(a) Show that the conditional variance formula (12.28) is equivalent to

$$M_{it}^2 - \int_0^t (\mathrm{d}M_{is})^2 \tag{12.1}$$

being a martingale.

Solution: Drop the i subscript. For $u > s$, we have

$$\begin{aligned}
\mathrm{var}_s(M_u - M_s) &= \mathsf{E}_s\left[(M_u - M_s)^2\right] - \mathsf{E}_s[M_u - M_s]^2 \\
&= \mathsf{E}_s\left[(M_u - M_s)^2\right] \\
&= \mathsf{E}_s\left[M_u^2 - 2M_sM_u + M_s^2\right] \\
&= \mathsf{E}_s[M_u^2] - 2M_s\mathsf{E}_s[M_u] + M_s^2 \\
&= \mathsf{E}_s[M_u^2] - M_s^2\,,
\end{aligned}$$

using the martingale property $\mathsf{E}_s[M_u] = M_s$. For

$$M_{it}^2 - \int_0^t (\mathrm{d}M_{is})^2$$

to be a martingale means that, for $u > s$,

$$\mathsf{E}_s\left[M_u^2 - \int_0^u (\mathrm{d}M_t)^2\right] = M_s^2 - \int_0^s (\mathrm{d}M_t)^2\,.$$

By the calculation above, this is equivalent to

$$\mathrm{var}_s(M_u - M_s) = \mathsf{E}_s\left[\int_s^u (\mathrm{d}M_t)^2\right]\,,$$

which is (12.28).

(b) Show that the conditional covariance formula (12.30) is equivalent to

$$M_{1t}M_{2t} - \int_0^t (\mathrm{d}M_{1s})\,(\mathrm{d}M_{2s}) \tag{12.2}$$

being a martingale. Note: A more general fact, which does not require the finite variance assumption, and which can be used as the definition of $(\mathrm{d}M_i)^2$ and $(\mathrm{d}M_i)(\mathrm{d}M_j)$, is that $\int_0^t (\mathrm{d}M_{is})^2$ is the finite-variation process such that (12.32) is a local martingale, and $\int_0^t (\mathrm{d}M_{1s})\,(\mathrm{d}M_{2s})$ is the finite-variation process such that (12.33) is a local martingale.

Solution: We have, for $u > s$,

$$\begin{aligned}
\operatorname{cov}_s(M_{1u} - M_{1s}, M_{2u} - M_{2s}) &= \mathsf{E}_s\left[(M_{1u} - M_{1s})(M_{2u} - M_{2s})\right] - \mathsf{E}_s[M_{1u} - M_{1s}]\mathsf{E}_s[M_{2u} - M_{2s}] \\
&= \mathsf{E}_s\left[(M_{1u} - M_{1s})(M_{2u} - M_{2s})\right] \\
&= \mathsf{E}_s[M_{1u}M_{2u} - M_{1u}M_{2s} - M_{1s}M_{2u} + M_{1s}M_{2s}] \\
&= \mathsf{E}_s[M_{1u}M_{2u}] - M_{2s}\mathsf{E}_s[M_{1u}] - M_{1s}\mathsf{E}_s[M_{2u}] + M_{1s}M_{2s} \\
&= \mathsf{E}_s[M_{1u}M_{2u}] - M_{1s}M_{2s}\,,
\end{aligned}$$

using the martingale property $\mathsf{E}_s[M_{iu}] = M_{is}$. For

$$M_{1t}M_{2t} - \int_0^t (\mathrm{d}M_{1s})\,(\mathrm{d}M_{2s})$$

to be a martingale means that, for $u > s$,

$$\mathsf{E}_s\left[M_{1u}M_{2u} - \int_0^u (\mathrm{d}M_{1t})(\mathrm{d}M_{2t})\right] = M_{1s}M_{2s} - \int_0^s (\mathrm{d}M_{1t})(\mathrm{d}M_{2t})\,.$$

By the calculation above, this is equivalent to

$$\operatorname{cov}_s(M_{1u} - M_{1s}, M_{2u} - M_{2s}) = \mathsf{E}_s\left[\int_s^u (\mathrm{d}M_{1t})(\mathrm{d}M_{2t})\right]\,,$$

which is (12.30).

