Interval Estimation for Discrete-Time Linear Parameter-Varying System with Unknown Inputs

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Abstract— **This paper proposes a new interval observer for joint estimation of the state and unknown inputs of a discrete-time linear parameter-varying (LPV) system with an unmeasurable parameter vector. This system is assumed to be subject to unknown inputs and unknown but bounded disturbances and measurement noise, while the parameter-varying matrices are elementwise bounded. Considering the unknown inputs as auxiliary states, the dynamics are rewritten as discrete-time LPV descriptor dynamics. A new structure of interval observer is then used, providing more degrees of freedom than the classical change of coordinates-based structure. The observer gains are computed by solving linear matrix inequalities derived from cooperativity condition and L_∞ norm. Numerical simulations are run to show the efficiency of the proposed observer.**

Index Terms— **Linear parameter-varying system, Unknown input observer, Interval observer, Discrete-time systems**

I. Introduction

Linear parameter-varying (LPV) systems are a powerful tool to develop control algorithms for nonlinear systems, since many of them can be represented as LPV systems. Due to their partial linearity, LPV dynamics allow for the use of methods developed for linear systems [1]. Most control algorithms are based on the knowledge of the system's state at all time. However, in real-life applications, the vector of scheduling parameters is not always available, the system is subject to perturbations, the state is not completely measured, and measurements are noisy. Observers are then needed to reconstruct the system's state from this incomplete and biased information. In addition, real-life systems can be subject to model uncertainties or faults, which can be represented by additive unknown inputs in their dynamics [2]. To mitigate such faults, it is often necessary to also reconstruct these unknown inputs [3]. In this case, an unknown input observer (UIO) [4]–[6] is used. However, these works do not consider the presence of external perturbations or measurement noise, which limits the performance of the proposed UIOs in a more general context [7].

To overcome this issue, set-based estimation algorithms have been developed, based on the assumption that the noise, perturbations and initial state of the systems are unknown, but bounded. These algorithms can be separated into two categories: set-membership estimation [8]–[10], where the set of all states consistent with the system's dynamics and the uncertainties' bounds is approximated by a geometrical set; and interval observers [11]–[14], where two sub-observers provide an upper and a lower bound for the state consistent with the dynamics and uncertainties' bounds. Due to its computational efficiency, this paper considers interval observers to provide guaranteed bounds to both the state of a LPV system and the unknown inputs acting on it. Moreover, interval observers are ideal to deal with unavailable scheduling parameters [15].

In the literature, several unknown input interval observers have been proposed for linear time-invariant systems [7], [16], [17], but relatively few have been proposed for LPV systems [18], the other observers being mainly based on set-membership strategies [10], [19]. These strategies are based on a change of coordinates decoupling the state from the unknown input. Considering a second change of coordinates to satisfy the cooperativity condition (i.e. the estimation error state matrix is elementwise nonnegative), an interval observer for the state is then designed. The resulting interval is then used to compute an interval bounding the unknown inputs. The performance of such an observer is heavily influenced by the choice of target coordinates [20]. For these reasons, [17] and [10] propose a different approach by augmenting the state vector with the unknown input vector, thus considering descriptor dynamics. Then, based on the TNL approach (named after the notation for the different matrices used) introduced in [13], additional gain matrices are introduced to ensure the cooperativity condition of the interval observer that provides guaranteed bounds simultaneously for the state and the unknown inputs. This approach provides more degrees of freedom for the observer design than the ones proposed, for example, in [7] or [18].

This paper then proposes an unknown input interval observer providing guaranteed bounds to the state and unknown inputs of a discrete-time linear parameter varying system with an unavailable vector of scheduling parameters and subject to unknown but bounded perturbations. Following [10], [13] and [15], the main contributions of this study are twofold: (i) a novel interval observer structure for a class of discrete-time LPV systems with unmeasurable parameters, allowing for more degrees of freedom in the computation of the observer's gains thanks to the TNL approach; (ii) a new modular gain design procedure based on the cooperativity of the dynamics and L_∞ norm of the estimation error.

The remainder of this paper is organized as follows. General

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prerequisites and assumptions are given in Section II. Section III presents the proposed structure and design procedure for the interval observer. In Section IV, numerical simulation results are introduced to assess the efficiency of the proposed estimation strategy. Finally, Section V draws concluding remarks and perspectives.

