Cost of Capital

BUSI 721: Data-Driven Finance I

Kerry Back, Rice University

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Overview

- Equations for efficient portfolio with cash
- Risk premium formula based on beta with respect to efficient portfolio
- Capital Asset Pricing Model (CAPM)
- Weighted Average Cost of Capital

Efficient portfolio with cash

minimize

$$\frac{1}{2} w' \Sigma w$$

subject to

$$\mu' w + \left(1 - \sum_{i=1}^{n} w_i\right) r_f = r$$

where $r =$ target expected return and $r_f =$ borrowing and saving rate.

Target expected return constraint

- Expected return is

$$\mu' w + (1 - \iota' w) r_f = r_f + (\mu - r_f \iota')w$$

- Equals target expected return r if and only if

$$(\mu - r_f \iota)' w = r - r_f$$

- So, minimize

$$\frac{1}{2} w' \Sigma w$$

subject to

$$(\mu - r_f \iota)' w = r = r_f$$

Marginal benefit-cost ratios

- Benefit = risk premium = $(\mu - r_f\iota)'w$
- Cost = half of variance = $(1/2)w'\Sigma w$
- Minimize cost subject to benefit $= r - r_f$
- The optimum is such that the ratio of marginal benefit to marginal cost is the same for all assets.

$$\frac{\partial \text{Benefit}/\partial w_i}{\partial \text{Cost}/\partial w_i} = \text{constant depending on } r - r_f$$

Related problem

- Outputs 1, ..., n. Quantities = x_i
- Output prices p_i
- Efficient production plan miminizes cost given the revenue it produces:

$$\text{minimize} \quad \frac{1}{2} C(x_1, \ldots, x_n) \quad \text{subject to} \sum_{i=1}^{n} p_i x_i = \text{revenue target}$$

- Optimum is such that

$$\frac{p_i}{\partial C / \partial x_i} = \text{constant depending on revenue target}$$

- as long as the x_i are ≥ 0 at the solution of these equations.

Why marginal benefit-cost ratios the same?

- Suppose two outputs both have marginal cost = 1. One has price = 1 The other has price = 2.
- Reduce production of the low-price output by Δ.
- Increase production of the high-price output by Δ.
- Cost goes down from reducing production of one and up from increasing producing of other, and these offset due to equal marginal costs.
- Make more revenue.

Back to finance

- Marginal benefit is the risk premium of asset i = $\mu_i - r_f$.
- To calculate marginal cost, assume only two assets for simplicity.

$$\text{Variance} = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho\sigma_1\sigma_2 w_1 w_2$$

- Differentiate with respect to w_1 for example:

$$\frac{\partial \text{Variance}}{\partial w_1} = 2w_1\sigma_1^2 + 2\rho\sigma_1\sigma_2 w_2$$

$$
\begin{aligned}
\frac{\partial \text{Cost}}{\partial w_1} &= w_1\sigma_1^2 + \rho\sigma_1\sigma_2 w_2 \\
&= w_1 \text{cov}(r_1, r_1) + w_2 \text{cov}(r_1, r_2) \\
&= \text{cov}(r_1, w_1 r_1) + \text{cov}(r_1, w_2 r_2) \\
&= \text{cov}(r_1, w_1 r_1 + w_2 r_2)
\end{aligned}
$$

- With n assets, the marginal cost of asset i is

$$
\text{cov}\left(r_i, \sum_{i=1}^{n} w_i r_i\right)
$$

where $(w_1, \ldots, w_n)$ is the efficient portfolio.

- Call the constant k
- So, marginal benefit / marginal cost $= k$
- So, marginal benefit $= k \times \text{marginal cost}$
- So, for each asset,

$$\text{risk premium} = k \times \text{cov}(\text{asset return}, \text{efficient portfolio return})$$

- This provides some hope for estimating the return investors expect from a stock (expected return = risk premium $+ r_f$)

A few more steps

- The risk premium formula also holds for portfolios of assets, so use it for the efficient portfolio return:

$$\text{efficient portfolio risk premium} = k \times \text{var}(\text{efficient portfolio return})$$

- This is a formula for k:

$$k = \frac{\text{efficient portfolio risk premium}}{\text{var}(\text{efficient portfolio return})}$$

- Substituting this, we get, for each asset,

$$\text{risk premium} = \frac{\text{cov}(\text{asset return}, \text{efficient portfolio return})}{\text{var}(\text{efficient portfolio return})} \times \text{efficient portfolio risk premium}$$

- A covariance to variance ratio like this is the slope in a regression.
- Let r_p denote the efficient portfolio return.
- We are dealing with risk premia, so compute excess returns $r_i - r_f$ and $r_p - r_f$.
- Run the regression:

$$r_i - r_f = \alpha_i + \beta_i(r_p - r_f) + \varepsilon_i$$

- Covariance to variance ratio is β_i. So, for each asset,

$$\text{risk premium} = \beta \ \times \text{efficient portfolio risk premium.}$$

- All of this is tautological and so far doesn't help us.
- To find the efficient portfolio by calculation, we already need to know all of the risk premia.
- But suppose we could guess a efficient portfolio. Then,
    - We can run regressions to estimate betas.
    - We can estimate the efficient portfolio risk premium using hopefully a very long data series of returns.
    - We get a formula for the return investors expect from a stock.

$$\text{expected return} = r_f + \beta \times \text{efficient portfolio risk premium}$$

Capital Asset Pricing Model (CAPM)

- Assume the market portfolio of stocks is an efficient portfolio.
- Why? People should hold efficient portfolios - maximize the Sharpe ratio.
- A weighted average of portfolios on the maximum Sharpe ratio line is also on the line.
- So the market is efficient if everyone holds an efficient portfolio.
- CAPM:

$$\text{expected return} = r_f + \beta \times \text{market risk premium}$$

Weighted average cost of capital (WACC)

- Compute equity / (equity + debt) and debt / (equity + debt) on a market value basis.
- Compute the expected return on your stock using probably the CAPM = cost of equity.
- Determine the coupon at which you could issue debt at face value (market rate for your debt) = cost of debt.
- Determine the marginal tax rate. It matters because interest is tax deductible but dividends are not.

$$\begin{aligned} \text{WACC} &= \backslash\% \text{ equity} \times \text{cost of equity} \\ &+ \backslash\% \text{ debt} \times (1 - \text{tax rate}) \times \text{cost of debt} \end{aligned}$$