12.13. Let $\mathrm{d}M_i = \theta_i\,\mathrm{d}B_i$ for $i = 1, 2$ and Brownian motions B_1 and B_2. Suppose θ_1 and θ_2 satisfy condition (12.5), so M_1 and M_2 are finite-variance martingales. Consider discrete dates $s = t_0 < t_1 < \cdots < t_N = u$ for some $s < u$. Show that

$$\operatorname{cov}_s(M_{1u} - M_{1s}, M_{2u} - M_{2s}) = \mathsf{E}_s\left[\sum_{j=1}^{N}(M_{1t_j} - M_{1t_{j-1}})(M_{2t_j} - M_{2t_{j-1}})\right]\,.$$

Hint: This is true of discrete-time finite-variance martingales, and the assumption that the M_i are stochastic integrals is neither necessary nor helpful in this exercise. However, it is interesting to compare this to (12.30).

Solution: As calculated in Exercise 12.12,

$$\text{cov}_s(M_{1u} - M_{1s}, M_{2u} - M_{2s}) = \mathsf{E}_s[M_{1u}M_{2u}] - M_{1s}M_{2s}\,.$$

Also, the calculation in Exercise 12.12 shows that

$$\mathsf{E}_{t_{j-1}}[(M_{1t_j} - M_{1t_{j-1}})(M_{2t_j} - M_{2t_{j-1}})] = \mathsf{E}_{t_{j-1}}[M_{1t_j}M_{2t_j}] - M_{1t_{j-1}}M_{2t_{j-1}}\,.$$

Therefore, by iterated expectations,

$$\begin{aligned}
\mathsf{E}_s\left[\sum_{j=1}^N (M_{1t_j} - M_{1t_{j-1}})(M_{2t_j} - M_{2t_{j-1}})\right] &= \sum_{j=1}^N \mathsf{E}_s[\mathsf{E}_{t_{j-1}}[(M_{1t_j} - M_{1t_{j-1}})(M_{2t_j} - M_{2t_{j-1}})]] \\
&= \sum_{j=1}^N \mathsf{E}_s[\mathsf{E}_{t_{j-1}}[M_{1t_j}M_{2t_j}]] - \mathsf{E}_s[M_{1t_{j-1}}M_{2t_{j-1}}] \\
&= \sum_{j=1}^N \mathsf{E}_s[M_{1t_j}M_{2t_j}] - \mathsf{E}_s[M_{1t_{j-1}}M_{2t_{j-1}}] \\
&= \mathsf{E}_s[M_{1t_N}M_{2t_N}] - \mathsf{E}_s[M_{1t_0}M_{2t_0}] \\
&= \mathsf{E}_s[M_{1u}M_{2u}] - M_{1s}M_{2s}\,.
\end{aligned}$$

Chapter 13

Continuous-Time Markets

13.1. For constants $\delta > 0$ and $\rho > 0$, assume

$$M_t \stackrel{\text{def}}{=} \mathrm{e}^{-\delta t}\left(\frac{C_t}{C_0}\right)^{-\rho} \tag{13.55}$$

is an SDF process, where C denotes aggregate consumption. Assume that

$$\frac{\mathrm{d}C}{C} = \alpha\,\mathrm{d}t + \theta'\,\mathrm{d}B \tag{13.56}$$

for stochastic processes α and θ.

(a) Apply Itô's formula to calculate $\mathrm{d}M/M$.

Solution: Using Exercise 12.2(a) and 12.1(c), we have

$$\begin{aligned}\frac{\mathrm{d}M}{M} &= -\delta\,\mathrm{d}t - \rho\,\frac{\mathrm{d}C}{C} + \frac{\rho(1+\rho)}{2}\left(\frac{\mathrm{d}C}{C}\right)^2 \\ &= -\left(\delta - \rho\alpha - \frac{\rho(1+\rho)\theta'\theta}{2}\right)\mathrm{d}t - \rho\theta'\,\mathrm{d}B\,.\end{aligned}$$

(b) Explain why the result of Part (a) implies that the instantaneous risk-free rate is

$$r = \delta + \rho\alpha - \frac{\rho(\rho+1)}{2}\theta'\theta \tag{13.57}$$

and the price of risk process is $\lambda = \rho\theta$.

Solution: The drift of $\mathrm{d}M/M$ is $-r\,\mathrm{d}t$, so the result of part (a) shows that r is as stated. The stochastic part of $\mathrm{d}M/M$ is $-\lambda'\,\mathrm{d}B$, so the result of part (a) shows that λ is as stated.

