Option Valuation

BUSI 722: Data-Driven Finance II

Kerry Back, Rice University

Comparison pricing

- How do you decide if a house is fairly priced?
- An analogue to price / square foot for valuing companies is price-to-earnings.
- For valuing an option, we could use other options or we can just start with the underlying price.
- Option value = intrinsic value + adjustment for time and uncertainty.

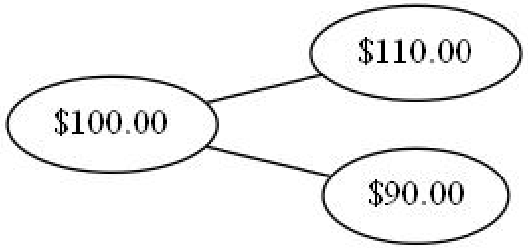
Replication

- If we could create the option value at maturity by dynamically trading the underlying, then the value of the option should be the cost of the underlying portfolio.
- Call $\sim$ long underlying with leverage, so value of call = cost of underlying minus amount borrowed
- Put $\sim$ short underlying not fully collateralized, so value of put = implicit collateral

Replication in a single period two-state example

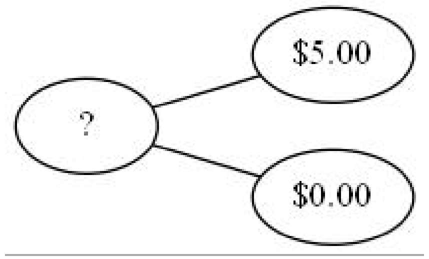
Simple Example

Suppose a stock priced at $100 will either go up by 10% or down by 10%.



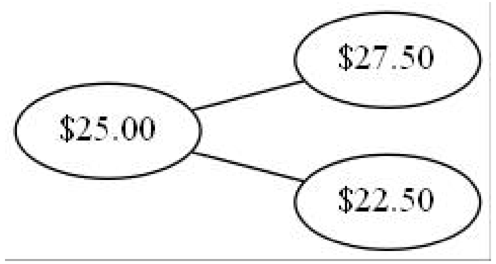
Call Option

- Consider a call option with a strike of 105.
- It ends with a value of 5 if the stock goes to 110 and a value of 0 if the stock goes to 90.
- We want to find its value at the beginning.



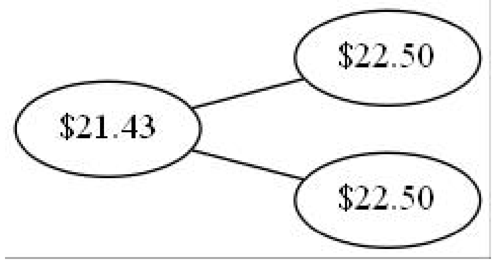
Delta

- Define Δ as the difference in the option values divided by the difference in the stock values.
- This is $(5 - 0)/(110 - 90) = 1/4$.
- Here is the value of 1/4 share of the stock.



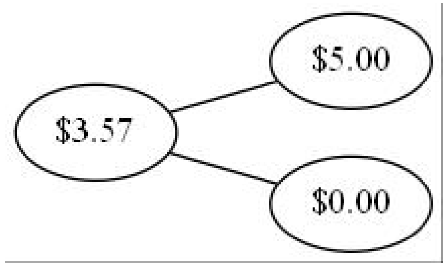
Debt

- Consider borrowing the PV of the bottom value from the previous figure.
- Suppose the interest rate is 5%. The PV of 22.50 is 21.43. Here is how the debt evolves.



Buy Δ shares on margin

The value of delta shares less the value of the debt is:



Conclusion

- In this simple example, we can get the call option payoff at maturity by investing 3.57, borrowing 21.43, and buying 1/4 share.
- The value of the call must be 3.57.

Risk-neutral probability

- If there were no risk premium, the call value would be the expected value discounted at the risk-free rate:

$$C = \frac{p \times \$5 + (1-p) \times \$0}{1.05}$$

- where $p =$ up probability. The stock price would also be the discounted expected value:

$$\$100 = \frac{p \times \$110 + (1-p) \times \$90}{1.05} \quad \Leftrightarrow \quad S = \frac{p(1+r_u)S + (1-p)(1+r_d)S}{1+r_f}$$

- Solve the stock equation for p:

$$p = \frac{r_f - r_d}{r_u - r_d} = \frac{0.05 - (-0.1)}{0.1 - (-0.1)} = \frac{.15}{.2} = 0.75$$

- Substitute into the call option equation. Obtain

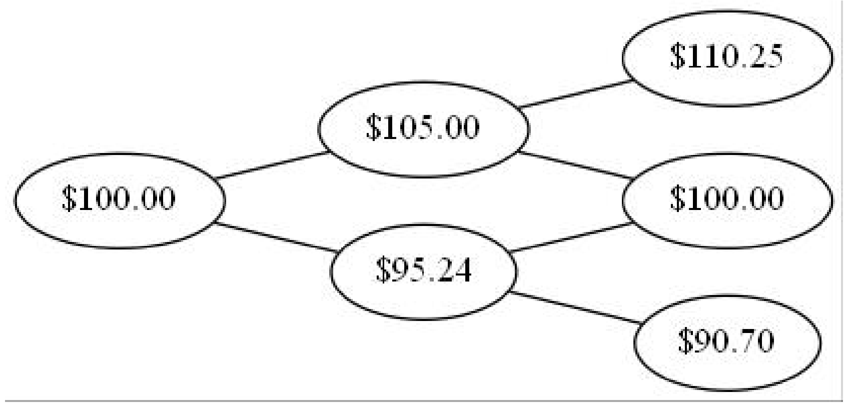
$$C = \frac{0.75 \times \$5 + 0.25 \times \$0}{1.05} = \$3.57$$

- Why does this work? Delta hedge argument didn't depend on risk preferences, so we can act as if investors don't require risk premia.

