Chapter 8: Dynamic Securities Markets

Kerry Back
BUSI 521/ECON 505
Rice University

Assets and Returns

- Dates $t = 0, 1, 2, \ldots$. No tildes anymore for random things. Information grows over time as random variables are observed.
- $D_{it} =$ dividend of asset i at date t. Ex-dividend price $P_{it} > 0$.
- Return from t to $t+1$ is

$$R_{i,t+1} := \frac{P_{i,t+1} + D_{i,t+1}}{P_{it}}$$

- Risk-free return from t to $t+1$ is $R_{f,t+1}$. Known at t (so risk-free from t to $t+1$) but maybe not known until t (randomly evolving interest rates).

Iterated Expectations

- Let E_t denote expectation given information at date t.
- Assume information is nondecreasing over time.
- For any $s < t < u$ and random variable X_u known at date u,

$$\mathsf{E}_s[X_u] = \mathsf{E}_s\Big[\mathsf{E}_t[X_u]\Big]$$

SDFs

One-Period SDFs

- SDF at t for pricing at $t+1$ is a r.v. Z_{t+1} depending on date $t+1$ information such that

$$\mathsf{E}_t[Z_{t+1}R_{i,t+1}] = 1$$

for all assets i.

- Equivalently, price at t of any portfolio payoff X_{t+1} at $t+1$ is

$$\mathsf{E}_t[Z_{t+1}X_{t+1}]$$

- With no uncertainty or with risk neutrality,

$$Z_{t+1} = \frac{1}{R_{f,t+1}} := \frac{1}{1+r_{f,t+1}}$$

Two-Period SDFs

- How to price at $t-1$ a payoff at $t+1$?

Two-Period SDFs

- How to price at $t-1$ a payoff at $t+1$?
- Same as price at $t-1$ of enough dollars at t to buy the payoff. So, is price at $t-1$ of

$$\mathsf{E}_t[Z_{t+1}X_{t+1}]$$

Two-Period SDFs

- How to price at $t-1$ a payoff at $t+1$?
- Same as price at $t-1$ of enough dollars at t to buy the payoff. So, is price at $t-1$ of

$$\mathsf{E}_t[Z_{t+1}X_{t+1}]$$

- So price at $t-1$ is

$$\mathsf{E}_{t-1}\Big[Z_t\mathsf{E}_t[Z_{t+1}X_{t+1}]\Big] = \mathsf{E}_{t-1}\Big[\mathsf{E}_t[Z_tZ_{t+1}X_{t+1}]\Big] = \mathsf{E}_{t-1}\Big[Z_tZ_{t+1}X_{t+1}\Big]$$

- We're compounding discount factors.
- With no uncertainty, price is

$$\frac{X_{t+1}}{(1+r_{f,t})(1+r_{f,t+1})}$$

SDF Process

- Define M by compounding discount factorrs:

$$M_t := Z_1 \times Z_2 \times \cdots \times Z_t$$

- Set $M_0 = 1$.
- Price at date 0 of a payoff X_t at date t is $\mathsf{E}[M_t X_t]$.

SDF Process

- Define M by compounding discount factorrs:

$$M_t := Z_1 \times Z_2 \times \cdots \times Z_t$$

- Set $M_0 = 1$.
- Price at date 0 of a payoff X_t at date t is $\mathsf{E}[M_t X_t]$.
- Price at date $s < t$ of payoff X_t at date t is

$$\mathsf{E}_s[Z_{s+1} \cdots Z_t X_t] = \mathsf{E}_s\left[\frac{Z_1 \cdots Z_t}{Z_1 \cdots Z_s} X_t\right] = \mathsf{E}_s\left[\frac{M_t}{M_s} X_t\right]$$

Factor Model

Dynamic Factor Model

- From
$$1 = \mathsf{E}_t\left[\frac{M_{t+1}}{M_t}R_{i,t+1}\right]$$
we get
$$1 = \frac{\mathsf{E}_t[R_{i,t+1}]}{R_{f,t+1}} + \mathsf{cov}_t\left(\frac{M_{t+1}}{M_t}, R_{i,t+1}\right)$$
- So
$$\mathsf{E}_t[R_{i,t+1}] - R_{f,t+1} = -R_{f,t+1}\,\mathsf{cov}_t\left(\frac{M_{t+1}}{M_t}, R_{i,t+1}\right)$$

Portfolio Choice

Portfolio Choice

- Stack returns into an n–vector R_{t+1}. One may be risk-free (return $= R_{f,t+1}$).
- Investor chooses consumption C_t and a portfolio $\pi_t \in \mathbb{R}^n$. $\iota' \pi_t = 1$. Labor income Y_t.
- Suppose investor seeks to maximize

$$\sum_{t=0}^{\infty} \delta^t u(C_t)$$

Wealth (actually financial wealth) W satisfies the intertemporal budget constraint

$$W_{t+1} = (W_t - C_t)\pi_t' R_{t+1} + Y_{t+1}$$

Euler Equation

- A necessary condition for consumption/investment optimality is that, for all dates t and assets i,

$$\mathsf{E}_t \left[\frac{\delta u'(C_{t+1})}{u'(C_t)} R_{i,t+1} \right] = 1$$

- This is called the Euler equation. It is derived by the same logic as in a single-period model.
- The Euler equation is equivalent to:

$$M_t := \frac{\delta^t u'(C_t)}{u'(C_0)}$$

  is an SDF process.
- The one-period SDFs are one-period marginal rates of substitution:

