```python
import numpy as np
import matplotlib.pyplot as plt
from typing import Tuple
import matplotlib as mpl
from typing import List
import pandas as pd
```

Neural Modelling exercise 2: Prediction learning

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Hand-in by Friday, 8.11.24 (midnight) to neuralmodelling24@gmail.com If you are handing in as a two person team, make sure to put both of your names on your solution (please hand in only one report per team). Besides your responses to the questions in text form, your submission should contain your code. You could e.g. link to a Github repo or submit your report as a Jupyter notebook.

Conditioning (Lecture 2)

1. Simulate data for each of the following conditioning paradigms:
    - Blocking
    - Inhibitory conditioning
    - Overshadowing
    - Secondary conditioning
    - Explaining away

    To do this, generate arrays which correspond to the sequences of presented stimuli and the sequences of presented rewards (ignore time within a trial for this, each entry is one trial). Also generate an idealised sequence of expectations of rewards of the animal (the expectation occurs after the presentation of the CS but before the potential presentation of a reward).

2. Now use the Rescorla-Wagner rule to learn these expectations, based on the stimulus and reward vectors you generated.

3. For which paradigms do your learned expectations qualitatively agree with your idealised predictions? For the non-matching paradigms, speculate why the Rescorla-Wagner rule failed to produce the correct expectations.

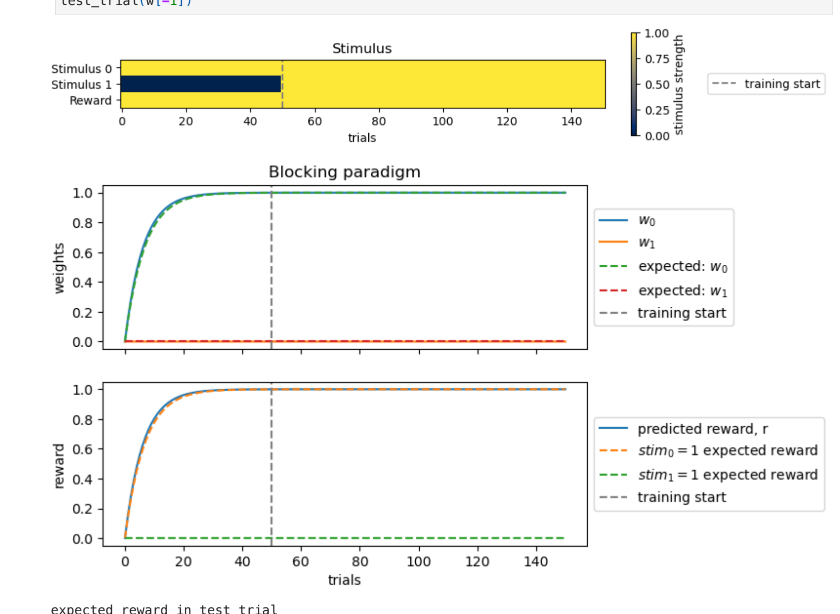
```python
cmap = mpl.colormaps["tab10"]


def v(w: np.ndarray, u: np.ndarray) -> float:
    """compute expected value

    Args:
        w (np.ndarray): (n_stimuli)
        u (np.ndarray): (n_stimuli)

    Returns:
        float: expected value
    """
    return np.dot(w, u)


def plot_stimulus(stim_0, stim_1, reward, n_pretraining: int = 0):
    fig, ax = plt.subplots(figsize = (8, 5))
    mapable = ax.imshow(np.stack([stim_0, stim_1, reward]), cmap = "cividis", vmin = 0, vmax = 1)

    ax.axvline(n_pretraining, linestyle = "--", color = "gray", label = "training start")
    ax.legend(loc = "center left", bbox_to_anchor = (1.2, 0.5))

    ax.set_aspect(5)
    ax.set_yticks(np.arange(0, 3), labels = ["Stimulus 0", "Stimulus 1", "Reward"])
    ax.set_xlabel("trials")
    fig.colorbar(mapable, location = "right", label = "stimulus strength", fraction = 0.01)
    ax.set_title("Stimulus")


def learn(u: np.ndarray, reward, learning_rate) -> Tuple[np.ndarray, np.ndarray]:
    v_pred = np.empty(len(u))
    w = np.zeros_like(u)
    for i in range(u.shape[0] - 1):
        v_pred[i] = v(w[i], u[i])
        delta = learning_rate * (reward[i] - v_pred[i]) * u[i]
        w[i + 1] = w[i] + delta

    v_pred[-1] = v(w[-1], u[-1])

    return w, v_pred


def test_trial(w):
    expected_reward = v(np.array([[0, 0], [0, 1], [1, 0], [1, 1]]), w).reshape(2, 2)

    print("expected reward in test trial")
    print(pd.DataFrame(expected_reward, columns = ["stim_0 = 0", "stim_0 = 1"], index = ["stim_1 =


def plot_riscorla_wagner(
        w,
        v_pred: np.ndarray,
        v_exp: List = None,
        v_exp_labels: List[str] = [],
        expectations: List[np.ndarray] = [],
        title: str = "",
        n_pretraining = 0
):
    fig, axs = plt.subplots(nrows = 2, sharex = True)

    for i in range(w.shape[1]):
        axs[0].plot(w[:, i], label = rf"$w_{i}$")

    for idx, expect in enumerate(expectations):
        axs[0].plot(expect, label = rf"expected: $w_{idx}$", linestyle = "--", color = cmap(w.shape

    axs[0].set_title(title)

    axs[0].set_ylabel("weights")

    axs[1].plot(v_pred, label = "predicted reward, r")

    if v_exp is not None:
        for exp, lab in zip(v_exp, v_exp_labels):
            axs[1].plot(exp, label = lab, linestyle = "--")

    axs[1].set_xlabel("trials")
    axs[1].set_ylabel("reward")

    axs[0].axvline(n_pretraining, linestyle = "--", color = "gray", label = "training start")
    axs[1].axvline(n_pretraining, linestyle = "--", color = "gray", label = "training start")

    axs[0].legend(loc = "center left", bbox_to_anchor = (1, 0.5))
    axs[1].legend(loc = "center left", bbox_to_anchor = (1, 0.5))

    plt.show()


n_pretraining = 50
n_trials = 100 + 1
learning_rate = 0.15
```

Blocking

The **blocking paradigm** is a phenomenon where prior learning interferes with new learning about a stimulus when that stimulus is paired with an already familiar cue.

