Robust Interval Observer for Systems Described by the Fornasini-Marchesini Second Model

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***Abstract*—This letter proposes a novel robust interval observer for a two-dimensional (treated as a synonym for a double-indexed system) linear time-invariant discrete-time system described by the Fornasini-Marchesini second model. This system is subject to unknown but bounded state disturbances and measurement noise. Built on recent interval estimation strategies designed for one-dimensional systems, the proposed observer is based on the introduction of weighting matrices which provide additional degrees of freedom in comparison with the classical structure relying on a change of coordinates. Linear matrix inequality conditions for the exponential stability and peak-to-peak performance of a two-dimensional system described by the Fornasini-Marchesini second model are then proposed, and applied to the design of a robust interval observer. Numerical simulation results are provided to show the efficiency of the proposed estimation strategy.**

***Index Terms*—Fornasini-Marchesini second model, Interval observer, Robust observer, Two-dimensional systems, Discrete-time systems**

I. Introduction

SINCE their introduction in the second half of the nineteen seventies, two-dimensional[1] (2D) systems have been widely studied [1]. Such systems are described by different state-space models such as the ones introduced by Roesser [2], Fornasini and Marchesini [3], [4] or Kurek [5]. 2D systems can be used to represent many physical processes [6] such as image processing [2], [7], repetitive industrial processes [8], spatio-temporal systems of which the behavior is governed by hyperbolic partial differential equations [9] or the task of iterative learning control synthesis [10]. Extensive studies of 2D system properties such as stability, controllability, observability, etc. have been conducted [1], [6]. Finally, several control [7], [8], [11] and estimation [7], [12]–[16] strategies have been investigated.

In the one-dimensional, i.e., single-indexed, case, set-based observers have been developed to compute sets guaranteed to contain the state vector when a system is subject to bounded disturbances and measurement noise [17]–[19]. In this context, interval observers have been widely studied [20]–[23]. They consist in the design of two sub-observers computing respectively an upper and a lower bound to the state vector. Often, the design of such observers is then based on combining a change of coordinates and gain tuning methods to ensure that the estimation error system is positive [24] and stable. However, the change of coordinates heavily impacts the performance of interval observers [25]. To overcome this issue, the work [22] introduces the TNL approach (named after the notation for the different matrices used) employing weighting matrices in addition to the traditional observer gain. With these additional degrees of freedom, it is easier to ensure that the error system becomes positive without requiring a change of coordinates.

To the authors' knowledge, interval observers for partial differential equations have been investigated [26], [27], but no interval observers for 2D systems have been proposed. Therefore, this letter proposes to build a robust interval observer for a 2D discrete-time linear time-invariant (LTI) system described by the Fornasini-Marchesini second (FM-II) model (defined in Section II.B). The definition of such an observer, derived from [28] where the one-dimensional case is studied, is based on the exponential stability of the estimation error dynamics and the peak-to-peak performance of these estimation errors. In [14], a filter for stochastic FM-II systems that is asymptotically stable and peak-to-peak norm bounded is introduced. Therefore, the proposed interval observer can be based on the pointwise observer of [14]. However, it has been shown in [29] that for FM-II systems, exponential stability implies asymptotic stability while the converse is not true. It is therefore necessary to ensure that the conditions proposed in [14] also entail exponential stability. To do so, conditions for the exponential stability of a 2D nonlinear FM-II system, which requires the boundary conditions of the system to be bounded by an exponentially decreasing function, have been given in [30]. In the meantime, [31] proposed a definition and a condition for exponential stability of a 2D system described by the Roesser model which can be adapted, as shown by [29], to 2D systems described by the FM-II model. This condition requires that the boundary conditions of the system have a finite infinity norm, thus relaxing the requirement of [30].

Based on what precedes, the contribution of the present letter is twofold: (i) a Lyapunov function-based condition for the exponential stability and peak-to-peak performance of a 2D LTI discrete-time system described by the FM-II model with boundary conditions having finite infinity norm; (ii) the

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[1]Following the majority of the literature on the subject, the name "two-dimensional system" is used here to refer to double-indexed systems.

definition and construction of a novel robust interval observer for such a system.

The remainder of this letter is organized as follows. General prerequisites and assumptions on the considered model are presented in Section II. In Section III, the proposed structure and design procedure for the interval observer as well as the proposed stability and boundedness conditions are introduced. Section IV proposes numerical simulation results, which assess the efficiency of the proposed method. Finally, Section V gathers concluding remarks and perspectives.

II. Prerequisites and Problem Formulation

A. Notations

The set of nonnegative integers, real numbers, and nonnegative real numbers are denoted by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$, respectively. The matrices I_n and $\mathbf{0}$ are respectively the identity matrix of size $n \in \mathbb{N}$ and the matrix of appropriate dimension filled with zeros. The matrices $A^\top$ and $A^\dagger$ denote respectively the transpose and the Moore-Penrose pseudo-inverse of a matrix $A \in \mathbb{R}^{n\times m}$. Any inequality involving vectors or matrices has to be understood elementwise. The positive or negative definiteness (resp. semi-definiteness) of a matrix $A \in \mathbb{R}^{n\times n}$ are denoted by $A \succ 0$ and $A \prec 0$ (resp. $A \succeq 0$ and $A \preceq 0$), respectively. If $x\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}^n$ is a 2D signal, its Euclidean norm is $\|x(k,l)\| = \sqrt{x(k,l)^\top x(k,l)}$ and its L_∞ norm is the supremum over the two indices of its Euclidean norm, i.e., $\|x\|_\infty = \sup\{\|x(k,l)\| | k,l \in \mathbb{N}\}$. The set of all signals $x\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}^n$ satisfying $\|x\|_\infty < \infty$ is denoted by $\mathcal{L}_\infty^n$. Finally, $\star$ is a placeholder denoting the transpose of a term placed symmetrically in a matrix.

