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On total weight choosability of graphs

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Abstract For a graph G with vertex set V and edge set E, a (k,k')-total list assignment L of G assigns to each vertex v a set $L(v)$ of k real numbers as permissible weights, and assigns to each edge e a set $L(e)$ of k' real numbers as permissible weights. If for any (k,k')-total list assignment L of G, there exists a mapping $f: V \cup E \to \mathbb{R}$ such that $f(y) \in L(y)$ for each $y \in V \cup E$, and for any two adjacent vertices u and v, $\sum_{y \in N(u)} f(uy) + f(u) \neq \sum_{x \in N(v)} f(vx) + f(v)$, then G is (k,k')-total weight choosable. It is conjectured by Wong and Zhu that every graph is $(2,2)$-total weight choosable, and every graph with no isolated edges is $(1,3)$-total weight choosable.

In this paper, it is proven that a graph G obtained from any loopless graph H by subdividing each edge with at least one vertex is $(1,3)$-total weight choosable and $(2,2)$-total weight choosable. It is shown that s-degenerate graphs (with $s \geq 2$) are $(1,2s)$-total weight choosable. Hence planar graphs are $(1,10)$-total weight choosable, and outerplanar graphs are $(1,4)$-total weight choosable. We also give a combinatorial proof that wheels are $(2,2)$-total weight choosable, as well as $(1,3)$-total weight choosable.

Keywords Total weighting · Edge weighting · Vertex coloring

Dedicated to Prof. Gerard Jennhwa Chang on the occasion of his 60th birthday.

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1 Introduction

Let G be a graph with vertex set V and edge set E. For a vertex v, we denote the neighborhood of v in G by $N_G(v)$, $d_G(v) = |N_G(v)|$; or simply $N(v)$, $d(v)$, if G is clear from the context. An *edge weighting* of G is a mapping that assigns to each edge e of G a real number $f(e)$. An edge weighting f induces a vertex coloring $\varphi_f : V \to \mathbb{R}$ of G, defined as $\varphi_f(v) = \sum_{y \in N(v)} f(vy)$. We say f is a *proper edge weighting* if the induced vertex coloring φ_f is proper, i.e., for any edge uv of G, $\varphi_f(u) \neq \varphi_f(v)$. The study of edge weighting was initiated by Karoński et al. (2004). They proposed the following interesting conjecture:

Conjecture 1.1 *Every graph with no isolated edges has a proper edge weighting f such that $f(e) \in \{1, 2, 3\}$ for every edge e.*

This conjecture is also referred as the 1-2-3-Conjecture. In Addario-Berry et al. (2007), it was proven that every graph with no isolated edges has a proper edge weighting f such that $f(e) \in \{1, 2, \dots, 30\}$. In Addario-Berry et al. (2005), the result is improved to $f(e) \in \{1, 2, \dots, 16\}$. In Wang and Yu (2008), the result is improved to $f(e) \in \{1, 2, \dots, 13\}$. In Kalkowski et al. (2010), the result is further improved to $f(e) \in \{1, 2, \dots, 5\}$. In Chang et al. (2011), it was shown that $\{1, 2\}$ is enough for k-regular bipartite graphs with $k \geq 3$. In Lu et al. (2011), it was shown that $\{1, 2\}$ is enough 3-connected bipartite graphs.

Bartnicki et al. (2009) considered the choosability version of edge weighting. A graph is said to be *k-edge weight choosable* if the following is true: For any list assignment L which assigns to each edge e a set $L(e)$ of k real numbers, G has a proper edge weighting f such that $f(e) \in L(e)$ for each edge e. They proposed the following conjecture:

Conjecture 1.2 *Every graph with no isolated edges is 3-edge weight choosable.*

In Bartnicki et al. (2009), by using permanents of matrices and Combinatorial Nullstellensatz (Alon and Tarsi 1999), it was proven that Conjecture 1.2 is true for complete graphs, trees, complete bipartite graphs except K_2.

A *total weighting* of G is a mapping $f : V(G) \cup E(G) \to \mathbb{R}$ which assigns to each vertex and each edge a real number as its weight. For a total weighting f, let $\varphi_f : V \to \mathbb{R}$ be defined as $\varphi_f(v) = \sum_{y \in N(v)} f(vy) + f(v)$. A total weighting is proper if the induced vertex coloring φ_f is proper. Przybyło and Woźniak (2008, 2010, 2011) studied total weighting of graphs. They proposed the following conjecture:

Conjecture 1.3 *Every simple graph G has a proper total weighting f such that $f(y) \in \{1, 2\}$ for all $y \in V \cup E$.*

This conjecture is also referred as the 1-2-Conjecture. Przybyło and Woźniak verified this conjecture for some special graphs, including complete graphs, graphs with $\Delta(G) \leq 3$, 4-regular graph, 3-colorable graph. They also proved that every simple graph G has a proper total weighting f such that $f(y) \in \{1, 2, \dots, 11\}$ for all $y \in V \cup E$.

The choosability version of total weighting was studied in Wong and Zhu (2011), Przybyło (2008), Wong et al. (2010). For positive integers k and k', a (k,k')-*total list assignment* L of a graph G assigns to each vertex v a set $L(v)$ of k real numbers as permissible weights, and assigns to each edge e a set $L(e)$ of k' real numbers as permissible weights. A graph is called (k,k')-*total weight choosable* if for any (k,k')-total list assignment L, G has a proper total weighting f such that $f(y) \in L(y)$ for all $y \in V \cup E$. Wong and Zhu (2011) proposed the following conjectures:

Conjecture 1.4 *Every graph with no isolated edges is* $(1,3)$*-total weight choosable.*

Conjecture 1.5 *Every graph G is* $(2,2)$*-total weight choosable.*

In Wong and Zhu (2011), it was proven that complete graphs, trees, complete bipartite graphs except K_2 are $(1,3)$-total weight choosable. They also proved that, for any graph H, a graph G obtained from H by subdividing each edge with at least two vertices is $(1,3)$-total weight choosable, as well as $(2,2)$-total weight choosable.

In Przybyło (2008), by using Combinatorial Nullstellensatz, it was shown that the Conjecture 1.5 is true for trees, wheels, unicyclic and complete graphs; it was also proven that complete bipartite graphs are $(3,3)$-total weight choosable.

In Wong et al. (2010), by using the so called "max-min weighting method", it was proven that complete bipartite graphs except K_2 are $(1,2)$-total weight choosable; complete multipartite graphs $K_{n,m,1,1,\dots,1}$ are $(2,2)$-total weight choosable; generalized theta graphs, cycles are $(2,2)$-total weight choosable.

In Przybyło and Woźniak (2010), it was proven that wheels are $(2,2)$-total weight choosable by using permanents of matrices and Combinatorial Nullstellensatz.

In Sect. 2 of this paper, we give a combinatorial proof that wheels are $(2,2)$-total weight choosable; we also show that wheels are $(1,3)$-total weight choosable. This section also serves as a warm-up for the rest of this paper.

In Sect. 3, we will show that, for a star $K_{1,s}$ (where $s \geq 2$, and v_0 is the center of the star), for $k \geq 2$, there are at least $2k-3$ choices such that $K_{1,s}$ is $(1,k)$-total weight choosable, and the total weight of v_0 is different for any two of such choices. This result will serve as a lemma for the rest of the paper.

In Sect. 4, we will prove that, a graph G obtained from any loopless graph H by subdividing each edge with at least one vertex is $(1,3)$-total weight choosable and $(2,2)$-total weight choosable. These improve the corresponding known results in Wong and Zhu (2011).

In Sect. 5, we will prove that s-degenerate graphs (with $s \geq 2$) are $(1,2s)$-total weight choosable; hence planar graphs are $(1,10)$-total weight choosable, and outerplanar graphs are $(1,4)$-total weight choosable.

We will end the introduction with the following two lemmas, which are heavily used throughout this paper. For $y \in V(G) \cup E(G)$, suppose a list assignment assigns $L(y) = \{L_1(y), \cdots, L_k(y)\}$, and we assume $L_1(y) < L_2(y) < \cdots < L_k(y)$.

Lemma 1.6 *Suppose that* $|L(y_1)| = \cdots = |L(y_i)| = k$. *Then* $f(y_1) + \cdots + f(y_i)$ *has at least* $(k-1)i+1$ *different values, where* $f(y_j) \in L(y_j)$ *for* $1 \leq j \leq i$.

Proof For every $j \in \{1, \dots, k-1\}$, note that $L_j(y_1) + L_j(y_2) + \cdots + L_j(y_i) < L_{j+1}(y_1) + L_j(y_2) + \cdots + L_j(y_i) < L_{j+1}(y_1) + L_{j+1}(y_2) + L_j(y_3) + \cdots + L_j(y_i) < \cdots < L_{j+1}(y_1) + L_{j+1}(y_2) + \cdots + L_{j+1}(y_i)$. Then simple counting will confirm the result. $\square$

By the same arguments, we have following more general statement.