Phases: (1) Pre-training: a stimulus 0, s_0, is associated with the reward, r. (2) Training: s_0 is presented alongside a new stimulus 1, s_1. Both cues continue to predict the reward, r. (3) Result: conditioned stimulus 0, s_0, continues to predict the reward, while conditioned stimulus 1, s_1, is not associated with the reward. The previously learned stimulus 0 essentially "blocks" stimulus 1 from forming a significant association with the reward.

```python
# 1st stimulus is constant during pretraining and trials
stim_0 = np.ones(n_pretraining + n_trials)

# 2nd stimulus is constant during trials only
stim_1 = np.ones_like(stim_0)
stim_1[:n_pretraining] = 0

# Reward is given always
reward = np.ones_like(stim_0)

# Final stimulus matrix
u = np.stack([stim_0, stim_1]).T  # shape (trials, n_stimuli)

plot_stimulus(stim_0, stim_1, reward, n_pretraining)

# Expected weights
expect_0 = 1 - np.exp(-np.arange(n_pretraining + n_trials) * learning_rate)
expect_1 = np.zeros(n_pretraining + n_trials)
# Expected reward
v_exp_0 = 1 - np.exp(-np.arange(n_pretraining + n_trials) * learning_rate)
v_exp_1 = np.zeros(n_pretraining + n_trials)

# Apply Rescorla-Wagner
w, v_pred = learn(u, reward, learning_rate)

plot_riscorla_wagner(
    w = w,
    v_pred = v_pred,
    v_exp = [v_exp_0, v_exp_1],
    v_exp_labels = [rf"$stim_0=1$ expected reward", rf"$stim_1=1$ expected reward"],
    expectations = [expect_0, expect_1],
    title = "Blocking paradigm",
    n_pretraining = n_pretraining
)
test_trial(w[-1])
```




```
expected reward in test trial
          stim_0 = 0  stim_0 = 1
stim_1 = 0    0.000000    0.000148
stim_1 = 1    0.999852    1.000000
```

The table shows expected reward in the test trial for all stimulus combinations. When both stimuli are not presented or when only the 1st stimulus is presented, no reward is expected. When the 2nd stimulus is presented (whether in combination with stimulus 1 or not), the full reward is expected.

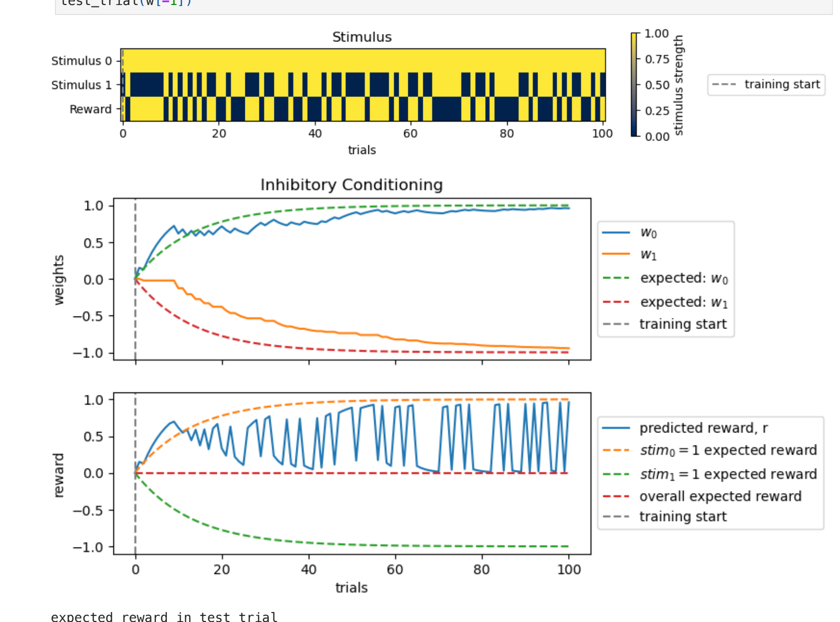
In the case of blocking paradigm, the Rescorla-Wagner rule predicts the expected result: the first stimulus is associated with the cue, while the second stimulus fails to form this connection.

Inhibitory Conditioning

The **inhibitory conditioning** pairs the stimulus signal with the absence of a reward.

Phases: (1) Training: stimulus 0, s_0, and stimulus 1, s_1, is presented. The s_0 cue predicts reward, while both cues predict no reward. (2) Result: conditioned stimulus 0, s_0, predicts the reward, while conditioned stimulus 1, s_1, predicts negative reward.

```python
# 1st stimulus is constantly presented in all trials
stim_0 = np.ones(n_trials)
# 2nd stimulus is randomly presented in some of the trials
np.random.seed(1)
stim_1 = np.random.sample(n_trials) > 0.5

# Reward is given when the 2nd stimulus is not presented
reward = (~stim_1).astype(int)

# Final stimulus matrix
u = np.stack([stim_0, stim_1]).T  # shape (trials, n_stimuli)

plot_stimulus(stim_0, stim_1, reward)

# Expected result
expect_0 = 1 - np.exp(-np.arange(n_trials) * learning_rate * 0.5)
expect_1 = np.exp(-np.arange(n_trials) * learning_rate * 0.5) - 1
# Expected reward
v_exp_0 = 1 - np.exp(-np.arange(n_trials) * learning_rate * 0.5)
v_exp_1 = np.exp(-np.arange(n_trials) * learning_rate * 0.5) - 1
v_exp = v_exp_0 + v_exp_1

# Apply Rescorla-Wagner
w, v_pred = learn(u, reward, learning_rate)

plot_riscorla_wagner(
    w = w,
    v_pred = v_pred,
    v_exp = [v_exp_0, v_exp_1, v_exp],
    v_exp_labels = [rf"$stim_0=1$ expected reward", rf"$stim_1=1$ expected reward", "overall expect
    expectations = [expect_0, expect_1],
    title = "Inhibitory Conditioning"
)
test_trial(w[-1])
```