B. Results on positive 2D systems

Any matrix $A \in \mathbb{R}^{n\times m}$ can be decomposed into two nonnegative matrices $A^+ = \max\{\mathbf{0}, A\} \geq \mathbf{0}$ (where the maximum is understood elementwise) and $A^- = A^+ - A \geq \mathbf{0}$. The matrix A is called nonnegative if $A^- = \mathbf{0}$.

Lemma 1 ([32]). *Let $x \in \mathbb{R}^m$ be a vector satisfying $\underline{x} \leq x \leq \overline{x}$, with $\underline{x}, \overline{x} \in \mathbb{R}^m$. Let $A \in \mathbb{R}^{n\times m}$ be a constant matrix. Then,*
$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}.$$

Now, consider a 2D LTI discrete-time system described by the FM-II model,
$$\begin{cases} x(k+1,l+1) = F_1x(k,l+1) + F_2x(k+1,l) \\ \qquad\qquad\qquad + G_1\upsilon(k,l+1) + G_2\upsilon(k+1,l) \\ \qquad\quad y(k,l) = Hx(k,l) + J\upsilon(k,l), \end{cases} \tag{1}$$
where $x \in \mathbb{R}^n$, $\upsilon \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and F_1, F_2, G_1, G_2, H, J are constant matrices of appropriate dimensions.

Definition 1 ([24]). *The system* (1) *is called internally positive if, for all boundary conditions $x(k,0), x(0,l) \in \mathbb{R}_+^n$ and sequences of inputs $\upsilon(k,l) \in \mathbb{R}_+^m$, with $k,l \in \mathbb{N}$, $x(k,l) \in \mathbb{R}_+^n$ and $y(k,l) \in \mathbb{R}_+^p$, $\forall k,l \in \mathbb{N}$.*

Lemma 2 ([24]). *The system* (1) *is internally positive if $F_1, F_2 \in \mathbb{R}_+^{n\times n}$, $H \in \mathbb{R}_+^{p\times n}$, and $x(k,0), x(0,l) \in \mathbb{R}_+^n$, $G_1\upsilon(k,l), G_2\upsilon(k,l) \in \mathbb{R}_+^n$, $J\upsilon(k,l) \in \mathbb{R}_+^p$, $\forall k,l \in \mathbb{N}$.*

C. Problem formulation

Consider a 2D LTI discrete-time system described by the following FM-II model affected by disturbances and measurement noise
$$\begin{cases} x(k+1,l+1) = A_1x(k,l+1) + A_2x(k+1,l) \\ \qquad\qquad\qquad + B_1u(k,l+1) + B_2u(k+1,l) \\ \qquad\qquad\qquad + D_1w(k,l+1) + D_2w(k+1,l) \\ \qquad\quad y(k,l) = Cx(k,l) + Ev(k,l), \end{cases} \tag{2}$$
where $x \in \mathbb{R}^{n_x}$, $y \in \mathbb{R}^{n_y}$, $u \in \mathbb{R}^{n_u}$, $w \in \mathbb{R}^{n_w}$, and $v \in \mathbb{R}^{n_v}$ are respectively the state, output, input, disturbance, and measurement noise vectors. The matrices $A_1, A_2, B_1, B_2, C, D_1, D_2$, and E have appropriate dimensions. The system (2) admits for boundary conditions
$$x(k,0) = \psi_1(k),\ k \in \mathbb{N}, \qquad x(0,l) = \psi_2(l),\ l \in \mathbb{N}. \tag{3}$$

Assumption 1. *The boundary conditions ψ_1 and ψ_2, the disturbance w, and the measurement noise v are unknown but bounded and satisfy $\underline{\psi}_1(k) \leq \psi_1(k) \leq \overline{\psi}_1(k)$, $\underline{\psi}_2(l) \leq \psi_2(l) \leq \overline{\psi}_2(l)$, $\underline{w}(k,l) \leq w(k,l) \leq \overline{w}(k,l)$, and $\underline{v}(k,l) \leq v(k,l) \leq \overline{v}(k,l)$, $\forall k,l \in \mathbb{N}$, with $\underline{\psi}_1, \overline{\psi}_1, \underline{\psi}_2, \overline{\psi}_2 \in \mathcal{L}_\infty^{n_x}$, $\underline{w}, \overline{w} \in \mathcal{L}_\infty^{n_w}$, and $\underline{v}, \overline{v} \in \mathcal{L}_\infty^{n_v}$.*

This letter's purpose is to define a new interval observer based on the TNL approach of [22]. This interval observer provides two signals $\underline{x}, \overline{x}\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}^{n_x}$ satisfying $\underline{x}(k,l) \leq x(k,l) \leq \overline{x}(k,l), \forall k,l \in \mathbb{N}$. In addition, the proposed observer is designed so that the ratio of the norm of the estimation error to the norm of the disturbances and measurement noise is lower than a prescribed value $\gamma > 0$.

III. Main Results

This section introduces the proposed framer enclosing the state of a 2D system described by the FM-II model subject to disturbances and measurement noise. To compute the observer's gains and to guarantee that the framer is a robust interval observer for the system (2), a linear matrix inequality-based design strategy is provided.

A. Stability and robustness for 2D systems described by the Fornasini-Marchesini second model

In the one-dimensional case, the work [28] defines an interval observer as two signals $\underline{x} : \mathbb{N} \to \mathbb{R}^{n_x}$ and $\overline{x} : \mathbb{N} \to \mathbb{R}^{n_x}$ satisfying $\underline{x} \leq x \leq \overline{x}$ at all time so that the upper and lower estimation errors $\overline{e} = \overline{x} - x$ and $\underline{e} = x - \underline{x}$ are input-to-state stable. These conditions can be adapted to the 2D case.