Lemma 1.7 *Suppose* $|L(y_1)| = k_1$, $|L(y_2)| = k_2$, $\dots$, $|L(y_i)| = k_i$. *Then* $f(y_1) + f(y_2) + \cdots + f(y_i)$ *has at least* $k_1 + k_2 + \cdots + k_i - i + 1$ *different values, where* $f(y_j) \in L(y_j)$ *for* $1 \le j \le i$.

2 Notations, and the total weight choosability of wheels

The following notations and Lemma 2.1 were similar to those in Wong et al. (2010). We will use them throughout this paper.

Suppose L is a (k,k')-total list assignment of a graph G. Let $f : V(G) \cup E(G) \to \mathbb{R}$ be a map such that $f(y) \in L(y)$ for all $y \in V \cup E$. We call f a *L-weighting of G*.

For a fixed vertex v, let f_0 be a map from $\{v\} \cup \{vy \in E: \ y \in N(v)\}$ to $\mathbb{R}$ such that for each $y \in N(v)$, $f_0(vy) \in L(vy)$ and $f_0(v) \in L(v)$. We call f_0 a *partial L-weighting of G for vertex v*. If f_0 is a partial L-weighting of G for v, let $\varphi_{f_0}(v) = \sum_{y \in N(v)} f_0(vy) + f_0(v)$. Then for any total weighting f of G which coincides with f_0 on $\{v\} \cup \{vy \in E: \ y \in N(v)\}$, we have $\varphi_f(v) = \varphi_{f_0}(v)$. I.e., the color of v is determined by the partial L-weighting f_0.

Given a partial L-weighting f_0 of G for vertex v. We define a (k,k')-total list assignment L_1 for $G_1 = G - v$ as follows:

$$L_1(u) = \begin{cases} L(u), & \text{if } u \text{ is a vertex not adjacent to } v, \\ \{\alpha + f_0(vu) \colon \ \alpha \in L(u)\}, & \text{if } u \text{ is a vertex adjacent to } v, \\ L(u), & \text{if } u \text{ is an edge of } G_1. \end{cases}$$

We call L_1 *the (k,k')-total list assignment induced by L and f_0*.

Suppose L_1 is the (k,k')-total list assignment of G_1 induced by L and f_0. If f_1 is an L_1-weighting of G_1, then we combine f_1 and f_0 to obtain a weighting f of G as follows:

$$f(y) = \begin{cases} f_1(y) - f_0(vy), & \text{if } y \text{ is a vertex adjacent to } v, \\ f_1(y), & \text{if } y \text{ is a vertex not adjacent to } v \text{ or } y \text{ is an edge of } G_1, \\ f_0(y), & \text{if } u \in N(v) \text{ and } y = vu; \text{ or } y = v. \end{cases}$$

We call f the L-weighting of G *induced by f_1 and f_0*.

Similarly, for a fixed edge $e = \{v_1v_2\}$, let f_1 be a map from $\{v_1v_2\}$ to $\mathbb{R}$ such that $f_1(v_1v_2) \in L(v_1v_2)$, we call f_1 a *partial L-weighting of G for edge $e = \{v_1v_2\}$*.

Given a partial L-weighting f_1 of G for edge v_1v_2. We define a (k,k')-total list assignment L_2 for $G_2 = G - \{v_1v_2\}$ as follows:

$$L_2(u) = \begin{cases} L(u), & \text{if } u \text{ is a vertex not adjacent to } v_1v_2, \\ \{\alpha + f_0(vu)\colon\ \alpha \in L(u)\}, & \text{if } u = v_1, \text{ or } u = v_2, \\ L(u), & \text{if } u \text{ is an edge of } G_2. \end{cases}$$

We call L_2 *the* (k,k')*-total list assignment induced by* L *and* f_1.

Suppose L_2 is the (k,k')-total list assignment of G_2 induced by L and f_1. If f_2 is an L_2-weighting of G_2, then we combine f_2 and f_1 to obtain a weighting f of G as follows:

$$f(y) = \begin{cases} f_2(y) - f_1(v_1v_2), & \text{if } y = v_1 \text{ or } y = v_2, \\ f_2(y), & \text{if } y \in V - \{v_1, v_2\} \text{ or } y \text{ is an edge of } G_2, \\ f_1(y), & \text{if } y = v_1v_2. \end{cases}$$

We call f the L-weighting of G *induced by* f_2 *and* f_1.

The following lemma is obvious, similar to that from Wong et al. (2010), and its proof is omitted.

Lemma 2.1 *Suppose L is a (k,k')-total list assignment of a graph G.*

- *Suppose f_0 is a partial L-weighting of G for vertex v, and L_1 is the (k,k')-total list assignment of $G_1 = G - v$ induced by L and f_0. If f_1 is an L_1-weighting of G_1, and f is the L-weighting of G induced by f_1 and f_0, then for each $u \in V(G_1)$, $\varphi_f(u) = \varphi_{f_1}(u)$ and $\varphi_f(v) = \varphi_{f_0}(v)$.*
- *Suppose f_1 is a partial L-weighting of G for edge $e = v_1v_2$, and L_2 is the (k,k')-total list assignment of $G_2 = G - \{v_1v_2\}$ induced by L and f_1. If f_2 is an L_2-weighting of G_2, and f is the L-weighting of G induced by f_2 and f_1, then for each $u \in V(G)$, $\varphi_f(u) = \varphi_{f_2}(u)$.*

For each $y \in V \cup E$, let $L_{\max}(y) = \max L(y)$ be the maximum element of $L(y)$, and let $L_{\min}(y) = \min L(y)$ be the minimum element of $L(y)$. If $|L(y)| = 3$, then let $L_{\mathrm{med}}(y)$ be the median element of $L(y)$. Note that if $|L(y)| = 1$, then $L_{\max}(y) = L_{\min}(y)$. Otherwise, $L_{\max}(y) > L_{\min}(y)$.

Let

$$\varphi_{L,\max}(v) = \sum_{y \in N(v)} L_{\max}(vy) + L_{\max}(v),$$

$$\varphi_{L,\min}(v) = \sum_{y \in N(v)} L_{\min}(vy) + L_{\min}(v).$$

It follows from the definition that $\varphi_{L,\max}(v) = \max \varphi_f(v)$ and $\varphi_{L,\min}(v) = \min \varphi_f(v)$, where the maximum and minimum is taken over all L-weightings f of G. In other words, for any L-weighting f of G, we have

$$\varphi_{L,\min}(v) \le \varphi_f(v) \le \varphi_{L,\max}(v).$$

For a (k,k')-total list assignment L of G, let $V_{L,\max}(G)$ be the set of globally maximum vertices of G and let $V_{L,\min}(G)$ be the set of globally minimum vertices of G, that is:

$$V_{L,\max}(G) = \big\{v \in V(G)\colon\ \forall u \in V(G),\ \varphi_{L,\max}(v) \geq \varphi_{L,\max}(u)\big\},$$
$$V_{L,\min}(G) = \big\{v \in V(G)\colon\ \forall u \in V(G),\ \varphi_{L,\min}(v) \leq \varphi_{L,\min}(u)\big\}.$$

A wheel W_n is a graph that is composed of one vertex v_0 and one cycle $C_n = v_1v_2\cdots v_n$ (the length of the cycle is n), and the vertex v_0 is adjacent to every vertex of the cycle. In Przybyło and Woźniak (2010), it was proven that W_n is $(2,2)$-total weight choosable by using permanents of matrices and Combinatorial Nullstellensatz. Here, first we give a combinatorial proof that W_n is $(2,2)$-total weight choosable; then we show that W_n is also $(1,3)$-total weight choosable.

Theorem 2.2 *A wheel W_n is $(2,2)$-total weight choosable.*

Proof Because W_3 is K_4, which is $(2,2)$-total weight choosable; W_4 is $k_{2,2,1}$, in Wong et al. (2010), it is proven that $K_{n,m,1,1,\ldots,1}$ is $(2,2)$-total weight choosable, so W_4 is $(2,2)$-total weight choosable; hence next we assume $n \geq 5$. Suppose L is a $(2,2)$-total list assignment of W_n.

Case 1. Suppose $v_0 \in V_{L,\min}(G)$. Let $f(v_1v_2) = L_{\max}(v_1v_2)$; $f(v_{i-1}v_i) = L_{\max}(v_{i-1}v_i)$, $i = 4,5,\ldots,n$ (skip edge v_2v_3 here); $f(v_iv_0) = L_{\min}(v_iv_0)$, $i = 1,2,\ldots,n$; $f(v_0) = L_{\min}(v_0)$ and $f(v_1) = L_{\min}(v_1)$, then $\varphi_f(v_0) < \varphi_f(v_i)$, for $i = 1,2,\ldots,n$.