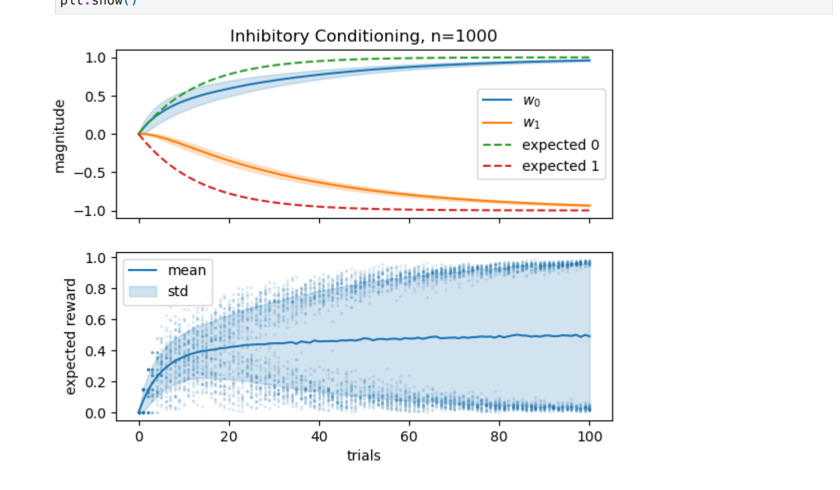



```
expected reward in test trial
          stim_0 = 0  stim_0 = 1
stim_1 = 0    0.000000   -0.944756
stim_1 = 1    0.959466    0.014710
```

In the case of inhibitory conditioning, the Rescorla-Wagner rule predicts the expected result: only the first stimulus predicts the reward ($stim_0 = 0$, $stim_1 = 1$ in the table above); the second stimulus, on the other hand, predicts negative reward ($stim_0 = 1$, $stim_1 = 0$ in the table above).

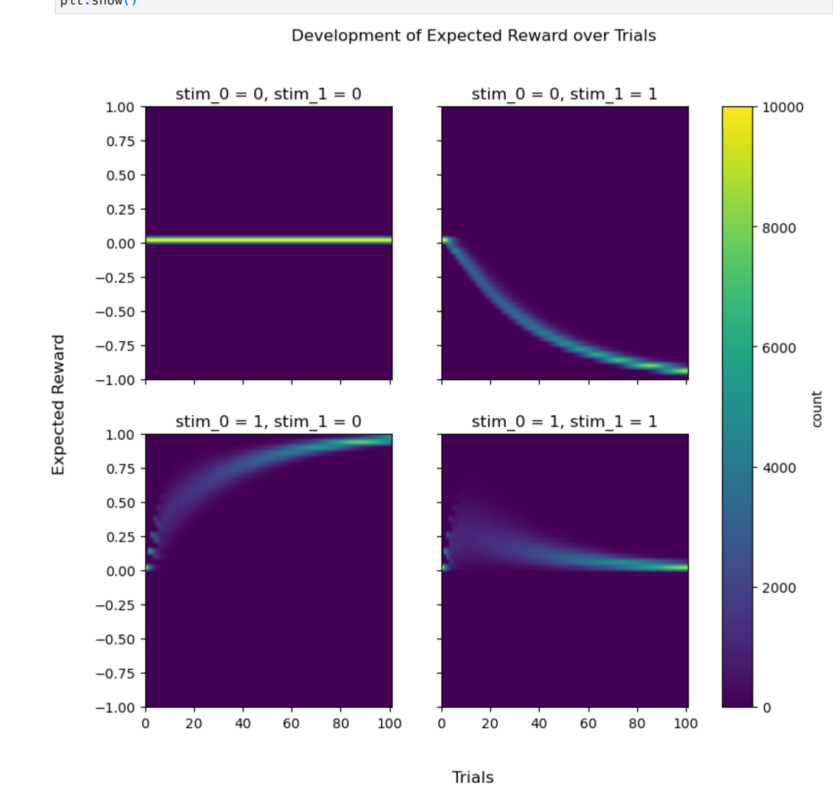
```python
# Since the process is stochastic, we sample 10k trials

w_stack = []
v_exp_stack = []
for _ in range(10000):
    stim_0 = np.ones(n_trials)
    stim_1 = np.random.sample(n_trials) > 0.5
    reward = (~stim_1).astype(int)

    u = np.stack([stim_0, stim_1]).T  # shape (trials, n_stimuli)

    w, v_exp = learn(u, reward, learning_rate)
    w_stack.append(w)
    v_exp_stack.append(v_exp)

w_stack = np.stack(w_stack)
v_exp_stack = np.stack(v_exp_stack)
```

```python
mean = w_stack.mean(axis = 0)
std = w_stack.std(axis = 0)

fig, ax = plt.subplots(nrows = 2, sharex = True)
ax[0].plot(mean[:, 0], c = cmap(0), label = rf"$w_0$")
ax[0].plot(mean[:, 1], c = cmap(1), label = rf"$w_1$")

ax[0].fill_between(np.arange(len(mean)), mean[:, 0] - std[:, 0], mean[:, 0] + std[:, 0], alpha = 0.
ax[0].fill_between(np.arange(len(mean)), mean[:, 1] - std[:, 1], mean[:, 1] + std[:, 1], alpha = 0.

expect_0 = 1 - np.exp(-np.arange(n_trials) * learning_rate * 0.5)
expect_1 = np.exp(-np.arange(n_trials) * learning_rate * 0.5) - 1
ax[0].plot(expect_0, linestyle = "--", color = cmap(2), label = "expected 0")
ax[0].plot(expect_1, linestyle = "--", color = cmap(3), label = "expected 1")

ax[0].set_title("Inhibitory Conditioning, n=1000")
ax[0].legend()
ax[0].set_ylabel("magnitude")
ax[1].set_xlabel("trials")

mean = v_exp_stack.mean(axis = 0)
std = v_exp_stack.std(axis = 0)

for v_ in v_exp_stack[:50]:
    ax[1].scatter(np.arange(n_trials), v_, color = cmap(0), alpha = 0.1, s = 2)

ax[1].plot(mean, label = "mean")
ax[1].fill_between(np.arange(len(mean)), mean - std, mean + std, alpha = 0.2, color = cmap(0), labe
ax[1].legend()
ax[1].set_ylabel("expected reward")
plt.show()
```



