

# Chapter 2: Portfolio Choice

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# Simplest Problem

- Single risky asset, price  $p$  per share at date 0, price  $\tilde{x}$  per share at date 1.
- Risk-free asset with interest rate  $r_f$ .
- Investor has  $w_0$  to invest. Allocates between risk-free and risky assets.
- Let  $\theta$  = number of shares of risky asset. Then  $w_0 - p\theta$  is invested risk-free (this could be negative, meaning borrowing).
- $\theta$  is chosen to maximize

$$E[u(\theta\tilde{x} + (w_0 - p\theta)(1 + r_f))]$$

- FOC is

$$E[u'(\theta^*\tilde{x} + (w_0 - p\theta^*)(1 + r_f))\{\tilde{x} - p(1 + r_f)\}] = 0$$

## More on the FOC

- Date-1 wealth is

$$\tilde{w}^* := \theta^* \tilde{x} + (w_0 - p\theta^*)(1 + r_f)$$

- Divide the FOC by  $p$ . Set  $\tilde{R} = \tilde{x}/p$ . This is the (gross) return on the risky asset, meaning  $1 + \text{rate of return}$ .
- Set  $R_f = 1 + r_f$ . This is the (gross) risk-free return.
- The FOC is

$$E[u'(\tilde{w}^*)\{\tilde{R} - R_f\}] = 0$$

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- The FOC is

$$E[u'(\tilde{w}^*)\{\tilde{R} - R_f\}] = 0$$

- In words, marginal utility at the optimum is orthogonal to the excess return.

## Preview of Chapter 3

- Set  $\tilde{m} = u'(\tilde{w}^*)$ . The FOC is  $E[\tilde{m}(\tilde{R} - R_f)] = 0$ .
- By the definition of covariance,

$$E[\tilde{m}(\tilde{R} - R_f)] = E[\tilde{m}]E[\tilde{R} - R_f] + \text{cov}(\tilde{m}, \tilde{R} - R_f)$$

- So, the risk premium is

$$E[\tilde{R} - R_f] = -\frac{1}{E[\tilde{m}]} \text{cov}(\tilde{m}, \tilde{R})$$

- What sign should the covariance with marginal utility have?

# Notation

- Single consumption good at each of two dates 0 and 1
- Date-0 wealth  $w_0$  (in units of consumption good)
- Assets
  - Assets  $i = 1, \dots, n$
  - Date-0 prices  $p_i$  (in units of consumption good)
  - Date-1 payoffs  $\tilde{x}_i$  (in units of consumption good)
- Returns
  - Returns  $\tilde{R}_i = \tilde{x}_i/p_i$  (assuming  $p_i > 0$ )
  - Rates of return  $(\tilde{x} - p_i)/p_i = \tilde{R}_i - 1$
  - If there is a risk-free asset ( $\tilde{x}$  constant) then return is  $R_f$
- Portfolios
  - $\theta_i$  = number of shares held in portfolio
  - $\phi_i = \theta_i p_i$  = units of consumption good invested
  - $\pi_i = \theta_i p_i / w_0$  = fraction of wealth invested

# Portfolio Choice Problem

- Choose  $\theta_1, \dots, \theta_n$  to

$$\max E \left[ u \left( \sum_{i=1}^n \theta_i \tilde{X}_i \right) \right] \quad \text{subject to} \quad \sum_{i=1}^n p_i \theta_i = w_0 .$$

- Choose  $\phi_1, \dots, \phi_n$  to

$$\max E \left[ u \left( \sum_{i=1}^n \phi_i \tilde{R}_i \right) \right] \quad \text{subject to} \quad \sum_{i=1}^n \phi_i = w_0 .$$

- Choose  $\pi_1, \dots, \pi_n$  to

$$\max E \left[ u \left( w_0 \sum_{i=1}^n \pi_i \tilde{R}_i \right) \right] \quad \text{subject to} \quad \sum_{i=1}^n \pi_i = 1 .$$

- Short sales are allowed ( $\theta_i < 0$ )
- There are no margin requirements.
  - In the U.S. stock market, an investor with \$100 cash can only buy \$200 of stock (borrowing \$100).
  - In our formulation, there are no limits on borrowing, except that  $\sum \theta_i \tilde{x}_i$  must be in the domain of  $u(\cdot)$ —for example, positive if  $u = \log$ .
  - In real markets, collateral (margin) also has to be posted against short sales, but we do not require that in our formulation.
- Here, we take date-0 consumption and investment as given and optimize over the portfolio. We can also optimize over date-0 consumption and investment.
- We can sometimes allow for other non-portfolio income  $\tilde{y}$  at date-1 (for example, labor income).



