Thus,

$$\mu - R_f p = \alpha \begin{pmatrix} H & 0 \\ H_U\sigma_{12}/\sigma_{11} & H_I \end{pmatrix}^{-1} \Sigma\bar{\theta}$$

$$= \alpha \begin{pmatrix} 1/H & 0 \\ -\frac{H_U\sigma_{12}}{HH_I\sigma_{11}} & 1/H_I \end{pmatrix} \Sigma\bar{\theta}$$

$$= \alpha \left[\frac{1}{H} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{H_U}{HH_I} \begin{pmatrix} 0 & 0 \\ -\frac{\sigma_{12}}{\sigma_{11}} & 1 \end{pmatrix} \right] \Sigma\bar{\theta}$$

$$= \frac{\alpha}{H}\Sigma\bar{\theta} + \frac{H_U\alpha}{HH_I} \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}} \end{pmatrix} \bar{\theta}$$

Hence,

$$p = \frac{1}{R_f}\mu - \frac{\alpha}{HR_f}\Sigma\bar{\theta} - \frac{\alpha}{HR_f}\left(\frac{H_U}{H_I}\right)\left(\sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}\right)\begin{pmatrix} 0 \\ \bar{\theta}_2 \end{pmatrix} .$$

Note that $p_2 < p_2^*$ because $\sigma_{11}\sigma_{12} > \sigma_{12}^2$.

(b) Show that there exist $A > 0$ and λ such that

$$\mathsf{E}[\tilde{R}_1] = R_f + \lambda\frac{\text{cov}(\tilde{R}_1, \tilde{R}_m)}{\text{var}(\tilde{R}_m)}\,, \tag{6.35a}$$

$$\mathsf{E}[\tilde{R}_2] = A + R_f + \lambda\frac{\text{cov}(\tilde{R}_2, \tilde{R}_m)}{\text{var}(\tilde{R}_m)}\,, \tag{6.35b}$$

$$\lambda = \mathsf{E}[\tilde{R}_m] - R_f - A\pi_2\,, \tag{6.35c}$$

where $\pi_2 = p_2\bar{\theta}_2/(p_1\bar{\theta}_1 + p_2\bar{\theta}_2)$ is the relative date–0 market capitalization of the second risky asset. (Note that λ is less than in the CAPM, and the second risky asset has a positive alpha, relative to λ.)

Solution: Set $\tilde{R}_i = \tilde{x}_i/p_i$ and

$$\tilde{R}_m = \frac{\bar{\theta}_1\tilde{x}_1 + \bar{\theta}_2\tilde{x}_2}{\bar{\theta}_1 p_1 + \bar{\theta}_2 p_2}\,.$$

We have

$$\operatorname{cov}(\tilde{R}_i, \tilde{R}_m) = \frac{p_i}{\bar{\theta}_1 p_1 + \bar{\theta}_2 p_2}(\sigma_{i1}\bar{\theta}_1 + \sigma_{i2}\bar{\theta}_2)\,.$$

The formula for p in the previous part implies

$$\mu = R_f p + \frac{\alpha}{H}\Sigma\bar{\theta} + \frac{\alpha}{H}\left(\frac{H_U}{H_I}\right)\left(\frac{\sigma_{11}\sigma_2^2 - \sigma_{12}^2}{\sigma_{11}}\right)\begin{pmatrix} 0 \\ \\ \bar{\theta}_2 \end{pmatrix}\,.$$

Thus,

$$\begin{aligned}\mathsf{E}[\tilde{R}_1] &= R_f + \frac{\alpha}{H}(\bar{\theta}_1 p_1 + \bar{\theta}_2 p_2)\operatorname{cov}(\tilde{R}_1, \tilde{R}_m) \\ &= R_f + \lambda\frac{\operatorname{cov}(\tilde{R}_1, \tilde{R}_m)}{\operatorname{var}(\tilde{R}_m)}\,,\end{aligned}$$

where

$$\lambda = \frac{\alpha}{H}(\bar{\theta}_1 p_1 + \bar{\theta}_2 p_2)\operatorname{var}(\tilde{R}_m)\,.$$

