ELSEVIER

European Journal of Operational Research 115 (1999) 459–471

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Theory and Methodology

Estimating and bootstrapping Malmquist indices [1]

Léopold Simar [a,2], Paul W. Wilson [b,*,3]

[a] *Institut de Statistique and CORE, Université Catholique de Louvain, Voie du Roman Pays 20, Louvain-la-Neuve, Belgium*
[b] *Department of Economics, University of Texas at Austin, Austin, TX 78712-1173, USA*

Received 1 August 1996; accepted 1 October 1997

Abstract

This paper develops a consistent bootstrap estimation procedure for obtaining confidence intervals for Malmquist indices of productivity and their decompositions. Although the exposition is in terms of input-oriented indices, the techniques can be trivially extended to the output orientation. The bootstrap methodology is an extension of earlier work described in Simar and Wilson (Simar, L., Wilson, P.W., 1998, Management Science). Some empirical examples are also given, using data on Swedish pharmacies.

Keywords: DEA; Productivity; Resampling; Bootstrap; Malmquist indices

1. Introduction

Färe et al. (1992) merge ideas on measurement of efficiency from Farrell (1957) and on measurement of productivity from Caves et al. (1982) to develop a Malmquist index of productivity change. Caves et al. define their input-based Malmquist productivity index as the ratio of two input distance functions, while assuming no technical inefficiency in the sense of Farrell. Färe et al. extend the Caves et al. approach by dropping the assumption of no technical inefficiency and developing a Malmquist index of productivity that can be decomposed into indices describing changes in technology and efficiency. We extend the Färe et al. approach by giving a statistical interpretation to their Malmquist productivity index and its components, and by presenting a bootstrap algorithm which may be used to estimate confidence intervals for the indices. This work will allow researchers to speak in terms of whether changes in productivity, efficiency, or technology are significant in a statistical sense. In other words, our methods can be used to determine whether indicated changes in productivity, efficiency, or

* Corresponding author. Tel.: 512 471 3211; fax: 512 471 3510; e-mail: wilson@eco.utexas.edu.

[1] Pontus Roos graciously provided the data used in the empirical examples. We alone, of course, are responsible for any remaining errors or omissions.

[2] Research support from the contract "Projet d'Actions de Recherche Concertées" (PARC No. 93/98–164) of the Belgian Government is gratefully acknowledged.

[3] Research support from the Management Science Group, US Department of Veterans Affairs, is gratefully acknowledged.

0377-2217/99/$ – see front matter
PII: S0377-2217(97)00450-5

technology are real, or merely artifacts of the fact that we do not know the true production frontiers and must estimate them from finite samples.

The input-based Malmquist index of productivity developed by Färe et al. measures productivity change between times t_1 and t_2. This index, as well as its component indices describing changes in technology and efficiency, consists of ratios of input distance functions (a more rigorous description appears in Section 2). However, Färe et al. do not distinguish between the underlying *true* distance functions and their *estimates*. For example, as a prelude to their Eq. (4) (page 88), they state that "the value of the distance function... is obtained as the solution to the linear programming problem...". In fact, solving their linear programming problem yields an *estimate* of the distance function, not the distance function itself. Färe et al. are not alone in this regard; indeed, the literature on nonparametric efficiency measurement is filled with such statements. Lovell (1993) and others have labeled nonparametric, linear-programming based approaches to efficiency measurement as *deterministic*, which seems to suggest that these approaches have no statistical underpinnings. Yet, if one views production data as having been generated from a distribution with bounded support over the true production set, then efficiency, and changes in productivity, technology, and efficiency, are always measured relative to *estimates* of underlying, *true* frontiers, conditional on observed data resulting from the underlying (and unobserved) data-generating process. Consequently, the estimates researchers are interested in involved uncertainty due to sampling variation. [4]

[4] The literature is also filled with references to the "observed best-practice frontier", typically taken to be the boundary of the convex, conical, or free-disposal hull of the observed data. But, one may always ask what would happen if another observation were obtained. Clearly, an additional observation might push the observed best-practice frontier outward (but never inward), although not beyond underlying boundary of the true production set. Authors who use such terminology have merely conditioned their analysis on an estimate of the true production frontier. While possible, this is unnecessary.

Simar and Wilson (1998) develop a bootstrap procedure which may be used to estimate confidence intervals for distance functions used to measure technical efficiency, and demonstrate that the key to statistically consistent estimation of these confidence intervals lies in the replication of the unobserved data-generating process. This paper extends those ideas to the case of Malmquist indices constructed from nonparametric distance function estimates using data from different time periods.

In the next section, we define the input-based Malmquist productivity index and the distance functions from which it is constructed, and describe how the productivity index can be decomposed into indices of efficiency change and technical shift. We also briefly discuss how these measures can be estimated nonparametrically using linear programming techniques. While we focus on input-based indices, one may trivially extend our results to output-based measures by merely modifying our notation. The bootstrap procedure is presented in Section 3. In Section 4, we illustrate the bootstrap estimation using a panel of data on Swedish pharmacies previously examined by Färe et al. Conclusions are given in Section 5.

2. Estimating Malmquist indices

To begin, consider firms which produce m outputs from n inputs. Let $\boldsymbol{x} \in \mathbb{R}^n_+$ and $\boldsymbol{y} \in \mathbb{R}^m_+$ denote input and output vectors, respectively. The production possibilities set at time t is given by the closed set

$$\mathscr{P}^t = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x} \text{ can produce } \boldsymbol{y} \text{ at time } t\}, \tag{1}$$

which may be described in terms of its sections

$$\mathscr{X}^t(\boldsymbol{y}) = \left\{\boldsymbol{x} \in \mathbb{R}^n_+ \mid (\boldsymbol{x}, \boldsymbol{y}) \in \mathscr{P}^t\right\}, \tag{2}$$

i.e., its corresponding input requirement sets. Shephard (1970) discusses assumptions one may reasonably make regarding $\mathscr{X}^t(\boldsymbol{x})$ (and hence $\mathscr{P}^t$); typical assumptions, which we adopt, are (i) $\mathscr{X}^t(\boldsymbol{y})$ is convex for all $\boldsymbol{y}$, t; (ii) all production requires use of some inputs, i.e., $0 \notin \mathscr{X}^t(\boldsymbol{y})$ if $\boldsymbol{y} \geqslant 0, \boldsymbol{y} \neq 0$;

and (iii) both inputs and outputs are strongly disposable, i.e.,

$$\tilde{\boldsymbol{x}} \geqslant \boldsymbol{x} \in \mathscr{X}^t(\boldsymbol{y}) \;\Rightarrow\; \tilde{\boldsymbol{x}} \in \mathscr{X}^t(\boldsymbol{y})$$

and

$$\tilde{\boldsymbol{y}} \leqslant \boldsymbol{y} \Rightarrow \mathscr{X}^t(\tilde{\boldsymbol{y}}) \subseteq \mathscr{X}^t(\boldsymbol{y}).$$

Let subscript i, $i = 1, \ldots, N$, denote a particular firm i; the N firms are each observed at (the same) two points in time. The Shephard (1970) input distance function for firm i at time t_1, relative to the technology existing at time t_2, is defined as

$$D_i^{t_1|t_2} \equiv \sup\{\theta > 0 \mid \boldsymbol{x}_{it_1}/\theta \in \mathscr{X}^{t_2}(\boldsymbol{y}_{it_1})\}. \tag{3}$$

The distance function $D_i^{t_1|t_2}$ gives a normalized measure of distance from the ith firm's position in the input/output space at time t_1 to the boundary of the production set at time t_2 in the hyperplane where outputs remain constant. If $t_1 = t_2$, then we have a measure of efficiency relative to the contemporaneous technology, and $D_i^{t|t} \geqslant 1$. If $t_1 \neq t_2$, then $D_i^{t_1|t_2}(<, =, >)1$.

