Chapter 5: Mean-Variance Analysis

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Standard Deviation – Mean Plots

Notation

- n risky assets with returns $\widetilde{R}_i$. $\widetilde{\mathbf{R}} = (\widetilde{R}_1 \cdots \widetilde{R}_n)'$
- $\mu =$ vector of expected returns. At least two of the assets have different expected returns.
- $\Sigma =$ covariance matrix. Assume no redundant assets, so Σ is positive definite.
- $\iota =$ n–vector of 1's.
- $\pi \in \mathbb{R}^n$ is a portfolio (of risky assets). If the portfolio is fully invested in risky assets, then $\iota'\pi = 1$. Otherwise, $1 - \iota'\pi$ is the fraction of wealth invested in the risk-free asset.

Portfolio Mean and Standard Deviation

- Two assets with expected returns μ_i, standard deviations σ_i, and correlation ρ.
- Portfolio (π_1, π_2) with $\pi_1 + \pi_2 = 1$.
- Portfolio return is

$$\pi' \widetilde{\mathbf{R}} \;=\; \pi_1 \tilde{R}_1 + \pi_2 \tilde{R}_2$$

- Portfolio expected return is

$$\pi' \mu \;=\; \pi_1 \mu_1 + \pi_2 \mu_2$$

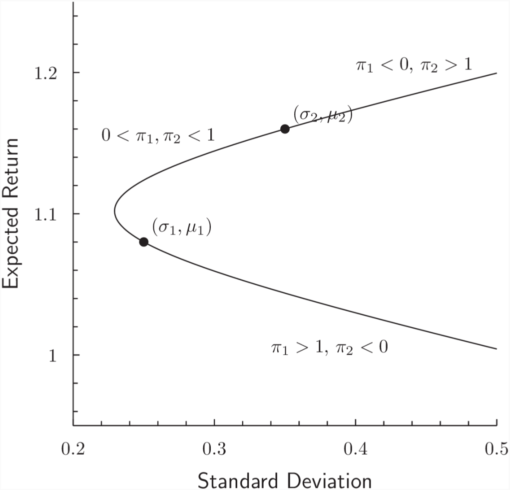
- Write the covariance matrix as

$$\Sigma \;=\; \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

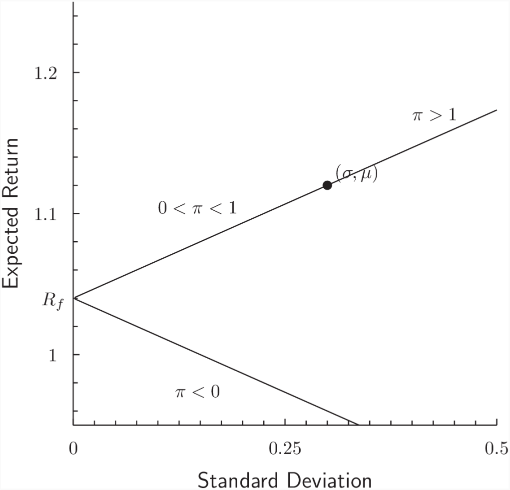
- The portfolio variance is

$$\pi' \Sigma \pi \;=\; \pi_1^2\sigma_1^2 + \pi_2^2\sigma_2^2 + 2\pi_1\pi_2\rho\sigma_1\sigma_2$$

Portfolios of Two Risky Assets



Portfolios of a Risky and Risk-Free Asset



GMV Portfolio

Global Minimum Variance Portfolio

- The portfolio of risky assets with minimum variance is called the Global Minimum Variance (GMV) portfolio.
- It solves the optimization problem

$$\min \quad \frac{1}{2}\pi'\Sigma\pi \quad \text{subject to} \quad \iota'\pi = 1$$

- The Lagrangean for this problem is

$$\frac{1}{2}\pi'\Sigma\pi - \gamma(\iota'\pi - 1)$$

- The FOC is

$$\Sigma\pi = \gamma\iota \quad \Leftrightarrow \quad \pi = \gamma\Sigma^{-1}\iota$$

- Impose the constraint $\iota'\pi = 1$ and solve for γ to obtain

$$\pi = \frac{1}{\iota'\Sigma^{-1}\iota}\Sigma^{-1}\iota$$

- In other words, take the vector $\Sigma^{-1}\iota$ and divide by the sum of its elements, so the rescaled vector sums to 1.