II. Preliminaries and problem formulation

A. Notations

The sets of positive integers and real numbers are denoted respectively by $\mathbb{N}$ and $\mathbb{R}$. The matrix I_n is the identity matrix of size $n \in \mathbb{N}$. The vector $\mathbf{1}_n$ is the column vector of size $n \in \mathbb{N}$ filled with ones. The matrix $\mathbf{0}$ is the matrix of appropriate size filled with zeros. The matrices $A^\top$ and $A^\dagger$, with $A \in \mathbb{R}^{n\times m}$, denote respectively the transpose and the Moore-Penrose pseudo-inverse of matrix A. The notations $A \succeq 0$ and $A \preceq 0$ (respectively $A \succ 0$ and $A \prec 0$) mean that A is positive or negative semidefinite (resp. positive or negative definite). The matrix $\operatorname{diag}(A_1, \ldots, A_n)$ is the block diagonal matrix with diagonal blocks $A_1, \ldots, A_n$. Given a signal $x : \mathbb{N} \to \mathbb{R}^n$, its Euclidean norm is defined as $\|x_k\|_2^2 = x_k^\top x_k$ and its L_∞ norm is the supremum over time of its Euclidean norm, i.e. $\|x\|_\infty = \sup\{\|x_k\|_2 | k \in \mathbb{N}\}$. The set of all signals $x : \mathbb{N} \to \mathbb{R}^n$ satisfying $\|x\|_\infty < \infty$ is denoted by $\mathcal{L}_\infty^n$. The Kronecker product of two matrices A and B is denoted by $A \otimes B$. Finally, $\star$ is a placeholder denoting the transpose of a term placed symmetrically in a matrix.

B. Preliminary results on interval analysis

Let $A_1, A_2 \in \mathbb{R}^{n\times m}$ be two matrices. Then, the relation $A_1 \leq A_2$ is understood elementwise. Moreover, a matrix $A \in \mathbb{R}^{n\times m}$ can be decomposed into two nonnegative matrices $A^+ = \max\{\mathbf{0}, A\}$ (where the maximum is understood elementwise) and $A^- = A^+ - A$. The matrix A is said to be nonnegative if $A^- = \mathbf{0}$. The same decomposition can be applied to any vector $x \in \mathbb{R}^n$.

Lemma 1 ([21])**.** *Let $x \in \mathbb{R}^n$ be a vector satisfying $\underline{x} \leq x \leq \overline{x}$, with $\underline{x}, \overline{x} \in \mathbb{R}^n$.*

1) *Let $A \in \mathbb{R}^{m\times n}$ be a constant matrix. Then*
$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}.$$
2) *Let $A \in \mathbb{R}^{m\times n}$ be a matrix satisfying $\underline{A} \leq A \leq \overline{A}$, with $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$. Then*
$$\begin{aligned}\underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- &\leq Ax\\ &\leq \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-.\end{aligned}$$

Remark 1. *If x is a constant vector and A a matrix satisfying $\underline{A} \leq A \leq \overline{A}$, the first item of Lemma 1 becomes*
$$\underline{A}x^+ - \overline{A}x^- \leq Ax \leq \overline{A}x^+ - \underline{A}x^-.$$

Lemma 2 ([22])**.** *Let $F(z,\rho) = M(\rho)z$ be a function of z and ρ, with $M(\rho)$ a matrix and $\underline{z} \leq z \leq \overline{z}$. If there exist two matrices $\underline{M}$ and $\overline{M}$ and two functions $\underline{F}(\underline{z},\overline{z})$ and $\overline{F}(\underline{z},\overline{z})$ satisfying, for all possible values of ρ and z, $\underline{M} \leq M(\rho) \leq \overline{M}$ and $\underline{F}(\underline{z},\overline{z}) \leq F(z,\rho) \leq \overline{F}(\underline{z},\overline{z})$ then*
$$\begin{cases}\left\|\overline{F}(\underline{z},\overline{z}) - F(z,\rho)\right\|_2 \leq \bar{l}_F\|\overline{z}-z\|_2 + \underline{l}_F\|\underline{z}-z\|_2 + \overline{m}_F\\ \left\|\underline{F}(\underline{z},\overline{z}) - F(z,\rho)\right\|_2 \leq \bar{l}_F\|\overline{z}-z\|_2 + \underline{l}_F\|\underline{z}-z\|_2 + \underline{m}_F\end{cases}$$
where $\overline{m}_F$ and $\underline{m}_F$ are positive constants depending on the values of M and z, and
$$\begin{cases}\bar{l}_F = \left\|\overline{M}^+\right\|_2 + \left\|\underline{M}^+\right\|_2\\ \underline{l}_F = \left\|\overline{M}^-\right\|_2 + \left\|\underline{M}^-\right\|_2\end{cases}$$

C. Problem formulation

Consider the following discrete-time LPV system
$$\begin{cases}x_{k+1} = A(\rho_k)x_k + B(\rho_k)u_k + Dd_k + D_w(\rho_k)w_k\\ y_k = Cx_k + D_v v_k\end{cases} \tag{1}$$
where $x_k \in \mathbb{R}^{n_x}$ is the state vector, $u_k \in \mathbb{R}^{n_u}$ is the known input vector, $y_k \in \mathbb{R}^{n_y}$ is the output vector, $d_k \in \mathbb{R}^{n_d}$ is the unknown input vector, $w_k \in \mathbb{R}^{n_w}$ and $v_k \in \mathbb{R}^{n_v}$ are respectively the disturbance and measurement noise vectors and $\rho_k \in \mathbb{R}^{n_\rho}$ is the unmeasurable parameter vector. The matrices $A(\rho_k)$, $B(\rho_k)$, C, D, $D_w(\rho_k)$ and D_v are matrices of appropriate dimensions such that
$$M(\rho_k) = M_0 + \Delta M(\rho_k), \tag{2}$$
with $M \in \{A, B, D_w\}$.