Definition 2. *Let $\overline{x} : \mathbb{N}\times\mathbb{N} \to \mathbb{R}^{n_x}$ and $\underline{x} : \mathbb{N}\times\mathbb{N} \to \mathbb{R}^{n_x}$ be two 2D signals. The pair $(\underline{x}, \overline{x})$ is a robust interval observer for system* (2) *with boundary conditions* (3) *if:*

(i) *$(\underline{x}, \overline{x})$ is a framer for system* (2)*, i.e., $\underline{x}(k,l) \leq x(k,l) \leq \overline{x}(k,l)$, $\forall k,l \in \mathbb{N}$;*

(ii) *the upper and lower estimation errors $\overline{e} = \overline{x} - x$ and $\underline{e} = x - \underline{x}$ are exponentially stable for zero input, having the peak-to-peak norm bound $\gamma > 0$, under zero boundary conditions, i.e., $\|\overline{e}\|_\infty \leq \gamma\left(\|\overline{\phi}\|_\infty + \|\phi\|_\infty\right)$*

and $\|\underline{e}\|_\infty \leq \gamma\left(\|\underline{\phi}\|_\infty + \|\phi\|_\infty\right)$ *where* ϕ *is the total disturbance acting on* (2)*, satisfying* $\underline{\phi}(k,l) \leq \phi(k,l) \leq \overline{\phi}(k,l)$*,* $\forall k, l \in \mathbb{N}$.

As stated in the introduction, in most works dealing with control or estimation for 2D systems described by the FM-II model [14], only asymptotic stability requirements are made. However, as shown in [29], for the FM-II model with boundary conditions given as in (3), exponential stability implies asymptotic stability while the converse is not true. In addition, ensuring exponential stability gives some control over the convergence speed of the interval observer. Therefore, it is necessary to obtain conditions guaranteeing exponential stability of a 2D system described by the FM-II model to propose an interval observer for such a system.

A condition for exponential stability of systems described by the Roesser model is proposed in [31]. Then, [29] shows that the definition of exponential stability given in [31] can be adapted to systems described by the FM-II model. Therefore, based on the asymptotically stable peak-to-peak norm bounded filter introduced in [14], the following theorem gives conditions for exponential stability and peak-to-peak performance of 2D systems described by the FM-II model.

Theorem 1. *If there exist scalars* $\alpha \in (0,1)$*,* $\gamma \geq 0$*, and* $\mu \geq 0$*, with* $\gamma - \mu \geq 0$*, and two matrices* $P \in \mathbb{R}^{n\times n}$ *and* $Q \in \mathbb{R}^{n\times n}$*, with* $P = P^\top \succ 0$*,* $Q = Q^\top \succ 0$*, and* $P - Q \succ 0$*, such that*

$$V_0(k+1,l+1) \leq (1-\alpha)(V_1(k,l+1) + V_2(k+1,l)) + \frac{1}{2}\mu\big(\|\upsilon(k,l+1)\|^2 + \|\upsilon(k+1,l)\|^2\big), \quad (4)$$

where $V_0(k,l) = x(k,l)^\top P x(k,l)$*,* $V_1(k,l) = x(k,l)^\top (P - Q)x(k,l)$*, and* $V_2(k,l) = x(k,l)^\top Q x(k,l)$*, then system* (1) *with boundary conditions* (3) *is exponentially stable for zero input, i.e., for* $\upsilon \equiv \mathbf{0}$*. In addition to condition* (4)*, if*

$$\frac{1}{2}\|x(k,l+1)\|^2 + \frac{1}{2}\|x(k+1,l)\|^2 \leq \gamma\big(\alpha^2\big(V_1(k,l+1) + V_2(k+1,l)\big) + \frac{1}{2}(\gamma-\mu)\big(\|\upsilon(k,l+1)\|^2 + \|\upsilon(k+1,l)\|^2\big)\big), \quad (5)$$

system (1) *satisfies* $\|x\|_\infty \leq \gamma\|\upsilon\|_\infty$ *under zero boundary conditions.*

Proof. Since $V_0(k,l) = V_1(k,l) + V_2(k,l)$, inequality (4) can be rewritten, when $\upsilon \equiv \mathbf{0}$, as

$$V_1(k+1,l+1) \leq (1-\alpha)V_1(k,l+1) + (1-\alpha)V_2(k+1,l) - V_2(k+1,l+1), \quad (6)$$

which implies, as in [31], by recursively bounding $V_1(i, l+1)$, with $0 < i \leq k$, on the right hand side of (6)

$$V_1(k+1,l+1) \leq (1-\alpha)^{k+1}V_1(0,l+1) + (1-\alpha)W_2(k,l) - W_2(k,l+1), \quad (7)$$

where $W_2(k,l) = \sum_{i=0}^{k}(1-\alpha)^{k-i}V_2(i+1,l)$. The function $W_2(k,j)$, with $0 < j \leq l$, can also be recursively bounded in the same way as V_1 so that

$$W_1(k+1,l) + W_2(k,l+1) \leq (1-\alpha)^{k+1}W_1(0,l) + (1-\alpha)^{l+1}W_2(k,0),$$

where $W_1(k,l) = \sum_{j=0}^{l}(1-\alpha)^{l-j}V_1(k,j+1)$. The term $W_1(k+1,l) + W_2(k,l+1)$ is a finite sum of nonnegative terms, implying that it is greater than or equal to $V_1(k+1,l+1) + V_2(k+1,l+1) = V_0(k+1,l+1)$. Therefore, factoring on the right hand side by $(1-\alpha)^2$ and $(1-\alpha)^{k+l+1}$,

$$x(k+1,l+1)^\top P x(k+1,l+1) \leq (1-\alpha)^{k+l+3}\left(\sum_{i=0}^{k}\frac{V_2(i+1,0)}{(1-\alpha)^{i+2}} + \sum_{j=0}^{l}\frac{V_1(0,j+1)}{(1-\alpha)^{j+2}}\right).$$

