Chapter 1: Utility and Risk Aversion

Kerry Back
BUSI 521/ECON 505
Rice University

Utility Functions

Utility and Risk Aversion

- Expected utility $\mathsf{E}[u(\tilde{w})]$
  - Utility function u is unique up to monotone affine transform: $f(w) = a + bu(w)$ for $b > 0$.
- Risk aversion: $\mathsf{E}[\tilde{\varepsilon}] = 0 \Rightarrow \mathsf{E}[u(w + \tilde{\varepsilon})] < \mathsf{E}[u(w)]$.
  - Equivalent to concavity (Jensen's inequality)
  - Equivalent to decreasing marginal utility: $u'' \leq 0$.
  - Invariant under monotone affine transformations.

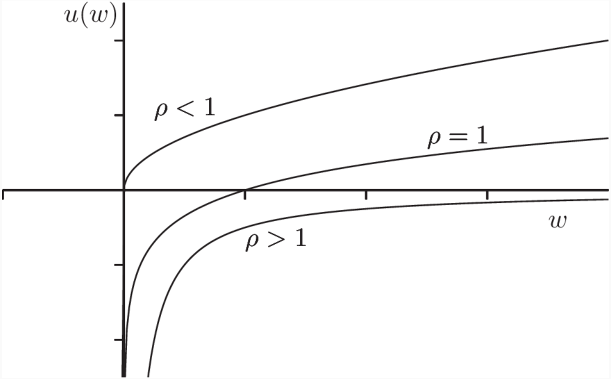
Coefficients of Risk Aversion

- Absolute risk aversion: $\alpha(w) = -u''(w)/u'(w)$
- Risk tolerance: $\tau(w) = 1/\alpha(w)$
- Relative risk aversion: $\rho(w) = w\alpha(w) = -wu''(w)/u'(w)$.

Some Special Utility Functions

- CARA = Constant Absolute Risk Aversion
  - $u(w) = -\mathrm{e}^{-\alpha w}$
  - α is the coefficient of absolute risk aversion.
- CRRA = Constant Relative Risk Aversion
  - $u(w) = \log w \Rightarrow$ relative risk aversion $= 1$
  - $u(w) = w^{1-\rho}/(1-\rho) \Rightarrow$ relative risk aversion $= \rho$

CRRA Utility Functions



Proof that CARA = $-\mathrm{e}^{-\alpha w}$

The coefficient of absolute risk aversion is

$$-\frac{u''(w)}{u'(w)} = -\frac{\mathrm{d}\log u'(w)}{\mathrm{d}w}$$

CARA means this is constant. Call it α. Then $\log u'$ is an affine (constant plus linear) function of w with slope $-\alpha$. This means

$$\begin{aligned}\log u'(w) = \log u'(0) - \alpha w &\Rightarrow u'(w) = u'(0)\mathrm{e}^{-\alpha w} \\ &\Rightarrow u(w) = u(0) + u'(0)\int_0^w \mathrm{e}^{-\alpha x}\,\mathrm{d}x \\ &\Rightarrow u(w) = u(0) - \frac{u'(0)}{\alpha}\left(\mathrm{e}^{-\alpha w} - 1\right)\end{aligned}$$

Linear Risk Tolerance

- CARA risk tolerance $= 1/\alpha$
- CRRA absolute risk aversion $= \rho/w$. Risk tolerance $= w/\rho$
- Linear risk tolerance: $\tau(w) = A + Bw$.
  - For CARA, $A = 1/\alpha$, $B = 0$
  - For CRRA, $A = 0$, $B = 1/\rho$
  - In general, B is called the cautiousness parameter.

LRT Utility Functions with $B > 0$

- CARA
- CRRA
- Shifted CRRA
  - Shifted log: $u(w) = \log(w - \zeta)$
  - Shifted power: $u(w) = \frac{1}{1-\rho}(w - \zeta)^{1-\rho}$

Quadratic Utility

$$u(w) = -\frac{1}{2}\left(\zeta - w\right)^2$$

- Quadratic utility is monotone increasing for $w < \zeta$ (ζ is bliss point).
- Risk tolerance: $\tau(w) = \zeta - w$, so $A = \zeta$ and $B = -1$.
- Implies mean-variance preferences:

$$\mathrm{E}[u(\tilde{w})] \ \sim \ \zeta\overline{w} - \frac{1}{2}\overline{w}^2 - \frac{1}{2}\,\mathrm{var}(\widetilde{w})$$

Decreasing Absolute Risk Aversion

- An investor has DARA utility if absolute risk aversion $\alpha(w)$ is a decreasing function of w.
- CRRA utility is DARA: $\alpha(w) = \rho/w$ for a constant ρ (relative risk aversion).
- LRT utility with $B > 0$ is DARA.
- Quadratic utility is not DARA.