Assumption 1. *The initial state vector x_0, the disturbance vector w_k and the measurement noise vector v_k are unknown but bounded and satisfy $\underline{x}_0 \leq x_0 \leq \overline{x}_0$, $\underline{w}_k \leq w_k \leq \overline{w}_k$ and $\underline{v}_k \leq v_k \leq \overline{v}_k$, $\forall k \geq 0$, with $\underline{w}_k, \overline{w}_k \in \mathcal{L}_\infty^{n_w}$ and $\underline{v}_k, \overline{v}_k \in \mathcal{L}_\infty^{n_v}$.*

Moreover, since the parameter vector ρ_k is unmeasurable, some conditions have to be imposed to the matrices ΔA, ΔB and ΔD_w to design an observer.

Assumption 2. *The matrices ΔA, ΔB and ΔD_w are unknown but bounded and satisfy $\underline{\Delta A} \leq \Delta A(\rho_k) \leq \overline{\Delta A}$, $\underline{\Delta B} \leq \Delta B(\rho_k) \leq \overline{\Delta B}$ and $\underline{\Delta D}_w \leq \Delta D_w(\rho_k) \leq \overline{\Delta D}_w$, $\forall k \geq 0$.*

Finally, a condition has to be imposed on the evolution of the state vector over time.

Assumption 3. *The known input vector u_k, the unknown input vector d_k and the state vector x_k are such that $u_k \in \mathcal{L}_\infty^{n_u}$, $d_k \in \mathcal{L}_\infty^{n_d}$ and $x_k \in \mathcal{L}_\infty^{n_x}$. As a direct consequence, $y_k \in \mathcal{L}_\infty^{n_y}$.*

The goal of the present paper is to propose a new interval observer derived from the interval estimation strategies for discrete-time LPV systems proposed in [15] and the unknown input interval observer for discrete-time linear time invariant system proposed in [17]. The proposed interval observer must provide simultaneously guaranteed bounds $\underline{x}_k, \overline{x}_k$ and $\underline{d}_k, \overline{d}_k$ to the state vector and the unknown input vector such that $\underline{x}_k \leq x_k \leq \overline{x}_k$ and $\underline{d}_k \leq d_k \leq \overline{d}_k$, $\forall k > 0$.

III. Main result

This section presents the proposed framer for the linear parameter-varying system as well as the L_∞-based design strategy for the interval observer gains.

A. System augmentation

In [7], [16], [18] are proposed different strategies using a change of coordinates to decouple the state and the unknown inputs. Guaranteed bounds can then be computed for the state vector and this information is used to compute bounds for the unknown inputs. In this paper, the dynamical system (1) is instead rewritten into a discrete-time LPV descriptor system by considering the unknown inputs as auxiliary states [17]. Then, consider the equivalent system

$$\begin{cases} Ez_{k+1} = F(\rho_k)z_k + G(\rho_k)u_k + W(\rho_k)w_k \\ \quad y_k = Hz_k + D_v v_k \end{cases} \tag{3}$$

where $z_k^\top = \begin{bmatrix} x_k^\top & d_{k-1}^\top \end{bmatrix}$, $H = \begin{bmatrix} C & \mathbf{0} \end{bmatrix}$,

$$E = \begin{bmatrix} I_{n_x} & -D \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad G(\rho_k) = \begin{bmatrix} B(\rho_k) \\ \mathbf{0} \end{bmatrix},$$

$$F(\rho_k) = \begin{bmatrix} A(\rho_k) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad W(\rho_k) = \begin{bmatrix} D_w(\rho_k) \\ \mathbf{0} \end{bmatrix},$$

and, for the definition of z_0, d_{-1} is chosen to be $\mathbf{0}$. The matrices $F(\rho_k)$, $G(\rho_k)$ and $W(\rho_k)$ can then be decomposed into two parts as in (2) and the bounds of $\Delta F(\rho_k)$, $\Delta G(\rho_k)$ and $\Delta W(\rho_k)$ are immediately deduced from Assumption 2.

Assumption 4. *The matrices E and H satisfy the rank condition*

$$\operatorname{rank} \begin{bmatrix} I_{n_x} & -D \\ C & \mathbf{0} \end{bmatrix} = n_x + n_d = n_z.$$

With the descriptor formulation given in (3), the goal of the proposed strategy is to find two bounds $\underline{z}_k$ and $\overline{z}_k$ for the augmented state vector z_k such that $\underline{z}_k \leq z_k \leq \overline{z}_k$, $\forall k \in \mathbb{N}$. The interval observer design is then based on the following lemma.

Lemma 3 ([23]). *Given matrices $X \in \mathbb{R}^{n\times m}$, $Y \in \mathbb{R}^{m\times p}$ and $Z \in \mathbb{R}^{n\times p}$ with $\operatorname{rank} Y = p$, the general solution X of the equation $XY = Z$ is*

$$X = ZY^\dagger + \Xi\left(I_m - YY^\dagger\right)$$

where $\Xi \in \mathbb{R}^{n\times m}$ is an arbitrary matrix.