(c) Explain why the risk premium of any asset with price S is

$$\rho\left(\frac{\mathrm{d}S}{S}\right)\left(\frac{\mathrm{d}C}{C}\right).$$

Note: This is a preview of the CCAPM (Section 14.6), which holds under more general assumptions.

Solution: The risk premium of an asset is

$$-\left(\frac{\mathrm{d}S}{S}\right)\left(\frac{\mathrm{d}M}{M}\right) = \rho\left(\frac{\mathrm{d}S}{S}\right)\left(\frac{\mathrm{d}C}{C}\right).$$

13.2. Consider an asset paying dividends D over an infinite horizon. Assume D is a geometric Brownian motion:

$$\frac{\mathrm{d}D}{D} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$$

for constants μ and σ and a Brownian motion B. Assume the instantaneous risk-free rate r is constant, and assume there is an SDF process M such that

$$\left(\frac{\mathrm{d}D}{D}\right)\left(\frac{\mathrm{d}M}{M}\right) = -\sigma\lambda\,\mathrm{d}t \tag{13.58}$$

for a constant λ. Assume $\mu - \sigma\lambda < r$, and assume there are no bubbles in the price of the asset. **This exercise is simpler if it is assumed that B is the only Brownian motion in the economy. In this case (13.58) is equivalent to**

$$\frac{\mathrm{d}M}{M} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B.$$

If there are other Brownian motions, then some regularity condition is needed to ensure a local martingale is a martingale. See Exercise 15.2 for such a result.

(a) Show that the asset price is

$$P_t = \frac{D_t}{r + \sigma\lambda - \mu}\,.$$

Show that the Sharpe ratio of the asset is λ. Note: This is a continuous-time version of the Gordon growth model (Section 10.4). This exercise is continued in Exercise 15.2.

Solution: The asset price is

$$\begin{aligned} P_t &= \mathsf{E}_t \int_t^\infty \frac{M_u}{M_t} D_u \,\mathrm{d}u \\ &= D_t\, \mathsf{E}_t \int_t^\infty \frac{M_u D_u}{M_t D_t}\,\mathrm{d}u \\ &= D_t \int_t^\infty \mathsf{E}_t\left[\frac{M_u D_u}{M_t D_t}\right]\mathrm{d}u\,. \end{aligned}$$

We want to show that

$$\mathsf{E}_t\left[\frac{M_u D_u}{M_t D_t}\right] = \mathrm{e}^{(\mu - r - \sigma\lambda)(u-t)}\,. \qquad (*)$$

The result will then follow. We have

$$\begin{aligned} \frac{\mathrm{d}(DM)}{DM} &= \frac{\mathrm{d}D}{D} + \frac{\mathrm{d}M}{M} + \left(\frac{\mathrm{d}D}{D}\right)\left(\frac{\mathrm{d}M}{M}\right) \\ &= (\mu - r - \sigma\lambda)\,\mathrm{d}t + (\sigma - \lambda)\,\mathrm{d}B\,. \end{aligned}$$

Thus, MD is a geometric Brownian motion with drift with drift $\mu - r - \sigma\lambda$. This implies $(*)$.

(b) Assume (13.55) is an SDF for constants $\delta > 0$ and $\rho > 0$, where $C = D$. Show that (13.58) holds. What is λ? Referencing Exercise 12.4, calibrate to the following statistics reported by Mehra and Prescott (1985): $r = \log 1.008$, $\mathsf{E}_t[C_{t+1}/C_t] = 1.018$, $\mathrm{stdev}_t(C_{t+1}/C_t) = 0.036$, $\mathsf{E}_t[(P_{t+1} + C_{t+1})/P_t] = 1.0698$, and $\mathrm{stdev}_t(P_{t+1} + C_{t+1})/P_t) = 0.1654$. Calculate ρ and δ.

