

# Chapter 4: Equilibrium and Efficiency

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# Equilibrium

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# Competitive Equilibrium

- $n$  assets, possibly including a risk-free asset, with values  $\tilde{x}_i$ .
- Investors  $h = 1, \dots, H$  have endowments of shares  $\bar{\theta}_h \in \mathbb{R}^n$  at date 0.
- Choose portfolios  $\theta_h \in \mathbb{R}^n$  subject to budget constraint

$$p' \theta_h \leq p' \bar{\theta}_h$$

- Maximize expected utility of date-1 wealth  $\tilde{w}_h := \sum_{i=1}^n \theta_{hi} \tilde{x}_i$ .
- Equilibrium is  $(p, \theta_1^*, \dots, \theta_H^*)$  such that  $\theta_h^*$  is optimal for each investor  $h$  given  $p$ , and markets clear:

$$\sum_{h=1}^H \theta_h^* = \sum_{h=1}^H \bar{\theta}_h.$$

- Existence? Optimality? Equilibrium risk premia?

# Arrow-Debreu Model

- $k$  states.
- Assets are Arrow securities: pay 1 in single state and 0 otherwise.
- Denote price vector by  $q \in \mathbb{R}^k$ .
- Portfolio  $\theta$  determines date-1 wealth as  $w_j = \theta_j$  for  $j = 1, \dots, k$ . In other words, the wealth vector  $w \in \mathbb{R}^k$  is the portfolio.
- Existence: standard result
- Optimality: standard welfare theorems

# Security Markets vs. Arrow-Debreu Model

- When is a competitive equilibrium in a securities market equivalent to an equilibrium in an Arrow-Debreu model?
- Answer: if the securities market is complete.
- Asset prices  $p = (p_1, \dots, p_n)$  and state prices  $(q_1, \dots, q_k)$  correspond as  $Xq = p$ .
- So, equilibria in complete markets are Pareto optimal.

# Efficiency

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# Pareto Optimum

- Let  $\tilde{w}_m$  denote end-of-period market wealth.
- An allocation is  $\tilde{w}_1, \dots, \tilde{w}_H$  such that  $\sum_h \tilde{w}_h = \tilde{w}_m$ .
- A Pareto optimum is an allocation such that any other allocation that makes at least one person better off also makes at least one person worse off.
- A Pareto optimum solves a social planner's problem: for some weights  $\lambda_1, \dots, \lambda_H$ ,

$$\max \sum_{h=1}^H \lambda_h E[u_h(\tilde{w}_h)] \quad \text{subject to} \quad \sum_{h=1}^H \tilde{w}_h = \tilde{w}_m.$$

# Social Planner's Problem

- The resource constraint is a separate constraint for each state. And, expected utility is additive across states.
- So, we can solve the maximization problem state-by-state.
  - What does this mean?
  - Consider the problem  $\max a + b$  subject to  $a \leq 3$  and  $b \leq 5$ .
  - We can solve this as separate problems:  $\max a$  s.t.  $a \leq 3$  and  $\max b$  s.t.  $b \leq 5$ .
- In each state of the world  $\omega$ , the social planner solves

$$\max \sum_{h=1}^H \lambda_h u_h(\tilde{w}_h(\omega)) \quad \text{subject to} \quad \sum_{h=1}^H \tilde{w}_h(\omega) = \tilde{w}_m(\omega).$$



# Equality of MRS's

- Let  $\tilde{\eta}(\omega)$  denote the Lagrange multiplier in state  $\omega$ .
- Then, for all  $h$ ,

$$\lambda_h u'_h(\tilde{w}_h(\omega)) = \tilde{\eta}(\omega).$$

- Consider another state  $\hat{\omega}$  and divide the equations (for the same  $h$ ):

$$\frac{u'_h(\tilde{w}_h(\omega))}{u'_h(\tilde{w}_h(\hat{\omega}))} = \frac{\tilde{\eta}(\omega)}{\tilde{\eta}(\hat{\omega})}$$

- So, for any other investor  $\ell$ ,

$$\frac{u'_h(\tilde{w}_h(\omega))}{u'_h(\tilde{w}_h(\hat{\omega}))} = \frac{u'_\ell(\tilde{w}_\ell(\omega))}{u'_\ell(\tilde{w}_\ell(\hat{\omega}))}$$

