Chapter 15: Continuous-Time Topics

Kerry Back
BUSI 521/ECON 505
Rice University

Risk Neutral Probabilities

Risk-Neutral Probabilities in Continuous Time

- Consider $T < \infty$.
- Let R denote the money market account price with $R_0 = 1$.
- Let M be an SDF process. Assume MR is a martingale so

$$\mathsf{E}[M_T R_T] \;=\; R_0 \;=\; 1$$

- Define

$$\mathbb{Q}(A) \;=\; \mathsf{E}\left[M_T R_T 1_A\right]$$

  for each event A that is distinguishable at date T, where $1_A = 1$ when the state of the world is in A and 0 otherwise.
- It follows that $\mathbb{Q}$ is a probability (measure) and

$$\mathsf{E}^*[X_T] \;=\; \mathsf{E}\left[M_T R_T X_T\right]$$

  for any random variable X_T depending on date$-T$ information, where E^* denotes expectation with respect to $\mathbb{Q}$.

Risk-Neutral Valuation

- Let W be such that MW is a martingale under the physical probability. Because we changed the probability using MR, a theorem in probability theory tells us that

$$\frac{MW}{MR}$$

  is a $\mathbb{Q}$–martingale.
- So, W/R is a $\mathbb{Q}$–martingale. Thus,

$$W_t \;=\; R_t \mathsf{E}_t^* \left[\frac{W_T}{R_T}\right] \;=\; \mathsf{E}_t^* \left[\exp\left(-\int_t^T r_u \,\mathrm{d}u\right) W_T\right] .$$

- In other words, asset values are expected discounted values, taking expectations with respect to the risk neutral probability and discounting at the instantaneous risk-free rate.
- It follows that expected returns under the RNP equal the risk-free rate.

Girsanov's Theorem

- Let M be an SDF process with

$$\frac{\mathrm{d}M}{M} = -r\,\mathrm{d}t - \lambda'\,\mathrm{d}B$$

  Here, r and λ can be stochastic processes.
- Define the risk-neutral probability $\mathbb{Q}$ using the martingale MR.
- The vector B is not a vector of Brownian motions under $\mathbb{Q}$
  - Its drift is nonzero.
  - But, we still have quadratic variation $(\mathrm{d}B)(\mathrm{d}B)' = I\,\mathrm{d}t$, so it is "close" to being a vector of Brownian motions.
- Girsanov's theorem states that B^* defined by $B_0^* = 0$ and

$$\mathrm{d}B^* = \mathrm{d}B + \lambda\,\mathrm{d}t$$

  is a vector of independent Brownian motions under the risk-neutral probability $\mathbb{Q}$.

Asset Returns under a Risk-Neutral Probability

- Recall that the vector of asset returns is

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$$

- Define $\mathrm{d}B^* = \mathrm{d}B + \lambda\,\mathrm{d}t$. Substitute to obtain

$$\begin{aligned}\frac{\mathrm{d}S}{S} &= \mu\,\mathrm{d}t + \sigma\,(\mathrm{d}B^* - \lambda\,\mathrm{d}t) \\ &= (\mu - \sigma\lambda)\,\mathrm{d}t + \sigma\,\mathrm{d}B^* \\ &= r\iota\,\mathrm{d}t + \sigma\,\mathrm{d}B^*\end{aligned}$$

Fundamental PDE

Fundamental PDE

- Suppose S is a univariate GBM that is a dividend-reinvested price: $\mathrm{d}S/S = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B = r\,\mathrm{d}t + \sigma\,\mathrm{d}B^*$.
- Let $f(t, S_t)$ denote the value at t of some at $T > t$ that depends on S_T.
- The risk-neutral expected rate of return is the risk-free rate, so

$$\frac{\text{drift of } f \text{ under RNP}}{f(t, S_t)} = r$$

Fundamental PDE

- Suppose S is a univariate GBM that is a dividend-reinvested price: $\mathrm{d}S/S = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B = r\,\mathrm{d}t + \sigma\,\mathrm{d}B^*$.
- Let $f(t, S_t)$ denote the value at t of some at $T > t$ that depends on S_T.
- The risk-neutral expected rate of return is the risk-free rate, so

$$\frac{\text{drift of } f \text{ under RNP}}{f(t, S_t)} = r$$

$$\begin{aligned}
\mathrm{d}f &= f_t\,\mathrm{d}t + f_S\,\mathrm{d}S + \frac{1}{2} f_{SS}(\mathrm{d}S)^2 \\
&= f_t\,\mathrm{d}t + f_S(rS\,\mathrm{d}t + \sigma S\,\mathrm{d}B^*) + \frac{1}{2} f_{SS}(S^2\sigma^2\,\mathrm{d}t)
\end{aligned}$$

Fundamental PDE

- Suppose S is a univariate GBM that is a dividend-reinvested price: $\mathrm{d}S/S = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B = r\,\mathrm{d}t + \sigma\,\mathrm{d}B^*$.
- Let $f(t, S_t)$ denote the value at t of some at $T > t$ that depends on S_T.
- The risk-neutral expected rate of return is the risk-free rate, so

$$\frac{\text{drift of } f \text{ under RNP}}{f(t, S_t)} = r$$

$$\begin{aligned}\mathrm{d}f &= f_t\,\mathrm{d}t + f_S\,\mathrm{d}S + \frac{1}{2}f_{SS}(\mathrm{d}S)^2 \\ &= f_t\,\mathrm{d}t + f_S(rS\,\mathrm{d}t + \sigma S\,\mathrm{d}B^*) + \frac{1}{2}f_{SS}(S^2\sigma^2\,\mathrm{d}t)\end{aligned}$$

- So

$$\text{drift of } f \text{ under RNP} = f_t + rSf_S + \frac{1}{2}\sigma^2 S^2 f_{SS}$$

The fundamental PDE is

$$f_t + rSf_S + \frac{1}{2}\sigma^2 S^2 f_{SS} = rf \qquad (\star)$$