Frontier Portfolios

Mean-Variance Frontier of Risky Assets

- We continue to look at only risky assets so we continue to require portfolio weights to sum to 1 ($\iota'\pi = 1$).
- A frontier portfolio is a portfolio that achieves a target expected return with minimum risk.
- It solves an optimization problem

$$\min \quad \frac{1}{2}\pi'\Sigma\pi \quad \text{subject to} \quad \mu'\pi = \mu_{\text{targ}} \quad \text{and} \quad \iota'\pi = 1$$

  where μ_{targ} denotes the given target expected return.
- By varying the target expected return, we trace out the frontier.

Solving for Frontier Portfolios

- The Lagrangean for the optimization problem is

$$\frac{1}{2}\pi'\Sigma\pi - \delta(\mu'\pi - \mu_{\text{targ}}) - \gamma(\iota'\pi - 1)$$

- The FOC is

$$\Sigma\pi - \delta\mu - \gamma\iota \;=\; 0\,.$$

- The solution is

$$\pi \;=\; \delta\Sigma^{-1}\mu + \gamma\Sigma^{-1}\iota$$

- This means that π is a linear combination of the two vectors $\Sigma^{-1}\mu$ and $\Sigma^{-1}\iota$.
- Use constraints to solve for δ and γ.

More Notation

- Denote the GMV portfolio by π_{gmv}. It is $\Sigma^{-1}\iota$ rescaled to sum to 1:

$$\pi_{\text{gmv}} = \frac{1}{\iota\Sigma^{-1}\iota}\Sigma^{-1}\iota$$

- Let's also rescale the vector $\Sigma^{-1}\mu$ to sum to 1 and call it π_μ:

$$\pi_\mu = \frac{1}{\iota\Sigma^{-1}\mu}\Sigma^{-1}\mu$$

- To simplify, define $A = \mu'\Sigma^{-1}\mu$, $B = \mu'\Sigma^{-1}\iota$, and $C = \iota'\Sigma^{-1}\iota$.
- Then,

$$\begin{aligned}\pi_{\text{gmv}} &= \frac{1}{C}\Sigma^{-1}\iota \\ \pi_\mu &= \frac{1}{B}\Sigma^{-1}\mu\end{aligned}$$

Solution of Frontier Portfolio

- We saw that a frontier portfolio is

$$\pi = \delta\Sigma^{-1}\mu + \gamma\Sigma^{-1}\iota$$

  for some δ and γ.

- We can write this as

$$\begin{aligned}\pi &= \delta B\frac{1}{B}\Sigma^{-1}\mu + \gamma C\frac{1}{C}\Sigma^{-1}\mu \\ &= \delta B\pi_\mu + \gamma C\pi_{\text{gmv}}\end{aligned}$$

- The constraint $\iota'\pi = 1$ implies

$$\delta B + \gamma C = 1$$

- Set $\lambda = \delta B$. Then, $\gamma C = 1 - \lambda$, so the frontier portfolio is

$$\pi = \lambda\pi_\mu + (1-\lambda)\pi_{\text{gmv}}$$

- To find the particular frontier portfolio meeting the target return constraint, we can calculate

$$\mu'\pi = \lambda\mu'\pi_\mu + (1-\lambda)\mu'\pi_{\mathsf{gmv}} \;=\; \lambda\frac{A}{B} + (1-\lambda)\frac{B}{C}$$

- Set this equal to μ_{targ} to obtain

$$\lambda \;=\; \frac{\mu_{\mathsf{targ}} - B/C}{A/B - B/C} \;=\; \frac{BC\mu_{\mathsf{targ}} - B^2}{AC - B^2}$$

