_d C_d] = e^{-rt}\mathbb{E}^R[C]$$

where $\mathbb{E}^R$ is the expectation under risk-neutral probabilities

Risk-Neutral Probability Formula

$$p_u = \frac{e^{rt} - S_d/S}{S_u/S - S_d/S}$$

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- The probability that makes the expected return equal to r
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Verification:

$$\mathbb{E}^R[S_T] = p_u S_u + p_d S_d = S e^{rt}$$

Section 4

Multiple States

Generalizing to J States

With J possible states at time t, we need J equations:

$$\begin{bmatrix} e^{rt} & e^{rt} & \cdots & e^{rt} \\ S_1^{(1)} & S_2^{(1)} & \cdots & S_J^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ S_1^{(J-1)} & S_2^{(J-1)} & \cdots & S_J^{(J-1)} \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_J \end{bmatrix} = \begin{bmatrix} 1 \\ S^{(1)} \\ \vdots \\ S^{(J-1)} \end{bmatrix}$$

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Need J traded assets with linearly independent payoffs

Valuation with State Prices

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Or equivalently with risk-neutral probabilities ($p_j = \pi_j e^{rt}$):

$$V = e^{-rt} \sum_{j=1}^{J} p_j V_j = e^{-rt} \mathbb{E}^p[V_t]$$

Section 5

Continuous Time

Stochastic Discount Factor

In continuous time with infinitely many states:

Fundamental Principle

If there are no arbitrage opportunities, there exists a strictly positive random variable m_t (stochastic discount factor) such that:

$$P_0 = \mathbb{E}[m_t P_t]$$

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The stochastic discount factor generalizes state prices:

- For finite states: $m_t = \pi_j/\text{prob}_j$ in state j
- Prices all assets consistently

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Result: Under the risk-neutral measure $\mathbb{P}^R$:

$$P_0 = e^{-rt}\mathbb{E}^R[P_t]$$

Price $=$ discounted expected payoff under risk-neutral measure

Martingale Property

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Key Result: Discounted asset prices are martingales under the risk-neutral measure:

$$\frac{X(0)}{R(0)} = \mathbb{E}^R\left[\frac{X_t}{R_t}\right]$$

where $R_t = e^{rt}$

Section 6

Change of Numeraire

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Benefits of changing numeraire:

- Simplifies certain calculations
- Provides different perspectives on the same problem
- Essential for pricing certain exotic options

Defining New Probability Measures

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Then any asset price satisfies:

$$S(0) = Y(0)\mathbb{E}^Y\left[\frac{S_t}{Y_t}\right]$$

Fundamental Pricing Formula

The Fundamental Formula

For any two dividend-reinvested assets P and Y:

$$Y_s = P_s \mathbb{E}_s^P \left[\frac{Y_t}{P_t} \right]$$

for all times $s < t$

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Interpretation:

- The ratio Y_t / P_t is a martingale under the P-numeraire measure
- Price of Y at time $s =$ price of P times expected ratio

Special Cases

When P is the risk-free asset:

$$Y_s = e^{-r(t-s)}\mathbb{E}_s^R[Y_t]$$

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When P is the stock S:

$$V_0 = S_0 \mathbb{E}^S\left[\frac{V_T}{S_T}\right]$$

Useful for pricing options with payoffs relative to stock price

Section 7

Stock as Numeraire

Stock Numeraire in Binomial Model

Define probabilities using stock as numeraire:

$$\text{prob}_u^S = \frac{\pi_u S_u}{S}, \quad \text{prob}_d^S = \frac{\pi_d S_d}{S}$$

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For a call option:

$$\frac{C}{S} = \text{prob}_u^S \frac{C_u}{S_u} + \text{prob}_d^S \frac{C_d}{S_d}$$

The ratio C/S is a martingale under stock numeraire

Stock Numeraire in Continuous Time

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Stock numeraire measure ($\mathbb{P}^S$):

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Key difference: Drift increases by σ^2 when using stock as numeraire

Section 8

Girsanov's Theorem

Changing Probability Measures

When we change probability measures:

- A process that was a Brownian motion **may not remain** a Brownian motion
- It becomes an Ito process with **different drift**
- The **diffusion coefficient** (volatility) **remains unchanged**

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Intuition:

- Changing probabilities changes expected movements (drift)
- But doesn't change how much paths "wiggle" (volatility)

Example: From Actual to Risk-Neutral

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where B is a Brownian motion

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Under the risk-neutral measure $\mathbb{P}^R$:

$$\frac{\mathrm{d}S}{S} = r \mathrm{d}t + \sigma \mathrm{d}B^*$$

where B^* is a Brownian motion under $\mathbb{P}^R$

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Under the risk-neutral measure $\mathbb{P}^R$:

$$\frac{\mathrm{d}S}{S} = r\mathrm{d}t + \sigma \mathrm{d}B^*$$

where B^* is a Brownian motion under $\mathbb{P}^R$

Note: Same volatility σ, different drift ($\mu \to r$)

Section 9

Applications

Three Ways to Price a Derivative

Consider a European call option with payoff $(S_T - K)^+$

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$$C = e^{-rT} \mathbb{E}^R[(S_T - K)^+]$$

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$$C = \sum_j \pi_j (S_j - K)^+$$

Method 2: Risk-Neutral Valuation

$$C = e^{-rT} \mathbb{E}^R[(S_T - K)^+]$$

Method 3: Stock Numeraire

$$C = S_0 \mathbb{E}^S \left[\frac{(S_T - K)^+}{S_T} \right]$$

All three give the **same answer**!

When to Use Each Method

State Prices:

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Stock Numeraire:

- Options with payoffs relative to stock price
- Digital options
- Lookback and Asian options

Section 10

Key Takeaways

Summary: The Big Picture

1. **No arbitrage** $\Rightarrow$ **Linear pricing** $\Rightarrow$ **State prices**
2. State prices $\Rightarrow$ **Risk-neutral probabilities**
3. Asset price ratios are **martingales** under appropriate measures
4. **Any positive asset** can serve as numeraire
5. **Fundamental pricing formula:** Values are expectations of discounted payoffs
6. **Girsanov's Theorem:** Changing measures changes drift, not volatility

Practical Implications

For Derivatives Pricing:

- Don't need to know actual probabilities
- Don't need to estimate expected returns μ
- Only need risk-free rate r and volatility σ
- Can price using expectations (Monte Carlo)
- Can choose convenient numeraire for each problem

What's Next?

This framework is the foundation for:

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- **Monte Carlo simulation** methods
- **Exotic option pricing**
- **Interest rate derivatives**
- **Credit derivatives**

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Key Message

Understanding arbitrage pricing is essential for understanding all of modern derivatives pricing theory

Section 11

Questions?

Discussion

Think about:

1. Why are risk-neutral probabilities useful if they're not "real"?
2. What happens to derivative prices if arbitrage opportunities exist?
3. How does the choice of numeraire affect computation?
4. What assumptions are we making (and are they realistic)?

Additional Resources

For deeper understanding:

- Read the chapter exercises (especially the binomial examples)
- Practice calculating state prices in simple models
- Simulate risk-neutral paths in Python
- Explore different numeraire choices for the same option

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Next class: We'll apply these concepts to derive the Black-Scholes formula and understand delta hedging