Färe et al. (1992) write their Malmquist productivity index as

$$\mathscr{M}_i(t_1, t_2) \equiv \frac{D_i^{t_2|t_2}}{D_i^{t_1|t_1}} \times \left(\frac{D_i^{t_2|t_1}}{D_i^{t_2|t_2}} \times \frac{D_i^{t_1|t_1}}{D_i^{t_1|t_2}} \right)^{(1/2)}, \tag{4}$$

where $t_2 > t_1$. Values $\mathscr{M}_i(t_1, t_2) < 1$ indicate improvements in productivity between t_1 and t_2, while values $\mathscr{M}_i(t_1, t_2) > 1$ indicate decreases in productivity from time t_1 to t_2 ($\mathscr{M}_i(t_1, t_2) = 1$ would indicate no change in productivity). The ratio $D_i^{t_2|t_2}/D_i^{t_1|t_1}$ in Eq. (4) measures the change in input technical efficiency between periods t_1 and t_2, and defines an input-based index of efficiency change:

$$\mathscr{E}_i(t_1, t_2) \equiv \frac{D_i^{t_2|t_2}}{D_i^{t_1|t_1}}. \tag{5}$$

Values of $\mathscr{E}_i(t_1, t_2)$ less than (greater than) unity indicate improvements (decreases) in efficiency between t_1 and t_2. Similarly, the remaining part of the right-hand side of Eq. (4) defines an input-based measure of technical change:

$$\mathscr{T}_i(t_1, t_2) \equiv \left(\frac{D_i^{t_2|t_1}}{D_i^{t_2|t_2}} \times \frac{D_i^{t_1|t_1}}{D_i^{t_1|t_2}} \right)^{(1/2)}. \tag{6}$$

As with $\mathscr{M}_i(t_1, t_2)$ and $\mathscr{E}_i(t_1, t_2)$, values of $\mathscr{T}_i(t_1, t_2)$ less than (greater than) unity indicate technical progress (regress) between times t_1 and t_2. [5]

Unfortunately, the production set $\mathscr{P}^t$ is typically unobserved; similarly, $\mathscr{X}^t(\boldsymbol{x})$ is also unobserved, as are the values of the distance functions which appear in the Malmquist index in Eq. (4) and its components in Eqs. (5) and (6). The indices in Eqs. (4)–(6) represent true values which must be estimated. Substituting estimators for the corresponding true distance function values in Eqs. (4)–(6) yields estimators $\hat{\mathscr{M}}_i(t_1, t_2)$, $\hat{\mathscr{E}}_i(t_1, t_2)$, and $\hat{\mathscr{T}}_i(t_1, t_2)$ of the productivity, efficiency, and technology change indices, respectively.

Estimation of the input distance functions comprising Eqs. (4)–(6) requires estimation of $\mathscr{P}^t$ and $\mathscr{X}^t(\boldsymbol{y})$. Given a sample

$$\mathscr{S} = \{(\boldsymbol{x}_{it}, \boldsymbol{y}_{it}) \mid i = 1, \ldots, N;\ t = 1, 2\}$$

of observations on N firms in 2 periods, there are several ways in which $\mathscr{P}^t$ may be estimated. A common approach is to estimate $\mathscr{P}^t$ by the conical hull of the sample observations, which is tantamount to assuming constant returns to scale for the production technology. The corresponding estimate of the input requirement set is

$$\hat{\mathscr{X}}^t(\boldsymbol{y}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{y} \leqslant \boldsymbol{Y}^t\boldsymbol{q},\ \boldsymbol{x} \geqslant \boldsymbol{X}^t\boldsymbol{q},\ \boldsymbol{q} \in \mathbb{R}_+^N\}, \tag{7}$$

where $\boldsymbol{Y}^t = [\boldsymbol{y}_{1t} \ldots \boldsymbol{y}_{Nt}]$, $\boldsymbol{X}^t = [\boldsymbol{x}_{1t} \ldots \boldsymbol{x}_{Nt}]$, with $\boldsymbol{x}_{it}$ and $\boldsymbol{y}_{it}$ denoting $(n \times 1)$ and $(m \times 1)$ vectors of observed inputs and outputs, respectively, and $\boldsymbol{q}$ is an $(N \times 1)$ vector of intensity variables. This implies the distance function estimator

$$\hat{D}_i^{t_1|t_2} \equiv \sup\left\{\lambda > 0 \mid \boldsymbol{x}_{it_1}/\lambda \in \hat{\mathscr{X}}^{t_2}(\boldsymbol{y}_{it_1})\right\}, \tag{8}$$

[5] Note that output distance functions can be defined similar to the input distance function in Eq. (3). Hence output-based measures of productivity, efficiency, and technical change can be constructed by merely replacing the input distance functions in Eqs. (4)–(6) with the corresponding output distance functions.

which may be computed by solving the linear program

$$\left(\hat{D}_i^{t_1|t_2}\right)^{-1} = \min\left\{\lambda \mid \boldsymbol{y}_{it_1} \leqslant \boldsymbol{Y}^{t_2}\boldsymbol{q}_i,\ \lambda\boldsymbol{x}_{it_1} \geqslant \boldsymbol{X}^{t_2}\boldsymbol{q}_i,\ \boldsymbol{q}_i \in \mathbb{R}_+^N\right\}, \quad (9)$$

where $t_1(<,=,>)t_2$.

Alternatively, the production set $\mathscr{P}^t$ could be estimated by the convex hull of the sample observations, which amounts to adding the constraint $\vec{\mathbf{1}}\boldsymbol{q}_i = 1$ in Eqs. (7)–(9), where $\vec{\mathbf{1}}$ is a $(1 \times N)$ vector of ones. This yields a distance function estimator $\tilde{D}_i^{t_1|t_2}$ which, when substituted into Eqs. (4)–(6), yields estimators $\tilde{\mathscr{M}}_i(t_1,t_2)$, $\tilde{\mathscr{E}}_i(t_1,t_2)$, and $\tilde{\mathscr{T}}_i(t_1,t_2)$ of the productivity, efficiency, and technology change indices, respectively. Other estimators of $\mathscr{P}^t$ are also possible; for instance, one could use the free disposal hull of the sample observations suggested by Deprins et al. (1984). Use of the conical hull as in Färe et al. (1992) implicitly assumes that the production technology exhibits constant returns to scale, allowing the Malmquist index in Eq. (4) to be interpreted as an index of total factor productivity. [6]

3. Bootstrapping the Malmquist indices

The methodology for bootstrapping distance function estimators such as Eq. (9) presented in Simar and Wilson (1998) are easily adapted to the present case, except here the possible time-dependence structure of the data must be taken into account. As in our earlier work, we assume a data-generating process where firms randomly deviate from the underlying true frontier in a radial input direction. These random deviations from the contemporaneous frontier at time t, measured by the Shephard input distance function $D_i^{t|t}$ in Eq. (3), are further assumed to result from inefficiency. At a given point in time, the marginal density of these deviations has support bounded on the left at unity.

Bootstrapping involves replicating this data-generating process, generating an appropriately large number B of pseudosamples

$$\mathscr{S}^* = \left\{(\boldsymbol{x}_{it}^*, \boldsymbol{y}_{it}^*) \mid i = 1,\ldots,N;\ t = 1,2\right\},$$

and applying the original estimators to these pseudosamples. For each bootstrap replication $b = 1,\ldots,B$, we use Eq. (9) to measure the distance from each observation in the original sample $\mathscr{S}$ to the frontiers estimated for either period from the pseudodata in $\mathscr{S}^*$. This is accomplished by solving, for the case of the conical hull estimator,

$$\left(\hat{D}_i^{t_1|t_2*}\right)^{-1} = \min\left\{\lambda \mid \boldsymbol{y}_{it_1} \leqslant \boldsymbol{Y}^{t_2*}\boldsymbol{q}_i,\ \lambda\boldsymbol{x}_{it_1} \geqslant \boldsymbol{X}^{t_2*}\boldsymbol{q}_i,\ \boldsymbol{q}_i \in \mathbb{R}_+^N\right\}, \quad (10)$$

where $\boldsymbol{Y}^{t*} = [\boldsymbol{y}_{1t}^* \ldots \boldsymbol{y}_{Nt}^*]$ and $\boldsymbol{X}^{t*} = [\boldsymbol{x}_{1t}^* \ldots \boldsymbol{x}_{Nt}^*]$. For two time periods t_1, t_2, this yields bootstrap estimates $\{\hat{D}_i^{t_1|t_1*}(b), \hat{D}_i^{t_2|t_2*}(b), \hat{D}_i^{t_1|t_2*}(b), \hat{D}_i^{t_2|t_1*}(b)\}_{b=1}^B$ for each firm $i = 1,\ldots,N$. [7] These estimates can then be used to construct bootstrap estimates $\hat{\mathscr{M}}_i^*(t_1,t_2)(b)$, $\hat{\mathscr{E}}_i^*(t_1,t_2)(b)$, and $\hat{\mathscr{T}}_i^*(t_1,t_2)(b)$ (where $i = 1,\ldots,N$ and $b = 1,\ldots,B$) corresponding to Eqs. (4)–(6), respectively, by replacing the true distance function values in Eqs. (4)–(6) with their corresponding bootstrap estimates.

