Supplementary Material to "Uncertainty Quantification of Bifurcations in Random Ordinary Differential Equations"

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Here, we apply our methodology from the manuscript "Uncertainty Quantification of Bifurcations in Random Ordinary Differential Equations" (available on arXiv: `https://arxiv.org/abs/2101.05581`) to two additional examples: we analyze the Lorenz example in equation (3) in the manuscript with a truncated power law distribution as input distribution in Section 1 and we consider a model from computational neursoscience, the Hindmarsh-Rose model [1], in Section 2.

1 Lorenz system with truncated power law distribution input

We refer to Example 4.1 (continued) from the manuscript and remember from equation (38) that the Mellin transform of the bifurcation normal from coefficient of the Lorenz system can be decomposed as follows:

$$\mathcal{M}\left(r_1/r_2(1+r_1)\right)(s) = \mathcal{M}(r_2)(-s+2) \cdot \sum_{i=0}^{2\cdot(s-1)} \frac{\hat{c}_i(s)}{i+1}. \tag{1}$$

Here, we modify PS D from Table 3 by using a truncated power law distribution for the uncertain parameter r_2 instead of the Gamma distribution and denote the modified parameter setting by *PS E*. Note that the calculation of the Mellin transform containing r_1 in (1) stays exactly the same as in the manuscript.

It remains to calculate the Mellin transform of the truncated power law density given by

$$\rho_{r_2}(x) = \begin{cases} ak^a x^{-1+a}, & \text{for } 0 < x \leq 1/k, \\ 0 & \text{else,} \end{cases} \tag{2}$$

where k is the domain parameter and a is the shape parameter. Note that there exist several definitions of a power law density. We took the definition that is implemented in Mathematica*. For $\mathfrak{Re}(a) > \mathfrak{Re}(s)$, the Mellin transform of the random variable r_2 can be calculated as

$$\mathcal{M}(r_2)(s) = \frac{1}{s-1+a} ak^a \left(\frac{1}{k}\right)^{s-1+a}. \tag{3}$$

By plugging (3) in (1), we obtain again a closed-form expression of the bifurcation normal form coefficient of the Lorenz system.

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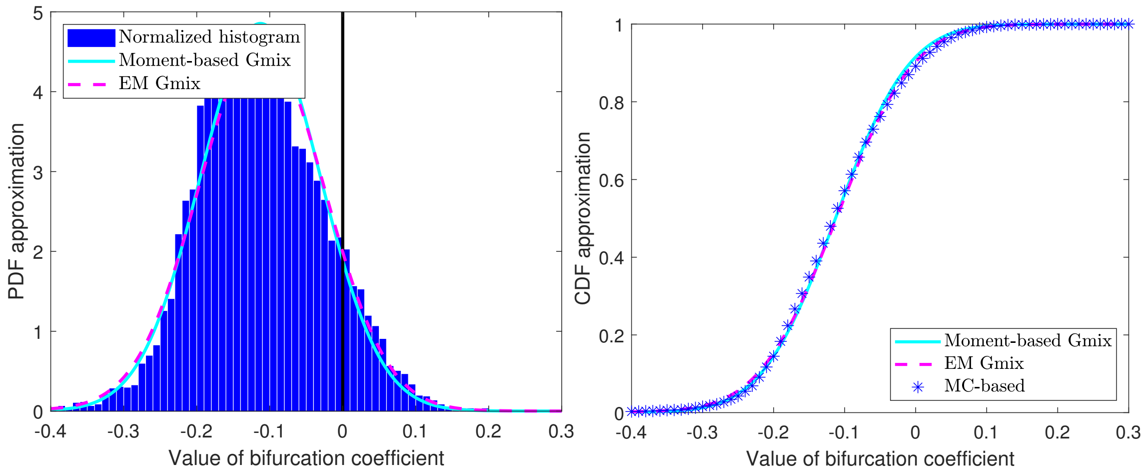
*`https://reference.wolfram.com/language/ref/PowerDistribution.html`, last checked: June 16, 2021

In Figure 1, we plot the PDF and the CDF of the bifurcation normal form coefficient of the Lorenz system for **PS E** with domain parameter $k = 2$ and shape parameter $a = 5.5$. Note that the random numbers used to calculate the normalized histogram are drawn in Mathematica[†] and imported in MATLAB. The MATLAB code and the csv-file containing the random numbers can be found at `https://github.com/kerstinLux/UQbifurcation`.



(a) PDF PS E (b) CDF PS E

Figure 1: Reconstruction of PDF and CDF of X in the reduced Lorenz system (8) in the manuscript via GMM for PS E: our Mellin-moments-based GMM approximations of the PDF and CDF in turqoise are compared to the sample-based MATLAB solutions (using the EM algorithm) in magenta and the blue normalized histogram/MC estimates: results are in excellent agreement.

The probabilities to observe a subcritical pitchfork bifurcation for the different methods are given in Table 1 and are in very close agreement.