With Lemma 3 and Assumption 4, there exist pairs of matrices (T, N) satisfying

$$TE + NH = I_{n_z}, \tag{4}$$

where

$$T = \Theta^\dagger\alpha_1 + \Xi\Psi\alpha_1, \qquad N = \Theta^\dagger\alpha_2 + \Xi\Psi\alpha_2, \tag{5}$$

with $\Xi \in \mathbb{R}^{n_z\times(n_z+n_y)}$ a free matrix, and

$$\Theta = \begin{bmatrix} E \\ H \end{bmatrix}, \qquad \Psi = I_{n_z+n_y} - \Theta\Theta^\dagger,$$

$$\alpha_1 = \begin{bmatrix} I_{n_z} \\ \mathbf{0} \end{bmatrix}, \qquad \alpha_2 = \begin{bmatrix} \mathbf{0} \\ I_{n_y} \end{bmatrix}.$$

Assumption 5. *The pair (TF_0, H) is observable.*

B. Framer for LPV systems

The proposed framer for system (3) obeys the dynamics

$$\begin{cases} \underline{z}_{k+1} = \left(TF_0 - \underline{L}H\right)\underline{z}_k + TG_0u_k + Ny_{k+1} \\ \qquad\qquad + \underline{L}y_k + \underline{\phi}_k + \underline{\chi}_k + \underline{\psi}_k + \underline{\omega}_k \\ \overline{z}_{k+1} = \left(TF_0 - \overline{L}H\right)\overline{z}_k + TG_0u_k + Ny_{k+1} \\ \qquad\qquad + \overline{L}y_k + \overline{\phi}_k + \overline{\chi}_k + \overline{\psi}_k + \overline{\omega}_k \end{cases} \tag{6}$$

where

$$\begin{cases} \underline{\phi}_k = T^+\underline{\delta}_k(F, z) - T^-\overline{\delta}_k(F, z) \\ \overline{\phi}_k = T^+\overline{\delta}_k(F, z) - T^-\underline{\delta}_k(F, z) \end{cases} \tag{7}$$

$$\begin{cases} \underline{\omega}_k = T^+\underline{\delta}_k(W, w) - T^-\overline{\delta}_k(W, w) \\ \overline{\omega}_k = T^+\overline{\delta}_k(W, w) - T^-\underline{\delta}_k(W, w) \end{cases} \tag{8}$$

$$\begin{cases} \underline{\chi}_k = T^+\left(\underline{\Delta G}u_k^+ - \overline{\Delta G}u_k^-\right) - T^-\left(\overline{\Delta G}u_k^+ - \underline{\Delta G}u_k^-\right) \\ \overline{\chi}_k = T^+\left(\overline{\Delta G}u_k^+ - \underline{\Delta G}u_k^-\right) - T^-\left(\underline{\Delta G}u_k^+ - \overline{\Delta G}u_k^-\right) \end{cases} \tag{9}$$

$$\begin{cases} \underline{\psi}_k = (TW_0)^+\underline{w}_k - (TW_0)^-\overline{w}_k + (ND_v)^-\underline{v}_{k+1} \\ \qquad - (ND_v)^+\overline{v}_{k+1} + (\underline{L}D_v)^-\underline{v}_k - (\underline{L}D_v)^+\overline{v}_k \\ \overline{\psi}_k = (TW_0)^+\overline{w}_k - (TW_0)^-\underline{w}_k + (ND_v)^-\overline{v}_{k+1} \\ \qquad - (ND_v)^+\underline{v}_{k+1} + \left(\overline{L}D_v\right)^-\overline{v}_k - \left(\overline{L}D_v\right)^+\underline{v}_k \end{cases} \tag{10}$$

and, for given matrix M and vector a,

$$\begin{cases} \underline{\delta}_k(M, a) = \underline{\Delta M}^+a_k^+ - \overline{\Delta M}^+\underline{a}_k^- - \underline{\Delta M}^-\overline{a}_k^+ + \overline{\Delta M}^-\overline{a}_k^- \\ \overline{\delta}_k(M, a) = \overline{\Delta M}^+\overline{a}_k^+ - \underline{\Delta M}^+\overline{a}_k^- - \overline{\Delta M}^-\underline{a}_k^+ + \underline{\Delta M}^-\underline{a}_k^- \end{cases}$$

with $\underline{L}$ and $\overline{L}$ two observer gains such that $TF_0 - \underline{L}H$ and $TF_0 - \overline{L}H$ are nonnegative matrices.

Theorem 1. *Let Assumptions 1 and 2 hold, $TF_0 - \underline{L}H$, $TF_0 - \overline{L}H$ be nonnegative matrices and $d_{-1} = \mathbf{0}$. Then, $\underline{z}_k$ and $\overline{z}_k$ obeying the dynamics* (6) *satisfy*

$$\underline{z}_k \leq z_k \leq \overline{z}_k,\ \forall k \geq 0. \tag{11}$$

Proof. Let $\overline{e}_k = \overline{z}_k - z_k$ and $\underline{e}_k = z_k - \underline{z}_k$ be the upper and lower estimation errors. Using equation (4), the descriptor vector z_k satisfies

$$\begin{aligned} z_{k+1} = \left(TF(\rho_k) - \underline{L}H\right)z_k + TG(\rho_k)u_k + Ny_{k+1} \\ + \underline{L}y_k + TW(\rho_k)w_k - \underline{L}D_vv_k - ND_vv_{k+1}, \end{aligned} \tag{12}$$