Also,

$$\begin{aligned}\mathsf{E}[\tilde{R}_2] &= R_f + \frac{\alpha}{H}(\bar{\theta}_1 p_1 + \bar{\theta}_2 p_2)\operatorname{cov}(\tilde{R}_2, \tilde{R}_m) + \frac{\alpha}{H p_2}\left(\frac{H_U}{H_I}\right)\left(\frac{\sigma_{11}\sigma_2^2 - \sigma_{12}^2}{\sigma_{11}}\right)\bar{\theta}_2 \\ &= R_f + \lambda\frac{\operatorname{cov}(\tilde{R}_2, \tilde{R}_m)}{\operatorname{var}(\tilde{R}_m)} + \frac{\alpha}{H}\left(\frac{H_U}{H_I}\right)\left(\frac{\operatorname{var}(\tilde{R}_1)\operatorname{var}(\tilde{R}_2) - \operatorname{cov}(\tilde{R}_1, \tilde{R}_2)^2}{\operatorname{var}(\tilde{R}_1)}\right)\bar{\theta}_2 p_2 \\ &= R_f + \lambda\frac{\operatorname{cov}(\tilde{R}_2, \tilde{R}_m)}{\operatorname{var}(\tilde{R}_m)} + A\,,\end{aligned}$$

where

$$A = \frac{\alpha}{H}\left(\frac{H_U}{H_I}\right)\left(\frac{\operatorname{var}(\tilde{R}_1)\operatorname{var}(\tilde{R}_2) - \operatorname{cov}(\tilde{R}_1, \tilde{R}_2)^2}{\operatorname{var}(\tilde{R}_1)}\right)\bar{\theta}_2 p_2 > 0\,.$$

It follows that

$$\mathsf{E}[\tilde{R}_m] = \pi_1\mathsf{E}[\tilde{R}_1] + \pi_2\mathsf{E}[\tilde{R}_2] = R_f + \lambda + \pi_2 A\,.$$

Hence,

$$\lambda = \mathsf{E}[\tilde{R}_m] - R_f - \pi_2 A < \mathsf{E}[\tilde{R}_m] - R_f\,.$$

6.9. Suppose there is no risk-free asset and the minimum-variance return is different from the constant-mimicking return, that is, $b_{\mathrm{m}} \neq b_{\mathrm{c}}$. From Section 6.2, we know there is a factor model with the constant-mimicking return as the factor:

$$\mathsf{E}[\tilde{R}] = R_z + \psi \operatorname{cov}(\tilde{R}, \tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p) \tag{6.36}$$

for every return $\tilde{R}$. From Section 6.2, we can conclude there is an SDF that is an affine function of the constant-mimicking return unless $R_z = 0$. However, the existence of an SDF that is an affine function of the constant-mimicking return would contradict the result of Section 5.5. So, it must be that $R_z = 0$ in (6.36). Calculate R_z to demonstrate this.

Solution: We have

$$\begin{aligned}\operatorname{cov}(\tilde{R}, \tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p) &= \mathsf{E}[\tilde{R}(\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p)] - \mathsf{E}[\tilde{R}]\mathsf{E}[\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p] \\ &= b_{\mathrm{c}}\mathsf{E}[\tilde{R}] - \mathsf{E}[\tilde{R}]\mathsf{E}[\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p]\,.\end{aligned}$$

Thus,

$$\mathsf{E}[\tilde{R}] = R_z + \psi \operatorname{cov}(\tilde{R}, \tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p)$$

implies

$$\mathsf{E}[\tilde{R}]\{1 - \psi b_{\mathrm{c}} + \psi\mathsf{E}[\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p]\} = R_z\,.$$

This can be true for all $\tilde{R}$ only if

$$1 - \psi b_{\mathrm{c}} + \psi\mathsf{E}[\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p] = 0 \quad \text{and} \quad R_z = 0\,.$$

Note that this implies

$$\psi = \frac{1}{b_{\mathrm{c}} - \mathsf{E}[\tilde{R}_p + b_{\mathrm{c}}\tilde{e}_p]}\,.$$

The denominator of this expression is nonzero because $b_{\mathrm{c}} \neq b_{\mathrm{m}}$.