Let β be the largest eigenvalue of $\mathrm{diag}(P-Q, Q)$ and δ be the smallest eigenvalue of P. Remembering that $0 < 1-\alpha < 1$, the above inequality then implies

$$\|x(k+1,l+1)\|^2 \leq \frac{\beta}{\delta}(1-\alpha)^{k+l}\left(\sum_{i=1}^{k+1}\frac{\|\psi_1(i)\|^2}{(1-\alpha)^{i+1}} + \sum_{j=1}^{l+1}\frac{\|\psi_2(j)\|^2}{(1-\alpha)^{j+1}}\right) \leq \frac{\beta}{\delta}(1-\alpha)^{k+l}\left(\sum_{i=1}^{k+1}\frac{\|\psi_1(i)\|}{\sqrt{1-\alpha}^{i+1}} + \sum_{j=1}^{l+1}\frac{\|\psi_2(j)\|}{\sqrt{1-\alpha}^{j+1}}\right)^2$$

so that

$$\|x(k,l)\| \leq Mq^{k+l}\left(\sum_{i=1}^{k}\frac{\|\psi_1(i)\|}{q^{i+1}} + \sum_{j=1}^{l}\frac{\|\psi_2(j)\|}{q^{j+1}}\right),$$

where $M = \sqrt{\beta/\delta}$ and $q = \sqrt{1-\alpha}$, which is the definition of exponential stability for a 2D FM-II system [29].

Applying the previous recursive bounding procedure with non-zero input and remembering that $\|\upsilon(k,l+1)\|^2 + \|\upsilon(k+1,l)\|^2 \leq 2\|\upsilon\|_\infty^2$,

$$W_1(k+1,l) + W_2(k,l+1) \leq (1-\alpha)^{k+1}W_1(0,l) + (1-\alpha)^{l+1}W_2(k,0) + \mu\sum_{i=0}^{k}\sum_{j=0}^{l}(1-\alpha)^{k-i}(1-\alpha)^{l-j}\|\upsilon\|_\infty^2,$$

or, since $0 < 1-\alpha < 1$,

$$V_1(k+1,l+1) + V_2(k+1,l+1) \leq \frac{\mu}{\alpha^2}\|\upsilon\|_\infty^2 + (1-\alpha)^{k+1}W_1(0,l) + (1-\alpha)^{l+1}W_2(k,0). \quad (8)$$

Using (8) to bound the right hand side of (5),

$$\|x(k+1,l+1)\|^2 \leq \gamma\mu\|\upsilon\|_\infty^2 + \gamma(\gamma-\mu)\|\upsilon\|_\infty^2 + \gamma\alpha^2((1-\alpha)^{k+1}W_1(0,l) + (1-\alpha)^{l+1}W_2(k,0)). \quad (9)$$

With zero boundary conditions, inequality (9) is equivalent to $\|x(k,l)\| \leq \gamma\|\upsilon\|_\infty$. □

Remark 1. *With non-zero boundary condition, the state vector's norm is bounded over time by* $\gamma\|\upsilon\|_\infty$ *plus a vanishing term depending on the boundary conditions. Therefore, the set* $\{x \in \mathbb{R}^n \mid \|x\| \leq \gamma\|\upsilon\|_\infty\}$ *is an attractor for system* (1)*.*

B. Robust interval observer design

Inspired by the work of [22], the proposed structure for the framer of system (2) is

$$\begin{cases} \overline{x}(k+1,l+1) = (TA_1 - L_1C)\overline{x}(k,l+1) \\ \qquad + (TA_2 - L_2C)\overline{x}(k+1,l) \\ \qquad + TB_1u(k,l+1) + TB_2u(k+1,l) \\ \qquad + L_1y(k,l+1) + L_2y(k+1,l) \\ \qquad + Ny(k+1,l+1) + \overline{\phi}(k,l) \\ \underline{x}(k+1,l+1) = (TA_1 - L_1C)\underline{x}(k,l+1) \\ \qquad + (TA_2 - L_2C)\underline{x}(k+1,l) \\ \qquad + TB_1u(k,l+1) + TB_2u(k+1,l) \\ \qquad + L_1y(k,l+1) + L_2y(k+1,l) \\ \qquad + Ny(k+1,l+1) + \underline{\phi}(k,l), \end{cases} \tag{10}$$

where $\underline{\phi}(k,l)$ and $\overline{\phi}(k,l)$ are

$$\begin{cases} \overline{\phi}(k,l) = (TD_1)^+\overline{w}(k,l+1) - (TD_1)^-\underline{w}(k,l+1) \\ \qquad + (TD_2)^+\overline{w}(k+1,l) - (TD_2)^-\underline{w}(k+1,l) \\ \qquad - (L_1E)^+\underline{v}(k,l+1) + (L_1E)^-\overline{v}(k,l+1) \\ \qquad - (L_2E)^+\underline{v}(k+1,l) + (L_2E)^-\overline{v}(k+1,l) \\ \qquad - (NE)^+\underline{v}(k+1,l+1) + (NE)^-\overline{v}(k+1,l+1) \\ \underline{\phi}(k,l) = (TD_1)^+\underline{w}(k,l+1) - (TD_1)^-\overline{w}(k,l+1) \\ \qquad + (TD_2)^+\underline{w}(k+1,l) - (TD_2)^-\overline{w}(k+1,l) \\ \qquad - (L_1E)^+\overline{v}(k,l+1) + (L_1E)^-\underline{v}(k,l+1) \\ \qquad - (L_2E)^+\overline{v}(k+1,l) + (L_2E)^-\underline{v}(k+1,l) \\ \qquad - (NE)^+\overline{v}(k+1,l+1) + (NE)^-\underline{v}(k+1,l+1), \end{cases} \tag{11}$$