Next, choose $f(v_2)$ and $f(v_2v_3)$ such that $f(v_2) + f(v_2v_3) + f(v_2v_0) \neq f(v_1) + f(v_1v_n) + f(v_1v_0)$ for whichever $f(v_1v_n)$ is chosen, then $\varphi_f(v_1) \neq \varphi_f(v_2)$. (We can do this, because that the combination of $f(v_2) + f(v_2v_3)$ has at least 3 different values, and $f(v_1v_n) \in L(v_1v_n)$ has only 2 options.)

Then choose $f(v_i)$ such that $\varphi_f(v_i) \neq \varphi_f(v_{i-1})$, for $i = 3,4,\ldots,n-1$. Next, choose $f(v_n)$ such that $f(v_{n-1}v_n) + f(v_nv_0) + f(v_n) \neq f(v_1v_2) + f(v_1v_0) + f(v_1)$, this will guarantee that $\varphi_f(v_n) \neq \varphi_f(v_1)$. Finally, choose $f(v_nv_1)$ such that $\varphi_f(v_n) \neq \varphi_f(v_{n-1})$.

Case 2 and not case 1. Now $v_0 \notin V_{L,\min}(G)$. Let us assume that $v_2 \in V_{L,\min}(G)$. Let $f(v_2) = L_{\min}(v_2)$, $f(v_2v_3) = L_{\min}(v_2v_3)$, $f(v_1v_2) = L_{\min}(v_1v_2)$, $f(v_2v_0) = L_{\min}(v_2v_0)$, $f(v_1v_0) = L_{\max}(v_1v_0)$, then $\varphi_f(v_2) < \varphi_f(v_0)$, $\varphi_f(v_2) < \varphi_f(v_1)$.

Since $n \geq 5$, we arbitrarily choose $f(v_iv_0)$, $i = 6,7,\ldots,n$, and choose $f(v_0)$, $f(v_3v_0)$, $f(v_4v_0)$, $f(v_5v_0)$ such that $\varphi_f(v_0) \neq \varphi_f(v_1)$ whichever $f(v_nv_1)$ and $f(v_1)$ choose. (We can do this, because that the combination of $f(v_0) + f(v_3v_0) + f(v_4v_0) + f(v_5v_0)$ has at least 5 different values, and the combination of $f(v_nv_1) + f(v_1)$ has at most 4 different values.)

Next choose $f(v_i)$ and $f(v_iv_{i+1})$ such that $\varphi_f(v_i) \neq \varphi_f(v_{i-1})$ and $\varphi_f(v_i) \neq \varphi_f(v_0)$, for $i = 3,4,\ldots,n$. (Here $v_{n+1} = v_1$, and note that $\varphi_f(v_i) + \varphi_f(v_{i-1})$ has at least 3 different values.) Finally, choose $f(v_1)$ such that $\varphi_f(v_n) \neq \varphi_f(v_1)$. □

Lemma 2.3 *A wheel W_4 is $(1,3)$-total weight choosable.*

Proof Suppose L is a $(1,3)$-total list assignment of W_4.

Case 1. Suppose $v_0 \in V_{L,\min}(G)$. For $z \in \{v_0\} \cup \{v_0y\colon\ y \in N(v_0)\}$, let $f_0(z) = L_{\min}(z)$. Note that $G_1 = G - v_0$ is $K_{2,2}$. Suppose L_1 is the induced $(1,3)$-total list assignment of G_1 by L and f_0.

Next, for $i = 1,2,3,4$, (here $v_5 = v_1$), define $L_2(v_iv_{i+1}) = L_1(v_iv_{i+1}) - \{L_{\min}(v_iv_{i+1})\}$. If $L_2(y)$ is not specified, then $L_2(y) = L_1(y)$. Since $K_{2,2}$ is $(1,2)$-total weight choosable, there exists a proper L_2-weighting f_2 of G_1. Suppose f is the induced L-weighting of G by f_2 and f_0, then by Lemma 2.1, and $\varphi_f(v_i) > \varphi_f(v_0)$ (for $i = 1,2,3,4$), we concluded that f is a proper L-weighting of G.

Case 2 and not case 1. Now, we assume that $v_2 \in V_{L,\min}(G)$. For $z \in \{v_2\} \cup \{v_2y\colon\ y \in N(v_2)\}$, let $f_0(z) = L_{\min}(z)$.

Let $G_1 = G - v_0$. Suppose L_1 is the induced $(1,3)$-total list assignment of G_1 by L and f_0. Define $L_2(v_1v_4) = L_1(v_1v_4) - \{L_{\min}(v_1v_4)\}$, $L_2(v_0v_3) = L_1(v_0v_3) - \{L_{\min}(v_0v_3)\}$. If $L_2(y)$ is not specified, then $L_2(y) = L_1(y)$. Then for any L_2-weighting f_2 of G_1, we have $\varphi_{f_0}(v_2) < \varphi_{f_2}(v_1)$, $\varphi_{f_0}(v_2) < \varphi_{f_2}(v_3)$, $\varphi_{f_0}(v_2) < \varphi_{f_2}(v_0)$.

If v_1 or $v_3 \in V_{L_2,\max}(G_1)$, without loss of generality, suppose that $v_1 \in V_{L_2,\max}(G_1)$. Then, for $z \in \{v_1\} \cup \{v_1y\colon\ y \in N_{G_1}(v_1)\}$, let $f_2(z) = L_{2\max}(z)$, and $f_2(v_4v_0) = L_{2\min}(v_4v_0)$. Then for any L_2-weighting f_2 of G_1, we have $\varphi_{f_2}(v_1) > \varphi_{f_2}(v_0)$, $\varphi_{f_2}(v_1) > \varphi_{f_2}(v_4)$. Next, choose $f_2(v_0v_3)$ such that $f_2(v_0v_4) + f_2(v_4) \neq f_2(v_0v_3) + f_2(v_3)$, then $\varphi_{f_2}(v_3) \neq \varphi_{f_2}(v_4)$. Finally, choose $f_2(v_4v_3)$ such that $\varphi_{f_2}(v_0) \neq \varphi_{f_2}(v_4)$, $\varphi_{f_2}(v_0) \neq \varphi_f(v_3)$.

If v_4 or $v_0 \in V_{L_2,\max}(G_1)$, without loss of generality, suppose that $v_4 \in V_{L_2,\max}(G_1)$. Then, for $z \in \{v_4\} \cup \{v_4y\colon\ y \in N_{G_1}(v_4)\}$, let $f_2(z) = L_{2\max}(z)$. Next we define $L_2'(v_1v_0) = L_2(v_1v_0) - \{L_{2\max}(v_1v_0)\}$. (The others in L_2' are the same as in L_2.) Then for any L_2'-weighting f_2 of G_1, we have $\varphi_{f_2}(v_4) > \varphi_{f_2}(v_0)$, $\varphi_{f_2}(v_4) > \varphi_{f_2}(v_1)$. Since $v_3 \notin V_{L_2,\max}(G_1)$, $\varphi_{f_2}(v_3) < \varphi_{f_2}(v_4)$. Choose $f_2(v_0v_3)$ such that $\varphi_{f_2}(v_1) \neq \varphi_{f_2}(v_0)$. Choose $f_2(v_0v_1)$ such that $\varphi_{f_2}(v_3) \neq \varphi_{f_2}(v_0)$.

So f_2 is a proper L_2-weighting of G_1. Suppose f is the induced L-weighting of G by f_2 and f_0, then f is a proper L-weighting of G. $\square$

Theorem 2.4 *A wheel W_n is $(1,3)$-total weight choosable.*

Proof Because W_3 is K_4, which is $(1,3)$-total weight choosable; by Lemma 2.3, W_4 is $(1,3)$-total weight choosable; hence we assume that $n \geq 5$. Suppose L is a $(1,3)$-total list assignment of W_n.

Case 1. Suppose $v_0 \in V_{L,\min}(G)$. Suppose $v_0 \in V_{L,\min}(G)$. For $z \in \{v_0\} \cup \{v_0y\colon\ y \in N(v_0)\}$, let $f_0(z) = L_{\min}(z)$. Let $G_1 = G - v_0$. Suppose L_1 is the induced $(1,3)$-total list assignment of G_1 by L and f_0.

Next let $f_1(v_1v_2) = L_{1\mathrm{med}}(v_1v_2)$, suppose L_2 is the induced $(1,3)$-total list assignment of $G_2 = G_1 - \{v_1v_2\}$ by L_1 and f_1. Then for any L_2-weighting f_2 of G_2, we have $\varphi_{f_0}(v_0) < \varphi_{f_2}(v_1)$, and $\varphi_{f_0}(v_0) < \varphi_{f_2}(v_2)$.

Next, define $L_3(v_nv_1) = L_2(v_nv_1) - \{L_{\min}(v_nv_1)\}$. If $L_3(y)$ is not specified, then $L_3(y) = L_2(y)$. Then for any L_3-weighting f_3 of G_2, we have $\varphi_{f_0}(v_0) < \varphi_{f_3}(v_n)$.