The results resemble what we see in the case of a single trial: the weights for stimulus 0 approach 1, while the weights for stimulus 1 approach -1. The mean of predicted reward, in this case, approaches 0.5 with a standard deviation of 1 (because the predicted reward is either zero or one with a 50/50 distribution).

```python
v_test = np.stack([w @ np.array([[0, 0], [0, 1], [1, 0], [1, 1]]).T for w in w_stack])
v_test = v_test.reshape(-1, n_trials, 2, 2)

bins = np.linspace(-1, 1, 51)

fig, axs = plt.subplots(ncols = 2, nrows = 2, figsize = (9, 8), sharex = True, sharey = True)
counts = np.stack([np.histogram(v_test[..., i, 0, 0], bins = bins)[0] for i in range(v_test.shape[1
mappable = axs[0, 0].imshow(counts.T[::-1], extent = (0, v_test.shape[1], -1, 1), aspect = "auto")
axs[0, 0].set_title("stim_0 = 0, stim_1 = 0")
counts = np.stack([np.histogram(v_test[..., i, 0, 1], bins = bins)[0] for i in range(v_test.shape[1
axs[0, 1].imshow(counts.T[::-1], extent = (0, v_test.shape[1], -1, 1), aspect = "auto")
axs[0, 1].set_title("stim_0 = 0, stim_1 = 1")
counts = np.stack([np.histogram(v_test[..., i, 1, 0], bins = bins)[0] for i in range(v_test.shape[1
axs[1, 0].imshow(counts.T[::-1], extent = (0, v_test.shape[1], -1, 1), aspect = "auto")
axs[1, 0].set_title("stim_0 = 1, stim_1 = 0")
counts = np.stack([np.histogram(v_test[..., i, 1, 1], bins = bins)[0] for i in range(v_test.shape[1
axs[1, 1].set_title("stim_0 = 1, stim_1 = 1")
axs[1, 1].imshow(counts.T[::-1], extent = (0, v_test.shape[1], -1, 1), aspect = "auto")

fig.suptitle("Development of Expected Reward over Trials")
fig.supxlabel("Trials")
fig.supylabel("Expected Reward")
fig.colorbar(mappable, ax = axs, label = "count")
plt.show()
```



We now computed the expected reward at each training step across all sampled trial sequences. This means, in each case we look at how the animal would predict a reward if it was presented with a particular stimulus (e.g. $stim_0 = 1$ and $stim_1 = 0$). Again, because this process is stochastic we get a distribution of expected rewards.

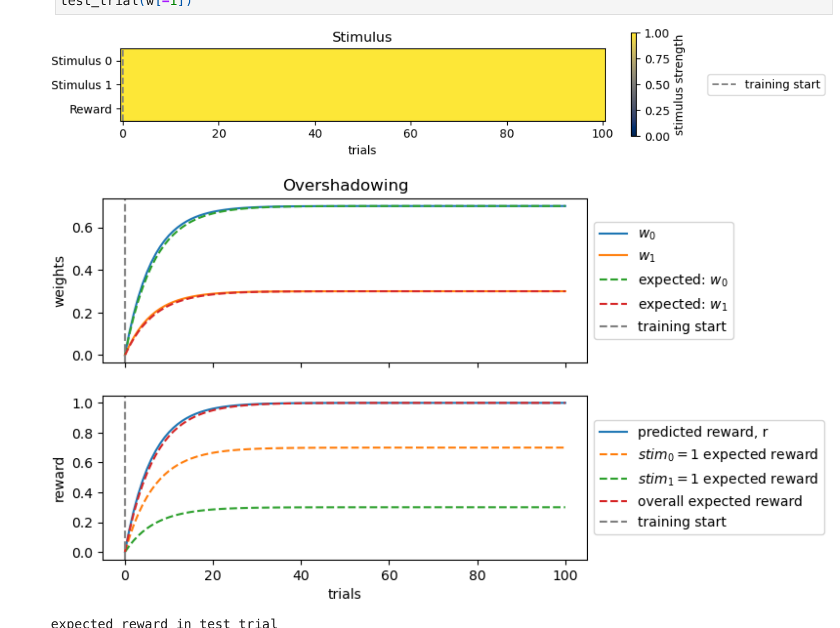
For each tile we can observe the mode approaching the expected reward in the test trial inferred by the Rescorla-Wagner rule.

Overshadowing

In overshadowing, the more intense or attention-grabbing cue dominates the learning process, preventing the other cue from becoming strongly associated with the outcome.

Phases: (1) Training: stimuli 0, s_0, and 1, s_1, are presented together with a reward, r. However, stimulus 0 is very clear while stimulus 1 is quite faint. (2) Result: both stimuli predict reward, however, their weights differ depending on their individual salience.

```python
# Both stimuli are constantly presented in all trials, together with the reward
stim_0 = np.ones(n_trials)
stim_1 = np.ones(n_trials)
reward = np.ones(n_trials)

u = np.stack([stim_0, stim_1]).T

plot_stimulus(stim_0, stim_1, reward)

# Expected weights
expect_0 = (1 - np.exp(-np.arange(n_trials) * learning_rate)) * 0.7
expect_1 = (1 - np.exp(-np.arange(n_trials) * learning_rate)) * 0.3
# Expected reward
v_exp_0 = (1 - np.exp(-np.arange(n_trials) * learning_rate)) * 0.7
v_exp_1 = (1 - np.exp(-np.arange(n_trials) * learning_rate)) * 0.3
v_exp = v_exp_0 + v_exp_1

w, v_pred = learn(u, reward, learning_rate = np.array([learning_rate * 0.7, learning_rate * 0.3]))

plot_riscorla_wagner(
    w = w,
    v_pred = v_pred,
    v_exp = [v_exp_0, v_exp_1, v_exp],
    v_exp_labels = [rf"$stim_0=1$ expected reward", rf"$stim_1=1$ expected reward", "overall expect
    expectations = [expect_0, expect_1],
    title = "Overshadowing"
)
test_trial(w[-1])
```




```
expected reward in test trial
          stim_0 = 0  stim_0 = 1
stim_1 = 0         0.0         0.3
stim_1 = 1         0.7         1.0
```

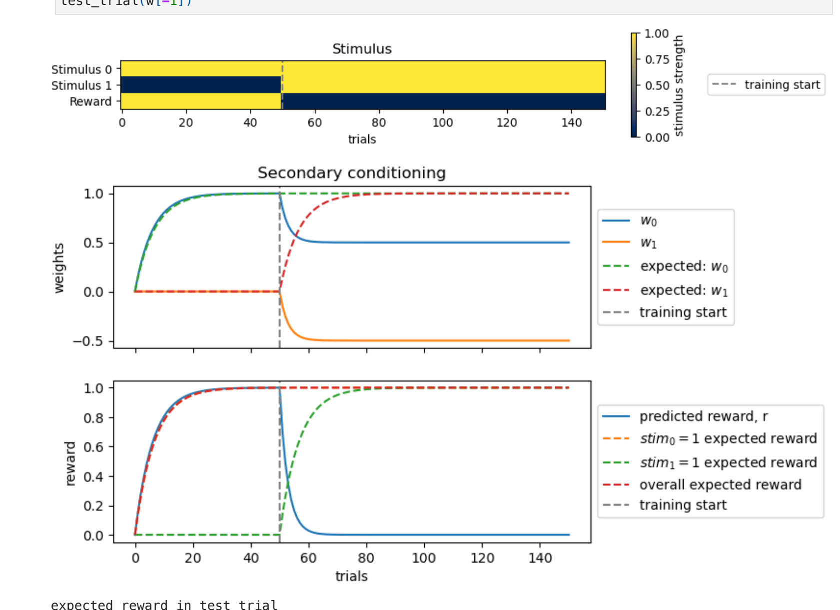
In the case of overshadowing, the Rescorla-Wagner rule predicts the expected result: both stimuli are associated with the reward with strength that corresponds to their salience (0.7 for stimulus 0, and 0.3 for stimulus 1). Salience values were chosen arbitrarily and in such a way that they sum up to one.