# First-Order Condition

- Lagrangean:

$$E \left[ u \left( \sum_{i=1}^n \theta_i \tilde{x}_i \right) \right] - \lambda \left( \sum_{i=1}^n p_i \theta_i - w_0 \right)$$

- Assume interior optimum and assume can interchange differentiation and expectation to obtain

$$(\forall i) \quad E \left[ u' \left( \sum_{i=1}^n \theta_i \tilde{x}_i \right) \tilde{x}_i \right] = \lambda p_i$$

- If  $p_i > 0$ ,

$$E \left[ u' \left( \sum_{i=1}^n \theta_i \tilde{x}_i \right) \tilde{R}_i \right] = \lambda$$

## First-Order Condition cont.

- If  $p_i > 0$  and  $p_j > 0$ ,

$$\mathbb{E} \left[ u' \left( \sum_{i=1}^n \theta_i \tilde{x}_i \right) (\tilde{R}_i - \tilde{R}_j) \right] = 0$$

- In words: marginal utility at the optimal wealth is orthogonal to excess returns.
  - A return is the payoff of a unit-cost portfolio.
  - An excess return is the payoff of a zero-cost portfolio (for example, a difference of returns).
- Why? Investing a little less in asset  $j$  and a little more in asset  $i$  (or the reverse) cannot increase expected utility at the optimum.

## Results for One Risky Asset

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# Go Long if Risk Premium is Positive

- Let  $\phi$  = amount invested in risky asset, so  $w_0 - \phi$  is invested in risk-free asset. Let  $\mu = E[\tilde{R}]$  and  $\sigma^2 = \text{var}(\tilde{R})$ .

- Date-1 wealth is

$$\tilde{w} = (w_0 - \phi)R_f + \phi\tilde{R} = w_0R_f + \phi(\tilde{R} - R_f)$$

- We will show:  $\mu > R_f \Rightarrow \phi^* > 0$  (by symmetry,  $\mu < R_f \Rightarrow \phi^* < 0$ ).

## Proof

We want to compare  $E[u(w_0 R_f + \phi(\tilde{R} - R_f))]$  to  $u(w_0 R_f)$ .

Define  $\bar{w} = w_0 R_f + \phi(\mu - R_f)$  and  $\tilde{\varepsilon} = \phi(\tilde{R} - \mu)$ , so

$$w_0 R_f + \phi(\tilde{R} - R_f) = \bar{w} + \tilde{\varepsilon}.$$

Define  $\pi$  by

$$u(\bar{w} - \pi) = E[u(\bar{w} + \tilde{\varepsilon})].$$

The variance of  $\tilde{\varepsilon}$  is  $\phi^2 \sigma^2$ , so by second-order risk aversion,

$$\pi \approx \frac{1}{2} \alpha(\bar{w}) \phi^2 \sigma^2 < (\mu - R_f) \phi$$

when  $\phi > 0$  and small, so

$$u(\bar{w} - \pi) > u(\bar{w} - (\mu - R_f) \phi) = u(w_0 R_f)$$

# DARA Implies Risky Asset is a Normal Good

- Normal good: demand rises when income (wealth) rises. Inferior: demand falls when income (wealth) rises.
- A single risky asset with  $\mu > R_f$  is a normal good if the investor has decreasing absolute risk aversion.
- Proof: The FOC is

$$E[u'(\tilde{w})(\tilde{R} - R_f)] = 0$$

Differentiate it:

$$\begin{aligned} 0 &= \frac{d}{dw_0} E[u'(w_0 R_f + \phi(\tilde{R} - R_f))(\tilde{R} - R_f)] \\ &= E[u''(\tilde{w})\{R_f + \phi'(w_0)(\tilde{R} - R_f)\}(\tilde{R} - R_f)] \end{aligned}$$

Rearrange as

$$\phi'(w_0) = -\frac{R_f E[u''(\tilde{w})(\tilde{R} - R_f)]}{E[u''(\tilde{w})(\tilde{R} - R_f)^2]}.$$

Can show: DARA  $\Rightarrow \phi' > 0$ .

## CARA-Normal with Single Risky Asset

Assume CARA utility  $E[-e^{-\alpha\tilde{w}}]$ . Assume  $\tilde{R} \sim \text{normal}(\mu, \sigma)$ . Then  $\tilde{w}$  is normally distributed.

