Introduction to Orbital Dynamics: the Earth-Sun System*

Kepler's First Law

For the next homework, we will look at problems involving the motion of planets under gravity. The simplest case, which we'll focus on in this exercise, is the problem of one planet orbiting a star. In our example, the planet is Earth and the star is our Sun. Let Earth have mass m_e and the sun m_s. We will consider the sun to be at point (0,0). At some point in time the Earth is located at position $\vec{r}$ and is moving with velocity $\vec{v}$.

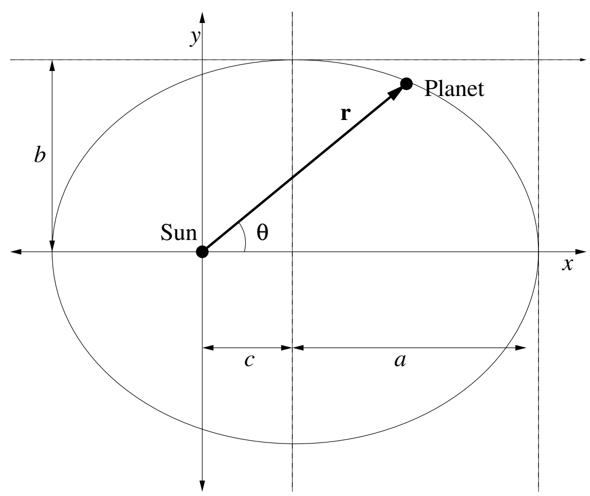

Figure 1: Elliptical orbit parameters.

*Modified from lecture notes and assignment developed by Gary Glatzmaier and Mark Krumholtz, UC Santa Cruz

The solution for this problem is known analytically and is described by Kepler's Laws. Kepler's first law states that the planet orbits are ellipses with the Sun at one focus, as shown in Figure 1.

The ellipse is described by two numbers: the semi-major axis a and the eccentricity $e = c/a$, where c is the distance from the Sun to the center of the ellipse. All the other properties of the orbit shape can be deduced from these numbers. The semi-minor axis is $b = \sqrt{a^2 - c^2}$, and the distance from the planet to the Sun r is related to its angle θ by

$$r(\theta) = \frac{a(1-e^2)}{1 - e\cos\theta}. \tag{1}$$

A perfectly circular orbit has $e = 0$, an orbit with $e = 1$ is a parabola rather than an ellipse, and one with $e > 1$ is a hyperbola. The case $e > 1$ describes an object that is not gravitationally bound to the Sun and will escape to infinity, and $e = 1$ is describes objects that are at the boundary between being bound and unbound. For Earth's orbit, $e = 0.0167$, so the shape is elliptical but only slightly so.

Simulating Orbits on a Computer

To compute orbits on a computer, we will use a numerical approach that solves coupled ordinary differential equations for the position and velocity. Let the Earth have mass m_e and the Sun m_s. At some point in time, Earth is located at position $\vec{r}$ and moving with velocity $\vec{v}$ relative to the stationary sun at coordinate (0,0). In general $\vec{r}$ is a three-element vector, but we can choose our coordinate system so that their orbit lies in the plane $z = 0$, and thus we will ignore the z components. We would like to find the position and velocity of Earth after some time t has passed. We therefore have a system with four unknowns: x- and y-positions (x, y) and x- and y-velocities (v_x, v_y) for Earth.