Now choose $f_3(v_2v_3)$, such that $f_3(v_2) + f_3(v_2v_3) \neq f_3(v_1) + f_3(v_1v_n)$ for whichever $f_3(v_nv_1)$ will be chosen, then $\varphi_{f_3}(v_2) \neq \varphi_{f_3}(v_1)$. (Note that we can do this, because $|L_3(v_nv_1)| = 2$ and $|L_3(v_2v_3)| = 3$.)

Then choose $f_3(v_i v_{i+1})$ between its median value and its maximum value, such that $\varphi_{f_3}(v_i) \neq \varphi_{f_3}(v_{i-1})$ for $i = 3, 4, \dots, n-2$, and then $\varphi_{f_3}(v_i) \neq \varphi_{f_0}(v_0)$ for $i = 3, 4, \dots, n-1$.

Next, choose $f_3(v_{n-1}v_n)$ such that $\varphi_{f_3}(v_{n-1}) \neq \varphi_{f_3}(v_{n-2})$ and $f_3(v_{n-1}v_n) + f_0(v_n v_0) + f(v_n) \neq f_1(v_1 v_2) + f_0(v_1 v_0) + f_3(v_1)$, then $\varphi_{f_3}(v_n) \neq \varphi_{f_3}(v_1)$. (Since $|L_3(v_{n-1}v_n)| = 3$, we can do this.)

Finally, choose $f_3(v_n v_1)$ such that $\varphi_{f_3}(v_n) \neq \varphi_{f_3}(v_{n-1})$.

We concluded that, suppose f is the induced L-weighting of G, induced by f_3 and f_1 and then f_0, then f is a proper L-weighting of G.

Case 2 and not case 1. Now $v_0 \notin V_{L,\min}(G)$. Let us assume that $v_1 \in V_{L,\min}(G)$, then let $f_0(v_1v_2) = L_{\min}(v_1v_2)$, $f_0(v_1v_0) = L_{\min}(v_1v_0)$, $f_0(v_1v_n) = L_{\min}(v_1v_n)$, $f_0(v_1) = L_{\min}(v_1)$. Let $G_1 = G - v_1$. Suppose L_1 is the induced $(1,3)$-total list assignment of G_1 by L and f_0.

Next, define $L_2(v_0v_2) = L_1(v_0v_2) - \{L_{\min}(v_0v_2)\}$; $L_2(v_0v_n) = L_1(v_0v_n) - \{L_{\min}(v_0v_n)\}$; if $L_2(y)$ is not specified, then $L_2(y) = L_1(y)$. Then for any L_2-weighting f_2 of G_1, we have $\varphi_{f_0}(v_1) < \varphi_{f_2}(v_n)$, $\varphi_{f_0}(v_1) < \varphi_{f_2}(v_2)$, and $\varphi_{f_0}(v_1) < \varphi_{f_2}(v_0)$.

Then choose $f_2(v_0v_n)$ and $f_2(v_0v_{n-1})$, such that $f_2(v_nv_1) + f_2(v_0v_n) + f_2(v_n) \neq f_2(v_0v_{n-1}) + f_2(v_{n-2}v_{n-1}) + f_2(v_{n-1})$ for whichever $f_2(v_{n-2}v_{n-1})$ will be chosen, this will make sure that $\varphi_{f_2}(v_{n-1}) \neq \varphi_{f_2}(v_n)$.

Since $n \geq 5$, we next choose $f_2(v_0v_2), \dots, f_2(v_0v_{n-2})$ such that

$$\sum_{i=1}^{n-1} f_2(v_0v_i) + f_2(v_0) \neq f_2(v_{n-1}v_n) + f_2(v_n) + f_2(v_nv_1)$$

for whichever $f_2(v_{n-1}v_n)$ will be chosen, then we have $\varphi_{f_2}(v_0) \neq \varphi_{f_2}(v_n)$.

Finally, choose $f_2(v_iv_{i+1})$ such that $\varphi_{f_2}(v_i) \neq \varphi_f(v_0)$ and $\varphi_{f_2}(v_i) \neq \varphi_{f_2}(v_{i-1})$ for $i = 2, \dots, n-1$.

We concluded that, suppose f is the induced L-weighting of G by f_2 and f_0, then f is a proper L-weighting of G. $\square$

3 A lemma on $(1, k)$-total weightings of stars

Let $K_{m,n}$ denote a complete bipartite graph with part size m and n respectively; especially, $K_{1,s}$ is a star.

Lemma 3.1 *Let $K_{1,s}$ be a star with $s \geq 2$, and v_0 is the center of the star. For any $k \geq 2$, there are at least $2k-3$ proper L-weightings f of $K_{1,s}$; moreover, for any two of these L-weightings f, their total weights $\varphi_f(v_0)$ of v_0 are different.*

Proof Suppose L is a $(1,k)$-total list assignment of $K_{1,s}$ with $s \geq 2$. For $e \in E(K_{1,s})$, suppose $L(e) = \{L_1(e), \dots, L_k(e)\}$, and $L_1(e) < L_2(e) < \cdots < L_k(e)$. The case of $s = 2$ is trivial. We assume that $s \geq 3$.

Firstly, we define a L-weighting f_0 of $K_{1,s}$ as following: for $e \in E(K_{1,s})$, let $f_0(e) = L_{\max}(e)$, that is $f_0(e) = L_k(e)$; and $f_0(z) = L_{\max}(z)$, for $z \in V(K_{1,s})$. Next,

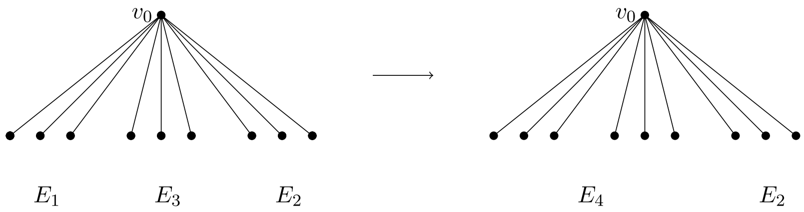


Fig. 1 Illustration for the algorithm (next)

for any L-weighting f of $K_{1,s}$, we define the following sets:

$$E_1 = \{e \mid e = vv_0,\ f(e) = L_k(e),\ \varphi_f(v) < \varphi_f(v_0)\};$$
$$E_2 = \{e \mid e = vv_0,\ f(e) = L_k(e),\ \varphi_f(v) > \varphi_f(v_0)\};$$
$$E_3 = \{e \mid e = vv_0,\ f(e) = L_k(e),\ \varphi_f(v) = \varphi_f(v_0)\};$$
$$E_4 = \{e \mid e = vv_0,\ f(e) \neq L_k(e),\ \varphi_f(v) \leq \varphi_f(v_0)\};$$
$$E_5 = \{e \mid e = vv_0,\ \varphi_f(v) = \varphi_f(v_0)\};$$
$$E_6 = \{e \mid e = vv_0,\ f(e) \neq L_k(e),\ \varphi_f(v) > \varphi_f(v_0)\}.$$

Note that $E_3 \subseteq E_5$, and $E_5 \backslash E_3 \subseteq E_4$.

We use the following algorithm for our proof (see Fig. 1).

Initialization: $i := 0$; $h := 0$; *count* $:= 0$; $f := f_0$;

While *count* $< 2k - 3$ and $E_1 \neq \emptyset$ do

1. Choose edge $e \in E_1$. If there exists j, $1 \leq j \leq k$, such that if we let $f(e) = L_j(e)$, then for such f, $E_5 \neq \emptyset$; then *count* $:=$ *count* $+ (k - j)$, record this j; else (that is, there is no j, such that $f(e) = L_j(e)$, and for this f, $E_5 \neq \emptyset$), then $j := 0$;
2. if $j \geq 3$, then let $f(e) := L_{j-1}(e)$;
3. if $j = 2$, then let $f(e) := L_1(e)$; $h := h + 1$;
4. if $j \leq 1$, then let $f(e) = L_1(e)$; $i = i + 1$;

First we note that the variable *count* is intended to denote the number of *good* L-weightings f of $K_{1,s}$ (as claimed in Lemma 3.1). Variables i, h, and j are introduced to help doing the counting for the variable *count*.

Secondly, we note that every time the function f is updated in the algorithm, we come up with a L-weighting f of $K_{1,s}$. Indeed, if $f(y)$ is not specified, then $f(y)$ is the same as the previous one.

Thirdly, note that $\varphi_{f_0}(v_0)$ is the maximum total weight of v_0 for all L-weightings f of $K_{1,s}$. And from the algorithm, we know that, every time the function f is updated in the algorithm, $\varphi_f(v_0)$ is smaller than the previous one.