Once these bootstrap values have been computed, we can correct for any finite-sample bias in the original estimators of the distance functions or

[6] Allowing for variable returns by using the convex hull or free-disposal hull estimators does not guarantee solutions to Eq. (8) for all observations when $t_1 \neq t_2$, and in addition prevents the Malmquist productivity index from being interpreted as an index of total factor productivity. However, if the true technology contains regions with either increasing or decreasing returns to scale, then the conical hull estimator will fail to converge to $\mathscr{P}^t$ as $N \to \infty$, and thus will give inconsistent estimates. The convex and free-disposal hull estimators are consistent regardless of whether returns to scale are constant or variable in the sense that they converge to $\mathscr{P}^t$ as $N \to \infty$, but these estimators may converge more slowly than the conical hull estimator if the true technology is one of constant returns to scale everywhere. See Korostelev et al. (1995a) and Korostelev et al. (1995b) for rates of convergence of the various estimators of $\mathscr{P}^t$.

[7] Note that $\hat{D}_i^{t_1|t_1*}$, $\hat{D}_i^{t_2|t_2*}$, and $\hat{D}_i^{t_2|t_1*}$ can be computed by (i) changing t_2 to t_1, (ii) changing t_1 to t_2, or (iii) reversing t_1 and t_2 in Eq. (10), respectively. For purposes of the Malmquist index and its components, we would typically order the time periods so that t_1 occurs before t_2.

the Malmquist indices, and confidence intervals at the desired level of significance can be constructed using the simple procedure outlined below. To illustrate, consider the set of bootstrap estimates for the Malmquist index for firm i: $\{\hat{\mathscr{M}}_i^*(t_1,t_2)(b)\}_{b=1}^B$. The efficiency and technology change indices can be analyzed similarly by merely changing $\mathscr{M}$ to either $\mathscr{E}$ or $\mathscr{T}$ in the notation below. The bootstrap bias estimate for the original estimator $\hat{\mathscr{M}}_i(t_1,t_2)$ is

$$\widehat{\text{bias}}_B[\hat{\mathscr{M}}_i(t_1,t_2)] = B^{-1}\sum_{b=1}^{B}\hat{\mathscr{M}}_i^*(t_1,t_2)(b) - \hat{\mathscr{M}}_i(t_1,t_2), \tag{11}$$

which is the empirical bootstrap analog of $E[\hat{\mathscr{M}}_i(t_1,t_2)] - \mathscr{M}_i(t_1,t_2)$. Therefore, a bias-corrected estimate of $\mathscr{M}_i(t_1,t_2)$ may be computed as

$$\hat{\hat{\mathscr{M}}}_i(t_1,t_2) = \hat{\mathscr{M}}_i(t_1,t_2) - \widehat{\text{bias}}_B[\hat{\mathscr{M}}_i(t_1,t_2)] = 2\hat{\mathscr{M}}_i(t_1,t_2) - B^{-1}\sum_{b=1}^{B}\hat{\mathscr{M}}_i^*(t_1,t_2)(b). \tag{12}$$

Unfortunately, as Efron and Tibshirani (1993) note, the bias-corrected estimator in Eq. (12) may have higher mean-square error than the original estimator. [8] The variance of the summation term on the right-hand side of the second line of Eq. (12) can be made arbitrarily small by increasing B; yet, even if $B \to \infty$, the bias-corrected estimator $\hat{\hat{\mathscr{M}}}_i(t_1,t_2)$ will have variance equal to four times that of the original estimator, $\hat{\mathscr{M}}_i(t_1,t_2)$, due to the first term on the second line of Eq. (12). The mean-square errors of the original and bias-corrected estimators can be compared by using the sample variance s_{*i}^2 of the bootstrap values $\{\hat{\mathscr{M}}_i^*(t_1,t_2)(b)\}_{b=1}^B$ to estimate the variance of $\hat{\mathscr{M}}_i(t_1,t_2)$. Then the estimated mean-square error of $\hat{\hat{\mathscr{M}}}_i(t_1,t_2)$ is $4s_{*i}^2$, and the estimated mean-square error of $\hat{\mathscr{M}}_i(t_1,t_2)$ is $[s_{*i}^2 + (\widehat{\text{bias}}_B[\hat{\mathscr{M}}_i(t_1,t_2)])^2]$. A bit of algebra reveals that the bias-corrected estimator will likely have higher mean-square error than the original estimator unless $s_{*i}^2 < \frac{1}{3}(\widehat{\text{bias}}_B[\hat{\mathscr{M}}_i(t_1,t_2)])^2$.

[8] Note that we refer to $\hat{\hat{\mathscr{M}}}_i(t_1,t_2)$ as a bias-corrected, rather than an unbiased, estimator, since Eq. (12) involves only a first-order correction of the bias in $\hat{\mathscr{M}}_i(t_1,t_2)$.

In the case of individual distance function estimates, the corresponding bias may be substantial; but since Malmquist indices are defined as ratios of distance functions, the overall bias of these statistics may be somewhat less than for individual distance function estimates since the terms in both the numerator and the denominator are biased in the same direction. In any case, whether bias corrections such as Eq. (12) should be employed is always an empirical question to be answered by the data in any application.

To estimate confidence intervals for the Malmquist index, note that the idea behind the bootstrap is to approximate the unknown distribution of $(\hat{\mathscr{M}}_i(t_1,t_2) - \mathscr{M}_i(t_1,t_2))$ by the distribution of $(\hat{\mathscr{M}}_i^*(t_1,t_2) - \hat{\mathscr{M}}_i(t_1,t_2))$ conditioned on the original data $\mathscr{S}$. As noted above, our bootstrap procedure yields bootstrap values $\{\hat{\mathscr{M}}_i^*(t_1,t_2)(b)\}_{b=1}^B$, which with the original estimate $\hat{\mathscr{M}}_i(t_1,t_2)$, can be used to obtain an empirical approximation to the second distribution.

If we knew the distribution of $(\hat{\mathscr{M}}_i(t_1,t_2) - \mathscr{M}_i(t_1,t_2))$, then it would be trivial to find values a_α, b_α such that

$$\text{Prob}\Big(-b_\alpha \leqslant \hat{\mathscr{M}}_i(t_1,t_2) - \mathscr{M}_i(t_1,t_2) \leqslant -a_\alpha\Big) = 1-\alpha \tag{13}$$

for some small value of α, say 0.10 or 0.05. Since we do not know this distribution, we can use the bootstrap values to find values a_α^*, b_α^* such that the statement

$$\text{Prob}\Big(-b_\alpha^* \leqslant \hat{\mathscr{M}}_i^*(t_1,t_2) - \hat{\mathscr{M}}_i(t_1,t_2) \leqslant -a_\alpha^* | \mathscr{S}\Big) = 1-\alpha \tag{14}$$

is true with high probability. Mechanically, this involves sorting the values

$$\Big(\hat{\mathscr{M}}_i^*(t_1,t_2)(b) - \hat{\mathscr{M}}_i(t_1,t_2)\Big), \quad b = 1,\ldots,B$$

by algebraic value, deleting $((\alpha/2)\times 100)$-percent of the elements at either end of this sorted array, and then setting $-b_\alpha^*$ and $-a_\alpha^*$ equal to the endpoints of the resulting (sorted) array, with $a_\alpha^* \leqslant b_\alpha^*$. When we say that Eq. (14) is "true with high probability", we mean that this can be made so by making the number of bootstrap replications, B,

large enough; as $B \to \infty$, the probability that Eq. (14) is true approaches 1. [9]