Here are the steps we need to solve this system:

1. Write down the underlying equations

   The distance between the two objects is

$$r = \sqrt{x^2 + y^2}, \tag{2}$$

   The force of gravity on Earth from the Sun is given by Newton's law of gravity:

$$\vec{F} = -\frac{G m_e m_s}{r^2}\left(\frac{\vec{r}}{r}\right). \tag{3}$$

G is the gravitational constant, $G \approx 6.67$ x$10^{-11} m^3 kg^{-1} s^{-2}$. We can write down the x and y components of this separately:

$$F_x = -\frac{G m_e m_s}{r^2}\left(\frac{x}{r}\right) \quad (4)$$

$$F_y = -\frac{G m_e m_s}{r^2}\left(\frac{y}{r}\right). \quad (5)$$

Given the force, we can compute the acceleration of Earth from Newton's 2nd law $F = ma$. Noting that the acceleration a is the time derivative of velocity v, we have $dv/dt = F/m$. We can then transform the force component equations above into:

$$a_x = \frac{dv_x}{dt} = -\frac{G m_s}{r^2}\left(\frac{x}{r}\right) \quad (6)$$

$$a_y = \frac{dv_y}{dt} = -\frac{G m_s}{r^2}\left(\frac{y}{r}\right). \quad (7)$$

The equations describing how the position changes in time are trivial:

$$\frac{dx}{dt} = v_x \quad (8)$$

$$\frac{dy}{dt} = v_y. \quad (9)$$

Thus we have written down a total of four equations for four unknowns: equations (6) – (9). This system of four equations, together with four initial conditions (the initial x and y positions and x and y velocities of Earth), are enough to fully calculate Earth's orbit as a function of time. Now let's look at the numerical techniques required to do this.

2. Discretize / Pick Update Strategy in Time: Adams-Bashforth Method

   As we have done previously, we will discretize time t in the usual way using a time step Δt and using a superscript k to denote the specific time t^k where $t^k = t^{k-1} + \Delta t$.

   In our previous homework, we used a very simple strategy for predicting future values. There, for some variable q, we assumed:

$$q^k = q^{k-1} + \Delta t \frac{dq}{dt}^{k-1}. \quad (10)$$

   This is the simplest way to "step forward" in time; use the function's derivative at time t^{k-1} to predict the value of the function at time t^k. That worked well enough in the mass-balance glacier equations and in the FTCS method for the diffusion equation for groundwater and thermal modeling. However, a critical requirement for this method is that the time step is small enough so that the derivative term remains accurate at projecting the solution forward in time (using the function slope defined by the derivative). For more complicated problems, this time step method is not accurate enough, and small errors will accumulate over each time step, leading to increasingly inaccurate solutions as time goes on. More complicated systems of equations require higher-order time stepping methods that are more accurate.

For this problem, we will use the Adams-Bashforth Method, which is in the category of linear-multistep methods. The method above is know as a single-step method, since the update is done using a single derivative so that the values at t^{k-1} are used to predict the new values at t^k. Conversely, multistep methods will use the derivatives from one or more previous steps to help improve accuracy.

Background and justification for this method will not be covered here but is readily available on a variety of web sites and text books. For our planetary orbit problem, the Adams-Bashforth Method uses values at t^{k-1} as well as at t^{k-2} to predict the new values for velocity. These new velocity values at t^k are then used along with the velocities at t^{k-1} to predict the new spatial locations. Although we are skipping over the derivation of the A-B method, you should know that it computes the position and velocities at different time grid points that leap-frog past each other with each time-step. Thus it is also know as a *leap-frog* technique.

Application of this concept to the equations laid out in the previous section goes as follows:

First, we use Newton's second law to calculate the current accelerations. Rearranging eqns 6 & 7 yields:

$$(a_x)^{k-1} = \left(\frac{dv_x}{dt}\right)^{k-1} = -\frac{Gm_s(x)^{k-1}}{(r^3)^{k-1}} \tag{11}$$

$$(a_y)^{k-1} = \left(\frac{dv_y}{dt}\right)^{k-1} = -\frac{Gm_s(y)^{k-1}}{(r^3)^{k-1}} \tag{12}$$

Here r is the distance between the particles. The quantities on the left hand side the accelerations at a given time step.[1]

Updated velocities can then be calculated using these accelerations as well as the acceleration values from the previous time step (this is the A-B method)

$$(v_x)^k = (v_x)^{k-1} + dt\left(\frac{3}{2}(a_x)^{k-1} - \frac{1}{2}(a_x)^{k-2}\right) \tag{13}$$

$$(v_y)^k = (v_y)^{k-1} + dt\left(\frac{3}{2}(a_y)^{k-1} - \frac{1}{2}(a_y)^{k-2}\right) \tag{14}$$

[1] We also use the notation $(\cdot)^k$ to denote the value of the quantity inside the parenthesis at time t^k. It does NOT mean to take the k-th power!