Next we show that in Step 1 of the while loop, "*count* $:=$ *count* $+ (k - j)$", the *count* is increased correctly. This is because there exists j, $1 \leq j \leq k$, such that $f(e) = L_j(e)$, and for such f, $E_5 \neq \emptyset$. This implies that, for $f(e) = L_t(e)$, where $t \in \{j + 1, \ldots, k\}$, since $L_t(e) > L_j(e)$, we have $\varphi_f(v_0) \neq \varphi_f(v)$ holds for all $v \in N(v_0)$. Moreover, for any two of these L-weightings f, their total weights $\varphi_f(v_0)$ of v_0 are different.

The same arguments as above show that, if i increases one, the *count* can be increased by $k-1$; if h increases one, the *count* can be increased by $k-2$.

Now if the algorithm is stopped by the condition of "*count* $< 2k-3$", then we complete the proof. Hence we assume that the algorithm is stopped by the condition of "$E_1 \neq \emptyset$". Since *count* $< 2k-3$, we have $i \leq 1$. Next, we prove the following claim.

Claim 1 *If* $i=1$, *then* $|E_4|-1 \leq |E_2|$; *if* $i=0$, *then* $|E_4| \leq |E_2|$.

Proof Initially, $f = f_0$; therefore, $|E_4|=0$ and $|E_2| \geq 0$, the claim is true.

Next, suppose at some running step of the algorithm, $e \in E_1$ is chosen.

If $j \geq 2$, then in the algorithm we let $f(e) = L_{j-1}(e)$. First note that in the new f, e leaves E_1 and enters E_4. Then, if $|E_3| \geq 1$, then all the edge(s) in E_3 enter(s) E_2; if $|E_3|=0$, then all the edge(s) in $\emptyset \neq E_5 \backslash E_3 \subset E_4$ leave(s) E_4 and enter(s) E_6. In other words, if E_4 increases by one, then E_2 increases by at least one; if E_4 is non-increasing, then E_2 is unchanged.

If $j \leq 1$, then in the algorithm we let $f(e) = L_1(e)$. If $E_5 \neq \emptyset$, arguments in the above paragraph still work. If $E_5 = \emptyset$, then in the new f, e enters E_4, but E_2 is unchanged; in other words, E_4 increases by one, but E_2 is unchanged.

If $i=0$, this means that all $j \geq 2$, then $|E_4| \leq |E_2|$. If $i=1$, means that all $j \geq 2$ except one $j \leq 1$, then $|E_4|-1 \leq |E_2|$. □

Next we distinguish two cases to finish the proof.

Case 1, $i=1$. Then $|E_4|-1 \leq |E_2|$, and there are at least $k-1$ good L-weightings f of $K_{1,s}$ before the algorithm stops. First we show that we can assume $E_3 = \emptyset$.

If $E_2 \neq \emptyset$ and $E_3 \neq \emptyset$, then choose one edge e from E_2, let $f(e) = L_{k-1}(e)$. Then in the new f, all the edge(s) in E_3 enter(s) E_2, hence $|E_2|$ is non-decreasing and $|E_4|$ is non-increasing. Therefore, $E_2 \neq \emptyset$, and $E_3 = \emptyset$, and $|E_4|-1 \leq |E_2|$ still holds.

If $E_2 = \emptyset$ and $|E_3| \geq 2$, then choose one edge from E_3, let $f(e) = L_{k-1}(e)$. Then in the new f, the edge e leaves E_3 and enters E_4; all the other edge(s) of E_3 enter(s) E_2; hence $|E_2|$ increases by at least one, and $|E_4|$ increases by at most one. Therefore, $E_2 \neq \emptyset$, and $E_3 = \emptyset$, and $|E_4|-1 \leq |E_2|$ still holds.

If $E_2 = \emptyset$ and $|E_3| = 1$. Since *count* $< 2k-3$ and $i=1$, then $h=0$. By the algorithm, there are no two edges satisfying $f(e) = L_1(e)$. Since $s \geq 3$, there exists $e \notin E_3$, such that $f(e) = L_j(e)$ and $j \geq 2$. Then in the new f, we let $f(e) = L_{j-1}(e)$, then the sole edge in E_3 enters E_2. Therefore, $E_2 \neq \emptyset$, and $E_3 = \emptyset$, and $|E_4|-1 \leq |E_2|$ still holds.

Hence from now on we assume that $E_1 = \emptyset$, $E_3 = \emptyset$.

Suppose $|E_2| = t \geq 1$, then $|E_4| \leq t+1$. By Lemma 1.6, by changing the edge weights $f(e)$ for $e \in E_2$, $\varphi_f(v_0)$ has at least $(k-1)t+1$ different total weights. By removing the possible bad ones of $\varphi_f(v_0) = \varphi_f(v)$ where $e = vv_0 \in E_4$, there are at least $(k-1)t+1-(t+1) = (k-2)t \geq k-2$ good L-weightings f. And note that these $k-2$ good L-weightings f are different from those $k-1$ good L-weightings f which are counted for $i=1$.

For the left of this case, we have $E_1 = E_2 = E_3 = \emptyset$. Then $|E_4| \leq 1$; since $s \geq 3$, we have $|E_6| \geq 2$. Since *count* $< 2k - 3$ and $i = 1$, then $h = 0$. Therefore there exists $e_0 \in E_6$, $f(e_0) = L_{j_0}(e_0)$ and $j_0 \geq 2$. Note that for $i = 1$, we count $k - 1$ good L-weightings f. Now consider edge e_0. Before the algorithm stopped and set $f(e_0) = L_{j_0}(e_0)$, there are $k - j_0 - 1$ good L-weightings f are added to variable *count*. After the algorithm stopped, we can set $f(e_0)$ as any one of $\{L_1(e_0), \dots, L_{j_0}(e_0)\}$, these are j_0 choices for $f(e_0)$; all these j_0 choices of f still have $\varphi_f(v) > \varphi_f(v_0)$ for $vv_0 \in E_6$. Hence there are at least $(k - 1) + (k - j_0 - 1) + j_0 - 1 = 2k - 3$ good L-weightings f.

Case 2, $i = 0$. Then $|E_4| \leq |E_2|$. First we show that we can assume $E_2 \neq \emptyset$. Suppose otherwise, that is when the algorithm finished running, $E_2 = \emptyset$.

If the algorithm stopped at $f = f_0$, that is $E_1 = \emptyset$ at the initial step, then $|E_3| = s \geq 3$. Then choose one edge e_0 from E_3, let $f(e_0) = L_{k-1}(e_0)$. Then in the new f, all the edges in $E_3 - \{e_0\}$ enter E_2, this makes $E_2 \neq \emptyset$. And note that $|E_4|$ is increased by one, $|E_4| \leq |E_2|$ still holds.

If the algorithm is not stopped at $f = f_0$, then $E_1 \neq \emptyset$ at the initial step. Then in Step 1 of the algorithm, it will search for j, $1 \leq j \leq k$, such that $f(e) = L_j(e)$, and for such f, $E_5 \neq \emptyset$. Note that this is the first running step of the algorithm, so we have $E_3 = E_5 \neq \emptyset$. Since $i = 0$, we have $j \geq 2$. Then in Step 2 or Step 3, we let $f(e) = L_{j-1}(e)$. In the new f, all the edge(s) in E_3 enter(s) E_2, this makes $E_2 \neq \emptyset$. And note that $|E_4|$ is increased by one, $|E_4| \leq |E_2|$ still holds.

Let us note that after the first step of the algorithm, since $\varphi_f(v_0)$ is decreasing with every new f, we know that $|E_2|$ is not decreased. Therefore, we can assume $E_2 \neq \emptyset$.

Next, if $E_2 \neq \emptyset$ and $E_3 \neq \emptyset$, then choose one edge from E_2, let $f(e) = L_{k-1}(e)$. Then in the new f, all the edge(s) in E_3 enter(s) E_2, hence $|E_2|$ is non-decreasing and $|E_4|$ is non-increasing. Therefore, $E_2 \neq \emptyset$, and $E_3 = \emptyset$, and $|E_4| \leq |E_2|$ still holds.

Hence from now on we assume that $E_1 = \emptyset$, $E_2 \neq \emptyset$, and $E_3 = \emptyset$.

Suppose $|E_2| = t \geq 2$, then $|E_4| \leq t$. By Lemma 1.6, by changing the edge weights $f(e)$ for $e \in E_2$, $\varphi_f(v_0)$ has at least $(k - 1)t + 1$ different total weights. By removing the possible bad ones of $\varphi_f(v_0) = \varphi_f(v)$ where $e = vv_0 \in E_4$, there are at least $(k - 1)t + 1 - t = (k - 2)t + 1 \geq 2k - 3$ good L-weightings f.