Since

$$\left[\hat{\mathscr{M}}_i(t_1,t_2) - \mathscr{M}_i(t_1,t_2)\right] \overset{\text{approx}}{\sim} \left[\hat{\mathscr{M}}_i^*(t_1,t_2) - \hat{\mathscr{M}}_i(t_1,t_2)\right] \mid \mathscr{S}, \tag{15}$$

we obtain the bootstrap approximation

$$\text{Prob}\left(-b_\alpha^* \leqslant \hat{\mathscr{M}}_i(t_1,t_2) - \mathscr{M}_i(t_1,t_2) \leqslant -a_\alpha^* \mid \mathscr{S}\right) \approx 1-\alpha \tag{16}$$

by substituting a_α^* and b_α^* for a_α and b_α in Eq. (13) and noting the conditioning in Eq. (15). Rearranging the terms in parentheses in Eq. (16) yields an estimated $(1-\alpha)$-percent confidence interval

$$\hat{\mathscr{M}}_i(t_1,t_2) + a_\alpha^* \leqslant \mathscr{M}_i(t_1,t_2) \leqslant \hat{\mathscr{M}}_i(t_1,t_2) + b_\alpha^*, \tag{17}$$

and we say that the estimated Malmquist index is significantly different from unity (which would indicate no productivity change) if the interval in Eq. (17) does not include unity. [10]

The key to obtaining consistent bootstrap estimates of the confidence intervals lies in consistent replication of the data-generating process. As discussed in Simar and Wilson (1998), resampling from the empirical distribution of the data (i.e., drawing with replacement from the set of original observations on inputs and outputs, or equivalently, from the set of original distance function estimates) to construct the pseudosamples $\mathscr{S}^*$ will lead to inconsistent bootstrap estimation of the confidence intervals. This results from the empirical distribution placing a positive probability mass at the boundary of the estimated production set; this mass does not disappear as $N \to \infty$, and so the empirical distribution provides an inconsistent estimate of the underlying distribution of inefficiencies measured by the input distance function.

Using a smooth bootstrap procedure as in Simar and Wilson (1998) overcomes this problem and yields consistent estimates. When bootstrapping distance function estimates from a single cross-section of data, this may be accomplished by using a univariate kernel estimator of the density of the original distance function estimates, and then drawing from this estimated density to construct the pseudosamples $\mathscr{S}^*$ as in and Simar and Wilson (1998). In the present case, however, we have panel data, with the possibility of temporal correlation. For example, an inefficient firm in period one may be more likely to be inefficient in period two than a firm that is relatively more efficient in period one. To preserve any temporal correlation present in the data, we use kernel methods to estimate the joint density of $\left\{(\hat{D}_i^{t_1,t_1}, \hat{D}_i^{t_2,t_2})\right\}_{i=1}^N$.

The bivariate kernel density estimator with bivariate kernel function $K(\cdot)$ and bandwidth h is given by

$$\hat{f}(\boldsymbol{z}) = N^{-1}h^{-2}\sum_{i=1}^{N} K\left(\frac{(\boldsymbol{z}-\boldsymbol{Z}_i)}{h}\right), \tag{18}$$

where $\boldsymbol{z}$ has dimension (1×2) and $\boldsymbol{Z}_i = [\hat{D}_i^{t_1|t_1} \quad \hat{D}_i^{t_2|t_2}]$ is the ith row of the $(N\times 2)$ matrix containing the original data. [11] Note, however, that both $\hat{D}_i^{t_1,t_1}$ and $\hat{D}_i^{t_2,t_2}$ are bounded from below by unity. The density estimated from Eq. (18) can be shown to be inconsistent and asymptotically

[9] Of course, one could also choose $-b_\alpha^*$ and $-a_\alpha^*$ as the $100\times\tau_1$th and $100\times\tau_2$th percentiles of the sorted bootstrap differences, where $\tau_1 = \Phi(2\hat{w}_0 + w^{(\alpha/2)})$, $\tau_2 = \Phi(2\hat{w}_0 + w^{(1-\alpha/2)})$, $\hat{w}_0 = \Phi^{-1}\left(\#\{\hat{\mathscr{M}}_i^*(t_1,t_2)(b) < \hat{\mathscr{M}}_i(t_1,t_2)\}/B\right)$, $\Phi(\cdot)$ is the standard normal distribution function, and $w^{(\alpha)}$ is the $100\times\alpha$th percentile of the standard normal distribution so that $\Phi(w^{(\alpha)}) = \alpha$. This procedure involves a median-bias correction, and was used in Simar and Wilson (1998), and is discussed in Efron and Tibshirani (1993).

[10] In Simar and Wilson (1998), we constructed confidence intervals for distance functions using an estimate of bias analogous to Eq. (11). The approach in this paper avoids introducing the extra noise contained in this estimate. The bias inherent in the distance function estimates is implicitly accounted for here since we use the bootstrap values to construct an empirical distribution of *differences* as in Eq. (15).

[11] One might prefer to use different bandwidths in each direction; however, this is not necessary if the kernel function is scaled by an estimate of the covariance matrix of the data as discussed below.

biased when the support of f is bounded, as in our case. [12]

To overcome this problem, we adapt the univariate reflection method described by Silverman (1986) to our bivariate case. For the case of univariate data $\{z_i\}_{i=1}^N$ bounded from below at unity, the reflection method involves using the univariate kernel density estimator to estimate the density of the original observations and their reflections $\{z_i'\}_{i=1}^N$ about unity, where $z_i' = 2 - z_i$, $\forall\, i = 1, \ldots, N$. Truncating the resulting density estimate (based on $2N$ unbounded observations) on the left at unity yields the desired density estimate for the univariate case (see Simar and Wilson (1998) for an illustration). In the bivariate case, we proceed similarly, except that there are now two boundaries in $\mathbb{R}^2$.

First, we form $(N \times 1)$ vectors

$$\boldsymbol{A} = \left[\hat{D}_1^{t_1,t_1} \quad \ldots \quad \hat{D}_N^{t_1,t_1}\right]', \tag{19}$$

and

$$\boldsymbol{B} = \left[\hat{D}_1^{t_2,t_2} \quad \ldots \quad \hat{D}_N^{t_2,t_2}\right]'. \tag{20}$$

The values in $\boldsymbol{A}$ and $\boldsymbol{B}$ are bounded from below at unity. To reflect the distance function values about the boundaries in two-dimensional space, we form the $(4N \times 2)$ matrix represented in partitioned form by

$$\boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ 2-\boldsymbol{A} & \boldsymbol{B} \\ 2-\boldsymbol{A} & 2-\boldsymbol{B} \\ \boldsymbol{A} & 2-\boldsymbol{B} \end{bmatrix}. \tag{21}$$

The matrix $\boldsymbol{\Delta}$ contains $4N$ pairs of values corresponding to the two time periods. [13] The temporal correlation of the *original* data $[\boldsymbol{A} \quad \boldsymbol{B}]$ is measured by the estimated covariance matrix

$$\hat{\boldsymbol{\Sigma}} = \begin{bmatrix} \hat{\sigma}_1^2 & \hat{\sigma}_{12} \\ \hat{\sigma}_{12} & \hat{\sigma}_2^2 \end{bmatrix} \tag{22}$$

of the columns of $[\boldsymbol{A} \quad \boldsymbol{B}]$. [14] Note that $\hat{\boldsymbol{\Sigma}}$ is also necessarily the estimated covariance matrix of the reflected data $[2-\boldsymbol{A} \quad 2-\boldsymbol{B}]$. Moreover,

$$\hat{\boldsymbol{\Sigma}}_R = \begin{bmatrix} \hat{\sigma}_1^2 & -\hat{\sigma}_{12} \\ -\hat{\sigma}_{12} & \hat{\sigma}_2^2 \end{bmatrix} \tag{23}$$

must be the corresponding estimate of the covariance matrix of $[2-\boldsymbol{A} \quad \boldsymbol{B}]$ and $[\boldsymbol{A} \quad 2-\boldsymbol{B}]$.