6.10. Suppose there is no risk-free asset and the minimum-variance return is different from the constant-mimicking return, that is, $b_{\mathrm{m}} \neq b_{\mathrm{c}}$. From Section 5.5, we know that there is an SDF that is an affine function of the minimum-variance return:

$$\tilde{m} = \gamma + \beta(\tilde{R}_p + b_{\mathrm{m}}\tilde{e}_p) \tag{6.37}$$

for some γ and β. From Section 6.2, we know that there is no factor model with the minimum-variance return as the factor. However, because there is an SDF that is an affine function of the minimum-variance return, we also know from Section 6.2 that there *is* a factor model with the minimum-variance return as the factor unless $\mathsf{E}[\tilde{m}] = 0$. So it must be that $\mathsf{E}[\tilde{m}] = 0$ for the SDF $\tilde{m}$ in (6.37). Calculate $\mathsf{E}[\tilde{m}]$ to demonstrate this.

Solution: We have

$$\begin{aligned}
\mathsf{E}[\tilde{m}] &= \gamma + \beta\mathsf{E}[\tilde{R}_p] + \beta b_{\mathrm{m}}\mathsf{E}[\tilde{e}_p] \\
&= \gamma + \beta\mathsf{E}[\tilde{R}_p] + \frac{\beta\mathsf{E}[\tilde{R}_p]\mathsf{E}[\tilde{e}_p]}{1-\mathsf{E}[\tilde{e}_p]} \\
&= \frac{\gamma(1-\mathsf{E}[\tilde{e}_p]) + \beta\mathsf{E}[\tilde{R}_p]}{1-\mathsf{E}[\tilde{e}_p]}\,.
\end{aligned}$$

Because $\tilde{m}$ is an SDF and $\tilde{e}_p$ is an excess return,

$$\begin{aligned}
0 &= \mathsf{E}[\tilde{m}\tilde{e}_p] \\
&= \gamma\mathsf{E}[\tilde{e}_p] + \beta\mathsf{E}[\tilde{R}_p\tilde{e}_p] + \beta b_{\mathrm{m}}\mathsf{E}[\tilde{e}_p^2] \\
&= \gamma\mathsf{E}[\tilde{e}_p] + \beta b_{\mathrm{m}}\mathsf{E}[\tilde{e}_p] \\
&= \gamma\mathsf{E}[\tilde{e}_p] + \frac{\beta\mathsf{E}[\tilde{R}_p]\mathsf{E}[\tilde{e}_p]}{1-\mathsf{E}[\tilde{e}_p]} \\
&= \mathsf{E}[\tilde{e}_p]\frac{\gamma(1-\mathsf{E}[\tilde{e}_p]) + \beta\mathsf{E}[\tilde{R}_p]}{1-\mathsf{E}[\tilde{e}_p]} \\
&= \mathsf{E}[\tilde{e}_p]\mathsf{E}[\tilde{m}]\,,
\end{aligned}$$

using Facts 8 and 16 for the third equality. Because $\mathsf{E}[\tilde{e}_p] \neq 0$ (by Fact 17), this implies $\mathsf{E}[\tilde{m}] = 0$.

Chapter 7

Representative Investors

7.1. Assume there is a representative investor with quadratic utility $u(w) = -(\zeta - w)^2$. Assume $\mathsf{E}[\tilde{w}_m] \neq \zeta$. Show that λ in the CAPM (6.11) equals

$$\frac{\text{var}(\tilde{w}_m)}{\mathsf{E}[\tau(\tilde{w}_m)]}\,,$$

where $\tau(w)$ denotes the coefficient of risk tolerance of the representative investor at wealth level w. (Thus, the risk premium is higher when market wealth is riskier or when the representative investor is more risk averse.)