Secondary Conditioning

The secondary conditioning paradigm is a form of learning where a neutral stimulus (stimulus 1) becomes associated with a conditioned response through its association with a previously conditioned stimulus (stimulus 0), rather than through direct pairing with an unconditioned stimulus.

Phases: (1) Pre-training (first-order conditioning): associate stimulus 0, s_0, with the reward, r. (2) Training (second-order conditioning): associate a new stimulus 1, s_1, with s_0. (3) Result: s_1 predicts s_0, which in turn predicts r.

```python
# Stimulus 0 is constantly on
stim_0 = np.ones(n_pretraining + n_trials)
# Stimulus 1 is off during pre-training and only on during training
stim_1 = np.ones(n_pretraining + n_trials)
stim_1[:n_pretraining] *= 0

# Reward is presented only during pre-training, to associate
# stimulus 0 with the reward
reward = np.zeros(n_pretraining + n_trials)
reward[:n_pretraining] = 1

u = np.stack([stim_0, stim_1]).T

plot_stimulus(stim_0, stim_1, reward, n_pretraining = n_pretraining)

# Expected weights
expect_0 = 1 - np.exp(-np.arange(n_pretraining + n_trials) * learning_rate)
expect_1 = np.zeros_like(expect_0)
expect_1[n_pretraining:] = 1 - np.exp(-np.arange(n_trials) * learning_rate)
# Expected rewards
v_exp_0 = 1 - np.exp(-np.arange(n_pretraining + n_trials) * learning_rate)
v_exp_1 = np.zeros(n_pretraining + n_trials)
v_exp_1[n_pretraining:] = 1 - np.exp(-np.arange(n_trials) * learning_rate)
v_exp = v_exp_0

w, v_pred = learn(u, reward, learning_rate)
# plot_riscorla_wagner(w=w, v_exp=v_exp, expectations = [expect_0, expect_1], title = "Secondary Co

plot_riscorla_wagner(
    w = w,
    v_pred = v_pred,
    v_exp = [v_exp_0, v_exp_1, v_exp],
    v_exp_labels = [rf"$stim_0=1$ expected reward", rf"$stim_1=1$ expected reward", "overall expect
    expectations = [expect_0, expect_1],
    title = "Secondary conditioning",
    n_pretraining = n_pretraining
)

test_trial(w[-1])
```




```
expected reward in test trial
          stim_0 = 0    stim_0 = 1
stim_1 = 0    0.000000 -4.998521e-01
stim_1 = 1    0.499852  3.330669e-16
```

In the case of secondary conditioning, the Rescorla-Wagner rule does not predict the expected result. We expected that stimulus 0 is associated with the reward during the pre-training phase; however, it is unclear what would happen to this association later on (i.e., whether the animal would forget it during the training phase or not), so we chose the alternative that the association between stimulus 0 and the reward is not forgotten. In the case of the stimulus 1, we expect that it will become associated with the reward during the training phase in the manner $stim_1 \rightarrow stim_0 \rightarrow \text{reward}$.

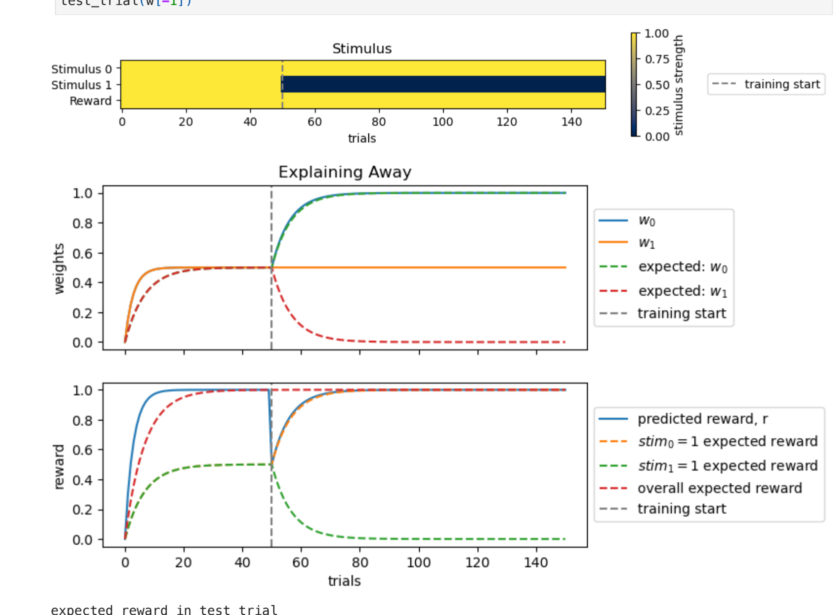
However, the Rescorla-Wagner rule does not support secondary conditioning (it only creates association from stimulus to reward and not stimulus to another stimulus), so when both stimuli are presented with no reward, the weights for both stimuli decrease to predict no reward. The weight for stimulus 0 decreases to 0.5, while the weight for stimulus 1 becomes negative (-0.5).

Explaining Away

In the explaining away paradigm, the animal is presented with two stimuli and a reward, however, later it is revealed that only one of the stimuli is enough to predict the reward.