Let $\boldsymbol{\Delta}_{j\cdot}$ denote the jth row of $\boldsymbol{\Delta}$. Then

$$\hat{g}(\boldsymbol{z}) = \frac{1}{4Nh^2}\sum_{j=1}^{4N} K_j\left(\frac{\boldsymbol{z} - \boldsymbol{\Delta}_{j\cdot}}{h}\right) \tag{24}$$

is a kernel estimator of the density of the $4N$ reflected data points represented by the rows of $\boldsymbol{\Delta}$, where $\boldsymbol{z} = [z_1 \quad z_2]$, and $K_j(\cdot)$ is the bivariate normal density function with shape $\hat{\boldsymbol{\Sigma}}$ for $j = 1, \ldots, N$; $2N+1, \ldots, 3N$ or shape $\hat{\boldsymbol{\Sigma}}_R$ for $j = N+1, \ldots, 2N$; $3N+1, \ldots, 4N$. Then a consistent estimate of the density of the original data $[\boldsymbol{A} \quad \boldsymbol{B}]$ with bounded support is given by

[12] The kernel function $K(\cdot)$ in Eq. (18) must integrate to one. At a point $\boldsymbol{z}$, Eq. (18) estimates the underlying density as the mean of N bivariate functions $K(\cdot)$ centered on the original data in $\boldsymbol{Z}$; consequently, if the data are unbounded, then $\hat{f}(\boldsymbol{z})$ necessarily integrates to unity. However, when the data are bounded as in our case, for data near the boundary, the corresponding bivariate kernel functions extend over the boundaries in one or both dimensions. Consequently, when the density estimate obtained from Eq. (18) is integrated over its bounded support, it will integrate to less than unity; the estimated density will be too small near the boundaries. Moreover, this problem remains as $N \to \infty$.

[13] To visualize the reflection, let a_i, b_i represent the ith elements of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively $(i = 1, \ldots, N)$. Then for N firms at times t_1 and t_2, we have N points (a_i, b_i) lying northeast of the point $(1,1)$ in two-dimensional Euclidean space. These points may be reflected by taking the "mirror images" about vertical and horizontal lines passing through $(1,1)$. This leaves a boundary along the horizontal line passing through $(1,1)$ to the left of this point, and along the vertical line passing through $(1,1)$ below this point. Hence, an additional reflection is required, obtained by taking the mirror image of the points to the southeast of $(1,1)$ about the vertical line passing through $(1,1)$, or equivalently, by taking the mirror image of the points to the northwest of $(1,1)$ about the horizontal line passing through $(1,1)$. Hence $\boldsymbol{\Delta}$ in Eq. (21) has $4N$ rows.

[14] We use the sample covariance matrix to estimate the covariance of $\boldsymbol{A}$ and $\boldsymbol{B}$. Alternatively, one might use a robust estimator of the covariance matrix, such as an M-estimator or the minimum volume ellipsoid estimator proposed by Rousseeuw (1985).

$$\hat{g}^*(z) = \begin{cases} 4\hat{g}(z) & \text{for } z_1 \geqslant 1,\ z_2 \geqslant 1, \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

To generate the random deviates needed for the bootstrap, we do not have to actually estimate the density of the observations in Δ; rather, we use the method suggested by Silverman (1986) and analogous to that used for the univariate case in Simar and Wilson (1998). First, we randomly draw with replacement N rows from Δ to form the $(N \times 2)$ matrix $\Delta^* = [\delta_{ij}]$, $i = 1, \ldots, N$, $j = 1, 2$ such that each row of Δ has equal probability of selection. Let $\bar{\delta}_{\cdot j} = N^{-1} \sum_{i=1}^{N} \delta_{ij}$ for $j = 1, 2$. Then compute the $(N \times 2)$ matrix

$$\Gamma = (1 + h^2)^{-1/2} \left(\Delta^* + h\epsilon^* - C \begin{bmatrix} \bar{\delta}_{\cdot 1} & 0 \\ 0 & \bar{\delta}_{\cdot 2} \end{bmatrix} \right) + C \begin{bmatrix} \bar{\delta}_{\cdot 1} & 0 \\ 0 & \bar{\delta}_{\cdot 2} \end{bmatrix}, \tag{26}$$

where C is an $(N \times 2)$ matrix of ones, which gives an $(N \times 2)$ matrix of bivariate deviates from the estimated density of Δ, scaled to have the first and second moment properties observed in the original sample represented by $[A \quad B]$. In addition, ϵ^* is an $(N \times 2)$ matrix containing N independent draws from the kernel functions $K_j(\cdot)$ in Eq. (24), with the ith row of ϵ^* representing (i) a draw from a normal density with shape $\hat{\Sigma}$ if $\Delta^*_{i\cdot}$ was drawn from $[A \quad B]$ or $[2 - A \quad 2 - B]$; or (ii) a draw from a normal density with shape $\hat{\Sigma}_R$ if $\Delta^*_{i\cdot}$ was drawn from $[2 - A \quad B]$ or $[A \quad 2 - B]$.

Draws from a bivariate $N(\mathbf{0}, \hat{\Sigma})$ density can be simulated by generating independent, identically distributed pseudorandom $N(0, 1)$ deviates (z_1, z_2) using the Box–Muller method (e.g., Press et al., 1986). The Cholesky decomposition of the (2×2) matrix $\hat{\Sigma}$ yields the lower triangular matrix

$$L = \begin{bmatrix} \ell_1 & 0 \\ \ell_2 & \ell_3 \end{bmatrix}, \tag{27}$$

where

$$LL' = \hat{\Sigma}, \quad \ell_1 = \hat{\sigma}_1, \quad \ell_2 = \hat{\sigma}_{12}/\hat{\sigma}_1, \quad \text{and} \quad \ell_3 = \left(\hat{\sigma}_2^2 - \hat{\sigma}_{12}^2/\hat{\sigma}_1^2 \right)^{1/2}.$$

Then $(\ell_1 z_1, \ell_2 z_1 + \ell_3 z_2) \sim N(\mathbf{0}, \hat{\Sigma})$. Draws from a $N(\mathbf{0}, \hat{\Sigma}_R)$ density can be simulated similarly by computing $(\ell_1 z_1, -\ell_2 z_1 + \ell_3 z_2) \sim N(\mathbf{0}, \hat{\Sigma}_R)$.

Finally, for each element γ_{ij} of Γ, set

$$\gamma^*_{ij} = \begin{cases} \gamma_{ij}, & \text{if } \gamma_{ij} \geqslant 1, \\ 2 - \gamma_{ij}, & \text{otherwise.} \end{cases} \tag{28}$$

The resulting $(N \times 2)$ matrix $\Gamma^* = [\gamma^*_{ij}]$ consists of two column-vectors of simulated distance function values. Pseudosamples $\mathcal{S}^*$ are then constructed by setting $x^*_{it_j} = \gamma^*_{ij} x_{it_j} / \hat{D}_i^{t_j|t_j}$ and $y^*_{it_j} = y_{it_j}$ for $i = 1, \ldots, N$, $j = 1, 2$. [15]

The only remaining issue is the choice of the bandwidth, h. Tapia and Thompson (1978), Silverman (1978, 1986), and Härdle (1990) discuss considerations relevant to the choice of h; in general, for a given sample size, larger values of h produce more diffuse (i.e., less efficient) estimates of the density, while very small values produce estimated densities with multiple modes. In the empirical examples which follow, we use Silverman's (1986) suggestion for bivariate data by setting $h = (4/5N)^{1/6}$ since we are using a bivariate normal kernel scaled to have the same shape as the data. [16]

Finally, we note that in many applications, one may wish to consider the evolution of efficiency, technology, and productivity over more than two periods. Such was the case considered by Färe et al. (1992) and in our examples in the next section. The

[15] Computing $x_{it_j} / \hat{D}_i^{t_j|t_j}$ scales the input vector back to the ostensibly efficient level indicated by the estimated frontier; multiplying by γ^*_{ij} simulates a random deviation away from this frontier. If we were using output distance functions and the output-based Malmquist index, we would retain the original input vector in the pseudosample and generate a new output vector.

[16] This bandwidth minimizes the approximate mean integrated square error of the density estimate when the data are bivariate normally distributed. Alternatively, one might use the least-squares cross-validation procedure discussed by Silverman (1986) to choose h. Since many of the elements in Δ will typically equal one, the least cross-validation function may suffer from degenerate behavior related to the discretization problem described by Silverman. The results in Simar and Wilson (1998) suggest, however, that the estimated confidence intervals are not very sensitive with respect to the choice of bandwidth.