and $T \in \mathbb{R}^{n_x \times n_x}$, $N \in \mathbb{R}^{n_x \times n_y}$, $L_1 \in \mathbb{R}^{n_x \times n_y}$, and $L_2 \in \mathbb{R}^{n_x \times n_y}$ are design parameters of the framer, with T and N satisfying

$$T + NC = I_{n_x}. \tag{12}$$

Theorem 2. *Let Assumption 1 hold. If there exist $T \in \mathbb{R}^{n_x \times n_x}$ and $L_1, L_2 \in \mathbb{R}^{n_x \times n_y}$ such that $TA_1 - L_1C$ and $TA_2 - L_2C$ are nonnegative matrices, then*

$$\underline{x}(k,l) \le x(k,l) \le \overline{x}(k,l),\ \forall k,l \in \mathbb{N}. \tag{13}$$

Proof. Let $\overline{e}(k,l) = \overline{x}(k,l) - x(k,l)$ and $\underline{e}(k,l) = x(k,l) - \underline{x}(k,l)$ be the upper and lower estimation errors. Let also

$$\begin{aligned}\phi(k,l) &= TD_1w(k,l+1) + TD_2w(k+1,l) \\ &\quad -L_1Ev(k,l+1) - L_2Ev(k+1,l) - NEv(k+1,l+1)\end{aligned}$$

be the total disturbance. Then,

$$\begin{aligned}\overline{e}(k+1,l+1) &= (TA_1 - L_1C)\overline{e}(k,l+1) \\ &\quad + (TA_2 - L_2C)\overline{e}(k+1,l) + \overline{\phi}(k,l) - \phi(k,l)\end{aligned}$$

and, by Lemma 1, $\overline{\phi}(k,l) \ge \phi(k,l)$. Therefore, by Lemma 2, $\overline{e}(k,l) \ge \mathbf{0}$, $\forall k,l \in \mathbb{N}$, if $TA_1 - L_1C$ and $TA_2 - L_2C$ are nonnegative matrices. With the same reasoning, $\underline{e}(k,l) \ge \mathbf{0}$, $\forall k,l \in \mathbb{N}$, if $TA_1 - L_1C$ and $TA_2 - L_2C$ are nonnegative matrices. □

As noted in [22], the matrices T and N provide additional degrees of freedom in the interval observer's tuning. Indeed, it might be difficult to obtain matrices L_i, with $i \in \{1,2\}$, such that the matrices $A_i - L_iC$ are both nonnegative and stable [21]. The matrix T relaxes this difficulty while allowing, in conjuction with the matrix N, a better performance tuning.

Then, from the results of Theorem 1 and Theorem 2, it is possible to derive linear matrix inequality conditions to ensure that (10) is an interval observer according to Definition 2. First, the following lemma recalls the structure of the matrices T and N.

Lemma 3 ([33])**.** *Given three matrices $X \in \mathbb{R}^{n\times m}$, $Y \in \mathbb{R}^{m \times p}$, and $Z \in \mathbb{R}^{n\times p}$, with $\operatorname{rank} Y = p$, the general solution X of the equation $XY = Z$ is*

$$X = ZY^\dagger + \Xi\left(I_m - YY^\dagger\right)$$

where $\Xi \in \mathbb{R}^{n\times m}$ is an arbitrary matrix.

With Lemma 3, T and N satisfying (12) are

$$T = \Theta^\dagger\lambda_1 + \Xi\Upsilon\lambda_1, \qquad N = \Theta^\dagger\lambda_2 + \Xi\Upsilon\lambda_2, \tag{14}$$

where $\Xi \in \mathbb{R}^{n_x\times(n_x+n_y)}$ is an arbitrary matrix, $\Theta^\top = \begin{bmatrix} I_{n_x} & C^\top\end{bmatrix}$, $\Upsilon = I_{n_x+n_y} - \Theta\Theta^\dagger$, $\lambda_1^\top = \begin{bmatrix} I_{n_x} & \mathbf{0}\end{bmatrix}$, $\lambda_2^\top = \begin{bmatrix}\mathbf{0} & I_{n_y}\end{bmatrix}$.

Theorem 3. *Let Assumption 1 hold. If there exist scalars $\alpha \in (0,1)$, $\gamma \ge 0$, and $\mu \ge 0$ and matrices $P \in \mathbb{R}^{n_x\times n_x}$ diagonal, $Q \in \mathbb{R}^{n_x \times n_x}$ diagonal, $X_1 \in \mathbb{R}^{n_x\times n_y}$, $X_2 \in \mathbb{R}^{n_x \times n_y}$, and $Y \in \mathbb{R}^{n_x\times(n_x+n_y)}$, with $P, Q, P-Q \succ 0$, such that*

$$S \ge 0, \tag{15}$$

$$\begin{bmatrix} (\alpha-1)\Delta & \star & \star \\ \mathbf{0} & -\mu I_{n_\phi}/2 & \star \\ S & \Phi & -P \end{bmatrix} \preceq 0, \tag{16}$$

$$\begin{bmatrix} \alpha^2\Delta & \star & \star \\ \mathbf{0} & (\gamma-\mu)I_{n_\phi}/2 & \star \\ I_{2n_x} & \mathbf{0} & 2\gamma I_{2n_x} \end{bmatrix} \succeq 0, \tag{17}$$

where $S = \begin{bmatrix} S_1 & S_2 \end{bmatrix}$, $S_1 = \Pi\lambda_1A_1 - X_1C$, $S_2 = \Pi\lambda_1A_2 - X_2C$, with $\Pi = P\Theta^\dagger + Y\Upsilon$, $\Delta = \operatorname{diag}(P-Q, Q)$, $n_\phi = 2n_w + 3n_v + n_x$, and

$$\Phi = \begin{bmatrix} \Pi\lambda_1D_1 & \Pi\lambda_1D_2 & -X_1E & -X_2E & -\Pi\lambda_2E & -P\end{bmatrix},$$

then (10) is a robust interval observer for system (1) with $L_1 = P^{-1}X_1$, $L_2 = P^{-1}X_2$, and $\Xi = P^{-1}Y$.

