Chapter 1: Utility and Risk Aversion

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Outline

- Quick overview of finance
- Utility functions and risk aversion coefficients
- Certainty equivalents

Overview of Finance

Examples of Finance Questions

1. A company can invest K to generate an uncertain cash flow of $\tilde{x}$ in one year. Should it make the investment?
2. If the company needs to raise the capital, under what conditions will it choose debt versus equity financing?
3. What is the optimal portfolio of stocks for an investor?
4. Which of two stocks should have the higher expected return?
5. What is the most efficient way to organize the buying and selling of stocks?

Utility Functions

Utility and Risk Aversion

- Expected utility $\mathrm{E}[u(\tilde{w})]$
  - Utility function u is unique up to monotone affine transform: $f(w) = a + bu(w)$ for $b > 0$.
- Risk aversion: $\mathrm{E}[\tilde{\varepsilon}] = 0 \Rightarrow \mathrm{E}[u(w + \tilde{\varepsilon})] < \mathrm{E}[u(w)]$.
  - Equivalent to concavity (Jensen's inequality)
  - Equivalent to decreasing marginal utility: $u'' \leq 0$.
  - Invariant under monotone affine transformations.

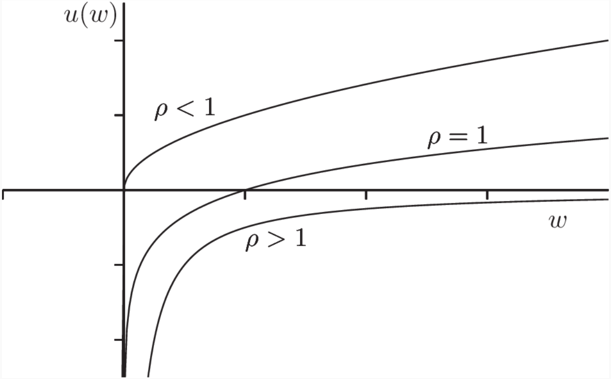
Coefficients of Risk Aversion

- Absolute risk aversion: $\alpha(w) = -u''(w)/u'(w)$
- Risk tolerance: $\tau(w) = 1/\alpha(w)$
- Relative risk aversion: $\rho(w) = w\alpha(w) = -wu''(w)/u'(w)$.

Some Special Utility Functions

- CARA = Constant Absolute Risk Aversion
  - $u(w) = -\mathrm{e}^{-\alpha w}$
  - α is the coefficient of absolute risk aversion.
- CRRA = Constant Relative Risk Aversion
  - $u(w) = \log w \Rightarrow$ relative risk aversion $= 1$
  - $u(w) = w^{1-\rho}/(1-\rho) \Rightarrow$ relative risk aversion $= \rho$

CRRA Utility Functions



Proof that CARA = $-\mathrm{e}^{-\alpha w}$

The coefficient of absolute risk aversion is

$$-\frac{u''(w)}{u'(w)} = -\frac{\mathrm{d}\log u'(w)}{\mathrm{d}w}$$

CARA means this is constant. Call it α. Then $\log u'$ is an affine (constant plus linear) function of w with slope $-\alpha$. This means

$$\begin{aligned}
\log u'(w) = \log u'(0) - \alpha w &\Rightarrow u'(w) = u'(0)\mathrm{e}^{-\alpha w} \\
&\Rightarrow u(w) = u(0) + u'(0)\int_0^w \mathrm{e}^{-\alpha x}\,\mathrm{d}x \\
&\Rightarrow u(w) = u(0) - \frac{u'(0)}{\alpha}\left(\mathrm{e}^{-\alpha w} - 1\right)
\end{aligned}$$

Linear Risk Tolerance

- CARA risk tolerance $= 1/\alpha$
- CRRA absolute risk aversion $= \rho/w$. Risk tolerance $= w/\rho$
- Linear risk tolerance: $\tau(w) = A + Bw$.
  - For CARA, $A = 1/\alpha$, $B = 0$
  - For CRRA, $A = 0$, $B = 1/\rho$
  - In general, B is called the cautiousness parameter.

LRT Utility Functions with $B > 0$

- CARA
- CRRA
- Shifted CRRA
  - Shifted log: $u(w) = \log(w - \zeta)$
  - Shifted power: $u(w) = \frac{1}{1-\rho}(w - \zeta)^{1-\rho}$

Quadratic Utility

$$u(w) = -\frac{1}{2}(\zeta - w)^2$$

- Quadratic utility is monotone increasing for $w < \zeta$ (ζ is bliss point).
- Risk tolerance: $\tau(w) = \zeta - w$, so $A = \zeta$ and $B = -1$.
- Implies mean-variance preferences:

$$\mathrm{E}[u(\tilde{w})] \;\sim\; \zeta\overline{w} - \frac{1}{2}\overline{w}^2 - \frac{1}{2}\,\mathrm{var}(\widetilde{w})$$

Decreasing Absolute Risk Aversion

- An investor has DARA utility if absolute risk aversion $\alpha(w)$ is a decreasing function of w.
- CRRA utility is DARA: $\alpha(w) = \rho/w$ for a constant ρ (relative risk aversion).
- LRT utility with $B > 0$ is DARA.
- Quadratic utility is not DARA.