Solution: We have $u'(w) = 2(\zeta - w)$, $u''(w) = -2$, and $\tau(w) = -u'(w)/u''(w) = \zeta - w$. There is an SDF $\tilde{m} = \gamma u'(\tilde{w}_m) = 2\gamma(\zeta - \tilde{w}_m)$ for some γ, and $\mathsf{E}[\tilde{m}] \neq 0$ by virtue of the assumption $\mathsf{E}[\tilde{w}_m] \neq \zeta$. Thus,

$$\mathsf{E}[\tilde{R}] = \frac{1}{\mathsf{E}[\tilde{m}]} - \frac{1}{\mathsf{E}[\tilde{m}]}\,\text{cov}(\tilde{m}, \tilde{R})$$

for each return $\tilde{R}$. We have $\mathsf{E}[\tilde{m}] = 2\gamma\mathsf{E}[\tau(\tilde{w}_m)]$ and $\text{cov}(\tilde{m}, \tilde{R}) = -2\gamma\,\text{cov}(\tilde{w}_m, \tilde{R})$. Therefore,

$$\begin{aligned}
\mathsf{E}[\tilde{R}] &= \frac{1}{2\gamma\mathsf{E}[\tau(\tilde{w}_m)]} + \frac{1}{\mathsf{E}[\tau(\tilde{w}_m)]}\,\text{cov}(\tilde{m}, \tilde{R}) \\
&= \frac{1}{2\gamma\mathsf{E}[\tau(\tilde{w}_m)]} + \frac{\text{var}(\tilde{w}_m)}{\mathsf{E}[\tau(\tilde{w}_m)]}\,\frac{\text{cov}(\tilde{m}, \tilde{R})}{\text{var}(\tilde{w}_m)}\,.
\end{aligned}$$

Therefore, $\lambda = \text{var}(\tilde{w}_m)/\mathsf{E}[\tau(\tilde{w}_m)]$.

7.2. Assume there is a risk-free asset and a representative investor with power utility, so (7.15) is an SDF. Let $\tilde{z} = \log(\tilde{c}_1/c_0)$ and assume $\tilde{z}$ is normally distributed with mean μ and variance σ^2. Let κ denote the maximum Sharpe ratio of all portfolios.

(a) Use the Hansen-Jagannathan bound (3.35) to show that

$$\rho \geq \frac{\sqrt{\log(1+\kappa^2)}}{\sigma} \Leftrightarrow \kappa \leq \sqrt{e^{\rho^2\sigma^2} - 1}\,. \tag{7.36}$$

Hint: Apply the result of Exercise 1.7. Note that (7.36) implies risk aversion must be larger if consumption volatility is smaller or the maximum Sharpe ratio is larger. Also, using the approximation $\log(1+x) \approx x$, the lower bound on ρ in (7.36) is approximately κ/σ, and, using the approximation $e^x \approx 1 + x$, the upper bound on κ is approximately $\rho\sigma$.

Solution: We have

$$\frac{\text{stdev}(\tilde{m})}{\mathsf{E}[\tilde{m}]} = \frac{\text{stdev}\left(\mathrm{e}^{-\rho\tilde{z}}\right)}{\mathsf{E}\left[\mathrm{e}^{-\rho\tilde{z}}\right]} = \sqrt{\mathrm{e}^{\rho^2\sigma^2} - 1}\,,$$

using the result of Exercise 1.7. Thus, the Hansen-Jagannathan bound (3.35) implies

$$\sqrt{\mathrm{e}^{\rho^2\sigma^2} - 1} \geq \kappa\,,$$

which implies

$$\mathrm{e}^{\rho^2\sigma^2} \geq 1 + \kappa^2\,,$$

and

$$\rho \geq \frac{\sqrt{\log(1+\kappa^2)}}{\sigma}\,.$$

(b) Explain why the weak inequalities in (7.36) must be equalities when the market is complete.

Solution: When the market is complete, there is a unique SDF and it is the SDF with the minimum standard deviation. For this SDF the Hansen-Jagannathan bound holds with equality.