Malmquist index (4) and its components in Eqs. (5) and (6) are defined over two periods; in our bootstrap procedure, we account for the covariance between successive periods in Eq. (22), but we ignore the covariance between periods that are not adjacent to each other in time. This suggests that one might increase statistical efficiency by accounting for this information in our estimation procedure. Unfortunately, however, doing so would require nonparametric estimation of a high dimensional density, rather than the bivariate estimation we employ in Eqs. (23) and (24). It is well known that kernel methods, as with most nonparametric estimation methods, suffer from the curse of dimensionality (see Silverman, 1986, for a discussion of the magnitude of this problem for the case of kernel density estimation). Therefore, any possible gains in statistical efficiency from considering the covariance between distance function estimates in nonsuccessive periods would almost certainly be overwhelmed by the increase in mean-square error resulting from estimating a high-dimensional density.

4. Empirical examples

Färe et al. (1992) describe annual data on 42 Swedish pharmacies from 1980 to 1989, which produce four outputs from four inputs. Färe et al. assume constant returns to scale and estimate distance functions using Eq. (10) to construct estimates $\hat{\mathscr{E}}_i(t_1,t_2)$, $\hat{\mathscr{T}}_i(t_1,t_2)$, and $\hat{\mathscr{M}}_i(t_1,t_2)$, which they report in Tables 1–3 of their article. Maintaining the assumption of constant returns to scale, we applied the bootstrap methods outlined in section three to obtain estimates of bias and variance, and to test for significant differences from unity, while setting $B = 2000$. In comparing the estimated biases and variances, we found that the bias correction in Eq. (12) for the efficiency, technology, and productivity change indices would increase mean-square error. Consequently, we do not report the bias corrected estimates $\hat{\hat{\mathscr{E}}}_i(t_1,t_2)$, $\hat{\hat{\mathscr{T}}}_i(t_1,t_2)$, and $\hat{\hat{\mathscr{M}}}_i(t_1,t_2)$. Consistent with Färe et al., we report the reciprocals of our original estimates (which are identical to the values reported by Färe et al.) in Tables 1–3 (respectively), so that numbers greater than unity denote progress while numbers less than unity denote regress. In addition, we use single asterisks (*) to indicate cases where the indices are significantly different from unity at the 0.10 level, and double asterisks (**) to indicate cases where the indices are significantly different from unity at the 0.05 level.

While examining changes in efficiency, Färe et al. find (page 96) "five pharmacies (nos. 15, 32, 33, 35, and 39) to be efficient in all time periods" as indicated by values of unity for efficiency change between all successive pairs of years in Table 1 for these five pharmacies. Our bootstrap reveals an additional six pharmacies (nos. 2, 14, 17, 20, 26, and 36) with statistically insignificant changes in efficiency (at either the 0.10 or 0.05 levels), for all pairs of years. Three more pharmacies had statistically insignificant changes in all but one pair of years (nos. 19, 31, and 38). In all, of the 237 estimates of efficiency change reported in Table 1 that are not equal to unity, only 113 (47.7%) are significantly different from unity at either 0.1 or 0.05.

Turning to our results for the technical change index in Table 2, our bootstrap results generally support the statements made by Färe et al. with respect to technical change. Färe et al. state that "between 1981 and 1982 almost all pharmacies showed technical progress", and our results show that our estimates of technical change are statistically significant at the 0.05 level in all but four instances; of these four pharmacies, three show significant technical change at the 0.01 level. Färe et al. find only one pharmacy (no. 33) whose radially efficient frontier point shows technical progress in all periods, but our results indicate that the changes for this pharmacy are not significant in four periods.

Similarly, our results for the index of productivity change in Table 3 generally support statements made by Färe et al. Where Färe et al. find productivity gains in 259 cases and productivity losses in 119 cases, we find significant (at 0.10) gains in 217 cases, and significant (at 0.10) losses in 91 cases; 81.5 percent of the estimates shown in Table 3 are significantly different from unity at the 0.10 level.

Table 1
Changes in efficiency (i.e., reciprocal of $\mathscr{E}_i(t_1, t_2)$), 42 Swedish pharmacies. Numbers greater than one indicate improvements (constant returns to scale)

No.	1980/1981	1981/1982	1982/1983	1983/1984	1984/1985	1985/1986	1986/1987	1987/1988	1988/1989
1	1.0022	0.8813**	1.1521**	1.0994**	0.9610*	0.9946	1.0205	1.0252	1.0000
2	1.0000	1.0000	1.0000	1.0000	1.0000	0.9807	1.0197	1.0000	1.0000
3	1.0000	0.9454	1.0578**	1.0000	1.0000	1.0000	1.0000	1.0000	0.8968**
4	1.0085	0.9132**	1.0326	0.9972	0.9816	0.9483**	0.9927	1.0541**	1.0205*
5	1.0686**	0.7166**	1.1822**	1.1875**	0.9659	0.9279**	0.9990	1.0348*	0.9756
6	1.0443**	0.8887**	1.2018**	0.9176*	1.0109	0.9984	0.9593*	1.0088	0.9667
7	1.1589**	1.0050	0.9380*	1.0833**	0.9984	0.9719*	0.9946	1.0897**	1.0000
8	1.5818**	0.7598**	1.3161**	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
9	0.9708	0.9709**	1.0063	1.2077**	0.9910	0.9117**	0.9704	0.9959	0.9970
10	1.0169	0.8248**	1.2096**	0.8933**	1.0987**	0.9667	1.0317	1.0108	1.0408
11	0.9656*	0.9300**	1.2463**	0.8941	1.0306	1.0175	1.0024	1.0138	1.0495
12	0.9949	1.0359	1.0000	0.9714	0.9696	0.9567*	1.0533	0.8924**	1.0342*
13	1.0000	1.0000	1.0000	0.9858	0.9428*	1.0446	0.9733	1.0583*	1.0000
14	1.0226	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
16	1.0317*	0.9457**	1.1173**	1.0010	0.9482**	0.9937	0.9594	1.0377*	1.0233
17	1.0000	0.9348	1.0697	1.0000	1.0000	1.0000	1.0000	0.9990	0.9990
18	0.9915	1.0086	1.0000	0.8828	1.0662*	1.0375	0.9851	1.0396	0.9132**
19	1.1690**	1.0000	0.9761	1.0245	1.0000	1.0000	1.0000	1.0000	1.0000
20	0.9634	1.0013	1.0209	1.0154	1.0000	1.0000	1.0000	1.0000	1.0000
21	1.0000	0.8986**	1.0704**	0.9956	0.9098**	0.9030**	1.0121	1.1707**	0.9927
22	0.9306	0.8546**	1.1616**	0.9297**	0.9623	0.9710	1.0481*	1.0203	1.0519**
23	1.0000	0.8787**	1.1380**	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
24	1.0838**	0.9348**	1.0139	0.9430**	1.0173	1.0803**	0.9406**	1.0831**	0.9229**
25	0.9918	0.9335**	1.0127	1.0666**	0.9386**	1.0654**	1.0000	1.0000	1.0000
26	1.0000	1.0000	1.0000	1.0000	1.0000	0.9363	1.0680	1.0000	1.0000
27	1.0908**	0.9265**	1.0013	0.9000**	1.0228	0.9368**	1.0401**	0.9870	1.0659**
28	1.0000	0.8143**	1.1384**	0.9974	0.9692**	0.9844	1.0040	1.0586**	1.0664
29	0.9353**	0.8031**	1.1935**	1.0476**	0.9695*	0.9878	0.9943	0.9679	1.1612**
30	1.0373	1.0000	1.0000	0.8570	1.0630**	0.9925	1.0489*	1.0544	1.0000
31	1.0000	1.0000	1.0000	0.8428	0.9913	1.0172	1.0553**	0.9703	1.0057
32	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
33	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
34	1.0838	1.0000	1.0000	1.0000	1.0000	1.0000	0.8548	1.0887**	1.0496**
35	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
36	1.0000	1.0000	1.0000	0.9057	1.0517	1.0498	1.0000	1.0000	1.0000
37	1.1145**	0.9227**	0.9912	1.0098	1.0251	0.7797**	1.3062**	0.9867	1.0818**
38	1.0000	1.0000	1.0000	0.9814	1.0190	1.0000	1.0000	0.9721	0.9338**
39	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
40	0.8441**	0.8560**	1.0969**	0.9782	1.3137**	0.9892	0.9970	1.0184	1.0137
41	1.0000	0.9266	0.9930	0.9668	1.0103	1.1126**	0.8243**	0.9030**	1.0625**
42	1.0000	0.9064**	1.1033	1.0000	1.0000	0.9509	0.9896	1.0609**	0.9520*

Note: Single asterisks (*) denote significant differences from unity at 0.10; double asterisks (**) denote significant differences from unity at 0.05.