For the left of this case, we have $E_1 = E_3 = \emptyset$, and $|E_2| = 1$. Then $|E_4| \leq 1$; since $s \geq 3$, we have $|E_6| \geq 1$. Assume $e_1 \in E_2$, $e_2 \in E_6$, $f(e_1) = L_k(e_1)$, and $f(e_2) = L_{j_0}(e_2)$ where $1 \leq j_0 \leq k - 1$. Consider edge e_2, before the algorithm stopped and set $f(e_2) = L_{j_0}(e_2)$, there are $k - j_0 - 1$ good L-weightings f are added to variable *count*. After the algorithm stopped, we can set $f(e_1)$ as any one of $\{L_1(e_1), \dots, L_k(e_1)\}$, and $f(e_2)$ as any one of $\{L_1(e_1), \dots, L_{j_0}(e_2)\}$. By Lemma 1.7, by changing the edge weights $f(e_1)$ and $f(e_2)$, $\varphi_f(v_0)$ has at least $k + j_0 - 1$ different total weights. By removing the possible bad one of $\varphi_f(v_0) = \varphi_f(v)$ where $e = vv_0 \in E_4$, there are at least $(k - j_0 - 1) + (k + j_0 - 1) - 1 = 2k - 3$ good L-weightings f. □

When $k = 2$, we have the following corollary:

Corollary 3.2 $K_{1,s}$ (*with* $s \geq 2$) *is* $(1, 2)$*-total weight choosable.*

4 Total weight choosability of subdivided graphs

Suppose H is a graph without loops (with parallel edges permitted). G is a subdivision of H. It follows from results in Bartnicki et al. (2009), if G is obtained from H by subdividing each edge with at least three vertices, then G is $(1,3)$-total weight choosable. It follows from results in Wong and Zhu (2011), if G is obtained from H by subdividing each edge with at least two vertices, then G is $(2,2)$-total weight choosable as well as $(1,3)$-total weight choosable. Both proofs from Bartnicki et al. (2009) and Wong and Zhu (2011) used permanents of matrices and Combinatorial Nullstellensatz. Here we give a combinatorial proof that, if G is obtained from H by subdividing each edge with at least one vertex, then G is $(1,3)$-total weight choosable as well as $(2,2)$-total weight choosable.

Theorem 4.1 *For any loopless graph H, a graph G is obtained from H by subdividing each edge with at least one vertex, then G is $(1,3)$-choosable.*

Proof We may assume that H is connected. We prove this by induction on $|V(H)|$. Suppose L is a $(1,3)$-total list assignment of G. Since H has no loops, the base step of $|V(H)|=1$ is trivial. If $|V(H)|=2$ and $H=K_2$ (a single edge), then G is a path, hence the result holds. Next, suppose $|V(H)|=n+1\geq 2$, assume $d_H(v_n)\geq 2$, let $H'=H-v_n$, and G' is obtained from H' by subdividing each edge with at least one vertex.

Assume $N_H(v_n)=\{v_1,\dots,v_i\}$, subdividing vertices on edge v_jv_n (and in this order), where $j=1,\dots,i$, are denoted by $v_1^j,\dots,v_{h_j}^j$. For every $j=1,\dots,i$, let $f_0(v_jv_1^j)=L_{\min}(v_jv_1^j)$. This defines a partial L-weighting f_0 of G for edges $v_jv_1^j$ $(j=1,\dots,i)$. From L and f_0, we define a $(1,3)$-total list assignment L_1 for G' as follows:

$$L_1(u)=\begin{cases} L(u), & \text{if } u \text{ is a vertex not adjacent to } v_jv_1^j,\ j=1,\dots,i,\\ \{\alpha+f_0(v_jv_1^j):\ \alpha\in L(u)\}, & \text{if } u=v_j, \text{ where } j=1,\dots,i,\\ L(u), & \text{if } u \text{ is an edge of } G'.\end{cases}$$

By induction hypothesis, G' is $(1,3)$-total weight choosable, therefore there exists a proper L_1-weighting f_1 of G'. First, we combine f_1 and f_0 to obtain a partial L-weighting f_2 of G as follows:

$$f_2(y)=\begin{cases} f_1(y)-f_0(v_jv_1^j), & \text{if } y=v_j, \text{ where } j=1,\dots,i,\\ f_1(y), & \text{if } y\in V(G')-\{v_j:\ j=1,\dots,i\} \text{ or } y \text{ is an edge of } G',\\ f_0(y), & \text{if } y=v_jv_1^j, \text{ where } j=1,\dots,i.\end{cases}$$

Note that for each $u\in V(G')$, $\varphi_{f_2}(u)=\varphi_{f_1}(u)$. Therefore, f_2 is a (partial) proper weighting for G'.

Next, for convenience, we let $v_0^j=v_j$. Choose $f_2(v_t^jv_{t+1}^j)$ such that $\varphi_{f_2}(v_t^j)\neq\varphi_{f_2}(v_{t-1}^j)$, $t=1,\dots,h_j-1$. Then from $L(v_{h_j}^jv_n)$, remove the value $f(v_{h_j}^jv_n)$, which satisfies $f(v_{h_j}^jv_n)+f_2(v_{h_j}^j)+f_2(v_{h_j}^jv_{h_j-1}^j)=\varphi_{f_2}(v_{h_j-1}^j)$, if such $f(v_{h_j}^j v_n) \in L(v_{h_j}^j v_n)$. Suppose the new list assignment for edge $v_{h_j}^j v_n$ is $L'(v_{h_j}^j v_n)$, and let $L'(v_n) = L(v_n)$. Then for any $f_2(v_{h_j}^j v_n) \in L'(v_{h_j}^j v_n)$, we have $\varphi_{f_2}(v_{h_j}^j) \neq \varphi_{f_2}(v_{h_j-1}^j)$.

Since $v_n, v_{h_1}^1, v_{h_2}^2, \dots, v_{h_i}^i$ induces a star S in G, and L' contains a $(1,2)$-total list assignment of S; by Corollary 3.2, there is a proper L'-weighting f_2 for S. This finishes the proof. □

Theorem 4.2 *For any loopless graph H, a graph G is obtained from H by subdividing each edge with at least one vertex, then G is $(2,2)$-choosable.*

Proof We may assume that H is connected. We prove this by induction on $|V(H)|$. Suppose L is a $(2,2)$-total list assignment of G. Since H has no loops, the base step of $|V(H)| = 1$ is trivial.

For the induction step, suppose $|V(H)| = n+1 \geq 2$. Let $V_{L,\max}^H = \{v \in V(H) \mid \varphi_{L,\max}(v) \geq \varphi_{L,\max}(u),\ \forall u \in V(H)\}$. Assume $v_0 \in V_{L,\max}^H$; $N_H(v_0) = \{v_1, \dots, v_k\}$; the subdividing vertices on edge $v_0 v_j$ (and in this order) are $v_1^j, \dots, v_{t_j}^j$, $j = 1, \dots, k$. Let $H' = H - v_0$; and G' be the induced subgraph of G, such that the branch vertices of G' is $V(H')$; then G' is obtained from H' by subdividing each edge with at least one vertex.

For $z \in \{v_0\} \cup \{v_0 y\colon\ y \in N_G(v_0)\}$, let $f_0(z) = L_{\max}(z)$. Suppose L_1 is the induced $(2,2)$-total list assignment of $G_1 = G - v_0$ by L and f_0. Next we will define a partial L_1-weighting f_1; and then define a $(2,2)$-total list assignment L_2 of G' which is induced by L_1 and the partial L_1-weighting f_1.

Case 1, $t_j = 1$. If $\varphi_{L,\max}(v_0) < \varphi_{L,\max}(v_1^j)$, then $\varphi_{L,\max}(v_j) < \varphi_{L,\max}(v_1^j)$. For $z \in \{v_1^j\} \cup \{v_1^j v_j\}$, let $f_1(z) = L_{1\max}(z)$. Let $L_2(v_j) = \{\alpha + L_{1\max}(v_1^j v_j)\colon\ \alpha \in L_1(v_j)\}$; if $L_2(y)$ is not specified, then $L_2(y) = L_1(y)$.

If $\varphi_{L,\max}(v_0) \geq \varphi_{L,\max}(v_1^j)$, let $f_1(v_1^j v_j) = L_{1\min}(v_1^j v_j)$, then $\varphi_{L,\max}(v_0) > \varphi_{f_1}(v_1^j)$, for whichever $f_1(v_1^j)$ will be chosen. Let $L_2(v_j) = \{\alpha + L_{1\min}(v_1^j v_j)\colon\ \alpha \in L_1(v_j)\}$; if $L_2(y)$ is not specified, then $L_2(y) = L_1(y)$.