(c) In the data analyzed by Mehra and Prescott (1985), the standard deviation of the market return is 16.54%. Derive a lower bound on the representative investor's risk aversion ρ by using this and the other Mehra-Prescott data in Section 7.3 in conjunction with (7.36).

Solution: An equity premium of 6.18% and a standard deviation of 16.54% imply a Sharpe ratio of 0.374. Using $\sigma^2 = 0.00125$, as stated in the text, we have

$$\rho \geq \sqrt{\frac{\log(1+0.374^2)}{0.00125}} = 10.23\,.$$

(d) Use $\delta = 0.99$ and $\rho = 10$ and the Mehra-Prescott data on the mean and standard deviation of consumption growth to compute the standard deviation of the theoretical market return (7.17). Note: This is a very simple illustration of the excess volatility puzzle.

Solution: It is calculated in Section 7.3 that the mean and standard deviation of consumption growth reported by Mehra and Prescott combined with the assumption of lognormality imply that $\log(\tilde{c}_1/c_0) = \mu + \sigma\tilde{\xi}$, where $\mu = 0.017$, $\sigma^2 = 0.00125$, and $\tilde{\xi}$ is a standard normal random variable. From (7.17), the definition of ν just above (7.17), and the Mehra-Prescott estimate that $\text{stdev}(\tilde{c}_1/c_0) = 0.036$, we have

$$\begin{aligned}
\text{var}(\tilde{R}_m) &= \nu^{-2}\,\text{var}(\tilde{c}_1/c_0) \\
&= \delta^{-2}\mathsf{E}[\mathrm{e}^{(1-\rho)\mu+(1-\rho)\sigma\tilde{\xi}}]^{-2}(0.036)^2 \\
&= (0.99)^{-2}\mathrm{e}^{-2(1-\rho)\mu-(1-\rho)^2\sigma^2}(0.036)^2 \\
&= (0.99)^{-2}\mathrm{e}^{18\times 0.017-81\times 0.00125}(0.036)^2 \\
&= 0.001622768\,.
\end{aligned}$$

This implies that the standard deviation of the market return is $\sqrt{0.001622768} = 0.04 = 4\%$. Note that this is considerably smaller than the estimated standard deviation of 16.54%.

7.3. Assume there is a risk-free asset, and let $\tilde{m}$ be an SDF.

(a) Show that each return $\tilde{R}$ satisfies

$$\mathsf{E}[\tilde{R}] - R_f = \frac{\text{var}^*(\tilde{R})}{R_f} - \text{cov}(\tilde{m}\tilde{R}, \tilde{R}),$$

where var* denotes variance under the risk-neutral probability corresponding to $\tilde{m}$.

Solution: By definition, we have

$$\text{var}^*(\tilde{R}) \;=\; \mathsf{E}^*[\tilde{R}^2] - \mathsf{E}^*[\tilde{R}]^2 \;=\; R_f \mathsf{E}[\tilde{m}\tilde{R}^2] - R_f^2$$

and

$$\text{cov}(\tilde{m}\tilde{R}, \tilde{R}) \;=\; \mathsf{E}[\tilde{m}\tilde{R}^2] - \mathsf{E}[\tilde{m}\tilde{R}]\mathsf{E}[\tilde{R}] \;=\; \mathsf{E}[\tilde{m}\tilde{R}^2] - \mathsf{E}[\tilde{R}]\,.$$

Therefore,

$$\frac{\text{var}^*(\tilde{R})}{R_f} - \text{cov}(\tilde{m}\tilde{R}, \tilde{R}) \;=\; \mathsf{E}[\tilde{m}\tilde{R}^2] - R_f - \mathsf{E}[\tilde{m}\tilde{R}^2] + \mathsf{E}[\tilde{R}] \;=\; \mathsf{E}[\tilde{R}] - R_f\,.$$

(b) Assume there is a representative investor with constant relative risk aversion ρ, so $\gamma u'(\tilde{R}_m) = \gamma \tilde{R}_m^{-\rho}$ is an SDF for some constant γ. Show that $u'(x)x$ is a decreasing function of x if and only if $\rho > 1$.