As noted in Section 3, assuming constant returns to scale and using the conical hull of the observed data to estimate the production set will yield statistically inconsistent distance function estimates when the true technology has nonconstant returns to scale. The convex hull estimator of the production set, however, converges to the true production set regardless of whether returns to scale are constant or otherwise. Therefore, lacking a formal test of returns to scale, using distance

Table 2
Changes in technology (i.e., reciprocal of $\mathcal{T}_i(t_1, t_2)$), 42 Swedish pharmacies. Numbers greater than one indicate improvements (constant returns to scale)

No.	1980/1981	1981/1982	1982/1983	1983/1984	1984/1985	1985/1986	1986/1987	1987/1988	1988/1989
1	0.9049**	1.1028**	0.9894	0.9366**	1.0498**	0.9792	1.0216	1.0164	1.0466*
2	1.0077	1.1665**	0.9641	0.9672	1.0365	1.0311	1.1684**	1.1221**	1.0588
3	0.9928	1.1403**	0.9585	0.9446**	1.0511*	1.1135**	1.0285	1.0811**	1.1033**
4	0.9601**	1.1093**	0.9808	0.9485**	1.0410*	1.0397**	1.0937**	1.0405**	1.0300*
5	0.9550**	1.4118**	0.8887**	0.7916**	1.0651**	1.1419**	1.0599	1.0711**	1.1242**
6	0.9140**	1.1741**	0.9398**	0.8206**	1.0485**	1.0477**	1.1035**	1.0592**	1.0250
7	0.8587**	1.0565**	1.0257	0.9020**	1.0411**	1.0490**	1.1089**	1.0351	1.0087
8	0.9270**	1.4031**	0.8190**	1.1204**	1.0555	1.1297**	0.9825	1.0550	1.0423
9	0.9739**	1.1573**	0.9866	0.8380**	1.0355	1.0960**	1.0861**	1.1067**	1.0861**
10	0.9335**	1.2912**	0.8815**	1.0536**	1.0560**	1.0936**	1.0597**	1.0307	1.0268
11	0.9129**	1.1526**	0.9860	0.7631**	1.0473**	1.0717**	1.0946**	1.0509**	1.1222**
12	1.0392*	1.7109**	0.6984**	1.0038	1.0927**	1.0810**	1.0928**	1.0257	1.0093
13	1.0152	1.1443**	0.9577**	0.9966	0.9832	1.0365*	1.1484**	0.9818	1.0494*
14	1.0673**	1.1855**	1.0425	1.0163	0.9446	1.0675*	1.0500	1.1283**	0.9680
15	1.0103	1.0694**	0.9213**	1.0335	1.0281	0.9859	0.9944	0.8965**	1.0686**
16	0.9909	1.1352**	0.9583**	0.9409**	1.0410**	1.0096	1.0508*	1.0513**	1.0342
17	1.0519	1.0607**	1.0158	0.9376	1.0316	0.9879	1.0583	0.9019**	1.0633**
18	0.9734**	1.1271**	0.9304**	0.9698	1.0435**	0.9623**	1.0262	0.9522*	1.0449
19	0.7989**	1.1181**	0.9996	1.0828**	1.0866	1.0730*	0.9842	0.9494	0.9882
20	0.9809*	1.1202**	1.0441	1.0377*	1.0855**	1.0316	1.0162	0.9800	1.0593**
21	0.9648	1.1718**	0.9460**	1.0669**	1.1073**	1.0714**	1.0694**	1.0110	1.0337**
22	1.0322	1.1324**	0.8821**	0.9954	1.0032	0.9742	1.0015	0.9526**	1.0800**
23	0.6706**	1.0935**	0.9154*	0.9819	1.0821*	0.9445*	1.0519*	1.0074	1.0324
24	0.9185**	1.0865**	0.9881	0.9443**	1.0488**	1.0234	1.1721**	1.0796**	1.1412**
25	0.8828**	1.2458**	0.9749*	0.9074**	1.0598**	0.9405**	1.1102**	1.0295	1.0710
26	1.0291	1.0205*	1.0122	0.9696	1.1527*	0.8504**	1.2002**	1.0896*	1.0917*
27	0.9231**	1.0838**	1.0381	0.9296**	1.0504**	1.0291*	1.0292	1.0524**	1.0317
28	0.9142**	1.2740**	0.8912**	0.9196**	1.0716**	0.9758	1.0407	1.0224	1.0589**
29	0.8945**	1.2615**	0.8677**	0.9186**	1.0452**	1.0170	1.0511*	1.0384*	1.0392**
30	0.9193**	1.1177**	1.1206**	0.9220**	1.0708**	1.0505**	1.0109	1.0401**	1.0132
31	0.9686	1.0889**	1.0560	0.8360**	1.0847**	1.0472**	1.1085**	1.0408**	1.0450**
32	0.9815	1.0591**	1.0603	0.9641	1.0787**	1.0178	1.1063**	1.0125	0.9756
33	1.0376	1.0371*	1.0646	1.0512**	1.0025	1.1112**	1.1718**	1.0164	1.1154*
34	0.8972**	1.1141**	1.0966	0.9307	0.9303	1.0105	0.9013**	1.0588**	1.0321
35	1.3316**	0.7906**	1.0253	1.0852*	0.9586	0.9601	0.9651	1.0019	0.9189*
36	0.9829	1.0327*	1.0409	0.9244*	1.0193	0.9823	1.0634	1.0328	0.9816
37	0.9568**	1.2327**	1.0028	0.9613*	1.0661**	1.0323*	1.0420*	0.9903	1.0466**
38	1.0247	1.1834**	1.0265	0.9246	1.0309	1.1279**	1.0477	1.0904**	1.1078**
39	1.0224	1.0889**	1.1285	0.8541**	0.9857	1.0212	1.1629**	1.0270	1.0096
40	1.1425**	1.2353**	0.9351**	1.0002	0.9746	1.0977**	1.0964**	1.1014**	1.0663**
41	1.0934	0.9847	0.9698**	1.0295*	0.9670*	1.0719**	1.1459**	1.0068	1.0423*
42	1.0830	1.3972**	0.9142	0.9993	1.0435	1.0525	1.0251	1.0691**	1.0557*

Note: Single asterisks (*) denote significant differences from unity at 0.10; double asterisks (**) denote significant differences from unity at 0.05.

function estimators based on the convex hull to construct the Malmquist index may be a safer approach. As noted earlier, though, the Malmquist productivity index loses its interpretation as an index of total factor productivity when we do this.

5. Conclusions

Malmquist indices have been widely used in recent years to examine changes in productivity, efficiency, and technology not only within a variety

Table 3
Changes in productivity (i.e., reciprocal of $\mathcal{M}_i(t_1, t_2)$), 42 Swedish pharmacies. Numbers greater than one indicate improvements (constant returns to scale)

