

# Chapter 5: Mean-Variance Analysis

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## Standard Deviation – Mean Plots

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# Notation

- $n$  risky assets with returns  $\tilde{R}_i$ .  $\tilde{\mathbf{R}} = (\tilde{R}_1 \cdots \tilde{R}_n)'$
- $\mu$  = vector of expected returns. At least two of the assets have different expected returns.
- $\Sigma$  = covariance matrix. Assume no redundant assets, so  $\Sigma$  is positive definite.
- $\iota$  =  $n$ -vector of 1's.
- $\pi \in \mathbb{R}^n$  is a portfolio (of risky assets). If the portfolio is fully invested in risky assets, then  $\iota'\pi = 1$ . Otherwise,  $1 - \iota'\pi$  is the fraction of wealth invested in the risk-free asset.

# Portfolio Mean and Standard Deviation

- Two assets with expected returns  $\mu_i$ , standard deviations  $\sigma_i$ , and correlation  $\rho$ .
- Portfolio  $(\pi_1, \pi_2)$  with  $\pi_1 + \pi_2 = 1$ .
- Portfolio return is

$$\pi' \tilde{\mathbf{R}} = \pi_1 \tilde{R}_1 + \pi_2 \tilde{R}_2$$

- Portfolio expected return is

$$\pi' \mu = \pi_1 \mu_1 + \pi_2 \mu_2$$

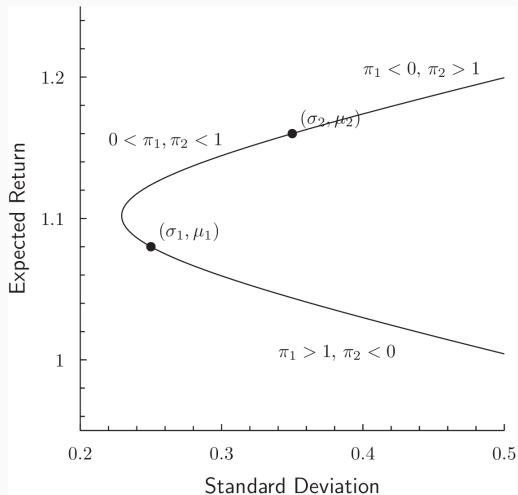
- Write the covariance matrix as

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

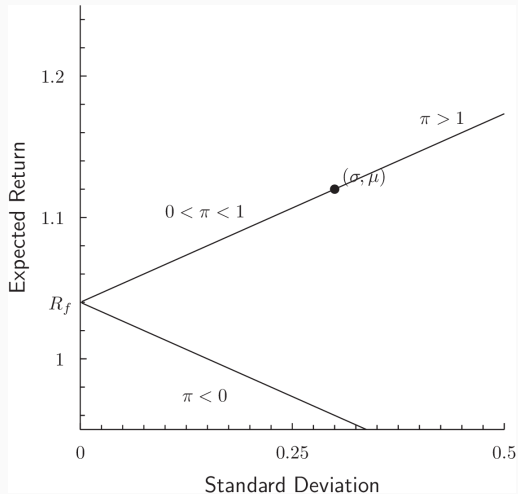
- The portfolio variance is

$$\pi' \Sigma \pi = \pi_1^2 \sigma_1^2 + \pi_2^2 \sigma_2^2 + 2\pi_1 \pi_2 \rho \sigma_1 \sigma_2$$

# Portfolios of Two Risky Assets



# Portfolios of a Risky and Risk-Free Asset



# GMV Portfolio

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# Global Minimum Variance Portfolio

- The portfolio of risky assets with minimum variance is called the Global Minimum Variance (GMV) portfolio.
- It solves the optimization problem

$$\min \quad \frac{1}{2} \pi' \Sigma \pi \quad \text{subject to} \quad \iota' \pi = 1$$

- The Lagrangean for this problem is

$$\frac{1}{2} \pi' \Sigma \pi - \gamma (\iota' \pi - 1)$$

- The FOC is

$$\Sigma \pi = \gamma \iota \quad \Leftrightarrow \quad \pi = \gamma \Sigma^{-1} \iota$$

- Impose the constraint  $\iota' \pi = 1$  and solve for  $\gamma$  to obtain

$$\pi = \frac{1}{\iota' \Sigma^{-1} \iota} \Sigma^{-1} \iota$$

- In other words, take the vector  $\Sigma^{-1} \iota$  and divide by the sum of its elements, so the rescaled vector sums to 1.