Proof. Since $P \succ 0$ and is diagonal, all its diagonal elements are strictly positive. Defining $X_1 = PL_1$, $X_2 = PL_2$, and $Y = P\Xi$, condition (15) is equivalent to the positivity of $TA_1 - L_1C$ and $TA_2 - L_2C$.

Moreover, applying the Schur complement to (16) and (17), pre-multiplying by $\overline{z}(k,l)^\top$ and post-multiplying by $\overline{z}(k,l)$, where

$$\begin{aligned}\overline{z}(k,l)^\top = \big[&\overline{e}(k,l+1)^\top\ \overline{e}(k+1,l)^\top\ w(k,l+1)^\top \\ &w(k+1,l)^\top\ v(k,l+1)^\top\ v(k+1,l)^\top \\ &v(k+1,l+1)^\top\ \overline{\phi}(k,l)^\top\big],\end{aligned}$$

yields conditions (4) and (5) for the upper estimation error, where the terms x and $G_1 v(k, l+1) + G_2 v(k+1, l)$ appearing in (1) are replaced by $\overline{e}$ and $\overline{\phi}(k,l) - \phi(k,l)$, respectively. Then, applying the Schur complement to (16) and (17), pre-multiplying by $\underline{z}(k,l)^\top$ and post-multiplying by $\underline{z}(k,l)$, where

$$\underline{z}(k,l)^\top = \big[\underline{e}(k,l+1)^\top \ \underline{e}(k+1,l)^\top \ w(k,l+1)^\top \ w(k+1,l)^\top \ v(k,l+1)^\top \ v(k+1,l)^\top \ v(k+1,l+1)^\top \ \underline{\phi}(k,l)^\top\big],$$

yields conditions (4) and (5) for the lower estimation error, where the terms x and $G_1 v(k, l+1) + G_2 v(k+1, l)$ appearing in (1) are replaced by $\underline{e}$ and $\phi(k,l) - \underline{\phi}(k,l)$, respectively.

Therefore, with conditions (15)–(17), the upper and lower estimation errors satisfy the conditions of Definition 2. $\square$

IV. Numerical Simulation

Consider a numerical example of system (2) adapted from [34], with

$$A_1 = \begin{bmatrix} 1 & -2.5 \\ 0.1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.6 \\ 0.3 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0.1 & -0.4 \\ 0.8 & -0.2 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.2 & 0.6 \\ -0.1 & -0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.2 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} -0.5 & 0.2 \end{bmatrix}, \quad E = 0.8,$$

with the boundary conditions

$$\psi_1(k) = \begin{bmatrix} \cos(k)\sin(0.3k) \\ \sin(0.5k) \end{bmatrix}, \quad \psi_2(l) = \begin{bmatrix} \cos(0.5l) \\ \sin(l)\cos(0.3l) \end{bmatrix}.$$

For simulation, the input signal is $u(k,l) = 0.5\sin(0.01k)\cos(0.01l)$. Moreover, the disturbance is a uniformly distributed random vector satisfying $\underline{w} \leq w(k,l) \leq \overline{w}$, with $\overline{w} = -\underline{w} = 0.05 \cdot \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$. Finally, the measurement noise is a uniformly distributed random scalar satisfying $\underline{v} \leq v(k,l) \leq \overline{v}$, with $\overline{v} = -\underline{v} = 0.1$.

The observer is initialized with $\overline{\psi}_1 = \overline{\psi}_2 = -\underline{\psi}_1 = -\underline{\psi}_2 = 3 \cdot \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$. In addition, $\alpha = 0.9$ is chosen. To obtain γ, T, N, L_1, and L_2, an optimization problem minimizing γ under the constraints (15)–(17) is solved using CVX [35], yielding $\gamma = \mu = 3.1689$ and

$$T = \begin{bmatrix} 0.0076 & 0.3970 \\ 0.0189 & 0.9924 \end{bmatrix}, \qquad N = \begin{bmatrix} -1.9849 \\ 0.0378 \end{bmatrix},$$
$$L_1 = \begin{bmatrix} -0.0945 \\ -0.2363 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 0.0272 \\ 0.0604 \end{bmatrix}.$$

For readability, Fig. 1 and 2 only present the evolution of the state variables for selected values of l and k, respectively. These figures show the efficiency of the proposed interval observer. Indeed, the interval defined by $\underline{x}$ and $\overline{x}$ tightens in a small number of steps, whether it be along the direction k or the direction l. In addition, the state remains contained between the bounds $\underline{x}$ and $\overline{x}$.

This can also be seen in Fig. 3, showing the surface plots of $\underline{e}_1$ and $\overline{e}_1$, respectively. Both the lower estimation error $\underline{e}_1(k,l)$ and the upper estimation error $\overline{e}_1(k,l)$ remain positive for all values of k and l, which is the desired behavior. In addition, in less than five steps along both directions, the estimation errors converge to bounded values.