Recall: If  $\tilde{x}$  is normally distributed with mean  $\mu_x$  and std dev  $\sigma_x$ , then

$$E[e^{\tilde{x}}] = e^{\mu_x + \sigma_x^2/2}$$

Given an investment  $\phi$  in the risky asset,  $-\alpha\tilde{w}$  is normal with mean  $-\alpha w_0 R_f - \alpha\phi(\mu - R_f)$  and std dev  $\alpha\phi\sigma$ . Hence,

$$E[-e^{-\alpha\tilde{w}}] = -e^{-\alpha[w_0 R_f + \phi(\mu - R_f) - \alpha\phi^2\sigma^2/2]}$$

Thus,

$$w_0 R_f + \phi(\mu - R_f) - \alpha\phi^2\sigma^2/2$$

is the certainty equivalent (mean minus one-half risk aversion times variance).

Optimal portfolio maximizes the certainty equivalent. Therefore, the optimum is

$$\phi^* = \frac{\mu - R_f}{\alpha \sigma^2}$$

The optimal fraction of wealth to invest is

$$\pi^* = \frac{\mu - R_f}{(\alpha w_0) \sigma^2}$$

Usually assume  $\alpha w_0$  is between 1 and 10.



## Multiple Risky Assets

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# Portfolio Mean and Variance

- $\tilde{\mathbf{R}} = n$ -vector of risky asset returns
- $\mu = n$ -vector of expected returns
- $\phi = n$ -vector of investments in consumption good units
- $\pi = (1/w_0)\phi$
- $\iota = n$ -vector of 1's
- $\Sigma = n \times n$  covariance matrix,  $\Sigma_{ij} = \text{cov}(\tilde{\mathbf{R}}_i, \tilde{\mathbf{R}}_j)$

$$\Sigma = E[(\tilde{\mathbf{R}} - \mu)(\tilde{\mathbf{R}} - \mu)']$$

- date-1 wealth  $\tilde{w} = w_0 R_f + \phi'(\tilde{\mathbf{R}} - R_f \iota)$
- expected wealth  $\bar{w} = w_0 R_f + \phi'(\mu - R_f \iota)$
- variance of wealth  $= \phi' \Sigma \phi$ . Proof:

$$E[(\tilde{w} - \bar{w})^2] = E[\{\phi'(\tilde{\mathbf{R}} - \mu)\}^2] = E[\phi'(\tilde{\mathbf{R}} - \mu)(\tilde{\mathbf{R}} - \mu)' \phi] = \phi' \Sigma \phi$$

# Diversification

- Portfolio variance is

$$\pi' \Sigma \pi = \sum_{i=1}^n \pi_i^2 \text{var}(\tilde{\mathbf{R}}_i) + 2 \sum_{i=1}^n \sum_{j=i+1}^n \pi_i \pi_j \text{cov}(\tilde{\mathbf{R}}_i, \tilde{\mathbf{R}}_j)$$

- We can generally make  $\sum_{i=1}^n \pi_i^2 \text{var}(\tilde{\mathbf{R}}_i)$  small by diversifying, if there are many assets.
- Suppose for example that the risky assets are uncorrelated and have the same variance  $\sigma^2$  ( $\Sigma = \sigma^2 I$ ). Then

$$\pi' \Sigma \pi = \sigma^2 \sum_{i=1}^n \pi_i^2$$

Among portfolios fully invested in risky assets ( $\pi_i$  sum to 1), this variance is minimized at  $\pi_i = 1/n$  and

$$\pi' \Sigma \pi = \sigma^2 \sum_{i=1}^n \left(\frac{1}{n}\right)^2 = \frac{\sigma^2}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

# CARA-Normal with Multiple Risky Assets

- Certainty equivalent is mean minus one-half risk aversion times variance:

$$w_0 R_f + \phi'(\mu - R_f \iota) - \frac{1}{2} \alpha \phi' \Sigma \phi$$

- FOC is

$$\mu - R_f \iota - \alpha \Sigma \phi = 0.$$

- Optimum is

$$\phi^* = \frac{1}{\alpha} \Sigma^{-1} (\mu - R_f \iota)$$

Note no wealth effects.

- Similar form to single risky asset case. Optimum investment in each asset depends on its covariances with other assets unless  $\Sigma$  is diagonal.

# Wealth Expansion Paths

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# Portfolio Return

- With a risk-free asset, date-1 wealth is

$$\sum_i \phi_i \tilde{\mathbf{R}}_i + \left( w_0 - \sum_i \phi_i \right) R_f$$

- Divide by  $w_0$  and write  $\pi_i = \phi_i / w_0$ .
- Date-1 wealth is

$$w_0 \left[ \sum_i \pi_i \tilde{\mathbf{R}}_i + \left( 1 - \sum_i \pi_i \right) R_f \right] := w_0 \tilde{R}_p$$

- In words, initial wealth times the (gross) portfolio return.