Solution: Substituting $C = D$, using the given dynamics for D, we have $\mathsf{E}[C_{t+1}/C_t] = \mathrm{e}^{\mu}$. So, $\mu = \log 1.018$. Also, from Exercise 12.4, $\mathrm{var}(C_{t+1}/C_t) = \mathrm{e}^{2\mu}(\mathrm{e}^{\sigma^2} - 1)$. So,

$$\sigma^2 = \log\left(\frac{0.036^2}{1.018^2} + 1\right) = 0.00125\,.$$

Using Exercise 13.1, we have

$$\frac{\mathrm{d}M}{M} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B$$

where $\lambda = \rho\sigma$. Note that the return on the market portfolio is

$$\begin{aligned}\frac{\mathrm{d}P + D\,\mathrm{d}t}{P} &= \frac{\mathrm{d}D}{D} + \frac{D}{P}\,\mathrm{d}t \\ &= \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B + (r + \sigma\lambda - \mu)\,\mathrm{d}t \\ &= (r + \sigma\lambda)\,\mathrm{d}t + \sigma\,\mathrm{d}B \\ &= (r + \rho\sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{d}B\,.\end{aligned}$$

It follows that the expected market return over the course of a year is $\mathrm{e}^{r+\rho\sigma^2}$. Matching this to the average market return gives $r + \rho\sigma^2 = \log 1.0698$, so

$$\rho = \frac{\log 1.0698 - \log 1.008}{\sigma^2} = 47.6\,.$$

Finally, from (13.57), we have

$$r = \delta + \rho\mu - \frac{\rho(1+\rho)\sigma^2}{2}\,,$$

so

$$\delta = r - \rho\mu + \frac{\rho(1+\rho)\sigma^2}{2} = 0.60\,.$$

This implies that the one-year discount factor is $\mathrm{e}^{-\delta} = 0.55$, as stated on p. 179.

13.3. Let r^d denote the instantaneous risk-free rate in the domestic currency, and let R^d denote the domestic currency price of the domestic money market account:

$$R^d_t = \exp\left(\int_0^t r^d_s\,\mathrm{d}s\right)\,.$$

As in Section 8.6, let X denote the price of a unit of a foreign currency in units of the domestic currency. Let r^f denote the instantaneous risk-free rate in the foreign currency, and let R^f denote the foreign currency price of the foreign money market account:

$$R_t^f = \exp\left(\int_0^t r_s^f \, \mathrm{d}s\right).$$

Suppose M^d is an SDF process for the domestic currency, so $M^f \overset{\text{def}}{=} M^d X/X_0$ is an SDF process for the foreign currency. Assume

$$\frac{\mathrm{d}X}{X} = \mu_x \, \mathrm{d}t + \sigma_x \, \mathrm{d}B$$

for a Brownian motion B.

(a) Show that

$$\frac{\mathrm{d}M^f}{M^f} = -r^f \, \mathrm{d}t + \mathrm{d}Z$$

for some local martingale Z.

Solution: Set $Y = M^f R^f$. Then Y is a local martingale, and

$$\frac{\mathrm{d}Y}{Y} = \frac{\mathrm{d}M^f}{M^f} + r^f \, \mathrm{d}t\,,$$

so

$$\frac{\mathrm{d}M^f}{M^f} = -r^f \, \mathrm{d}t + \mathrm{d}Z\,,$$

where Z is a local martingale defined by $\mathrm{d}Z = \mathrm{d}Y/Y$.

(b) Deduce from the previous result and Itô's formula that

$$\mu_x \, \mathrm{d}t = (r^d - r^f)\, \mathrm{d}t - \left(\frac{\mathrm{d}X}{X}\right)\left(\frac{\mathrm{d}M^d}{M^d}\right).$$

Note: This exercise is continued in Exercise 15.6.

Solution: From the previous part, the formula $M^f = M^d X/X_0$, and (13.17), we have

$$\begin{aligned}
-r^f \, \mathrm{d}t + \mathrm{d}Z &= \frac{\mathrm{d}M^f}{M^f} \\
&= \frac{\mathrm{d}M^d}{M^d} + \frac{\mathrm{d}X}{X} + \left(\frac{\mathrm{d}X}{X}\right)\left(\frac{\mathrm{d}M^d}{M^d}\right) \\
&= -r^d \, \mathrm{d}t - \lambda' \, \mathrm{d}B + \mu_x \, \mathrm{d}t + \sigma_x \, \mathrm{d}B_x + \left(\frac{\mathrm{d}X}{X}\right)\left(\frac{\mathrm{d}M^d}{M^d}\right).
\end{aligned}$$

For this to be true, the $\mathrm{d}t$ terms on each side must match, implying

$$\mu_x \,\mathrm{d}t = (r^d - r^f)\,\mathrm{d}t - \left(\frac{\mathrm{d}X}{X}\right)\left(\frac{\mathrm{d}M^d}{M^d}\right).$$

13.4. For a local martingale Y satisfying $\mathrm{d}Y/Y = \theta'\,\mathrm{d}B$ for some stochastic process θ, Novikov's condition is that

$$\mathsf{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta'\theta\,\mathrm{d}t\right)\right] < \infty\,.$$

Under this condition, Y is a martingale on $[0,T]$. Consider $Y = MW$, where M is an SDF process and W is a self-financing wealth process.