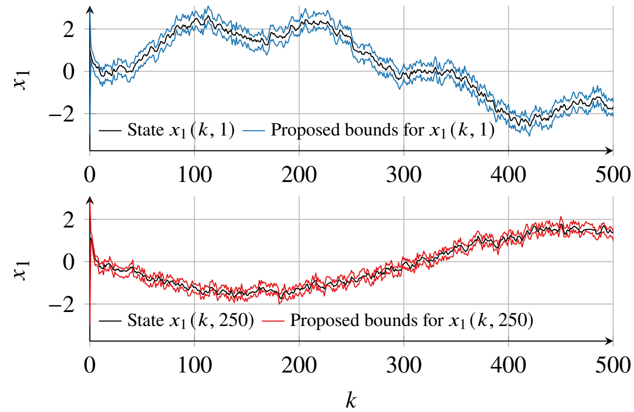

Fig. 1. First state variable and guaranteed bounds along the direction k for the two values $l = 1$ and $l = 250$.

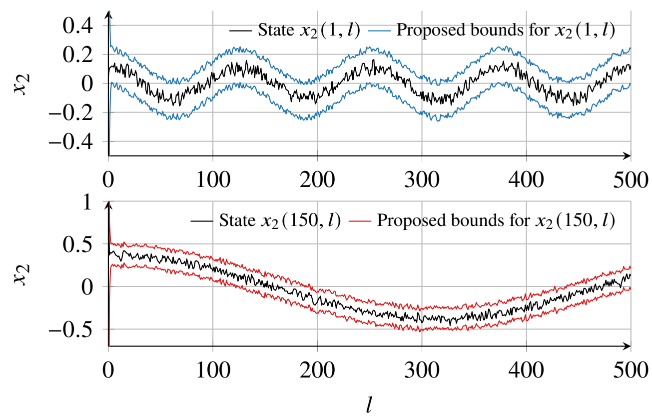

Fig. 2. Second state variable and guaranteed bounds along the direction l for the two values $k = 1$ and $k = 150$.

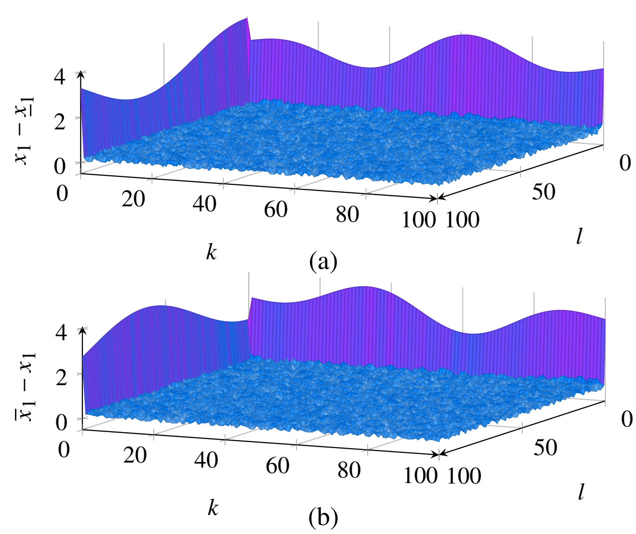

Fig. 3. Lower (a) and upper (b) estimation errors for the first state variable.

V. Conclusions and Future Work

This letter proposes a novel interval observer for two-dimensional systems described by the Fornasini-Marchesini second model. Building on existing interval observers for one-dimensional systems, the observer includes weighting matrices, in addition to the gains, in its design. These additional degrees of freedom are tuned along with the gain matrices to enforce the positivity and stability of the estimation errors and to attenuate the effect of the disturbances by considering a peak-to-peak norm criterion. The design parameters of the interval observer are then obtained by solving an optimization problem under linear matrix inequality constraints. The efficiency of the proposed estimation method is assessed by numerical simulation results. In future work, this interval observer can be adapted to other types of two-dimensional systems, such as the ones described by the Roesser model or the Fornasini-Marchesini first model. In addition, the proposed observer can be employed to evaluate the robustness of given control strategies against parameter uncertainties. Finally, interval-based (and more generally set-based) fault detection strategies for two-dimensional systems based on such an interval observer will be studied.

References

[1] K. Galkowski and E. Rogers, "Control systems analysis for the Fornasini-Marchesini 2D systems model – progress after four decades," *Int. J. Control*, vol. 91, no. 1, pp. 2801–2822, 2018.

[2] R. P. Roesser, "A discrete state-space model for linear image processing," *IEEE Trans. Autom. Control*, vol. 20, no. 1, pp. 1–10, 1975.

[3] E. Fornasini and G. Marchesini, "State-space realization theory for two-dimensional filters," *IEEE Trans. Autom. Control*, vol. 21, no. 4, pp. 484–492, 1976.

[4] ——, "Doubly-indexed dynamical systems: State-space models and structural properties," *Math. Syst. Theory*, vol. 12, pp. 59–72, 1978.

[5] J. E. Kurek, "The general state-space model for a two-dimensional linear digital system," *IEEE Trans. Autom. Control*, vol. 30, no. 6, pp. 600–602, 1985.

[6] T. Kaczorek, *Two-Dimensional Linear Systems*, ser. Lecture Notes in Control and Information Sciences. Berlin, Heidelberg, Germany: Springer, 1985, vol. 68.

[7] C. Du and L. Xie, *H_∞ Control and Filtering of Two-dimensional Systems*, ser. Lecture Notes in Control and Information Sciences. Berlin, Heidelberg, Germany: Springer, 2002, vol. 278.

[8] E. Rogers, K. Galkowski, and D. H. Owens, *Control Systems Theory and Applications for Linear Repetitive Processes*, ser. Lecture Notes in Control and Information Sciences. Berlin, Heidelberg, Germany: Springer, 2007, vol. 349.

[9] W. Marszalek, "Two-dimensional state-space discrete models for hyperbolic partial differential equations," *Appl. Math. Modelling*, vol. 8, no. 1, pp. 11–14, 1984.