$$\frac{M_{t+1}}{M_t} = \frac{\delta u'(C_{t+1})}{u'(C_t)}$$

Equity Premium Puzzle

Representative Investor and SDF Process

- Let C denote aggregate consumption.
- Assume there is a representative investor with CRRA utility and risk aversion ρ.
- Then, the one-period SDF is

$$\frac{M_{t+1}}{M_t} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho}$$

- The SDF process is

$$M_t = \delta^t \left(\frac{C_t}{C_0} \right)^{-\rho}$$

Market Price-Dividend Ratio

- Define the market portfolio as the claim to future consumption.
- Consumption is then the dividend of the market portfolio. Assume consumption growth C_{t+1}/C_t is iid lognormal.
- The ex-dividend date–t price of the market portfolio is

$$P_t := \mathsf{E}_t \sum_{u=t+1}^{\infty} \frac{M_u}{M_t} C_u = \mathsf{E}_t \sum_{u=t+1}^{\infty} \delta^{u-t} \left(\frac{C_u}{C_t} \right)^{-\rho} C_u$$

- So, the price-dividend ratio is

$$\begin{aligned} \frac{P_t}{C_t} &= \mathsf{E}_t \sum_{u=t+1}^{\infty} \delta^{u-t} \left(\frac{C_u}{C_t} \right)^{1-\rho} \\ &= \mathsf{E} \sum_{u=1}^{\infty} \delta^{u} \left(\frac{C_u}{C_0} \right)^{1-\rho} \end{aligned}$$

- Assume $\log C_{t+1} = \log C_t + \mu + \sigma\varepsilon_{t+1}$ for iid standard normals ε.
- Then

$$\log C_u = \log C_0 + u\mu + \sigma \sum_{n=1}^{u} \varepsilon_n$$

- Hence,

$$\begin{aligned}
\mathsf{E}\left[\left(\frac{C_u}{C_0}\right)^{1-\rho}\right] &= \mathsf{E}\left[\exp\left((1-\rho)\left\{u\mu + \sigma \sum_{n=1}^{u} \varepsilon_n\right\}\right)\right] \\
&= \exp\left((1-\rho)u\mu + \frac{1}{2}(1-\rho)^2 u\sigma^2\right) \\
&= \left(\mathrm{e}^{(1-\rho)\mu + (1-\rho)^2\sigma^2/2}\right)^u
\end{aligned}$$

- So, the price-dividend ratio is

$$\sum_{u=1}^{\infty} \left(\delta \mathrm{e}^{(1-\rho)\mu + (1-\rho)^2 \sigma^2 / 2} \right)^u = \frac{\nu_1}{1 - \nu_1}$$

  where

$$\nu_1 = \delta \mathsf{E} \left[\left(\frac{C_1}{C_0} \right)^{1-\rho} \right] = \delta \mathrm{e}^{(1-\rho)\mu + (1-\rho)^2 \sigma^2 / 2}$$

  provided $\nu_1 < 1$.
- This is the same ν_1 we saw in Chapter 7.
- Everything else—risk-free return, expected market return, log equity premium, equity premium puzzle—is exactly the same as in Chapter 7.

Risk-Neutral Probability

Risk-Neutral Probability

- Consider an arbitrary finite (possibly large) horizon T.
- Consider an event A that can be distinguished by date T (at date T, you know whether A happened or not).
- Define

$$Q(A) = \mathsf{E}[R_{f1} \cdots R_{fT} M_T 1_A]$$

- Then Q is a probability measure.
- Define E^* as expectation with respect to Q. Then for all assets i and dates t,

$$\mathsf{E}^*_t[R_{i,t+1}] = R_{f,t+1}$$

- And, the price at t of a payoff X_{t+1} at date $t+1$ is

$$\frac{\mathsf{E}^*_t[X_{t+1}]}{1 + r_{f,t+1}}$$

Martingales

Martingales

- A martingale is a sequence of random variables Y such that $Y_s = \mathsf{E}_s[Y_t]$ for all $s < t$.
- Equivalently, $E_s[Y_t - Y_s] = 0$.
- Consider any payoff at date u with value V_t at date t. Then
  1. The sequence $M_t V_t$ is a martingale (up to u).
  2. The sequence
  $$\frac{V_t}{(1 + r_{f1}) \cdots (1 + r_{ft})}$$
  is a Q-martingale.

Testing

Testing Conditional Models

- Suppose we have a model for an SDF. Call the model value $\hat{M}$. We want to test whether

$$(\forall\ t, i) \qquad \mathsf{E}_t\left[\frac{\hat{M}_{t+1}}{\hat{M}_t}\Big(R_{i,t+1} - R_{f,t+1}\Big)\right] = 0 \qquad (\star)$$

- Let I_t be any variable observed at t. Multiply by I_t to get:

$$(\forall\ t, i) \qquad \mathsf{E}_t\left[I_t\frac{\hat{M}_{t+1}}{\hat{M}_t}\Big(R_{i,t+1} - R_{f,t+1}\Big)\right] = 0$$

- Now use the law of iterated expectations to obtain

$$(\forall\ t, i) \qquad \mathsf{E}\left[I_t\frac{\hat{M}_{t+1}}{\hat{M}_t}\Big(R_{i,t+1} - R_{f,t+1}\Big)\right] \qquad (\star\star)$$

- The conditional model (**??**) implies the unconditional moment condition (**??**) for every instrument I. If we reject the unconditional moment conditions, then we reject the model.