# How does a Log Utility Investor's Portfolio Depend on Wealth?

- Utility is

$$\log(w_0 \tilde{R}_p) = \log w_0 + \log \tilde{R}_p = \log w_0 + \log \left( \sum_i \pi \tilde{R}_i + \left( 1 - \sum_i \pi_i \right) R_f \right)$$

- The optimal portfolio maximizes expected log of portfolio return and is independent of initial wealth.
- Optimal dollar investments are proportional to initial wealth:  $\phi_i^* = w_0 \pi_i^*$ .
- Optimal shares are also proportional to initial wealth:  $\theta_i^* = \phi_i^* / p_i = w_0 \pi_i^* / p_i$ .

- Utility is

$$\frac{1}{1-\rho}(w_0\tilde{R}_p)^{1-\rho} = w_0^{1-\rho} \times \frac{1}{1-\rho}\tilde{R}_p^{1-\rho}$$

- So  $w_0^{1-\rho}$  is a positive constant that multiplies the expected utility of the portfolio return.
- Optimal portfolio  $\pi^*$  maximizes expected utility of portfolio return and is independent of initial wealth.
- Optimal dollar investments are proportional to initial wealth:  $\phi_i^* = w_0\pi_i^*$ .
- Optimal shares are also proportional to initial wealth:  $\theta_i^* = \phi_i^*/p_i = w_0\pi_i^*/p_i$ .



- Stick with \$ investments. Date-1 wealth is

$$\sum_i \phi \tilde{\mathbf{R}}_i + \left( w_0 - \sum_i \phi_i \right) R_f = w_0 R_f + \phi'(\tilde{\mathbf{R}} - R_f \boldsymbol{\iota})$$

- Here,  $\phi$  is vector of  $\phi_i$ ,  $\tilde{\mathbf{R}}$  is vector of  $\tilde{\mathbf{R}}_i$  and  $\boldsymbol{\iota}$  is vector of 1's.
- Utility is

$$-e^{-\alpha \tilde{w}_1} = e^{-\alpha w_0 R_f} \times \left( -e^{-\alpha \phi'(\tilde{\mathbf{R}} - R_f \boldsymbol{\iota})} \right)$$

- Optimal dollar investments are independent of initial wealth (absence of wealth effects).
- Optimal shares are also independent of initial wealth.

- Optimal dollar investments are affine in initial wealth:  $\phi_i^* = a_i + b_i w_0$ .
- Optimal shares are also affine in initial wealth (divide  $a_i$  and  $b_i$  by  $p_i$ ).
- We say “wealth expansion paths are linear.”
- Slope coefficient depends on cautiousness parameter. (Recall LRT means  $\tau = A + Bw$  and  $B$  is called the cautiousness parameter.)
- CARA investors have horizontal (zero slope) expansion paths.
- CRRA investors with same relative risk aversion have parallel expansion paths (same  $b_i$ ).

# Euler Equation

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# Time-Additive Utility and the Euler Equation

- Date-0 and date-1 consumption. Utility function  $v(c_0, c_1)$ . Assume time-additive utility

$$v(c_0, c_1) = u(c_0) + \delta u(c_1)$$

- Consumption/investment problem: choose  $c_0, \phi_1, \dots, \phi_n$  to

$$\max u(c_0) + E \left[ \delta u \left( \sum_{i=1}^n \phi_i \tilde{\mathbf{R}}_i \right) \right] \quad \text{subject to} \quad c_0 + \sum_{i=1}^n \phi_i = w.$$

- FOC:  $u'(c_0) = \lambda$  and

$$(\forall i) \quad E \left[ \delta u' \left( \sum_{i=1}^n \phi_i \tilde{\mathbf{R}}_i \right) \tilde{\mathbf{R}}_i \right] = \lambda$$

- So

$$(\forall i) \quad E \left[ \frac{\delta u' \left( \sum_{i=1}^n \phi_i \tilde{\mathbf{R}}_i \right)}{u'(c_0)} \tilde{\mathbf{R}}_i \right] = 1$$

## Exercise 2.6

*With time-additive CRRA utility, the elasticity of intertemporal substitution is the reciprocal of relative risk aversion.*

Definition of EIS: for a utility function  $v(c_0, c_1)$ .

$$MRS = \frac{\partial v / \partial c_0}{\partial v / \partial c_1}$$

$$EIS = \frac{d \log(c_1/c_0)}{d \log MRS}$$

Set  $x = c_1/c_0$ . Assume time-additive CRRA utility:

$$v(c_0, c_1) = \frac{1}{1-\rho} c_0^{1-\rho} + \frac{\delta}{1-\rho} c_1^{1-\rho}$$

Then  $MRS = -\log \delta + \rho \log x$ . So,  $d \log MRS / d \log x = \rho$ . This implies

$$EIS = \frac{1}{\rho}$$