Risk-neutral probability in two-period example

Stock dynamics

- A $100 stock goes up by 5% or down by (1/1.05-1) = -4.76% in each of two periods.
- Interest rate is 3% per period



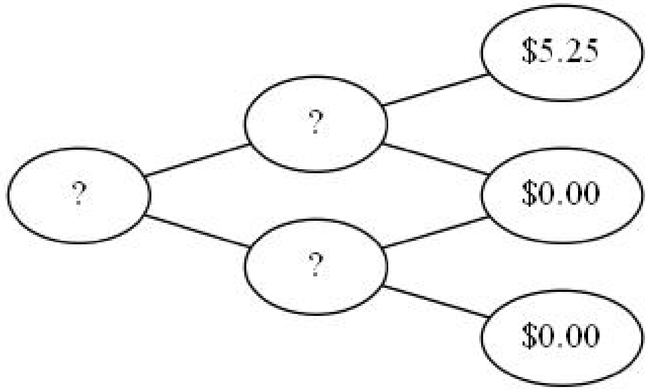
Risk-neutral probability

The risk-neutral probability of "up" is

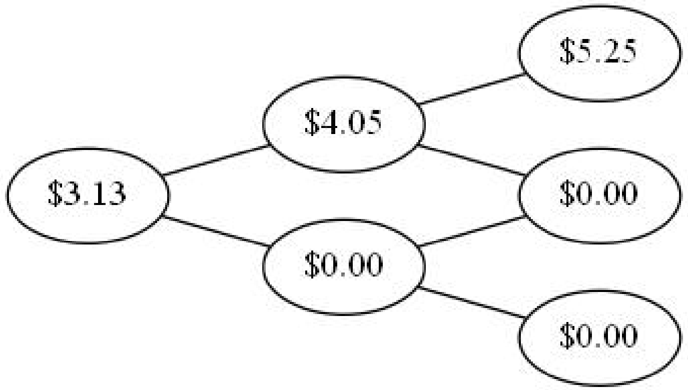
$$p = \frac{r_f - r_d}{r_u - r_d} = \frac{0.03 - (-0.0476)}{0.05 - (-0.0476)} = 0.795$$

Call option with strike = 105

The call evolves as



Discounting the expected call value (using prob of up = 0.795) at the risk-free rate yields



Exercise

Price a call with a strike of 95.

Calibration

- Estimate $\sigma =$ std dev of annual stock return
- Find $r_f =$ annualized continuously compounded interest rate = log(1+annual rate)
- $T =$ time to maturity of an option in years
- $N =$ number of periods in a binomial model

- $\Delta t = T/N =$ period length
- Set the up rate of return as $u = e^{\sigma\sqrt{\Delta t}} - 1$ and set $d = 1/(1+u) - 1$ as in the two-period example
- Set the 1-period interest rate as $r = e^{r_f \Delta t} - 1$
- The risk-neutral probability of "up" is

$$p = \frac{r-d}{u-d} = \frac{e^{r_f \Delta t} - e^{-\sigma \Delta t}}{e^{\sigma \Delta t} - e^{-\sigma \Delta t}}$$

Exercise

Price a six-month call option using a two-period model.

- $\sigma = 0.4$
- $r_f = 0.05$
- $T = 0.5$
- $N = 2$
- $S = 100$
- $K = 95$

Black-Scholes formulas

- As $N \to \infty$ the distribution of the stock price at date T converges to lognormal, meaning that the log stock price has a normal distribution.
- The values of European options converge to the Black-Scholes formulas.
- More about American options (and dividends) coming.

Black-Scholes call formula

In [13]:

```python
import numpy as np
from scipy.stats import norm

def BS_call(S, K, T, sigma, r, q=0):
    d1 = np.log(S/K) + (r-q+0.5*sigma**2)*T
    d1 /= sigma*np.sqrt(T)
    d2 = d1 - sigma*np.sqrt(T)
    N1 = norm.cdf(d1)
    N2 = norm.cdf(d2)
    return np.exp(-q*T)*S*N1 - np.exp(-r*T)*K*N2
```

Black-Scholes put formula

In [14]:

```python
def BS_put(S, K, T, sigma, r, q=0):
    d1 = np.log(S/K) + (r-q+0.5*sigma**2)*T
    d1 /= sigma*np.sqrt(T)
    d2 = d1 - sigma*np.sqrt(T)
    N1 = norm.cdf(-d1)
    N2 = norm.cdf(-d2)
    return np.exp(-r*T)*K*N2 - np.exp(-q*T)*S*N1
```

Example

In [15]:

```python
S, K, T, sigma, r = 100, 95, 0.5, 0.4, 0.05
print(f"Value of call option is ${BS_call(S, K, T, sigma, r):.2f}")
print(f"Value of put option is ${BS_put(S, K, T, sigma, r):.2f}")
```

```
Value of call option is $14.89
Value of put option is $7.55
```

Dividends and early exercise

Dividends

- To use binomial model, build a tree for "stock minus PV of future dividends," the future being until the option maturity.
- Try to set tree nodes near ex-dividend dates
- Everything else as before. As time passes, dividends get paid and "stock minus PV of future dividends" becomes "stock."

Early exercise

- It may be optimal to exercise an American put at any time (though just after ex-dividend date is better than just before).
- It may be optimal to exercise an American call just before a dividend is paid.
- To use binomial model, replace "discounted risk-neutral expectation of option value" with max of discounted risk-neutral expectation and intrinsic value.