Phases: (1) Pre-training: both stimuli 0 and 1 are associated with the reward. (2) Training: stimulus 1 is removed, and only stimulus 0 is presented with the reward. (3) Result: there is no association between stimulus 1 and the reward, because stimulus 0 alone is able to predict the reward.

```python
# Stimulus 0 is constantly present
stim_0 = np.ones(n_pretraining + n_trials)
# Stimulus 1 is present only during pre-training
stim_1 = np.ones(n_pretraining + n_trials)
stim_1[n_pretraining:] = 0

# Reward is constant
reward = np.ones_like(stim_0)

u = np.stack([stim_0, stim_1]).T  # shape (trials, n_stimuli)

plot_stimulus(stim_0, stim_1, reward, n_pretraining)

# Expected weight
expect_0 = np.zeros(n_pretraining + n_trials)
expect_0[:n_pretraining] = 0.5 - 0.5 * np.exp(-np.arange(n_pretraining) * learning_rate)
expect_0[n_pretraining:] = 1 - 0.5 * np.exp(-np.arange(n_trials) * learning_rate)
expect_1 = np.zeros(n_pretraining + n_trials)
expect_1[:n_pretraining] = 0.5 - 0.5 * np.exp(-np.arange(n_pretraining) * learning_rate)
expect_1[n_pretraining:] = (0.5 * np.exp(-np.arange(n_trials) * learning_rate))
# Expected reward
v_exp_0 = expect_0
v_exp_1 = expect_1
v_exp = v_exp_0 + v_exp_1

w, v_pred = learn(u, reward, learning_rate)

plot_riscorla_wagner(
    w = w,
    v_pred = v_pred,
    v_exp = [expect_0, expect_1, v_exp],
    v_exp_labels = [rf"$stim_0=1$ expected reward", rf"$stim_1=1$ expected reward", "overall expect
    expectations = [expect_0, expect_1],
    title = "Explaining Away",
    n_pretraining = n_pretraining
)
test_trial(w[-1])
```




```
expected reward in test trial
          stim_0 = 0  stim_0 = 1
stim_1 = 0         0.0         0.5
stim_1 = 1         1.0         1.5
```

In the case of explaining away, the Rescorla-Wagner rule does not predict the expected result. We expect that during the pre-training trials, both stimuli will be associated with the reward with the same weight (similar to overshadowing with equal salience). Then, during the trials, when stimulus 1 is removed and the reward is still present, the animal should learn that stimulus 0 is sufficient to explain the reward, thus, all the weight should go to stimulus 0 and decay for stimulus 1.

However, the Rescorla-Wagner rule does not update the weight for stimulus 1 when it is not present. So, after the stimulus 1 is removed, we notice a sudden dip in predicted reward (prediction drops to the currently predicted reward for stimulus 0). Over time, the Rescorla-Wagner rule learns the association between the stimulus 0 and the reward but the association for stimulus 1 is not forgotten and remains at 0.5, same level as before removing stimulus 1.

Extinction (Lecture 3)

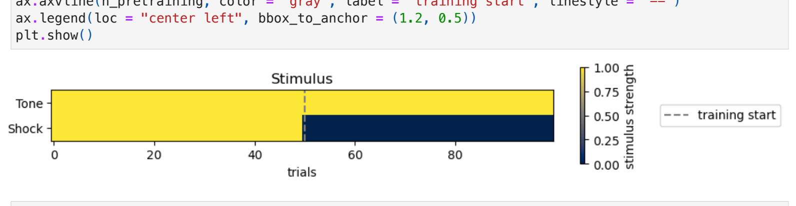
In an extinction protocol, animals repeatedly encounter a previously learned CS, but without receiving the corresponding punishment. However, in certain conditions, the learned association can resurface. We will work towards one way of modelling this, using state-based inference. (Remember: A state, or latent cause, is the context which an animal infers to make predictions about its environment.)

For this exercise please note that for this exercise we denote `state_0` as the state in which the animal is getting shocked after the tone and `state_1` is the state in which the animal only gets the tone.

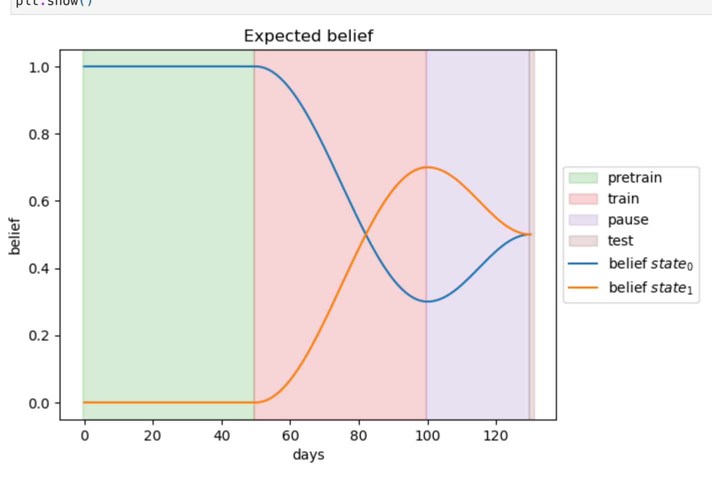
```python
n_pretraining = 50
n_training = 50
n_pause = 30

def plot_belief(belief: np.ndarray):
    # two states -> shocked and not shocked

    fig, ax = plt.subplots()
    ax.axvspan(-0.5, n_pretraining - 0.5, alpha = 0.2, color = cmap(2), label = "pretrain")
    ax.axvspan(
        -0.5 + n_pretraining,
        n_pretraining + n_pretraining - 0.5,
        alpha = 0.2,
        color = cmap(3),
        label = "train",
    )
    ax.axvspan(
        -0.5 + n_pretraining + n_pretraining,
        n_pretraining + n_pretraining + n_pause - 0.5,
        alpha = 0.2,
        color = cmap(4),
        label = "pause",
    )
    ax.axvspan(
        -0.5 + n_pretraining + n_pretraining + n_pause,
        n_pretraining + n_pretraining + n_pause + 1,
        alpha = 0.2,
        color = cmap(5),
        label = "test",
    )
    ax.plot(belief, label = rf"belief $state_0$", color = cmap(0))
    ax.plot(1 - belief, label = rf"belief $state_1$", color = cmap(1))

    ax.legend(loc = "center left", bbox_to_anchor = (1, 0.5))
    ax.set_xlabel("days")
    ax.set_ylabel("belief")

    return fig, ax
```

1. An animal first encounters 50 conditioning trials in a row, then, on the next day, 50 extinction trials in a row. Finally, after a substantial delay of 30 days, it encounter the CS again, for one single trial. Design an appropriate array which could describe the belief of the animal about the current state of the environment on each trial, right before receiving the CS. (Hint: How many states do you need? At which points in time do they change?)

```
In [12]: pretraining = np.ones(n_pretraining)
training = np.zeros(n_training)
test = np.ones(1)
shock = np.concatenate([pretraining, training])
tone = np.concatenate([np.ones_like(pretraining), np.ones_like(training)], )

fig, ax = plt.subplots(figsize = (8, 5))
mapable = ax.imshow(np.stack([tone, shock]), cmap = "cividis", vmin = 0, vmax = 1)
ax.set_aspect(5)
ax.set_yticks(np.arange(0, 2), labels = ["Tone", "Shock"])
ax.set_xlabel("trials")
fig.colorbar(mapable, location = "right", label = "stimulus strength", fraction = 0.009)
ax.set_title("Stimulus")
ax.axvline(n_pretraining, color = "gray", label = "training start", linestyle = "--")
ax.legend(loc = "center left", bbox_to_anchor = (1.2, 0.5))
plt.show()
```




```
In [13]: beta = 0.7 # learning rate
p_shocked = np.concatenate([
    np.ones(n_pretraining),
    beta / 2 * np.cos(np.arange(n_training) / n_training * np.pi) + 1 - (beta / 2),
    -(beta - 0.5) / 2 * np.cos(np.arange(n_pause + 1) / n_pause * np.pi) + (1 - beta) + (beta - 0.5
])

fig, ax = plot_belief(p_shocked)
ax.set_title("Expected belief")
ax.legend(loc = "center left", bbox_to_anchor = (1, 0.5))
plt.show()
```