No.	1980/1981	1981/1982	1982/1983	1983/1984	1984/1985	1985/1986	1986/1987	1987/1988	1988/1989
1	0.9069**	0.9719**	1.1399**	1.0297	1.0088	0.9740**	1.0425**	1.0419**	1.0466**
2	1.0077	1.1665**	0.9641	0.9672	1.0365**	1.0112	1.1913**	1.1221**	1.0588**
3	0.9928*	1.0781**	1.0138**	0.9446**	1.0511**	1.1135**	1.0285**	1.0811**	0.9893
4	0.9682**	1.0129**	1.0129*	0.9459**	1.0220**	0.9859	1.0857**	1.0969**	1.0511**
5	1.0206	1.0117**	1.0505**	0.9401**	1.0288**	1.0597**	1.0588**	1.1084**	1.0969**
6	0.9545**	1.0435**	1.1294**	0.7530**	1.0601**	1.0460**	1.0585**	1.0685**	0.9909
7	0.9951	1.0619**	0.9622**	0.9770**	1.0394**	1.0195	1.1029**	1.1279**	1.0087
8	1.4661**	1.0662**	1.0779**	1.1204**	1.0555**	1.1297**	0.9825	1.0550**	1.0423**
9	0.9454**	1.1236**	0.9930	1.0120	1.0263	0.9992	1.0541**	1.1022**	1.0830**
10	0.9493**	1.0650**	1.0663**	0.9411**	1.1601**	1.0573**	1.0933**	1.0420**	1.0687**
11	0.8816**	1.0718**	1.2288**	0.6822**	1.0793**	1.0904**	1.0972**	1.0654**	1.1777**
12	1.0338**	1.7724**	0.6984**	0.9751**	1.0595**	1.0341**	1.1510**	0.9153**	1.0438**
13	1.0152	1.1443**	0.9577**	0.9824**	0.9270**	1.0827**	1.1177**	1.0391**	1.0494**
14	1.0915**	1.1855**	1.0425**	1.0163**	0.9446**	1.0675**	1.0500**	1.1283**	0.9680**
15	1.0103**	1.0694**	0.9213**	1.0335**	1.0281**	0.9859**	0.9944	0.8965**	1.0686**
16	1.0223*	1.0735**	1.0708**	0.9418**	0.9871	1.0032	1.0082	1.0909**	1.0583**
17	1.0519**	0.9915	1.0867**	0.9376**	1.0316**	0.9879**	1.0583**	0.9010*	1.0622**
18	0.9652**	1.1368**	0.9304**	0.8562**	1.1126**	0.9983	1.0109	0.9899**	0.9542**
19	0.9340**	1.1181**	0.9757	1.1093**	1.0866**	1.0730**	0.9842*	0.9494**	0.9882
20	0.9450**	1.1217**	1.0659**	1.0536*	1.0855**	1.0316	1.0162	0.9800	1.0593**
21	0.9648**	1.0530**	1.0126	1.0622**	1.0075**	0.9674**	1.0825**	1.1836**	1.0261**
22	0.9605**	0.9677**	1.0246	0.9255**	0.9653**	0.9459**	1.0496**	0.9719**	1.1361**
23	0.6706**	0.9608	1.0419	0.9819	1.0821**	0.9445**	1.0519**	1.0074	1.0324**
24	0.9954	1.0156	1.0019	0.8904**	1.0669**	1.1057**	1.1024**	1.1693**	1.0533**
25	0.8756**	1.1629**	0.9873	0.9679**	0.9946	1.0020	1.1102**	1.0295**	1.0710**
26	1.0291**	1.0205**	1.0122**	0.9696**	1.1527**	0.7962**	1.2817**	1.0896**	1.0917**
27	1.0068	1.0041	1.0394**	0.8366**	1.0743**	0.9641**	1.0705**	1.0386*	1.0995**
28	0.9142**	1.0375**	1.0145**	0.9173**	1.0385**	0.9606**	1.0449**	1.0824**	1.1292**
29	0.8367**	1.0132	1.0355**	0.9623**	1.0133	1.0046	1.0450**	1.0050	1.2067**
30	0.9536**	1.1177**	1.1206**	0.7902**	1.1383**	1.0426**	1.0603**	1.0967**	1.0132
31	0.9686	1.0889**	1.0560**	0.7046**	1.0753**	1.0652**	1.1699**	1.0099	1.0510**
32	0.9815	1.0591**	1.0603**	0.9641	1.0787**	1.0178	1.1063**	1.0125	0.9756
33	1.0376**	1.0371**	1.0646**	1.0512	1.0025	1.1112**	1.1718**	1.0164**	1.1154**
34	0.9723**	1.1141**	1.0966**	0.9307**	0.9303**	1.0105**	0.7704**	1.1527**	1.0833**
35	1.3316**	0.7906**	1.0253**	1.0852**	0.9586**	0.9601**	0.9651**	1.0019	0.9189**
36	0.9829**	1.0327**	1.0409**	0.8372**	1.0719**	1.0312	1.0634**	1.0328**	0.9816**
37	1.0662**	1.1375**	0.9939	0.9707*	1.0929**	0.8049**	1.3609**	0.9771*	1.1322**
38	1.0247	1.1834**	1.0265**	0.9074**	1.0505**	1.1279**	1.0477**	1.0600**	1.0344**
39	1.0224**	1.0889**	1.1285**	0.8541**	0.9857**	1.0212**	1.1629**	1.0270**	1.0096**
40	0.9644**	1.0575**	1.0257	0.9784	1.2804**	1.0859**	1.0930**	1.1218**	1.0810**
41	1.0934**	0.9125**	0.9630**	0.9954	0.9769**	1.1926**	0.9446**	0.9093**	1.1074**
42	1.0830**	1.2663**	1.0086	0.9993	1.0435**	1.0009	1.0145	1.1342**	1.0050

Note: Single asterisks (*) denote significant differences from unity at 0.10; double asterisks (**) denote significant differences from unity at 0.05.

of industries, but across countries as well. In each case, researchers have provided point estimates, although clearly there must be uncertainty surrounding these estimates due to sampling variation. Our methodology outlined in the preceding sections provides a tractable approach for consistently estimating confidence intervals. In addition, as illustrated in our empirical examples, our bootstrap methodology provides a correction for the inherent bias in nonparametric distance func-

tion estimates (and hence in estimates of Malmquist indices), as well as a method for checking whether the bias-correction will increase mean-square error.

Confidence intervals such as those estimated in our empirical examples are essential in interpreting estimates of Malmquist indices. As with any estimator, it is not enough to know whether the Malmquist index estimator indicates increases or decreases in productivity, but whether the indicated changes are significant in a statistical sense; i.e., whether the result indicates a real change in productivity, or is an artifact of sampling noise. Our bootstrap procedure allows the researcher to make these distinctions.

References

Caves, D.W., Christensen, L.R., Diewert, W.E., 1982. The economic theory of index numbers and the measurement of input, output and productivity. Econometrica 5, 1393–1414.

Deprins, D., Simar, L., Tulkens, H., 1984. Measuring labor inefficiency in post offices. In: Marchand, M., Pestieau, P., Tulkens, H. (Eds.), The Performance of Public Enterprises: Concepts and Measurements. North-Holland, Amsterdam, pp. 243–267.

Efron, B., Tibshirani, R.J., 1993. An Introduction to the Bootstrap. Chapman and Hall, Inc., New York.

Färe, R., Grosskopf, S., Lindgren, B., Roos, P., 1992. Productivity changes in Swedish pharmacies 1980–1989: A nonparametric approach. Journal of Productivity Analysis 3, 85–101.

Farrell, M.J., 1957. The measurement of productive efficiency. Journal of the Royal Statistical Society A 120, 253–281.

Härdle, W., 1990. Applied Nonparametric Regression. Cambridge Univ. Press, Cambridge.

Korostelev, A.P., Simar, L., Tsybakov, A.B., 1995a. Efficient estimation of monotone boundaries. Annals of Statistics 23, 476–489.

Korostelev, A.P., Simar, L., Tsybakov, A.B., 1995b. On estimation of monotone and convex boundaries. Publications de l'Institut de Statistique de l'Université de Paris 39, 3–18.

Lovell, C.A.K., 1993. Production frontiers and productive efficiency. In: Fried, H., Lovell, C.A.K., Schmidt, S.S. (Eds.), The Measurement of Productive Efficiency: Techniques and Applications. Oxford Univ. Press, Oxford, pp. 3–67.

Press, W.H., Flannery, B.P., Teukolsky, S.A., Vetterling, W.T., 1986. Numerical Recipes. Cambridge University Press, Cambridge.

Rousseeuw, P.J., 1985. Multivariate estimation with high breakdown point. In: Grossmann, W., Pflug, G., Vincze, I., Wertz, W. (Eds.), Mathematical Statistics and Applications, vol. B. Reidel, Dordrecht, pp. 283–297.

Shephard, R.W., 1970. Theory of Cost and Production Functions. Princeton University Press, Princeton, NJ.

Silverman, B.W., 1978. Choosing the window width when estimating a density. Biometrika 65, 1–11.

Silverman, B.W., 1986. Density Estimation for Statistics and Data Analysis. Chapman and Hall, London.

Simar, L., Wilson, P.W., 1998. Sensitivity analysis of efficiency scores: How to bootstrap in nonparametric frontier models. Management Science 44, 49–61.

Tapia, R.A., Thompson, J.R., 1978. Nonparametric Probability Density Estimation. Johns Hopkins University Press, Baltimore, MD.