Using the difference equation (12), the dynamics of the lower estimation error are then

$$\begin{aligned} \underline{e}_{k+1} = \left(TF_0 - \underline{L}H\right)\underline{e}_k - \underline{\phi}_k + T\Delta F(\rho_k)z_k \\ - \underline{\chi}_k + T\Delta G(\rho_k)u_k - \underline{\omega}_k + T\Delta W(\rho_k)w_k \\ - \underline{\psi}_k + TW_0w_k - ND_vv_{k+1} - \underline{L}D_vv_k. \end{aligned} \tag{13}$$

Replacing $\underline{L}$ by $\overline{L}$ in (12), the dynamics of the upper estimation error are

$$\begin{aligned} \overline{e}_{k+1} = \left(TF_0 - \overline{L}H\right)\overline{e}_k + \overline{\phi}_k - T\Delta F(\rho_k)z_k \\ + \overline{\chi}_k - T\Delta G(\rho_k)u_k + \overline{\omega}_k - T\Delta W(\rho_k)w_k \\ + \overline{\psi}_k - TW_0w_k + ND_vv_{k+1} + \overline{L}D_vv_k \end{aligned} \tag{14}$$

By Lemma 1, knowing that Assumptions 1 and 2 hold, it is immediate that

$$\begin{aligned}\underline{\beta}_k &= T\Delta G(\rho_k)u_k - \underline{\chi}_k + T\Delta W(\rho_k)w_k - \underline{\omega}_k \\ &\quad + TW_0w_k - ND_v v_{k+1} - \underline{L}D_v v_k - \underline{\psi}_k \geq \mathbf{0},\end{aligned} \tag{15a}$$

$$\begin{aligned}\overline{\beta}_k &= \overline{\chi}_k - T\Delta G(\rho_k)u_k + \overline{\omega}_k - T\Delta W(\rho_k)w_k \\ &\quad + \overline{\psi}_k - TW_0w_k + ND_v v_{k+1} + \overline{L}D_v v_k \geq \mathbf{0}.\end{aligned} \tag{15b}$$

With Assumptions 1 and 2, for $k=0$,

$$\underline{\varepsilon}_k = T\Delta F(\rho_k)z_k - \underline{\phi}_k \geq \mathbf{0}, \tag{16a}$$

$$\overline{\varepsilon}_k = \overline{\phi}_k - T\Delta F(\rho_k)z_k \geq \mathbf{0}. \tag{16b}$$

Given that $TF_0 - \underline{L}H, TF_0 - \overline{L}H \geq \mathbf{0}$, then $\underline{e}_1, \overline{e}_1 \geq \mathbf{0}$ so that $\underline{z}_1 \leq z_1 \leq \overline{z}_1$ and (16a) and (16b) are true for $k=1$. Then, given that (15a) and (15b) are true at all time $k \geq 0$, by induction, (11) is satisfied at all time $k \geq 0$. □

C. Interval observer with L_∞ performance

In order for the framer (6) to be an interval observer for the system (3), the dynamics of the estimation errors $\underline{e}_k$ and $\overline{e}_k$ have to be input-to-state stable [24]. To guarantee this and, in addition, to reduce the impact of the system's uncertainties on the bounds $\underline{z}_k$ and $\overline{z}_k$, a L_∞-based design procedure is given in the following theorem to obtain the gain matrices.

Theorem 2. *Let all the conditions of Theorem 1 hold. For a given scalar μ satisfying $0 < \mu < 1$, if there exists a scalar $\gamma \geq 0$, a positive definite diagonal matrix $P \in \mathbb{R}^{2n_z \times 2n_z}$, and a block diagonal matrix $X \in \mathbb{R}^{2n_z \times 2n_y}$ such that*

$$S \geq \mathbf{0} \tag{17a}$$

$$P \succeq \mu I_{2n_z} \tag{17b}$$

$$\begin{bmatrix}(\mu-1)P + \gamma Q & \star & \star & \star \\ \mathbf{0} & -\gamma I_{2n_z} & \star & \star \\ \mathbf{0} & \mathbf{0} & -\gamma I_{2n_z} & \star \\ S & P & P & -P\end{bmatrix} \preceq 0 \tag{17c}$$

where $S = P(I_2 \otimes T)(I_2 \otimes F_0) - X\Upsilon$, $\Upsilon = I_2 \otimes H$, and $Q = 6 \cdot \operatorname{diag}\left(\underline{l}_\phi^2 I_{n_z}, \overline{l}_\phi^2 I_{n_z}\right)$, with $\overline{l}_\phi$ and $\underline{l}_\phi$ defined in Lemma 2, then (6) is a robust interval observer for system (1). This interval observer satisfies the performance

$$\|e_k\|_2^2 \leq \frac{(1-\mu)^k}{\mu}V_0 + \frac{\gamma}{\mu^2}\left(\|\beta\|_\infty^2 + \eta\right) \tag{18}$$

where $e_k^\top = \begin{bmatrix}\underline{e}_k^\top & \overline{e}_k^\top\end{bmatrix}$, $V_0 = e_0^\top P e_0$, $\beta_k^\top = \begin{bmatrix}\underline{\beta}_k^\top & \overline{\beta}_k^\top\end{bmatrix}$, with $\underline{\beta}_k$ and $\overline{\beta}_k$ defined in (15), $\|\beta\|_\infty$ is the L_∞ norm of β_k over time as defined in Section II-A, and $\eta = 3\left(\underline{m}_\phi^2 + \overline{m}_\phi^2\right)$, with $\underline{m}_\phi$ and $\overline{m}_\phi$ defined in Lemma 2.