# Frontier Portfolios

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# Mean-Variance Frontier of Risky Assets

- We continue to look at only risky assets so we continue to require portfolio weights to sum to 1 ( $\iota'\pi = 1$ ).
- A **frontier portfolio** is a portfolio that achieves a target expected return with minimum risk.
- It solves an optimization problem

$$\min \quad \frac{1}{2}\pi'\Sigma\pi \quad \text{subject to} \quad \mu'\pi = \mu_{\text{targ}} \quad \text{and} \quad \iota'\pi = 1$$

where  $\mu_{\text{targ}}$  denotes the given target expected return.

- By varying the target expected return, we trace out the frontier.

# Solving for Frontier Portfolios

- The Lagrangean for the optimization problem is

$$\frac{1}{2}\pi'\Sigma\pi - \delta(\mu'\pi - \mu_{\text{targ}}) - \gamma(\iota'\pi - 1)$$

- The FOC is

$$\Sigma\pi - \delta\mu - \gamma\iota = 0.$$

- The solution is

$$\pi = \delta\Sigma^{-1}\mu + \gamma\Sigma^{-1}\iota$$

- This means that  $\pi$  is a linear combination of the two vectors  $\Sigma^{-1}\mu$  and  $\Sigma^{-1}\iota$ .
- Use constraints to solve for  $\delta$  and  $\gamma$ .

## More Notation

- Denote the GMV portfolio by  $\pi_{\text{gmv}}$ . It is  $\Sigma^{-1}\iota$  rescaled to sum to 1:

$$\pi_{\text{gmv}} = \frac{1}{\iota'\Sigma^{-1}\iota}\Sigma^{-1}\iota$$

- Let's also rescale the vector  $\Sigma^{-1}\mu$  to sum to 1 and call it  $\pi_{\mu}$ :

$$\pi_{\mu} = \frac{1}{\mu'\Sigma^{-1}\mu}\Sigma^{-1}\mu$$

- To simplify, define  $A = \mu'\Sigma^{-1}\mu$ ,  $B = \mu'\Sigma^{-1}\iota$ , and  $C = \iota'\Sigma^{-1}\iota$ .
- Then,

$$\begin{aligned}\pi_{\text{gmv}} &= \frac{1}{C}\Sigma^{-1}\iota \\ \pi_{\mu} &= \frac{1}{B}\Sigma^{-1}\mu\end{aligned}$$

# Solution of Frontier Portfolio

- We saw that a frontier portfolio is

$$\pi = \delta \Sigma^{-1} \mu + \gamma \Sigma^{-1} \iota$$

for some  $\delta$  and  $\gamma$ .

- We can write this as

$$\begin{aligned}\pi &= \delta B \frac{1}{B} \Sigma^{-1} \mu + \gamma C \frac{1}{C} \Sigma^{-1} \mu \\ &= \delta B \pi_{\mu} + \gamma C \pi_{\text{gmv}}\end{aligned}$$

- The constraint  $\iota' \pi = 1$  implies

$$\delta B + \gamma C = 1$$

- Set  $\lambda = \delta B$ . Then,  $\gamma C = 1 - \lambda$ , so the frontier portfolio is

$$\pi = \lambda \pi_{\mu} + (1 - \lambda) \pi_{\text{gmv}}$$

- To find the particular frontier portfolio meeting the target return constraint, we can calculate

$$\mu' \pi = \lambda \mu' \pi_{\mu} + (1 - \lambda) \mu' \pi_{\text{gmV}} = \lambda \frac{A}{B} + (1 - \lambda) \frac{B}{C}$$

- Set this equal to  $\mu_{\text{targ}}$  to obtain

$$\lambda = \frac{\mu_{\text{targ}} - B/C}{A/B - B/C} = \frac{BC\mu_{\text{targ}} - B^2}{AC - B^2}$$