Certainty Equivalents

Certainty Equivalents

- A constant x is the certainty equivalent of a random $\tilde{w}$ if $u(x) = \mathsf{E}[u(\tilde{w})]$.
- Risk aversion implies $x < \mathsf{E}[\tilde{w}]$.
- Certainty equivalents are invariant under monotone affine transformations.

CARA Utility and Normal Gambles

- CARA investor with a normally distributed wealth having mean w. How much less than w is the certainty equivalent?
- Equivalent question: CARA investor with initial wealth w facing a zero-mean normally distributed gamble. How much would she pay to avoid the gamble?
- Answer: would pay $\alpha\sigma^2/2$ where σ^2 is the variance of the gamble.
- Higher risk aversion $\Rightarrow$ pay more. Higher risk $\Rightarrow$ pay more.

Proof

- Call the gamble $w + \sigma\tilde{\varepsilon}$ where $\tilde{\varepsilon}$ is a standard normal.
- Call the certainty equivalent x. Define $\pi = w - x$, so the certainty equivalent is $w - \pi$.
- Claim is that $\pi = \alpha\sigma^2/2$.

Proof

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- Call the certainty equivalent x. Define $\pi = w - x$, so the certainty equivalent is $w - \pi$.
- Claim is that $\pi = \alpha\sigma^2/2$.
- The definition of the certainty equivalent and negative exponential (CARA) utility tell us that π satisfies

$$\begin{aligned} & u(w-\pi) = \mathsf{E}\left[u(w+\sigma\tilde{\varepsilon})\right] \\ \Leftrightarrow \qquad & -\mathrm{e}^{-\alpha(w-\pi)} = \mathsf{E}\left[-\mathrm{e}^{-\alpha(w+\sigma\tilde{\varepsilon})}\right] \end{aligned}$$

- Use the following fact: if $\tilde{x}$ is normal (μ, σ), then

$$\mathsf{E}\left[\mathrm{e}^{\tilde{x}}\right] = \mathrm{e}^{\mu+\sigma^2/2}$$

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- Solution: $\pi = \alpha\sigma^2/2$.

Small Gambles

- Drop the CARA assumption and drop the normal distribution.
- How much would an investor pay to avoid a gamble?
- We can give a limiting result as the size of the gamble goes to zero: for small gambles, the amount an investor would pay is approximately $\alpha\sigma^2/2$.
- More precisely: Fix w. Let $\tilde{\varepsilon}$ be a bounded zero-mean random variable with unit variance. For $\sigma > 0$, define $\pi(\sigma)$ by

$$u(w - \pi(\sigma)) = \mathrm{E}[u(w + \sigma\tilde{\varepsilon})]\,.$$

Then,

$$\lim_{\sigma \to 0} \frac{\pi(\sigma)}{\sigma^2} = \frac{1}{2}\alpha(w)$$

Proof

Take an exact Taylor series approximation:

$$\pi(\sigma) = \pi(0) + \pi'(0)\sigma + \frac{1}{2}\pi''(x_\sigma)\sigma^2$$

for $0 < x_\sigma < \sigma$ Clearly, $\pi(0) = 0$. Differentiate both sides of

$$u(w - \pi(\sigma)) = \mathsf{E}[u(w + \sigma\tilde{\varepsilon})] \qquad (\star)$$

and evaluate at $\sigma = 0$ to obtain

$$-u'(w)\pi'(0) = \mathsf{E}[u'(w)\tilde{\varepsilon}] = u'(w)\mathsf{E}[\tilde{\varepsilon}] = 0$$

Hence,

$$\frac{\pi(\sigma)}{\sigma^2} = \frac{1}{2}\pi''(x_\sigma) \to \frac{1}{2}\pi''(0)$$

Differentiate both sides of $(\star)$ twice and evaluate at $\sigma = 0$ to obtain $\pi''(0) = \alpha(w)$.

Relative Risk Aversion

- Instead of a gambles in dollars, let's make it a percent of wealth. Investor's wealth is $w + w\sigma\tilde{\varepsilon}$ where $\tilde{\varepsilon}$ is a bounded zero-mean unit-variance random variable as before. Relative risk aversion: $\rho(w) = w\alpha(w)$.
- Write the certainty equivalent as $w - w\gamma(\sigma)$. So, $\gamma := \pi/w$ is the amount she would pay relative to wealth to avoid the gamble.
- The variance of the gamble is now $w^2\sigma^2$, so, for small σ, $\pi \approx \alpha w^2\sigma^2/2$.
- This implies $\gamma \approx \alpha w\sigma^2 = \rho\sigma^2$, where ρ is relative risk aversion..