Case 2, $t_j \geq 2$. For convenience, let $v_0^j = v_0$. For $3 \leq i \leq t_j$, choose $f_1(v_{i-2}^j)$, $f_1(v_{i-2}^j v_{i-1}^j)$ such that $\varphi_{f_1}(v_{i-2}^j) \neq \varphi_{f_1}(v_{i-3}^j)$. Then pick $f_1(v_{t_j-1}^j)$ and $f_1(v_{t_j}^j v_j)$, such that $f_1(v_{t_j-1}^j) + f_1(v_{t_j-2}^j v_{t_j-1}^j) \neq f_1(v_{t_j}^j) + f_1(v_{t_j}^j v_j)$ for whichever $f_1(v_{t_j}^j)$ will be chosen. (We can do this, because that the combination of $f_1(v_{t_j-1}^j) - f_1(v_{t_j}^j v_j)$ has at least 3 different values, and $f_1(v_{t_j}^j) \in L(v_{t_j}^j)$ has only 2 options.) This will make $\varphi_{f_1}(v_{t_j-1}^j) \neq \varphi_{f_1}(v_{t_j}^j)$. Then choose $f_1(v_{t_j-1}^j v_{t_j}^j)$, such that $\varphi_{f_1}(v_{t_j-1}^j) \neq \varphi_{f_1}(v_{t_j-2}^j)$. Let $L_2(v_j) = \{\alpha + f_1(v_j v_{t_j}^j)\colon\ \alpha \in L_1(v_j)\}$; if $L_2(y)$ is not specified, then $L_2(y) = L_1(y)$.

We regard L_2 as the induced $(2,2)$-total list assignment of G' by L_1 and the partial L_1-weighting f_1. By the induction hypothesis, there is a proper L_2-weighting f_2 for G'. Finally, if needed, choose $f_1(v_{t_j}^j)$, such that $\varphi_{f_1}(v_{t_j}^j) \neq \varphi_{f_2}(v_j)$, $j = 1, \dots, k$.

We concluded that, suppose f is the induced L-weighting of G by f_2, f_1, and f_0; then f is a proper L-weighting of G. □

5 A rank function and total weight choosability of graphs

In this section, we will first define a rank function, named as $h(G)$ of graphs G; then we will show that any graph G with no isolated edges is $(1, h(G))$-total weight choosable; this will give us a series of results.

Let $\Pi(G)$ be the set of linear orderings on the vertex set of a graph G. Let $L \in \Pi(G)$, x and y be two vertices of G. The orientation $G_L = (V, E_L)$ of G with respect to L is obtained by setting $E_L = \{(x, y) \mid (x, y) \in E,\ x >_L y\}$. For every $x \in V$, let $N^+_{G_L}(x)$ denote the outneighbor(s) of x in G_L, let $N^-_{G_L}(x)$ denote the inneighbor(s) of x, let $d^+_{G_L}(x)$ be the outdegree of x, i.e., $d^+_{G_L}(x) = |N^+_{G_L}(x)|$. Let $\Delta^+(G_L) = \max_{v\in V} d^+_{G_L}(v)$. A graph G is said to be s-degenerate, if $\min_{L\in\Pi(G)} \Delta^+(G_L) \le s$.

Suppose $N^+_{G_L}(v_i) = \{v_i^1, v_i^2, \dots, v_i^{g(v_i)}\}$; then $d^+_{G_L}(v_i) = g(v_i)$; and we always let $v_i^1 <_L v_i^2 <_L \cdots <_L v_i^{g(v_i)}$. So $v_i^{g(v_i)}$ is the last outneighbor of v_i in L, and $v_i^{g(v_i)-1}$ is the second to last outneighbor of v_i in L. Define (see Fig. 2)

$$N_1^-(v_i) = \bigl\{v \mid v \in N^-(v_i),\ d(v) = d^+_{G_L}(v) = g(v),\ \text{and } v^{g(v)} = v_i\bigr\}.$$

Let $L \in \Pi(G)$, the rank of G with respect to L, denoted by $h(G_L)$, is defined by:

$$h(G_L) = 2 + \max_{v_i\in V}\Bigl\{d^+_{G_L}(v_i) + \max_{v\in N_1^-(v_i)}\bigl(d^+_{G_L}(v) - 1\bigr)\Bigr\}.$$

Note that if $N_1^-(v_i) = \emptyset$, then define $\max_{v\in N_1^-(v_i)}(d^+_{G_L}(v) - 1) = 0$. The rank of a graph G, denoted by $h(G)$, is defined by:

$$h(G) = \min_{L\in\Pi(G)} h(G_L).$$

Theorem 5.1 *Any graph G with no isolated edges is $(1, h(G))$-total weight choosable.*

Proof Suppose $\tilde{L} \in \Pi(G)$ satisfies $h(G) = h(G_{\tilde{L}})$; $v_1 <_{\tilde{L}} v_2 <_{\tilde{L}} \cdots <_{\tilde{L}} v_n$; and suppose L is a $(1, h(G))$-total list assignment of G. Define (see Fig. 3 and Fig. 4)

$$\begin{aligned} N_2^-(v_i) = \bigl\{&v \mid v \in N^-(v_i), d(v) = d^+_{G_{\tilde{L}}}(v) = g(v), v^{g(v)-1} = v_i,\ \text{let } v' = v^{g(v)}, \\ &\text{then } d\bigl(v'\bigr) = d^+_{G_{\tilde{L}}}\bigl(v'\bigr) + 1 = g\bigl(v'\bigr) + 1,\ \bigl(v'\bigr)^{g(v')} \text{ is } v_i \text{ or before } v_i\bigr\}, \end{aligned}$$

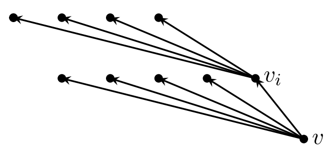


Fig. 2 Definition of $N_1^-(v_i)$

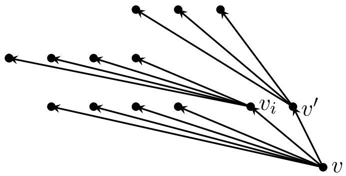
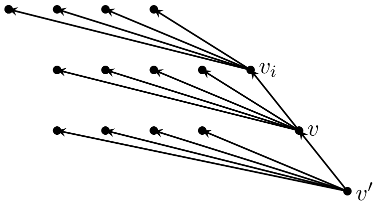


Fig. 3 Definition of $N_2^-(v_i)$

Fig. 4 Definition of $N_3^-(v_i)$



$$N_3^-(v_i) = \{v \mid v \in N^-(v_i),\ d(v) = d^+_{G_{\tilde{L}}}(v) + 1 = g(v) + 1,\ v^{g(v)} = v_i;$$
$$v' \text{ is the sole inneighbor of } v, \text{ and } d(v') = d^+_{G_{\tilde{L}}}(v') = g(v'),$$
$$\text{and } v = (v')^{g(v')}, \text{ and } (v')^{g(v')-1} \text{ is before } v_i\},$$
$$N_4^-(v_i) = N^-(v_i) \setminus \big(N_1^-(v_i) \cup N_2^-(v_i) \cup N_3^-(v_i)\big).$$

Next we will define a L-weighting f for G. The f is defined step by step according to the linear order $\tilde{L}$ of $V(G)$. At each step, we work on a vertex v_i $(1 \le i \le n)$. When we work on the vertex v_i, we will fix weights $f(z)$, where $z \in \{v_i\} \cup \{v_i v_j:\ v_j \in N^-(v_i)\}$, such that after these assignments, we have $\varphi_f(v_i) \neq \varphi_f(v)$ for all $v \in N^+(v_i)$. Since for some $v \in V(G)$, we may have $N^-(v) = \emptyset$. For such v, to make sure that $\varphi_f(v') \neq \varphi_f(v)$ holds for all $v' \in N^+(v)$, when we work on the vertex v_i, we take special considerations for how to pick $f(v_i v_j)$, where $v_j \in N_1^-(v_i) \cup N_2^-(v_i) \cup N_3^-(v_i)$. The $N_1^-(v_i)$, $N_2^-(v_i)$ and $N_3^-(v_i)$ are designed to take care of vertices v with $N^-(v) = \emptyset$.

First suppose that $N^-(v_i) \neq \emptyset$, where $1 \le i \le n$; now we begin defining the L-weighting $f(z)$, where $z \in \{v_i\} \cup \{v_i v_j:\ v_j \in N^-(v_i)\}$.

1. If $N_2^-(v_i) \cup N_3^-(v_i) \neq \emptyset$. Then, first for $v \in N_4^-(v_i)$, arbitrarily choose $f(v_i v) \in L(v_i v)$.
   (a) For $v \in N_3^-(v_i)$, from $L(v_i v)$, deleted the value $f(v_i v) = f(v^{g(v)} v)$, which satisfies $\sum_{j=1}^{g(v)} f(v^j v) + f(v) = \sum_{j=1}^{g(v')-1} f((v')^j v') + f(v')$, where v and v' are as defined in $N_3^-(v_i)$.

   Then, the resulting list $L(v_i v)$, where $v \in N_3^-(v_i)$, has at least $h(G) - 1$ choices left; and for any L-weighting f of this resulting list, we have $\varphi_f(v) \neq \varphi_f(v')$. Also note that $N^-(v') = \emptyset$.