[10] J. E. Kurek and M. B. Zaremba, "Iterative learning control synthesis based on 2-D system theory," *IEEE Trans. Autom. Control*, vol. 38, no. 1, pp. 121–125, 1993.

[11] R. Yang and W. X. Zheng, "Two-dimensional sliding mode control of discrete-time Fornasini-Marchesini systems," *IEEE Trans. Autom. Control*, vol. 64, no. 9, pp. 3943–3948, 2019.

[12] D. Zhao, Z. Lin, and Y. Wang, "Integrated state/disturbance observers for two-dimensional linear systems," *IET Control Theory Appl.*, vol. 9, no. 9, pp. 1373–1383, 2015.

[13] D.-W. Ding, H. Wang, and X. Li, "H_-/H_∞ fault detection observer design for two-dimensional Roesser systems," *Syst. Control Lett.*, vol. 82, pp. 115–120, 2015.

[14] C. K. Ahn, P. Shi, and M. V. Basin, "Two-dimensional peak-to-peak filtering for stochastic Fornasini-Marchesini systems," *IEEE Trans. Autom. Control*, vol. 63, no. 5, pp. 1472–1479, 2018.

[15] H. Alikhani, M. Aliyari Shoorehdeli, and N. Meskin, "A functional unknown input observer for linear singular Fornasini-Marchesini first model systems: with applications to fault diagnosis," *IEEE Syst. J.*, vol. 15, no. 3, pp. 4093–4104, 2021.

[16] J. Fu, Z. Duan, and Z. Xiang, "On mixed ℓ_1/ℓ_- fault detection observer design for positive 2D Roesser systems: Necessary and sufficient conditions," *J. Franklin Inst.*, 2021.

[17] J.-L. Gouzé, A. Rapaport, and M. Z. Hadj-Sadok, "Interval observers for uncertain biological systems," *Ecol. Modell.*, vol. 133, no. 1-2, pp. 45–56, 2000.

[18] C. Combastel, "A state bounding observer based on zonotopes," in *Proc. IEEE ECC*, 2003, pp. 2589–2594.

[19] T. Alamo, J. M. Bravo, and E. F. Camacho, "Guarateed state estimation by zonotopes," *Automatica*, vol. 41, no. 6, pp. 1035–1043, 2005.

[20] F. Mazenc and O. Bernard, "Interval observers for linear time-invariant systems with disturbances," *Automatica*, vol. 47, no. 1, pp. 140–147, 2011.

[21] T. Raïssi, D. Efimov, and A. Zolghadri, "Interval state estimation for a class of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 57, no. 1, pp. 260–265, 2012.

[22] Z. Wang, C.-C. Lim, and Y. Shen, "Interval observer design for uncertain discrete-time linear systems," *Syst. Control Lett.*, vol. 116, pp. 41–46, 2018.

[23] Z. Wang, H. Yin, T. N. Dinh, and T. Raïssi, "Interval estimation based on the reduced-order observer and peak-to-peak analysis," *Int. J. Control*, 2021.

[24] T. Kaczorek, *Positive 1D and 2D Systems*, ser. Communications and Control Engineering. London, United Kingdom: Springer, 2002.

[25] E. Chambon, L. Burlion, and P. Apkarian, "Overview of linear time-invariant interval observer design: towards a non-smooth optimisation-based approach," *IET Control Theory Appl.*, vol. 10, no. 11, pp. 1258–1268, 2016.

[26] T. Kharkovskaya, D. Efimov, A. Polyakov, and J.-P. Richard, "Design of interval observers and controls for PDEs using finite-element approximations," *Automatica*, vol. 93, pp. 302–310, 2018.

[27] T. Kharkovskaia, D. Efimov, E. Fridman, A. Polyakov, and J.-P. Richard, "Interval observer design and control of uncertain non-homogeneous heat equations," *Automatica*, vol. 111, p. 108595, 2020.

[28] T. N. Dinh, G. Marouani, T. Raïssi, Z. Wang, and H. Messaoud, "Optimal interval observers for discrete-time linear switched systems," *Int. J. Control*, vol. 93, no. 11, pp. 2613–2621, 2020.

[29] O. Bachelier, T. Cluzeau, R. David, F. J. Silva Álvarez, N. Yeganefar, and N. Yeganefar, "Structural stability, asymptotic stability and exponential stability for linear multidimensional systems: the good, the bad and the ugly," *Int. J. Control*, vol. 91, no. 12, pp. 2714–2725, 2018.

[30] P. Pakshin, J. Emelianova, K. Galkowski, and E. Rogers, "Stabilization of two-dimensional nonlinear systems described by Fornasini-Marchesini and Roesser models," *SIAM J. Control Optim.*, vol. 56, no. 5, pp. 3848–3866, 2018.

[31] N. Yeganefar, N. Yeganefar, M. Ghamgui, and E. Moulay, "Lyapunov theory for 2D nonlinear Roesser models: application to asymptotic and exponential stability," *IEEE Trans. Autom. Control*, vol. 58, no. 5, pp. 1299–1304, 2013.

[32] D. Efimov, T. Raïssi, S. Chebotarev, and A. Zolghadri, "Interval state observer for nonlinear time varying systems," *Automatica*, vol. 49, no. 1, pp. 200–205, 2013.

[33] C. R. Rao and S. K. Mitra, "Generalized inverse of a matrix and its applications," in *Proc. 6th Berkeley Symp. on Math. Statist. and Prob.*, vol. 1, 1972, pp. 601–620.

[34] V. Singh, "Stability analysis of 2-D linear discrete systems based on the Fornasini-Marchesini second model: Stability with asymmetric Lyapunov matrix," *Digit. Signal Process.*, vol. 26, pp. 183–186, 2014.

[35] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.1," http://cvxr.com/cvx, Mar 2014.