Solution: $u'(x) = x^{-\rho} \Rightarrow xu'(x) = x^{1-\rho}$

then

$$\frac{d}{dx}[xu'(x)] = \frac{d}{dx}[x^{1-\rho}] = (1-\rho)x^{-\rho} < 0 \;\Leftrightarrow\; \rho > 1\,.$$

(c) If $f(x)$ is a decreasing function of x, then $\text{cov}(f(\tilde{x}), \tilde{x}) < 0$ for any random variable $\tilde{x}$. Using this fact and the above results, explain why

$$\mathsf{E}[\tilde{R}_m] - R_f \geq \frac{\text{var}^*(\tilde{R}_m)}{R_f}$$

when $\rho > 1$.

Solution: Let $x = \tilde{R}$ and $f(x) = \tilde{m}\tilde{R}$. From part (b) we know that if $\rho > 1$ then $f(x)$ is a decreasing function. Therefore, using the fact in part (c) we have

$$\rho > 1 \Rightarrow \text{cov}(\tilde{m}\tilde{R}, \tilde{R}) < 0 \Rightarrow \mathsf{E}[\tilde{R}] - R_f > \frac{\text{var}^*(\tilde{R})}{R_f}$$

7.4. Assume there is a representative investor with constant relative aversion ρ. Assume there is a risk-free asset and the market is complete. Use the fact that $\tilde{R}_p$ and R_f span the mean-variance frontier to show that each mean-variance efficient return is of the form $a - b\tilde{R}_m^{-\rho}$ for $b > 0$.

Solution: It is shown in Section 5.4—see (5.26)—that each efficient return is of the form

$$(1+\delta)R_f - \delta\tilde{R}_p$$

for $\delta > 0$ (note $\delta = -\lambda$ in (5.26)). Because the market is complete, $\tilde{m}_p$ is the unique SDF, and the representative investor's marginal utility is proportional to an SDF, so $\tilde{m}_p$ is proportional to the representative investor's marginal utility $\tilde{R}_m^{-\rho}$. Furthermore, $\tilde{R}_p$ is proportional to $\tilde{m}_p$. Hence, we have

$$\delta\tilde{R}_p = b\tilde{R}_m^{-\rho}$$

for some b. This completes the proof.

7.5. Assume there is a representative investor with utility function u. The first-order condition

$$\mathsf{E}[u'(\tilde{R}_m)(\tilde{R}_1 - \tilde{R}_2)] = 0$$

must hold for all return $\tilde{R}_1$ and $\tilde{R}_2$. Assume there is a risk-free asset. Consider any return $\tilde{R}$. By orthogonal projection, we have

$$\tilde{R} - R_f = \alpha + \beta(\tilde{R}_m - R_f) + \tilde{\varepsilon}\,.$$

for α and β, where $\mathsf{E}[\tilde{\varepsilon}] = \mathsf{E}[\tilde{R}_m\tilde{\varepsilon}] = 0$.

(a) Use the first-order condition in conjunction with the returns $\tilde{R}_m$ and

$$\tilde{R}_* = \tilde{R} + (1-\beta)(\tilde{R_m} - R_f) = \tilde{R}_m + \alpha + \tilde{\varepsilon}\,.$$

to show that

$$\alpha = -\frac{\mathsf{E}[u'(\tilde{R}_m)\tilde{\varepsilon}]}{\mathsf{E}[u'(\tilde{R}_m)]}$$

Solution: Use the first-order condition with $\tilde{R}_1 = \tilde{R}_*$ and $\tilde{R}_2 = \tilde{R}_m$. We have

$$\mathsf{E}[u'(\tilde{R}_m)(\alpha + \tilde{\varepsilon})] = 0\,.$$

Hence,

$$\alpha\mathsf{E}[u'(\tilde{R}_m)] + \mathsf{E}[u'(\tilde{R}_m)\tilde{\varepsilon}] = 0\,,$$

which implies the claim.