Two Fund Spanning

Two Fund Spanning

- The characterization $\pi = \lambda\pi_\mu + (1-\lambda)\pi_{\text{gmv}}$ means that π lies on the line through π_μ and π_{gmv} in $\mathbb{R}^n$.
- Every frontier portfolio is a combination of π_μ and π_{gmv}. We say that these two portfolios span the frontier.
- We can consider the portfolios to be funds – like mutual funds. If you want a frontier portfolio, you can just invest in these two funds. We call this two-fund spanning.
- Any other two points on the line also span the line. So, any two frontier portfolios can serve as the funds.

Risk-Free Asset

Mean-Variance Frontier with a Risk-Free Asset

- Now, we add a risk-free asset. We continue to let $\pi \in \mathbb{R}^n$ denote the portfolio of risky assets.
- We no longer require $\iota'\pi = 1$. The weight on the risk-free asset is $1 - \iota'\pi$. This can be negative (borrowing).
- A portfolio's expected return is

$$(1 - \iota'\pi)R_f + \mu'\pi \;=\; R_f + (\mu - R_f\iota)'\pi$$

- A frontier portfolio solves the following for some μ_{targ}:

$$\min \quad \frac{1}{2}\pi'\Sigma\pi \quad \text{subject to} \quad R_f + (\mu - R_f\iota)'\pi = \mu_{\text{targ}}$$

- FOC is

$$\Sigma\pi - \delta(\mu - R_f\iota) = 0 \quad \Leftrightarrow \quad \pi = \delta\Sigma^{-1}(\mu - R_f\iota)$$

- So, all frontier portfolios are scalar multiples of the vector $\Sigma^{-1}(\mu - R_f\iota)$.
- In other words, the frontier portfolios form a line through the origin and the vector $\Sigma^{-1}(\mu - R_f\iota)$.

Tangency Portfolio

- We can probably divide the vector $\Sigma^{-1}(\mu - R_f \iota)$ by the sum of its elements to form a portfolio of purely risky assets (satisfying $\iota'\pi = 1$).
- We can do that as long as the sum is nonzero. That is, we need

$$\iota'\Sigma^{-1}(\mu - R_f \iota) \neq 0$$

- This expression is $B - R_f C$. It is nonzero if and only if $B/C \neq R_f$.

- The term B/C is the expected return of the GMV portfolio:

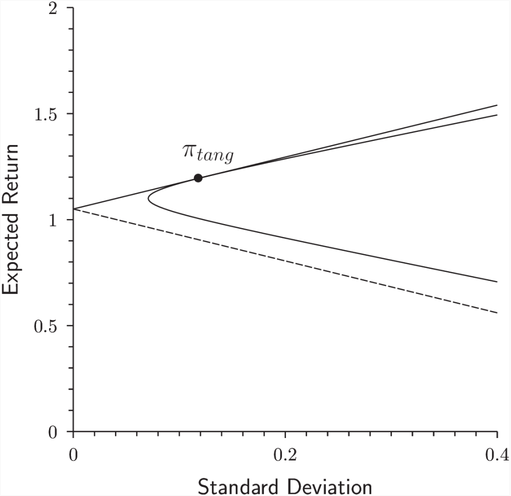
$$\mu'\pi_{\text{gmv}} \;=\; \mu'\left(\frac{1}{\iota'\Sigma^{-1}\iota}\Sigma^{-1}\iota\right) \;=\; \frac{1}{\iota'\Sigma^{-1}\iota}\mu'\Sigma^{-1}\iota \;=\; \frac{B}{C}$$

- So, when the expected return of the GMV portfolio is different from R_f, we can define