While this looks complicated, it simply says that the velocity is update at each time step by adding on a term that is essentially acceleration multiplied by time.

We can now calculate updated positions using an average of the velocities from t^k and t^{k-1}:

$$
\begin{aligned}
x^k &= x^{k-1} + \frac{dt}{2}\left((v_x)^k + (v_x)^{k-1}\right) && (15)\\
y^k &= y^{k-1} + \frac{dt}{2}\left((v_y)^k + (v_y)^{k-1}\right) && (16)
\end{aligned}
$$

Along with initial conditions described below, these equations provide a method by which we can model the change in position of each particle through time.

3. Pick a strategy for determining time step size

   The accuracy of our method will depend on the time step we use. Since our method amounts to assuming that the particle moves in straight lines between time steps k and $k+1$, the accuracy gets better the closer the true trajectory is to a straight line. Since we know that the real answer is an ellipse, which is traversed in one orbital period, we can ensure that the small segments of the ellipse each particle traces per time step are close to straight lines by making the time step much shorter than the orbital period (τ). A reasonable choice is

$$\Delta t = \frac{\tau}{1000}. \tag{17}$$

4. Specify initial conditions

   In addition to giving the mass of the Earth and Sun, the initial conditions for this problem are simple: we just need to specify the initial position and velocity (see Figure 2). For our problem, we can set the initial position at time $t = 0$ based on the average distance from Earth to the Sun (1 AU):

$$
\begin{aligned}
x^0 &= a(1+e) && (18)\\
y^0 &= 0 && (19)
\end{aligned}
$$

   We set the initial velocity v^0 at time $t = 0$ as:

$$
\begin{aligned}
v_x^0 &= 0 && (20)\\
v_y^0 &= \sqrt{\frac{Gm_s(1-e)}{a(1+e)}} && (21)
\end{aligned}
$$

   based on Kepler's laws.

Note that here we have defined initial conditions based on Earth's stable orbit around the sun, so the numerical solution you compute for the homework assignment should match Earth's orbit. What would happen if instead we gave Earth an initial velocity that was either much slower or much faster than its true velocity?

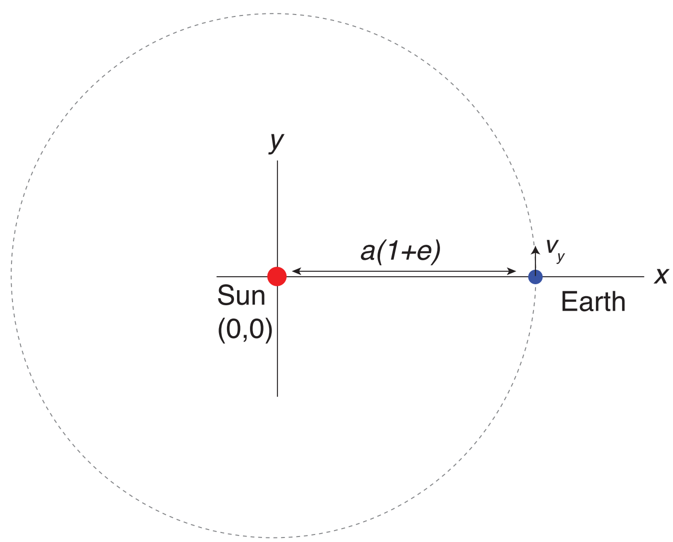

Figure 2: Initial conditions for the Earth-Sun orbit problem.