## Two Fund Spanning

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# Two Fund Spanning

- The characterization  $\pi = \lambda\pi_{\mu} + (1 - \lambda)\pi_{\text{gmV}}$  means that  $\pi$  lies on the line through  $\pi_{\mu}$  and  $\pi_{\text{gmV}}$  in  $\mathbb{R}^n$ .
- Every frontier portfolio is a combination of  $\pi_{\mu}$  and  $\pi_{\text{gmV}}$ . We say that these two portfolios span the frontier.
- We can consider the portfolios to be funds – like mutual funds. If you want a frontier portfolio, you can just invest in these two funds. We call this two-fund spanning.
- Any other two points on the line also span the line. So, any two frontier portfolios can serve as the funds.



## Risk-Free Asset

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# Mean-Variance Frontier with a Risk-Free Asset

- Now, we add a risk-free asset. We continue to let  $\pi \in \mathbb{R}^n$  denote the portfolio of risky assets.
- We no longer require  $\iota' \pi = 1$ . The weight on the risk-free asset is  $1 - \iota' \pi$ . This can be negative (borrowing).
- A portfolio's expected return is

$$(1 - \iota' \pi)R_f + \mu' \pi = R_f + (\mu - R_f \iota)' \pi$$

- A frontier portfolio solves the following for some  $\mu_{\text{targ}}$ :

$$\min \quad \frac{1}{2} \pi' \Sigma \pi \quad \text{subject to} \quad R_f + (\mu - R_f \iota)' \pi = \mu_{\text{targ}}$$

- FOC is

$$\Sigma\pi - \delta(\mu - R_f\iota) = 0 \quad \Leftrightarrow \quad \pi = \delta\Sigma^{-1}(\mu - R_f\iota)$$

- So, all frontier portfolios are scalar multiples of the vector  $\Sigma^{-1}(\mu - R_f\iota)$ .
- In other words, the frontier portfolios form a line through the origin and the vector  $\Sigma^{-1}(\mu - R_f\iota)$ .

- We can probably divide the vector  $\Sigma^{-1}(\mu - R_f \iota)$  by the sum of its elements to form a portfolio of purely risky assets (satisfying  $\iota' \pi = 1$ ).
- We can do that as long as the sum is nonzero. That is, we need

$$\iota' \Sigma^{-1}(\mu - R_f \iota) \neq 0$$

- This expression is  $B - R_f C$ . It is nonzero if and only if  $B/C \neq R_f$ .

- The term  $B/C$  is the expected return of the GMV portfolio:

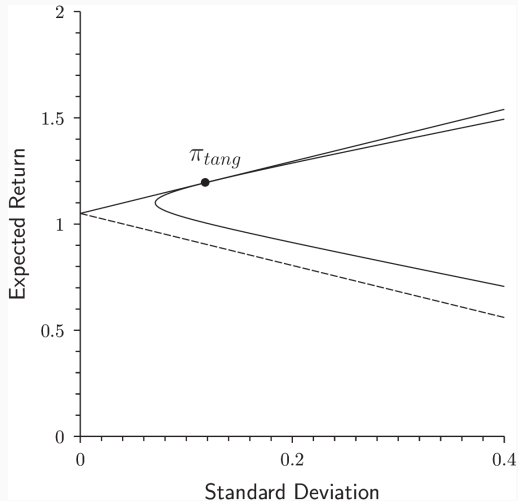
$$\mu' \pi_{\text{gmv}} = \mu' \left( \frac{1}{\iota' \Sigma^{-1} \iota} \Sigma^{-1} \iota \right) = \frac{1}{\iota' \Sigma^{-1} \iota} \mu' \Sigma^{-1} \iota = \frac{B}{C}$$

- So, when the expected return of the GMV portfolio is different from  $R_f$ , we can define