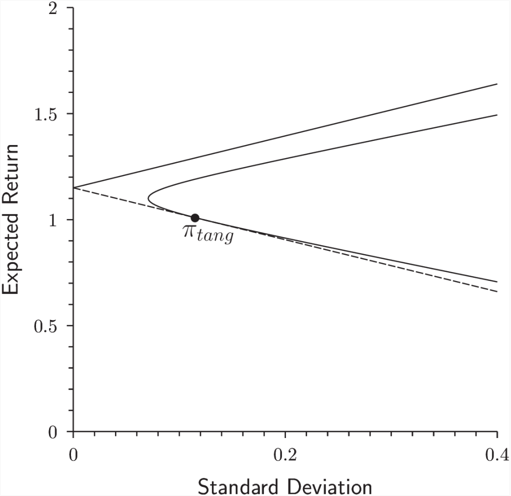
$$\pi_{\text{tang}} \;=\; \frac{1}{\iota'\Sigma^{-1}(\mu - R_f\,\iota)}\Sigma^{-1}(\mu - R_f\,\iota)$$

- We call this the tangency portfolio because it is on two frontiers: the frontier including the risk-free asset and the frontier of only risky assets.
- How do we know it is on the frontier of only risky assets?
    1. It is a portfolio constructed from the two vectors $\Sigma^{-1}\mu$ and $\Sigma^{-1}\iota$.
    2. Also, anything that solves a less constrained optimization problem (not requiring $\iota'\pi = 1$) and satisfies the constraints of a more constrained problem (satisfies $\iota'\pi = 1$ anyway) must solve the more constrained problem too.
- Thus, the two frontiers (in std dev/mean space) must be tangent at this point.

Mean-Variance Frontier: $B/C > R_f$



Mean-Variance Frontier: $B/C < R_f$



What if $B/C = R_f$?

- If $B/C = R_f$, then
  - The weights of each frontier portfolio sum to zero.
  - This means investing 100% in the risk-free asset and then go long and short equal dollars worth of risky assets.
- The cone and hyperbola never touch.

Two-Fund Spanning Again

Two Fund Spanning with a Risk-Free Asset

- All frontier portfolios lie on the line through the origin and the vector $\Sigma^{-1}(\mu - R_f\iota)$ in $\mathbb{R}^n$.
- Any vector on the line is a portfolio, because we are not requiring $\iota'\pi = 1$.
- The origin represents 100% in the risk-free asset.
- Any two portfolios on the line span the frontier in the sense that any frontier portfolio is a combination λ and $(1 - \lambda$ of the portfolios.

Maximum Sharpe Ratio

Maximum Sharpe Ratio

- What is the risk premium of the portfolio $\Sigma^{-1}(\mu - R_f\iota)$?
- What is the variance of the return of the portfolio $\Sigma^{-1}(\mu - R_f\iota)$?
- What is its Sharpe ratio (risk premium divided by standard deviation)?

SDFs and Mean-Variance Efficiency

- Project any SDF onto the span of the assets. There is a unique projection, and it is an SDF. Call it $\tilde{m}_p$.
- $\tilde{m}_p$ is the payoff of some portfolio (that's what it means to be in the span of the assets).
- Set $\widetilde{R}_p = \tilde{m}_p / \mathsf{E}[\tilde{m}_p^2]$. This is $\tilde{m}_p$ divided by its cost, so it has a cost of 1 and is a return.
- The return $\widetilde{R}_p$ is an inefficient frontier return.
- If there is a risk-free asset, then for any frontier return $\widetilde{R}$, $\widetilde{R}_p = \lambda R_f + (1-\lambda)\widetilde{R}$ for some λ (by two-fund spanning). So, $\tilde{m}_p = a + b\widetilde{R}$, where $a = \lambda \mathsf{E}[\tilde{m}_p^2]$ and $b = (1-\lambda)\mathsf{E}[\tilde{m}_p^2]$.
- Even without a risk-free asset (with one trivial exception),
  - SDF = affine function of return $\Rightarrow$ return is on MV frontier.
  - return $\widetilde{R}$ on MV frontier $\Rightarrow \tilde{m}_p = a + b\widetilde{R}$.