Proof. Since $P \succ 0$, all its diagonal elements are strictly positive. Defining the matrix $X = P\operatorname{diag}(\underline{L}, \overline{L})$, condition (17a) is then equivalent to the nonnegativity of $TF_0 - \underline{L}H$ and $TF_0 - \overline{L}H$.

Moreover, the dynamics of e_k is

$$e_{k+1} = \Pi e_k + \varepsilon_k + \beta_k$$

where $\Pi = (I_2 \otimes T)(I_2 \otimes F_0) - L\Upsilon$ and $\varepsilon_k^\top = \begin{bmatrix}\underline{\varepsilon}_k^\top & \overline{\varepsilon}_k^\top\end{bmatrix}$, with $L = \operatorname{diag}(\underline{L}, \overline{L})$ and $\underline{\varepsilon}_k$ and $\overline{\varepsilon}_k$ defined in (16). Consider the candidate Lyapunov function $V_k = e_k^\top P e_k$. The increment of V is

$$\begin{aligned}V_{k+1} - V_k &= e_k^\top(\Pi^\top P\Pi - P)e_k + \varepsilon_k^\top P\varepsilon_k + \beta_k^\top P\beta_k \\ &\quad + e_k^\top\Pi^\top P\varepsilon_k + e_k^\top\Pi^\top P\beta_k + \varepsilon_k^\top P\Pi e_k \\ &\quad + \varepsilon_k^\top P\beta_k + \beta_k^\top P\Pi e_k + \beta_k^\top P\varepsilon_k \\ &= e_k^\top(\Pi^\top P\Pi - (1-\mu)P)e_k - \mu e_k^\top P e_k \\ &\quad + \varepsilon_k^\top(P - \gamma I_{2n_z})\varepsilon_k + \gamma\varepsilon_k^\top\varepsilon_k \\ &\quad + \beta_k^\top(P - \gamma I_{2n_z})\beta_k + \gamma\beta_k^\top\beta_k \\ &\quad + e_k^\top\Pi^\top P\varepsilon_k + e_k^\top\Pi^\top P\beta_k + \varepsilon_k^\top P\Pi e_k \\ &\quad + \varepsilon_k^\top P\beta_k + \beta_k^\top P\Pi e_k + \beta_k^\top P\varepsilon_k.\end{aligned}$$

However, $\varepsilon_k^\top\varepsilon_k = \underline{\varepsilon}_k^\top\underline{\varepsilon}_k + \overline{\varepsilon}_k^\top\overline{\varepsilon}_k$. The functions $T\Delta F(\rho_k)z_k$, $\underline{\phi}_k$ and $\overline{\phi}_k$ satisfy the assumptions of Lemma 2 so that $\underline{\varepsilon}_k$ and $\overline{\varepsilon}_k$ are globally Lipschitz. Moreover, by Lemma 2

$$\begin{cases}\underline{\varepsilon}_k^\top\underline{\varepsilon}_k \leq \left(\overline{l}_\phi\|\overline{z}_k - z_k\|_2 + \underline{l}_\phi\|\underline{z}_k - z_k\|_2 + \underline{m}_\phi\right)^2 \\ \overline{\varepsilon}_k^\top\overline{\varepsilon}_k \leq \left(\overline{l}_\phi\|\overline{z}_k - z_k\|_2 + \underline{l}_\phi\|\underline{z}_k - z_k\|_2 + \overline{m}_\phi\right)^2\end{cases}$$

where $\|\overline{z}_k - z_k\|_2^2 = \overline{e}_k^\top\overline{e}_k$ and $\|\underline{z}_k - z_k\|_2^2 = \underline{e}_k^\top\underline{e}_k$. Then, by Cauchy-Schwarz inequality,

$$\begin{cases}\underline{\varepsilon}_k^\top\underline{\varepsilon}_k \leq 3\left(\overline{l}_\phi^2\overline{e}_k^\top\overline{e}_k + \underline{l}_\phi^2\underline{e}_k^\top\underline{e}_k + \underline{m}_\phi^2\right) \\ \overline{\varepsilon}_k^\top\overline{\varepsilon}_k \leq 3\left(\overline{l}_\phi^2\overline{e}_k^\top\overline{e}_k + \underline{l}_\phi^2\underline{e}_k^\top\underline{e}_k + \overline{m}_\phi^2\right)\end{cases}$$

so that

$$\varepsilon_k^\top\varepsilon_k \leq e_k^\top Q e_k + \eta. \tag{19}$$

With inequality (19), the increment of V now satisfies

$$V_{k+1} - V_k \leq \begin{bmatrix}e_k \\ \varepsilon_k \\ \beta_k\end{bmatrix}^\top \Lambda \begin{bmatrix}e_k \\ \varepsilon_k \\ \beta_k\end{bmatrix} - \mu V_k + \gamma\|\beta_k\|_2^2 + \gamma\eta \tag{20}$$

where

$$\Lambda = \begin{bmatrix}\Lambda_{11} & \Pi^\top P & \Pi^\top P \\ P\Pi & P - \gamma I_{2n_z} & P \\ P\Pi & P & P - \gamma I_{2n_z}\end{bmatrix} \tag{21}$$

with $\Lambda_{11} = \Pi^\top P\Pi - (1-\mu)P + \gamma Q$.