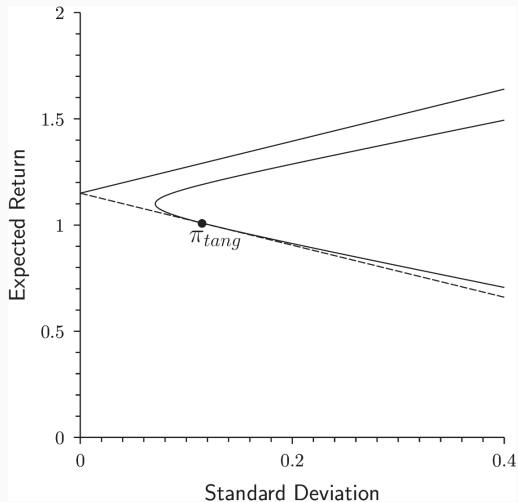
$$\pi_{\text{tang}} = \frac{1}{\iota' \Sigma^{-1} (\mu - R_f \iota)} \Sigma^{-1} (\mu - R_f \iota)$$

- We call this the tangency portfolio because it is on two frontiers: the frontier including the risk-free asset and the frontier of only risky assets.
- How do we know it is on the frontier of only risky assets?
  1. It is a portfolio constructed from the two vectors  $\Sigma^{-1}\mu$  and  $\Sigma^{-1}\iota$ .
  2. Also, anything that solves a less constrained optimization problem (not requiring  $\iota'\pi = 1$ ) and satisfies the constraints of a more constrained problem (satisfies  $\iota'\pi = 1$  anyway) must solve the more constrained problem too.
- Thus, the two frontiers (in std dev/mean space) must be tangent at this point.

## Mean-Variance Frontier: $B/C > R_f$



## Mean-Variance Frontier: $B/C < R_f$





## What if $B/C = R_f$ ?

- If  $B/C = R_f$ , then
  - The weights of each frontier portfolio sum to zero.
  - This means investing 100% in the risk-free asset and then go long and short equal dollars worth of risky assets.
- The cone and hyperbola never touch.

## Two-Fund Spanning Again

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## Two Fund Spanning with a Risk-Free Asset

- All frontier portfolios lie on the line through the origin and the vector  $\Sigma^{-1}(\mu - R_f \mathbf{1})$  in  $\mathbb{R}^n$ .
- Any vector on the line is a portfolio, because we are not requiring  $\mathbf{1}'\pi = 1$ .
- The origin represents 100% in the risk-free asset.
- Any two portfolios on the line span the frontier in the sense that any frontier portfolio is a combination  $\lambda$  and  $(1 - \lambda)$  of the portfolios.

# Maximum Sharpe Ratio

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# Maximum Sharpe Ratio

- What is the risk premium of the portfolio  $\Sigma^{-1}(\mu - R_f \mathbf{1})$ ?
- What is the variance of the return of the portfolio  $\Sigma^{-1}(\mu - R_f \mathbf{1})$ ?
- What is its Sharpe ratio (risk premium divided by standard deviation)?

# SDFs and Mean-Variance Efficiency

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- Project any SDF onto the span of the assets. There is a unique projection, and it is an SDF. Call it  $\tilde{m}_p$ .
- $\tilde{m}_p$  is the payoff of some portfolio (that's what it means to be in the span of the assets).
- Set  $\tilde{R}_p = \tilde{m}_p / E[\tilde{m}_p^2]$ . This is  $\tilde{m}_p$  divided by its cost, so it has a cost of 1 and is a return.
- The return  $\tilde{R}_p$  is an inefficient frontier return.
- If there is a risk-free asset, then for any frontier return  $\tilde{R}$ ,  $\tilde{R}_p = \lambda R_f + (1 - \lambda)\tilde{R}$  for some  $\lambda$  (by two-fund spanning). So,  $\tilde{m}_p = a + b\tilde{R}$ , where  $a = \lambda E[\tilde{m}_p^2]$  and  $b = (1 - \lambda)E[\tilde{m}_p^2]$ .
- Even without a risk-free asset (with one trivial exception),
  - SDF = affine function of return  $\Rightarrow$  return is on MV frontier.
  - return  $\tilde{R}$  on MV frontier  $\Rightarrow \tilde{m}_p = a + b\tilde{R}$ .