(b) For $v \in N_2^-(v_i)$, from $L(v_i v)$, delete value $f(v_i v) = f(v^{g(v)-1} v)$, which satisfies $\sum_{j=1}^{g(v)-1} f(v^j v) + f(v) = \sum_{j=1}^{g(v')} f((v')^j v') + f(v')$, where v and v' are as defined in $N_2^-(v_i)$.

Then, the resulting list $L(v_i v)$, where $v \in N_2^-(v_i)$, has at least $h(G) - 1$ choices left, and for any L-weighting f of this resulting list, we have $\varphi_f(v) \neq \varphi_f(v')$. Also note that $N^-(v) = \emptyset$.

2. If $|N_1^-(v_i)| = 0$. Then $N_2^-(v_i) \cup N_3^-(v_i) \cup N_4^-(v_i) \neq \emptyset$.
   (a) If $N_2^-(v_i) \cup N_3^-(v_i) \neq \emptyset$.

   Then after Step 1, for $f(v_i v)$, where $v \in N_2^-(v_i) \cup N_3^-(v_i)$, there are at least $(h(G) - 1) - g(v_i) \geq 1$ choices left, such that $\varphi_f(v_i) \neq \varphi_f(v_i^j)$, for all $j = 1, \ldots, g(v_i)$.

   (b) else, we have $N_2^-(v_i) \cup N_3^-(v_i) = \emptyset$ and $N_4^-(v_i) \neq \emptyset$.

   Hence for $f(v_i v)$, where $v \in N_4^-(v_i)$, there are at least $h(G) - g(v_i) \geq 1$ choices left, such that $\varphi_f(v_i) \neq \varphi_f(v_i^j)$, for all $j = 1, \ldots, g(v_i)$.
3. If $|N_1^-(v_i)| = 1$. Assume $N_1^-(v_i) = \{v'\}$.
   (a) If $N^-(v_i) \backslash N_1^-(v_i) \neq \emptyset$, then for $f(v_i v)$, where $v \in N^-(v_i) - N_1^-(v_i)$, there are at least $(h(G) - 1) - 1 = h(G) - 2$ choices left, such that $\sum_{j=1}^{g(v')-1} f((v')^j v') + f(v') \neq \sum_{v \in N(v_i) \backslash v'} f(v v_i) + f(v_i)$. And this will make $\varphi_f(v') \neq \varphi_f(v_i)$.
   (b) If $N^-(v_i) \backslash N_1^-(v_i) = \emptyset$, then we already have

$$\sum_{j=1}^{g(v')-1} f\big((v')^j v'\big) + f\big(v'\big) \neq \sum_{v \in N(v_i) \backslash v'} f(v v_i) + f(v_i). \tag{1}$$

   Indeed, Inequality (1) was guaranteed when we dealt with $v_i^{g(v_i)}$ or $(v')^{g(v')-1}$. To see this, note that if $(v')^{g(v')-1} < v_i^{g(v_i)}$, then $v_i \in N_3^-(v_i^{g(v_i)})$; Step 1 (a) guaranteed Inequality (1) is true. If $v_i^{g(v_i)} \leq (v')^{g(v')-1}$, then $v' \in N_2^-((v')^{g(v')-1})$; Step 1 (b) guaranteed Inequality (1) is true.

   Hence $\varphi_f(v_i) \neq \varphi_f(v')$.
   (c) Finally, choose $f(v_i v') \in L(v_i v')$ such that $\varphi_f(v_i) \neq \varphi_f(v_i^j)$, for $j = 1, \ldots, g(v_i)$; and $\varphi_f(v') \neq \varphi_f((v')^j)$, for $j = 1, \ldots, g(v') - 1$.

   Note that since $|L(v_i v')| = h(G) > g(v_i) + (g(v') - 1)$ (by the definition of $h(G)$), so there is a good choice for $f(v_i v')$.
4. Now $|N_1^-(v_i)| \geq 2$.

   For $v \in N^-(v_i) \backslash N_1^-(v_i)$, we arbitrarily choose $f(v_i v) \in L(v_i v)$.

   For $v \in N_1^-(v_i)$, delete those values from $L(v_i v)$ which satisfies $\varphi_f(v) = \varphi_f(v^j)$, $j = 1, 2, \ldots, g(v) - 1$. Then, the resulting list $L(v_i v)$, where $v \in N_1^-(v_i)$, has at least $h(G) - (g(v) - 1)$ choices left. Since $\{v_i\} \cup N_1^-(v_i)$ induce a star graph, by Lemma 3.1, there are

$$2\big(h(G) - \max_{v \in N_1^-(v_i)} \big(g(v) - 1\big)\big) - 3 \geq 2\big(g(v_i) + 2\big) - 3 = 2g(v_i) + 1 \tag{2}$$

choices, such that $\varphi_f(v_i) \neq \varphi_f(v)$, for $v \in N_1^-(v_i)$; and $\varphi_f(v) \neq \varphi_f(v^j)$, where $v \in N_1^-(v_i)$, $j = 1, 2, \dots, g(v) - 1$.

Now from the above $2g(v_i) + 1$ choices, remove at most $g(v_i)$ possible bad ones, where $\varphi_f(v_i) = \varphi_f(v_i^j)$, for $j = 1, \dots, g(v_i)$; then there are at least $g(v_i) + 1 \geq 1$ good choice(s) left.

Finally if $N^-(v_j) = \emptyset$, for some $1 \leq j \leq n$; then $v_j \in N_1^-(v_j^{g(v_j)})$. Note that in Step 3 and Step 4, we already take such vertices v_j into considerations. This finishes the proof. $\square$

For trees T with $|E(T)| > 1$, since there exists $L \in \Pi(V(T))$, such that $d^+_{T_L}(v_i) \leq 1$, for every $v_i \in V(T)$; therefore $h(T) = 3$.

Corollary 5.2 *Any tree except a single edge is* $(1, 3)$*-total weight choosable.*

Similarly to the proof of Theorem 5.1, we prove the following result.

Theorem 5.3 *Suppose* $\tilde{L} \in \Pi(G)$ *satisfies* $h(G) = h(G_{\tilde{L}})$. *If* $h(G) \geq d^+_{G_{\tilde{L}}}(v_i) + 3$, *for every* $v_i \in V(G)$; *then* G *is* $(1, h(G) - 1)$*-total weight choosable.*

Proof Suppose L is a $(1, h(G) - 1)$-total list assignment of G. The proof of this theorem is the same as Theorem 5.1, except that we need to check Inequality (2) (at Step 4 of the proof of Theorem 5.1), which is updated, because the list size for the edges is now $h(G) - 1$. We distinguish two cases for the proof.

Case 1. $g(v_i) \geq 2$. For the $(1, h(G) - 1)$-total list assignment L, Inequality (2) is updated to the following:

$$2\Big(\big(h(G) - 1\big) - \max_{v \in N_1^-(v_i)} \big(g(v) - 1\big)\Big) - 3 \geq 2\big(g(v_i) + 1\big) - 3 = 2g(v_i) - 1. \quad (3)$$

Then we have $(2g(v_i) - 1) - g(v_i) = g(v_i) - 1 \geq 1$.

Case 2. $g(v_i) \leq 1$. By the condition of the theorem, $h(G) \geq d^+_{G_{\tilde{L}}}(v_i) + 3 = g(v_i) + 3$, for every $v_i \in V(G)$; Therefore we have $g(v_i) - 1 \leq h(G) - 4$; Then Inequality (2) is updated to the following:

$$\begin{aligned} &2\Big(\big(h(G) - 1\big) - \max_{v \in N_1^-(v_i)} \big(g(v) - 1\big)\Big) - 3 \\ &\quad \geq 2\Big(h(G) - 1 - \big(h(G) - 4\big)\Big) - 3 = 3. \end{aligned} \quad (4)$$

Since $g(v_i) \leq 1$, there is still a good choice left. $\square$

Corollary 5.4 *For* $s \geq 2$, s*-degenerate graphs are* $(1, 2s)$*-total weight choosable.*

Proof Suppose $\tilde{L} \in \Pi(G)$ witnesses that graph G is s-degenerate, where $s \geq 2$. Then $\tilde{L}$ witnesses $h(G) \leq 2s + 1$.

If $h(G) \geq s+3$, then $h(G) \geq d^{+}_{G_{\tilde{L}}}(v_i)+3$, for every $v_i \in V(G)$; then by Theorem 5.3, G is $(1, h(G)-1)$-total weight choosable; since $h(G)-1 \leq 2s$, we know G is $(1, 2s)$-total weight choosable.

If $h(G) \leq d^{+}_{G_{\tilde{L}}}(v_i)+2$, then $h(G) \leq s+2 \leq 2s$; by Theorem 5.1, G is $(1, h(G))$-total weight choosable; Therefore G is $(1, 2s)$-total weight choosable. □

Corollary 5.5 *Every planar graph is* $(1, 10)$*-total weight choosable.*

Corollary 5.6 *Every outerplanar graph is* $(1, 4)$*-total weight choosable.*

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