BUSI 521 /ECON 505
Asset Pricing Theory / Financial Economics
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MIDTERM EXAM SOLUTION

This exam is closed book and closed notes. There are three questions, and they will be equally weighted.

1. Assume there are three states of the world that are equally likely. There are two assets with prices $p_1 = p_2 = 1$. The payoffs of the first asset across the three states of the world are $(1, 2, 1)$. The payoffs of the second asset across the three states of the world are $(0, 1, 3)$.

   (a) Describe the one-dimensional family of state price vectors.

   (b) Find the SDF that is spanned by the assets.

   Solution

   A state price vector is q satisfying $Xq = p$. This means

   $$\begin{aligned} q_1 + 2q_2 + q_3 &= 1 \\ q_2 + 3q_3 &= 1 \end{aligned}$$

   You can simplify this and describe the solutions in various ways. For example, $q_1 - 6q_3 = -1$. A state price vector is spanned by the assets if there exists θ such that $X'\theta = q$. Combining this with $Xq = p$, we have five equations in five unknowns. The solution is $q_1 = 1/7$, $q_2 = 11/35$, $q_3 = 8/35$.

2. Assume there is a risk-free asset and multiple risky assets with joint normal returns.

   (a) Derive the optimal portfolio for an investor with CARA utility.

   (b) Show that the return of the investor's optimal portfolio is a pricing factor.

Solution

Let μ denote the vector of expected returns of the risky assets, and let Σ denote the covariance matrix. We want to find a portfolio ϕ to maximize the certainty equivalent, which is

$$r_f + (\mu - r_f\iota)'\phi - \frac{\alpha}{2}\phi'\Sigma\phi\,.$$

The FOC is

$$\alpha\Sigma\phi = \mu - r_f\iota\,.$$

The solution is

$$\phi = \frac{1}{\alpha}\Sigma^{-1}(\mu - r_f\iota)\,.$$

In terms of the fractions of wealth invested in the risky assets, the solution is

$$\pi = \frac{1}{\alpha w_0}\Sigma^{-1}(\mu - r_f\iota)\,.$$

The FOC shows that risk premia are proportional to covariances with the return of the portfolio π, so the return is a pricing factor.

3. Use the Bellman equation to derive the optimal portfolio for a log utility investor with an infinite horizon. You can assume that returns are iid.

Solution

We first show that there is a solution of the Bellman equation of the form $J(w) = a + b\log w$. To show this, we need to find a and b such that

$$a + b\log w = \max_{c,\pi}\left\{\log c + \delta \mathsf{E}[a + b\log(w-c)\pi'\widetilde{\mathbf{R}}]\right\}.$$

This is equivalent to

$$a + b\log w = \delta a + \max_{c}\left\{\log c + \delta b\log(w-c)\right\} + \delta b\max_{\pi}\left\{\mathsf{E}[\log \pi'\widetilde{\mathbf{R}}]\right\}.$$

Call the second max A. Solve the first max. We obtain

$$a + b\log w = \delta a + \log\frac{w}{1+\delta b} + \delta b\log\frac{\delta b w}{1+\delta b} + \delta bA\,.$$

This equation is satisfied by $b = 1 + \delta b$ or $b = 1/(1-\delta)$ and by

$$a = \delta a + \delta b\log\delta b - b\log b + \delta bA\,.$$

The optimal portfolio is the one that solves the single-period optimization

$$\max_{\pi}\mathsf{E}[\log \pi'\widetilde{\mathbf{R}}]\,.$$