If $\Lambda \preceq 0$, inequality (19) implies that the estimation error e_k remains bounded over time [25] since, by Assumption 1, $\|\beta_k\|_2 < \infty, \forall k \geq 0$. Then, the framer (6) is a robust interval observer for the descriptor LPV system (3).

By using the Schur complement [26], the linear matrix inequality (LMI) $\Lambda \preceq 0$ is equivalent to the condition (17c).

Finally, since $\Lambda \preceq 0$, the quadratic term in the right hand side of (19) is negative. Therefore, there exists a scalar μ such that $0 < \mu < 1$ satisfying the inequality

$$V_{k+1} - V_k \leq -\mu V_k + \gamma\|\beta\|_\infty^2 + \gamma\eta$$

where $\|\beta\|_\infty$ is the L_∞ norm of β_k over time. Then,

$$V_{k+1} \leq (1-\mu)^{k+1}V_0 + \sum_{i=0}^{k}(1-\mu)^i\gamma\left(\|\beta\|_\infty^2 + \eta\right). \tag{22}$$

Since $0 < \mu < 1$, it is immediate that $0 < 1 - \mu < 1$. Therefore, the inequality (22) can be rewritten as

$$V_{k+1} \leq (1-\mu)^{k+1} V_0 + \frac{\gamma}{\mu}\left(\|\beta\|_\infty^2 + \eta\right).$$

With condition (17b), $\mu \|e_k\|_2^2 \leq V_k$, hence the performance (18). $\square$

With Theorem 2, the matrices $\underline{L}$ and $\overline{L}$ can be obtained as $\operatorname{diag}(\underline{L}, \overline{L}) = P^{-1}X$ while minimizing γ.

Remark 2. *Due to the symmetry of the constraints, if the observer gains are obtained by minimizing γ subject to the constraints (17a) to (17c), $\underline{L}$ and $\overline{L}$ might be equal. However, the problem has a modular structure, allowing for the introduction of additional constraints. The constraints on the upper estimation error can then be different from the constraints on the lower estimation error, leading to two different sets of gains.*

IV. Simulation Results

To assess the efficiency of the proposed interval observer, an academic example adapted from [27] is used. The considered system is

$$A_0 = 0.1\begin{bmatrix} -6 & 5 & 4 \\ 7 & 5 & 2 \\ 1 & 5 & 3 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad D_{w0} = I_3, \qquad D_v = I_2,$$

$\Delta B = \mathbf{0}$, $\Delta D_w = \mathbf{0}$, and

$$\Delta A(k) = 0.02 \cdot \begin{bmatrix} 0.1s(\omega_1 k) & s(\omega_2 k) & c(\omega_1 k) \\ c(\omega_2 k) & s(2\omega_1 k) & 0.1c(2\omega_1 k) \\ s(\omega_1 k/2) & 0.1c(\omega_2 k/2) & s(\omega_1 k)c(\omega_2 k) \end{bmatrix}$$

where $c(x)$ and $s(x)$ stand for $\cos(x)$ and $\sin(x)$, $\omega_1 = 0.02$ and $\omega_2 = 0.1/3$. With this definition of $\Delta A(k)$,

$$\overline{\Delta A} = -\underline{\Delta A} = 0.02 \cdot \begin{bmatrix} 0.1 & 1 & 1 \\ 1 & 1 & 0.1 \\ 1 & 0.1 & 1 \end{bmatrix}.$$

Moreover, the unknown input signal is $d_k = 0.5c(0.2k)$, the known input signal is $u_k = -\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} y_k$, and the disturbance and measurement noise vectors are two uniformly distributed random vectors so that $\overline{w} = -\underline{w} = 0.1 \cdot \mathbf{1}_3$ and $\overline{v} = -\underline{v} = 0.1 \cdot \mathbf{1}_2$. Finally, the bounds for the initial state are $\overline{x}_0 = 5 \cdot \mathbf{1}_3$ and $\underline{x}_0 = -2 \cdot \mathbf{1}_3$, with $x_0^\top = \begin{bmatrix} -1 & 4 & 2 \end{bmatrix}$, so that $\underline{z}_0^\top = \begin{bmatrix} \underline{x}_0^\top & 0 \end{bmatrix}$ and $\overline{z}_0^\top = \begin{bmatrix} \overline{x}_0^\top & 0 \end{bmatrix}$.