Mellin-moment-based	0.9152
EM (sample-based)	0.9030
MC (sample-based)	0.8915

Table 1: Quantitative comparison of probability estimate for the subcritical pitchfork bifurcation in the reduced Lorenz system (8) in the manuscript to be subcritical.

2 Hindmarsh-Rose model

We test the applicability of our methodology to the Hindmarsh-Rose model considered in [1] and given by

$$\begin{aligned} \dot{x} &= y - ax^3 + bx^2, \\ \dot{y} &= -c - dx^2 - y, \end{aligned} \tag{4}$$

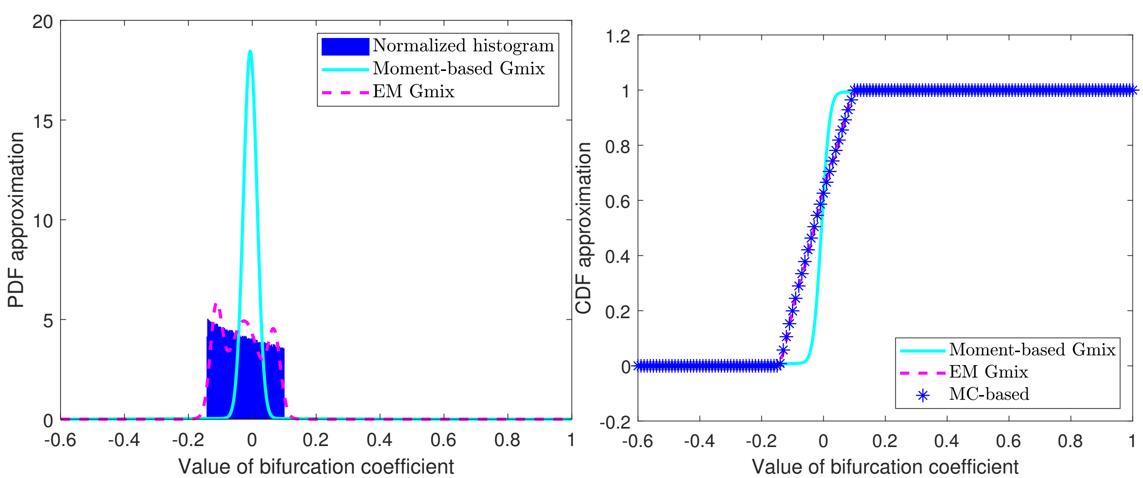
where a, b, c and d are positive parameters.
In [1], the authors calculate the first Lyapunov coefficient and argue that the sign of the expression

$$g_{a,d}(b) = -a - \frac{2}{3} \cdot d \cdot b + \frac{2}{3} \cdot b^2 \tag{5}$$

[†] `https://reference.wolfram.com/language/ref/RandomVariate.html`, last checked: June 16, 2021

determines the sign of the first Lyapunov coefficient and thus, the criticality of the bifurcation. Note that (5) is a nonlinear transformation of the input parameter b.

Here, we now assume b to be uncertain and assume a uniform distribution $b \sim \mathcal{U}(0.95, 1.2)$. Furthermore, we set $a = 0.3$ and $d = 0.7$. The range for the parameters is inspired by the parameter values used in [1].
To calculate the Mellin transform of (5) for integer values of s, we can use the same technique of iteratively applying the binomial formula as we did to calculate the Mellin transform of the Polynomial Chaos Expansion given in equation (31) of our manuscript.
The great benefit here is that the polynomial expression in (5) is not an approximation of a general non-polynomial nonlinear transformation but the coefficient itself is already a polynomial. Thus, the Mellin transform is exact and no error is induced.
However, as we restricted the transformation to integer values of s, an analytical inversion is not possible and we make use of the approximation of the probability density of (5) via a Gaussian mixture model (GMM) as described in Section 4.2.2 of the manuscript. Here, we use three component densities in the GMM.
In Figure 2, the Mellin-based GMM estimate, the Expectation-Maximization approximation and the normalized histogram are depicted. The agreement of the Mellin-based GMM estimate with the two other sample-based approximations is only very rough. However, our main interest lies in the probability of the sign of (5) being positive and thus facing a subcritical Hopf bifurcation. This probability is in close agreement for all three approximations (see Table 2). The MATLAB code can be found at `https://github.com/kerstinLux/UQbifurcation`.



(a) PDF PS E (b) CDF PS E

Figure 2: Reconstruction of PDF and CDF of $g_{a,d}(b)$ from equation (5) via GMM: our Mellin-moments-based GMM approximations of the PDF and CDF in turqoise are compared to the sample-based MATLAB solutions (using the EM algorithm) in magenta and the blue normalized histogram/MC estimates. There is only a moderate agreement of the PDFs.

Mellin-moment-based	0.3826
EM (sample-based)	0.3602
MC (sample-based)	0.3738

Table 2: Quantitative comparison of probability estimate for the Hopf bifurcation in the Hindmarsh-Rose model (4) to be subcritical.

References

[1] X. Liu and S. Liu, *Codimension-two bifurcation analysis in two-dimensional Hindmarsh–Rose model*, Nonlinear Dynamics, 67 (2012), pp. 847–857.