Chapter 4: Equilibrium and Efficiency

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Equilibrium

Competitive Equilibrium

- n assets, possibly including a risk-free asset, with values $\tilde{x}_i$.
- Investors $h = 1, \ldots, H$ have endowments of shares $\bar{\theta}_h \in \mathbb{R}^n$ at date 0.
- Choose portfolios $\theta_h \in \mathbb{R}^n$ subject to budget constraint

$$p'\theta_h \;\leq\; p'\bar{\theta}_h$$

- Maximize expected utility of date–1 wealth $\tilde{w}_h := \sum_{i=1}^n \theta_{hi}\tilde{x}_i$.
- Equilibrium is $(p, \theta_1^*, \ldots, \theta_H^*)$ such that θ_h^* is optimal for each investor h given p, and markets clear:

$$\sum_{h=1}^H \theta_h^* \;=\; \sum_{h=1}^H \bar{\theta}_h\,.$$

- Existence? Optimality? Equilibrium risk premia?

Arrow-Debreu Model

- k states.
- Assets are Arrow securities: pay 1 in single state and 0 otherwise.
- Denote price vector by $q \in \mathbb{R}^k$.
- Portfolio θ determines date–1 wealth as $w_j = \theta_j$ for $j = 1, \ldots, k$. In other words, the wealth vector $w \in \mathbb{R}^k$ is the portfolio.
- Existence: standard result
- Optimality: standard welfare theorems

Security Markets vs. Arrow-Debreu Model

- When is a competitive equilibrium in a securities market equivalent to an equilibrium in an Arrow-Debreu model?
- Answer: if the securities market is complete.
- Asset prices $p = (p_1, \ldots, p_n)$ and state prices $(q_1, \ldots, q_k)$ correspond as $Xq = p$.
- So, equilibria in complete markets are Pareto optimal.

Efficiency

Pareto Optimum

- Let $\tilde{w}_m$ denote end-of-period market wealth.
- An allocation is $\tilde{w}_1, \ldots, \tilde{w}_H$ such that $\sum_h \tilde{w}_h = \tilde{w}_m$.
- A Pareto optimum is an allocation such that any other allocation that makes at least one person better off also makes at least one person worse off.
- A Pareto optimum solves a social planner's problem: for some weights $\lambda_1, \ldots, \lambda_H$,

$$\max \quad \sum_{h=1}^{H} \lambda_h \mathrm{E}[u_h(\tilde{w}_h)] \quad \text{subject to} \quad \sum_{h=1}^{H} \tilde{w}_h = \tilde{w}_m\,.$$

Social Planner's Problem

- The resource constraint is a separate constraint for each state. And, expected utility is additive across states.
- So, we can solve the maximization problem state-by-state.
  - What does this mean?
  - Consider the problem $\max a + b$ subject to $a \leq 3$ and $b \leq 5$.
  - We can solve this as separate problems: $\max a$ s.t. $a \leq 3$ and $\max b$ s.t. $b \leq 5$.
- In each state of the world ω, the social planner solves

$$\max \quad \sum_{h=1}^{H} \lambda_h u_h(\tilde{w}_h(\omega)) \quad \text{subject to} \quad \sum_{h=1}^{H} \tilde{w}_h(\omega) = \tilde{w}_m(\omega)\,.$$

Equality of MRS's

- Let $\tilde{\eta}(\omega)$ denote the Lagrange multiplier in state ω.
- Then, for all h,

$$\lambda_h u_h'(\tilde{w}_h(\omega)) = \tilde{\eta}(\omega)\,.$$

- Consider another state $\hat{\omega}$ and divide the equations (for the same h):

$$\frac{u_h'(\tilde{w}_h(\omega))}{u_h'(\tilde{w}_h(\hat{\omega}))} = \frac{\tilde{\eta}(\omega)}{\tilde{\eta}(\hat{\omega})}$$

- So, for any other investor ℓ,

$$\frac{u_h'(\tilde{w}_h(\omega))}{u_h'(\tilde{w}_h(\hat{\omega}))} = \frac{u_\ell'(\tilde{w}_\ell(\omega))}{u_\ell'(\tilde{w}_\ell(\hat{\omega}))}$$