The value of the matrix Ξ is a design parameter of the proposed observer. It could be chosen so as to minimize the values of $\underline{l}_\phi$, $\overline{l}_\phi$, $\underline{m}_\phi$, and $\overline{m}_\phi$ or any other use case dependent criterion. For the sake of simplicity, the value $\Xi = \mathbf{0}$ is chosen so that

$$T = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & 0.5 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix},$$

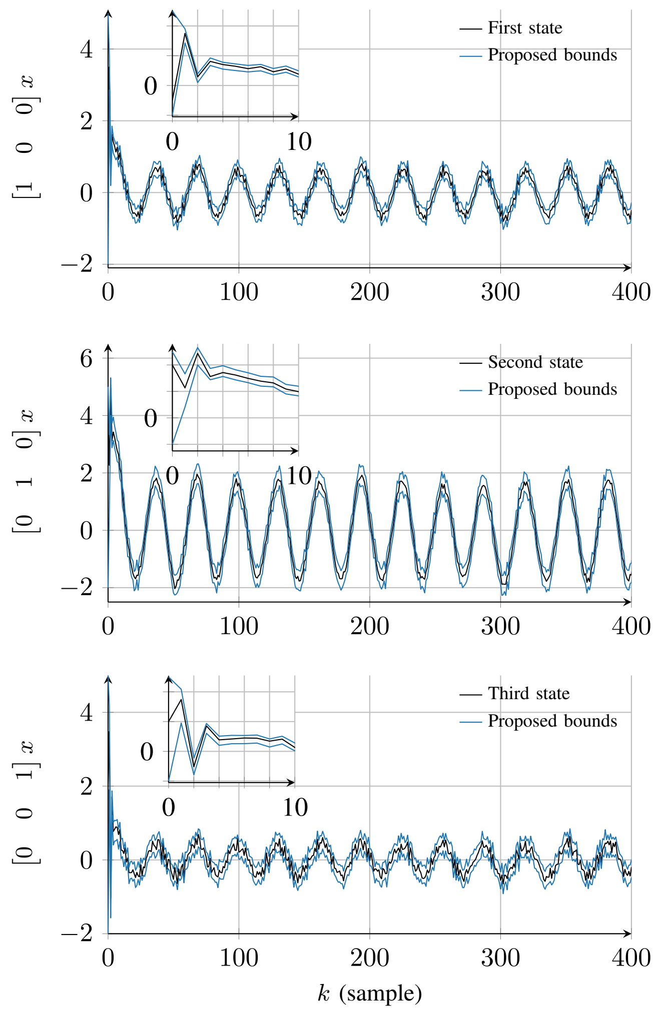

Fig. 1. States and guaranteed bounds from the proposed observer.

yielding $\underline{l}_\phi = \overline{l}_\phi = 0.0637$. Minimizing γ under the constraints (17) with $\mu = 0.1$, $\gamma = 0.2729$ and the observer gains are $\overline{L} = \underline{L}$, with

$$\underline{L} = \begin{bmatrix} 0.2 & -0.3006 \\ -0.5 & -0.1 \\ 0.3 & 0.1 \\ -1 & -0.8 \end{bmatrix}.$$

The intervals obtained with the proposed observer for the three states are presented in Figure 1 and the interval for the unknown inputs is presented in Figure 2. The real states and unknown inputs are contained in the computed intervals. Moreover, the interval width is not constant due to the effect of the parameter uncertainties as well as of the known and unknown input vectors.

V. Conclusion

This paper presents a new interval observer for discrete-time linear parameter-varying systems with unmeasurable parameter vector subject to unknown inputs and unknown but bounded disturbance and measurement noise. This observer

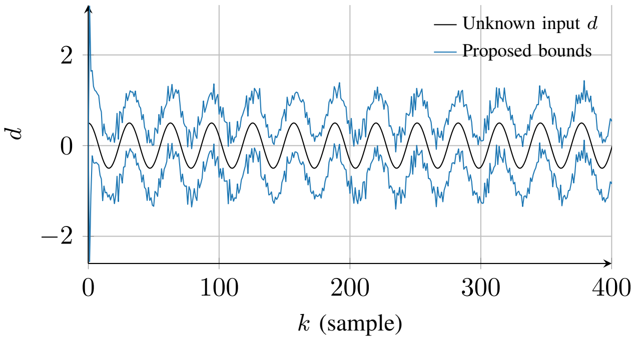

Fig. 2. Unknown input and guaranteed bounds from the proposed observer.

is used to compute simultaneously guaranteed bounds for the system's state and the unknown inputs. By considering the unknown inputs as auxiliary states, the system is rewritten as a linear parameter-varying descriptor system, allowing for the introduction of additional gains compared to the classical interval observers. These gains are tuned by enforcing the cooperativity of the observer and the effect of the perturbations is attenuated by considering a L_∞ norm criterion. All these conditions are written as linear matrix inequalities (LMI), such that additional constraints for the tuning of the observer's gains can be easily introduced as long as they can be written as LMIs. Numerical simulation results are presented to assess the efficiency of the proposed method. In future work, this method could be adapted to linear-parameter varying descriptor systems subject to unknown inputs or to linear parameter-varying systems with parameter dependent output matrix. In addition, the tuning of the weighting matrices could be integrated into the proposed L_∞ design procedure.

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