(b) Use the results of the previous part to derive the CAPM when there is a representative investor and the residual $\tilde{\varepsilon}$ of each asset return is mean independent of the market return (which is true, for example, when returns are jointly elliptically distributed; Chu, 1973).

Solution: By iterated expectations and the mean-independence assumption,

$$\begin{aligned}\mathsf{E}[u'(\tilde{R}_m)\tilde{\varepsilon}] &= \mathsf{E}\left[\mathsf{E}[u'(\tilde{R}_m)\tilde{\varepsilon} \mid \tilde{R}_m]\right]\\ &= \mathsf{E}\left[u'(\tilde{R}_m)\mathsf{E}[\tilde{\varepsilon} \mid \tilde{R}_m]\right] = 0\,.\end{aligned}$$

Hence, the result of part (a) implies $\alpha = 0$. Because this is true for each return, the CAPM holds.

7.6. Assume in (7.16) that $\log \tilde{R}$ and $\log(\tilde{c}_1/c_0)$ are joint normally distributed. Specifically, let $\log \tilde{R} = \tilde{y}$ and $\log(\tilde{c}_1/c_0) = \tilde{z}$ with $\mathsf{E}[\tilde{y}] = \mu_y$, $\text{var}(\tilde{y}) = \sigma_y^2$, $\mathsf{E}[\tilde{z}] = \mu$, $\text{var}(\tilde{z}) = \sigma^2$, and $\text{corr}(\tilde{y}, \tilde{z}) = \gamma$.

Note: $\gamma\sigma\sigma_y$ is the covariance of the continuously compounded rate of return $\tilde{y}$ with the continuously compounded consumption growth rate $\tilde{z}$, so (7.1) has the usual form

$$\text{Expected Return} = \text{Risk-Free Return} + \psi \times \text{Covariance}\,,$$

with $\psi = \rho$, except for the extra term $-\sigma_y^2/2$. The extra term, which involves the total and hence idiosyncratic risk of the return, is usually called a Jensen's inequality term, because it arises from the fact that $\mathsf{E}[\mathrm{e}^{\tilde{y}}] = \mathrm{e}^{\mu_y+\sigma_y^2/2} > \mathrm{e}^{\mu_y}$.

(a) Show that

$$\mu = -\log\delta + \rho\gamma\sigma\sigma_y + \rho\mu - \frac{1}{2}\rho^2\sigma^2 - \frac{1}{2}\sigma_y^2\,.$$

Solution: (7.16) states that

$$\mathsf{E}[\tilde{R}] = \frac{c_0^{-\rho}}{\delta\mathsf{E}\left[\tilde{c}_1^{-\rho}\right]} - \frac{1}{\mathsf{E}\left[\tilde{c}_1^{-\rho}\right]}\operatorname{cov}\left(\tilde{R}, \tilde{c}_1^{-\rho}\right)\,.$$

We have $\mathsf{E}[\tilde{R}] = \mathrm{e}^{\mu+\sigma^2/2}$. Also, $c_1^{-\rho} = c_0^{-\rho}\mathrm{e}^{-\rho\tilde{z}}$, so $\mathsf{E}\left[\tilde{c}_1^{-\rho}\right] = c_0^{-\rho}\mathrm{e}^{-\rho\mu_c+\rho^2\sigma_c^2/2}$.

Hence,

$$\mathrm{e}^{\mu+\sigma^2/2} = \frac{1}{\delta}\mathrm{e}^{\rho\mu_c-\rho^2\sigma_c^2/2} - \frac{1}{\mathsf{E}\left[\tilde{c}_1^{-\rho}\right]}\operatorname{cov}\left(\tilde{R}, \tilde{c}_1^{-\rho}\right)\,.$$