Sharing Rules

- If market wealth is higher in state ω than in state $\hat{\omega}$, then at any Pareto optimum (assuming strict risk aversion) all investors have higher wealth in state ω than in state $\hat{\omega}$:

$$\begin{aligned}\widetilde{w}_h(\omega) > \widetilde{w}_h(\hat{\omega}) &\Rightarrow \frac{u_h'(\widetilde{w}_h(\omega))}{u_h'(\widetilde{w}_h(\hat{\omega}))} < 1 \\ &\Rightarrow \frac{u_\ell'(\widetilde{w}_\ell(\omega))}{u_\ell'(\widetilde{w}_\ell(\hat{\omega}))} < \ 1 \\ &\Rightarrow \widetilde{w}_\ell(\omega) > \widetilde{w}_\ell(\hat{\omega})\,.\end{aligned}$$

- This implies each investor's wealth is a function of market wealth. The function is called a sharing rule.

Example

- Suppose there are two risk-averse investors and two possible states of the world, with $\widetilde{w}_m$ being the same in both states, say, $\widetilde{w}_m = 6$, and with the two states being equally likely.
- Can the allocation

$$\widetilde{w}_1 = \begin{cases} 2 & \text{in state 1} \\ 4 & \text{in state 2} \end{cases}$$

$$\widetilde{w}_2 = \begin{cases} 4 & \text{in state 1} \\ 2 & \text{in state 2} \end{cases}$$

  be Pareto optimal?

LRT Utility

Sharing Rules with Linear Risk Tolerance

- Assume $\tau_h(w) = A_h + Bw$ with same cautiousness parameter $B \geq 0$ for all individuals. Note $B > 0$ implies DARA. Then, either
  - Everyone has CARA utility: $-\mathrm{e}^{-\alpha_h w}$, or
  - Everyone has shifted log utility: $\log(w - \zeta_h)$, or
  - Everyone has shifted power utility with $\rho > 0$:

$$\frac{1}{1-\rho}(w - \zeta_h)^{1-\rho}$$

- In this case, Pareto optimal sharing rules are affine: $\tilde{w}_h = a_h + b_h \tilde{w}_m$ with
  - $\sum_{h=1}^{H} a_h = 0$, and
  - $\sum_{h=1}^{H} b_h = 1$.

Exposures to Market Risk

- With CARA utility, $b_h = \tau_h / \sum_{j=1}^{H} \tau_j$. Each person's exposure is proportional to her risk tolerance.
- With shifted log ($\rho = 1$) or shifted power,

$$b_h = \frac{\lambda_h^{1/\rho}}{\sum_{j=1}^{H} \lambda_j^{1/\rho}}$$

where the λ's are the weights in the social planning problem.

Proof for CARA Utility

- Social planner's problem is (in each state of the world)

$$\max \quad \sum_{h=1}^{H} \lambda_h \mathrm{e}^{-\alpha_h w_h} \quad \text{subject to} \quad \sum_{h=1}^{H} w_h = w$$

  where w denotes the value of $\tilde{w}_m$ in the given state.
- FOC is: $(\forall\, h)\ \lambda_h \alpha_h \mathrm{e}^{-\alpha_h w_h} = \eta$ where η is the Lagrange multiplier (in the given state).

Set $\tau = \sum_h \tau_h$. Take logs in the FOC:

$$w_h = -\tau_h \log \eta + \tau_h \log(\lambda_h \alpha_h)$$

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$$\Rightarrow \quad w_h = \frac{\tau_h}{\tau} w - \frac{\tau_h}{\tau} \sum_{\ell=1}^{H} \tau_\ell \log(\lambda_\ell \alpha_\ell) + \tau_h \log(\lambda_h \alpha_h)$$

Competitive Equilibria with LRT Utility

- Assume there is a risk-free asset. Assume all investors have linear risk tolerance $\tau_h(w) = A_h + Bw$ with the same cautiousness parameter B. Assume there are no $\tilde{y}_h$'s.
- The set of equilibrium prices does not depend on the distribution of wealth across investors.
    - Called Gorman aggregation
    - Due to wealth expansion paths being parallel (Chapter 2)
- Any Pareto optimal allocation can be implemented in the securities market.
    - Due to affine sharing rules, we only need the risk-free asset and market portfolio
    - Example of two-fund separation (Chapter 2)
- Any competitive equilibrium is Pareto optimal.