# Sharing Rules

- If market wealth is higher in state  $\omega$  than in state  $\hat{\omega}$ , then at any Pareto optimum (assuming strict risk aversion) all investors have higher wealth in state  $\omega$  than in state  $\hat{\omega}$ :

$$\begin{aligned}\tilde{w}_h(\omega) > \tilde{w}_h(\hat{\omega}) &\Rightarrow \frac{u'_h(\tilde{w}_h(\omega))}{u'_h(\tilde{w}_h(\hat{\omega}))} < 1 \\ &\Rightarrow \frac{u'_\ell(\tilde{w}_\ell(\omega))}{u'_\ell(\tilde{w}_\ell(\hat{\omega}))} < 1 \\ &\Rightarrow \tilde{w}_\ell(\omega) > \tilde{w}_\ell(\hat{\omega}).\end{aligned}$$

- This implies each investor's wealth is a function of market wealth. The function is called a sharing rule.

## Example

- Suppose there are two risk-averse investors and two possible states of the world, with  $\tilde{w}_m$  being the same in both states, say,  $\tilde{w}_m = 6$ , and with the two states being equally likely.
- Can the allocation

$$\tilde{w}_1 = \begin{cases} 2 & \text{in state 1} \\ 4 & \text{in state 2} \end{cases}$$

$$\tilde{w}_2 = \begin{cases} 4 & \text{in state 1} \\ 2 & \text{in state 2} \end{cases}$$

be Pareto optimal?

## LRT Utility

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# Sharing Rules with Linear Risk Tolerance

- Assume  $\tau_h(w) = A_h + Bw$  with same cautiousness parameter  $B \geq 0$  for all individuals. Note  $B > 0$  implies DARA. Then, either
  - Everyone has CARA utility:  $-e^{-\alpha_h w}$ , or
  - Everyone has shifted log utility:  $\log(w - \zeta_h)$ , or
  - Everyone has shifted power utility with  $\rho > 0$ :

$$\frac{1}{1-\rho}(w - \zeta_h)^{1-\rho}$$

- In this case, Pareto optimal sharing rules are affine:  $\tilde{w}_h = a_h + b_h \tilde{w}_m$  with
  - $\sum_{h=1}^H a_h = 0$ , and
  - $\sum_{h=1}^H b_h = 1$ .

# Exposures to Market Risk

- With CARA utility,  $b_h = \tau_h / \sum_{j=1}^H \tau_j$ . Each person's exposure is proportional to her risk tolerance.
- With shifted log ( $\rho = 1$ ) or shifted power,

$$b_h = \frac{\lambda_h^{1/\rho}}{\sum_{j=1}^H \lambda_j^{1/\rho}}$$

where the  $\lambda$ 's are the weights in the social planning problem.

# Proof for CARA Utility

- Social planner's problem is (in each state of the world)

$$\max \sum_{h=1}^H \lambda_h e^{-\alpha_h w_h} \quad \text{subject to} \quad \sum_{h=1}^H w_h = w$$

where  $w$  denotes the value of  $\tilde{w}_m$  in the given state.

- FOC is:  $(\forall h) \lambda_h \alpha_h e^{-\alpha_h w_h} = \eta$  where  $\eta$  is the Lagrange multiplier (in the given state).

Set  $\tau = \sum_h \tau_h$ . Take logs in the FOC:

$$w_h = -\tau_h \log \eta + \tau_h \log(\lambda_h \alpha_h)$$



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$$\Rightarrow w_h = \frac{\tau_h}{\tau} w - \frac{\tau_h}{\tau} \sum_{\ell=1}^H \tau_{\ell} \log(\lambda_{\ell} \alpha_{\ell}) + \tau_h \log(\lambda_h \alpha_h)$$

# Competitive Equilibria with LRT Utility

- Assume there is a risk-free asset. Assume all investors have linear risk tolerance  $\tau_h(w) = A_h + Bw$  with the same cautiousness parameter  $B$ . Assume there are no  $\tilde{y}_h$ 's.
- The set of equilibrium prices does not depend on the distribution of wealth across investors.
  - Called Gorman aggregation
  - Due to wealth expansion paths being parallel (Chapter 2)
- Any Pareto optimal allocation can be implemented in the securities market.
  - Due to affine sharing rules, we only need the risk-free asset and market portfolio
  - Example of two-fund separation (Chapter 2)
- Any competitive equilibrium is Pareto optimal.