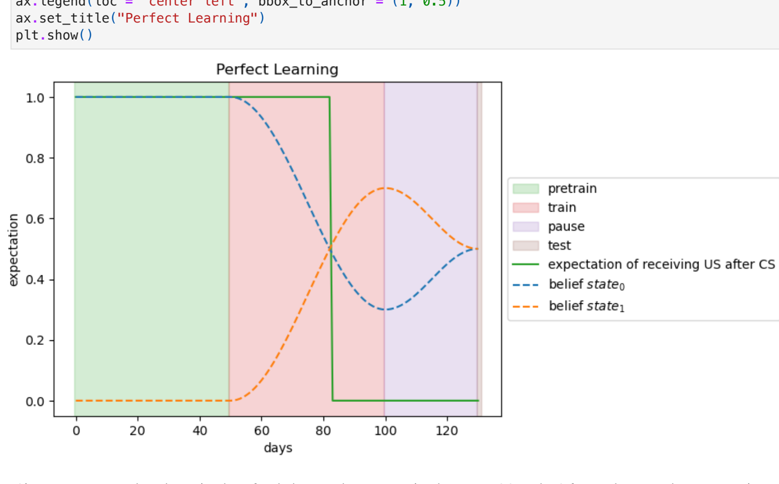


During the pre-training phase, animal already believes that it is in the state 0. Nothing changes, until it is presented with a tone and no shock at trial 51 (start of training phase). The animal observes this change in the environment and starts to consider a possibility of a new state (state 1). As the training trials progress, its belief about being in the state 1 increases, while the belief about being in state 0 decreases. After the training, during the pause period, the animal is presented with no stimuli and the uncertainty of being in one state or the other increases (in the context of this plot, being maximally uncertain about a state means a belief of 0.5 in either of them).

2. Using this belief array, plot the animal's expectation of receiving the US after the CS on each trial. (For this part of the exercise, ignore the learning of the CS-US relationship, assume that the animal perfectly knows the connection between CS and US for each state).

```
In [14]: perfect_learning = p_shocked > 0.5
```

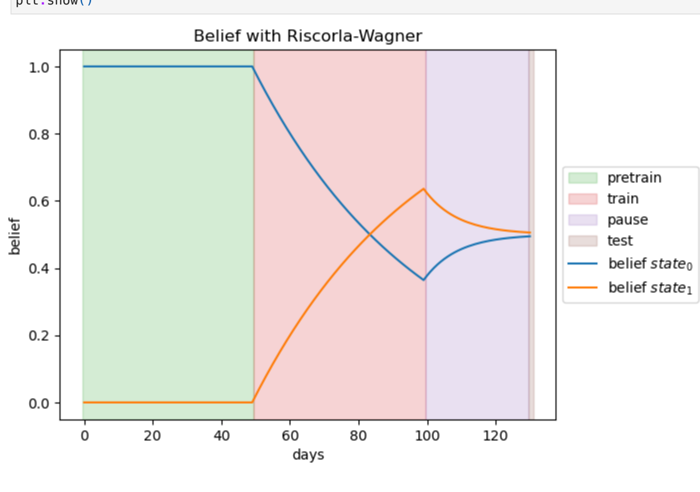
```
In [15]: _, ax = plt.subplots()
ax.axvspan(-0.5, n_pretraining - 0.5, alpha = 0.2, color = cmap(2), label = "pretrain")
ax.axvspan(
    -0.5 + n_pretraining,
    n_pretraining + n_pretraining - 0.5,
    alpha = 0.2,
    color = cmap(3),
    label = "train",
)
ax.axvspan(
    -0.5 + n_pretraining + n_pretraining,
    n_pretraining + n_pretraining + n_pause - 0.5,
    alpha = 0.2,
    color = cmap(4),
    label = "pause",
)
ax.axvspan(
    -0.5 + n_pretraining + n_pretraining + n_pause,
    n_pretraining + n_pretraining + n_pause + 1,
    alpha = 0.2,
    color = cmap(5),
    label = "test",
)

ax.plot(perfect_learning, label = "expectation of receiving US after CS", color = cmap(2))

ax.plot(p_shocked, label = rf"belief $state_0$", color = cmap(0), linestyle = "--")
ax.plot(1 - p_shocked, label = rf"belief $state_1$", color = cmap(1), linestyle = "--")

ax.legend()
ax.set_xlabel("days")
ax.set_ylabel("expectation")

ax.legend(loc = "center left", bbox_to_anchor = (1, 0.5))
ax.set_title("Perfect Learning")
plt.show()
```



Since we assume that the animal perfectly knows the connection between CS and US for each state, the expectation of receiving the US after the CS suddenly drops to zero after the belief of being in state 1 exceeds the belief of being in state 0.

3. Now, rather than assuming our state belief array, we will infer it ourselves with a simple heuristic. Write a function which takes three inputs: The state of the previous trial, the similarity of the previous trial to the one before it in terms of made observations, and the amount of time which passed since the last trial. Based on these arguments, return a belief over all states which are under consideration. On the first trial, the animal is 100% certain of being in state 1. (We don't expect any specific functional form for this function, but it should qualitatively recreate your belief array from the first part of this exercise).

```
In [16]: def update_state(prev_belief_state: np.ndarray, similarity: float, pause: int = 0, learning_rate =
    forget_rate = 0.1
    for _ in range(pause):
        delta = forget_rate * (0.5 - prev_belief_state)
        prev_belief_state += delta

    # assume 2 dimensional observation space
    delta = learning_rate * ((similarity - 1) - prev_belief_state)
    prev_belief_state += delta

    return prev_belief_state


observations = np.stack([tone, shock]).T
state_belief_state = np.empty(n_training + n_pretraining + n_pause + 1)
state_belief_state[0] = 1
for i in range(1, n_pretraining + n_training + n_pause + 1):
    # just for displaying purposes -> same thing as just putting in the 30 day pause into update_st
    if i >= n_pretraining + n_training:
        similarity = state_belief_state[i - 1] + 1
        pause = 1
    else:
        similarity = np.dot(*observations[i - 1: i + 1])
        pause = 0

    state_belief_state[i] = update_state(
        prev_belief_state = state_belief_state[i - 1],
        similarity = similarity,
        pause = pause,
    )

fig, ax = plot_belief(state_belief_state)
ax.set_title("Belief with Riscorla-Wagner")
plt.show()
```