(a) Show that $\mathrm{d}Y/Y = \theta'\,\mathrm{d}B$, where $\theta = \sigma'\pi - \lambda_p - \zeta$ and $\sigma\zeta = 0$.

Solution: For $Y = MW$, we have

$$\begin{aligned}
\frac{\mathrm{d}Y}{Y} &= \frac{\mathrm{d}M}{M} + \frac{\mathrm{d}W}{W} + \left(\frac{\mathrm{d}M}{M}\right)\left(\frac{\mathrm{d}W}{W}\right)\\
&= -r\,\mathrm{d}t - \lambda'\,\mathrm{d}B + r\,\mathrm{d}t + \pi'(\mu - r\iota)\,\mathrm{d}t + \pi'\sigma\,\mathrm{d}B - \pi'\sigma\lambda\,\mathrm{d}t\\
&= (\sigma'\pi - \lambda)'\,\mathrm{d}B\\
&= (\sigma'\pi - \lambda_p - \zeta)'\,\mathrm{d}B\,,
\end{aligned}$$

using $\sigma\lambda = \mu - r\iota$ and $\lambda = \lambda_p + \zeta$.

(b) Deduce that Novikov's condition is equivalent to (13.41).

Solution: Setting $\theta = \sigma'\pi - \lambda_p - \zeta$, we have

$$\theta'\theta = \pi'\Sigma\pi + \lambda_p'\lambda_p + \zeta'\zeta - 2\pi'\sigma\lambda_p - 2\pi'\sigma\zeta + 2\lambda_p'\zeta\,.$$

Substituting $\sigma\lambda_p = \mu - r\iota$, $\sigma\zeta = 0$ and

$$\lambda_p'\zeta = (\mu - r\iota)'\Sigma^{-1}\sigma\zeta = 0\,,$$

we obtain

$$\theta'\theta = \pi'\Sigma\pi + \lambda_p'\lambda_p + \zeta'\zeta - 2\pi'(\mu - r\iota)\,.$$

(c) By specializing (13.41), state sufficient conditions for MS_i to be a martingale for $i = 1, \ldots, n$.

Solution: To apply (13.41) to $W = S_i$, take the portfolio process π to be the i–th basis vector e_i. Condition (13.41) in this case is

$$\mathsf{E}\left[\exp\left\{\frac{1}{2}\int_0^T \lambda_p'\lambda_p + \zeta'\zeta + e_i'\Sigma e_i - 2(\mu_i - r)\,\mathrm{d}t\right\}\right] < \infty\,.$$

Note that $e_i'\Sigma e_i$ is the squared volatility of the asset return, i.e., the i–th diagonal element of Σ.

13.5. Suppose $W > 0$, C and π satisfy the intertemporal budget constraint (13.38). Define the consumption-reinvested wealth process $W^\dagger$ by (13.43).

(a) Show that $W^\dagger$ satisfies the intertemporal budget constraint (13.44).

Solution: From (13.43) and (13.38), we have

$$\begin{aligned}\frac{\mathrm{d}W^\dagger}{W^\dagger} &= \frac{\mathrm{d}W}{W} + \frac{C}{W}\,\mathrm{d}t \\ &= r\,\mathrm{d}t + \pi'(\mu - r\iota)\,\mathrm{d}t + \pi'\sigma\,\mathrm{d}B\,.\end{aligned}$$

(b) Show that

$$W_t^\dagger - W_t = W_t^\dagger \int_0^t \frac{C_s}{W_s^\dagger}\,\mathrm{d}s$$

for each t.

Hint: Define $Y = W/W^\dagger$ and use Itô's formula to show that

$$\mathrm{d}Y = -\frac{C}{W^\dagger}\,\mathrm{d}t\,.$$