Certainty Equivalents

Certainty Equivalents

- A constant x is the certainty equivalent of a random $\tilde{w}$ if $u(x) = \mathsf{E}[u(\tilde{w})]$.
- Risk aversion implies $x < \mathsf{E}[\tilde{w}]$.
- Certainty equivalents are invariant under monotone affine transformations.

CARA Utility and Normal Gambles

- CARA investor with a normally distributed wealth having mean w. How much less than w is the certainty equivalent?
- Equivalent question: CARA investor with initial wealth w facing a zero-mean normally distributed gamble. How much would she pay to avoid the gamble?
- Answer: would pay $\alpha\sigma^2/2$ where σ^2 is the variance of the gamble.
- Higher risk aversion $\Rightarrow$ pay more. Higher risk $\Rightarrow$ pay more.

Proof

- Call the gamble $w + \sigma\tilde{\varepsilon}$ where $\tilde{\varepsilon}$ is a standard normal.
- Call the certainty equivalent x. Define $\pi = w - x$, so the certainty equivalent is $w - \pi$.
- Claim is that $\pi = \alpha\sigma^2/2$.

Proof

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- Call the certainty equivalent x. Define $\pi = w - x$, so the certainty equivalent is $w - \pi$.
- Claim is that $\pi = \alpha\sigma^2/2$.
- The definition of the certainty equivalent and negative exponential (CARA) utility tell us that π satisfies

$$
\begin{aligned}
& u(w-\pi) = \mathsf{E}\left[u(w+\sigma\tilde{\varepsilon})\right] \\
\Leftrightarrow \quad & -\mathrm{e}^{-\alpha(w-\pi)} = \mathsf{E}\left[-\mathrm{e}^{-\alpha(w+\sigma\tilde{\varepsilon})}\right]
\end{aligned}
$$

- Use the following fact: if $\tilde{x}$ is normal (μ, σ), then

$$\mathsf{E}\left[\mathrm{e}^{\tilde{x}}\right] = \mathrm{e}^{\mu + \sigma^2/2}$$

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- Solve

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- Solution: $\pi = \alpha\sigma^2/2$.

Small Gambles

- Drop the CARA assumption and drop the normal distribution.
- How much would an investor pay to avoid a gamble?
- We can give a limiting result as the size of the gamble goes to zero: for small gambles, the amount an investor would pay is approximately $\alpha\sigma^2/2$.
- More precisely: Fix w. Let $\tilde{\varepsilon}$ be a bounded zero-mean random variable with unit variance. For $\sigma > 0$, define $\pi(\sigma)$ by

$$u(w - \pi(\sigma)) = \mathrm{E}[u(w + \sigma\tilde{\varepsilon})]\,.$$

Then,

$$\lim_{\sigma \to 0} \frac{\pi(\sigma)}{\sigma^2} = \frac{1}{2}\alpha(w)$$

Proof

Take an exact Taylor series approximation:

$$\pi(\sigma) = \pi(0) + \pi'(0)\sigma + \frac{1}{2}\pi''(x_\sigma)\sigma^2$$

for $0 < x_\sigma < \sigma$ Clearly, $\pi(0) = 0$. Differentiate both sides of

$$u(w - \pi(\sigma)) = \mathsf{E}[u(w + \sigma\tilde{\varepsilon})] \tag{$\star$}$$

and evaluate at $\sigma = 0$ to obtain

$$-u'(w)\pi'(0) = \mathsf{E}[u'(w)\tilde{\varepsilon}] = u'(w)\mathsf{E}[\tilde{\varepsilon}] = 0$$

Hence,

$$\frac{\pi(\sigma)}{\sigma^2} = \frac{1}{2}\pi''(x_\sigma) \to \frac{1}{2}\pi''(0)$$

Differentiate both sides of (**??**) twice and evaluate at $\sigma = 0$ to obtain $\pi''(0) = \alpha(w)$.

Relative Risk Aversion

- Instead of a gambles in dollars, let's make it a percent of wealth. Investor's wealth is $w + w\sigma\tilde{\varepsilon}$ where $\tilde{\varepsilon}$ is a bounded zero-mean unit-variance random variable as before. Relative risk aversion: $\rho(w) = w\alpha(w)$.
- Write the certainty equivalent as $w - w\gamma(\sigma)$. So, $\gamma := \pi/w$ is the amount she would pay relative to wealth to avoid the gamble.
- The variance of the gamble is now $w^2\sigma^2$, so, for small σ, $\pi \approx \alpha w^2\sigma^2/2$.
- This implies $\gamma \approx \alpha w\sigma^2 = \rho\sigma^2$, where ρ is relative risk aversion..