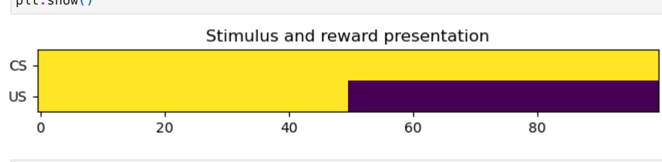
During the pre-training phase, the animal already believes that it is in the state 0. Nothing changes, until it is presented with CS and no US at trial 51 (start of training phase). Now, we apply the Rescorla-Wagner rule to update the belief about being in each state based on its similarity to the previous state. So, the belief about being in the state 1 increases, while the belief about being in state 0 decreases. After 50 training trials, the animal enters the pause phase, when it is not exposed to any of the stimuli. We model this using the Rescorla-Wagner rule, this time assuming that the belief about each state approaches equal probablities (in the case of two stimuli, this probability is 0.5).

This plot looks a bit different with respect to our expectation. We used a cosine function to come up with our expectation, however, this in turn causes it to be flawed as the rate of change in belief decreases before entering a new phase (clearly visible before the pause stage).

4. Finally, maintain a learned association strength between CS and US for each state separately. Update them after each trial according to the Rescorla-Wagner rule, but also weigh the magnitude of the update by the strength of the belief in the state. For simplicity, assume that the belief about the current state does not change between the beginning and the end of each trial.

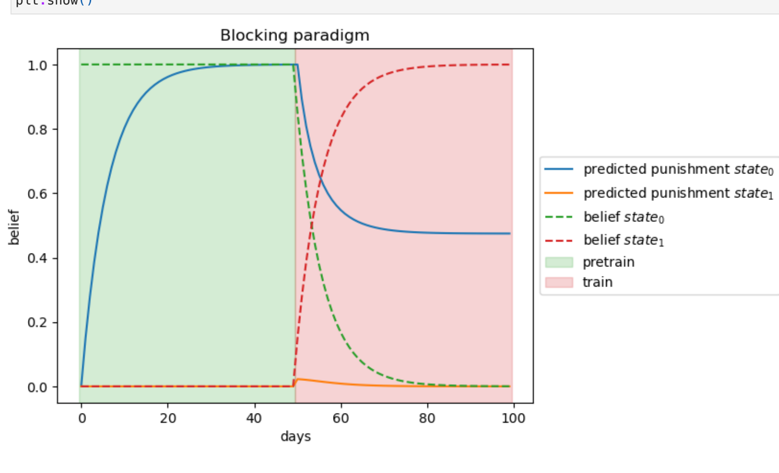
```
In [17]: # Define stimuli
tone = np.ones(n_pretraining + n_training).astype(bool)

# Define rewards
rewards = np.concatenate([
    np.ones(n_pretraining),
    np.zeros(n_training)
]).astype(bool)
```

```
In [18]: _, ax = plt.subplots(figsize = (8, 5))

plt.imshow([tone, rewards], aspect = "equal")

ax.set_aspect(5)
ax.set_yticks(np.arange(0, 2), labels = ["CS", "US"])
plt.title("Stimulus and reward presentation")

plt.show()
```




```
In [19]: ### More complicated Rescorla-Wagner (for two states, weighted by state belief)

# stimuli
u = tone
alpha = learning_rate
r = shock

# predicted rewards
vA = np.zeros(shape = (n_pretraining + n_training, 1))
vB = np.zeros(shape = (n_pretraining + n_training, 1))
# weights for state A - tone -> shock
wA = np.zeros(shape = (n_pretraining + n_training, 1))
wB = np.zeros(shape = (n_pretraining + n_training, 1))

# Belief about the state A
state_belief_state = np.empty(n_training + n_pretraining)
state_belief_state[0] = 1

observations = np.stack([tone, shock]).T

for i in range(1, n_pretraining + n_training):
    similarity = np.dot(*observations[i - 1: i + 1])
    state_belief_state[i] = update_state(
        prev_belief_state = state_belief_state[i - 1],
        similarity = similarity,
        learning_rate = 0.15
    )

    v_exp = np.sum(vA[i - 1])
    delta_w = alpha * (r[i - 1] - v_exp) * u[i - 1]
    wA[i] = wA[i - 1] + delta_w * state_belief_state[i]
    vA[i] = wA[i] * u[i]

    v_exp = np.sum(vB[i - 1])
    delta_w = alpha * (r[i - 1] - v_exp) * u[i - 1]
    wB[i] = wB[i - 1] + delta_w * (1 - state_belief_state[i])
    vB[i] = wB[i] * u[i]
```

```
In [20]: fig, ax = plt.subplots()

ax.plot(wA, label = rf"predicted punishment $state_0$")
ax.plot(wB, label = rf"predicted punishment $state_1$")
ax.plot(state_belief_state, "--", label = rf"belief $state_0$")
ax.plot(1 - state_belief_state, "--", label = rf"belief $state_1$")

ax.axvspan(-0.5, n_pretraining - 0.5, alpha = 0.2, color = cmap(2), label = "pretrain")
ax.axvspan(
    -0.5 + n_pretraining,
    n_pretraining + n_pretraining - 0.5,
    alpha = 0.2,
    color = cmap(3),
    label = "train"
)

ax.legend()
ax.set_xlabel("days")
ax.set_ylabel("belief")

plt.legend(loc = "center left", bbox_to_anchor = (1, 0.5))
plt.title("Blocking paradigm")

plt.show()
```



Now, we updated the learning rate for the state belief so that by the end of the training trials the animal is certain it is in the state one. We consider the association strength between the CS and the US to be variable. As the exercise dictates, we update this association strength with the Rescorla-Wagner rule. As expected, the association strength in state zero decreases because the probability of being in state zero is still quite high when the CS is presented with no US. A similar connection we can see for state one. Despite the animal believing that it is in state zero it also updates its association on not receiving a punishment in state one because there is now a chance that it might have changed the state.