Substituting $\operatorname{cov}\left(\tilde{R}, \tilde{c}_1^{-\rho}\right) = \mathsf{E}\left[\tilde{R}\tilde{c}_1^{-\rho}\right] - \mathsf{E}\left[\tilde{R}\right]\mathsf{E}\left[\tilde{c}_1^{-\rho}\right]$ yields

$$\mathrm{e}^{\mu+\sigma^2/2} = \frac{1}{\delta}\mathrm{e}^{\rho\mu_c-\rho^2\sigma_c^2/2} - \frac{\mathsf{E}\left[\tilde{R}\tilde{c}_1^{-\rho}\right]}{\mathsf{E}\left[\tilde{c}_1^{-\rho}\right]} + \mathrm{e}^{\mu+\sigma^2/2}\,.$$

Also,

$$\begin{aligned}
\frac{\mathsf{E}\left[\tilde{R}\tilde{c}_1^{-\rho}\right]}{\mathsf{E}\left[\tilde{c}_1^{-\rho}\right]} &= \frac{\mathsf{E}\left[\mathrm{e}^{\tilde{y}-\rho\tilde{z}}\right]}{\mathsf{E}\left[\mathrm{e}^{-\rho\tilde{z}}\right]} \\
&= \frac{\exp\left(\mu - \rho\mu_c + \frac{1}{2}(\sigma^2 - 2\rho\gamma\sigma\sigma_c + \rho^2\sigma_c^2)\right)}{\exp\left(-\rho\mu_c + \frac{1}{2}\rho^2\sigma_c^2\right)} \\
&= \mathrm{e}^{\mu-\rho\gamma\sigma\sigma_c+\sigma^2/2}\,.
\end{aligned}$$

Thus,

$$\mathrm{e}^{\mu-\rho\gamma\sigma\sigma_c+\sigma^2/2} = \frac{1}{\delta}\mathrm{e}^{\rho\mu_c-\rho^2\sigma_c^2/2},$$

which implies

$$\mu = -\log\delta + \rho\gamma\sigma\sigma_c + \rho\mu_c - \frac{1}{2}\rho^2\sigma_c^2 - \frac{1}{2}\sigma^2.$$

(b) Let $r = \log R_f$ denote the continuously compounded risk-free rate. Using (7.22'), show that

$$\mu = r + \rho\gamma\sigma\sigma_y - \frac{1}{2}\sigma_y^2. \tag{7.1}$$

Solution: From

$$R_f = \frac{1}{\mathsf{E}[\tilde{m}]} = \frac{c_0^{-\rho}}{\delta\mathsf{E}\left[\tilde{c}_1^{-\rho}\right]},$$

and $\mathsf{E}\left[\tilde{c}_1^{-\rho}\right] = c_0^{-\rho}\mathrm{e}^{-\rho\mu_c+\rho^2\sigma_c^2/2}$, we obtain $R_f = \mathrm{e}^{\rho\mu_c-\rho^2\sigma_c^2/2}/\delta$. Hence,

$$r = -\log\delta + \rho\mu_c - \frac{1}{2}\rho^2\sigma_c^2.$$

Substituting r in the formula for μ obtained in part (a) yields $\mu = r + \rho\gamma\sigma\sigma_c - \sigma^2/2$.

7.7. Show that if u_{h0} and u_{h1} are concave for each h, then the social planner's utility functions u_0 and u_1 are concave.

Solution: Consider $\hat{u}_0$. The argument is identical for $\hat{u}_1$. For convenience, drop the subscript 0 from $\hat{u}_0$ and u_{h0}. Consider two aggregate consumption levels c_1 and c_2. Let $\{c_{h1} \mid h = 1,\ldots,H\}$ and $\{c_{h2} \mid h = 1,\ldots,H\}$ satisfy $\sum_{h=1}^H c_{h1} = c_1$ and $\sum_{h=1}^H c_{h2} = c_2$. Let $0 < \delta < 1$. We have

$$\begin{aligned}\hat{u}(\delta c_1 + (1-\delta)c_2) &\geq \sum_{h=1}^H \lambda_h u_h(\delta c_{h1} + (1-\delta)c_{h2}) \\ &\geq \delta\sum_{h=1}^H \lambda_h u_h(c_{h1}) + (1-\delta)\sum_{h=1}^H \lambda_h u_h(c_{h1}),\end{aligned}$$