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Neighbor sum distinguishing total colorings via the Combinatorial Nullstellensatz

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Abstract Let $G = (V, E)$ be a graph and ϕ be a total coloring of G by using the color set $\{1, 2, \dots, k\}$. Let $f(v)$ denote the sum of the color of the vertex v and the colors of all incident edges of v. We say that ϕ is neighbor sum distinguishing if for each edge $uv \in E(G)$, $f(u) \neq f(v)$. The smallest number k is called the neighbor sum distinguishing total chromatic number, denoted by $\chi''_{\rm nsd}(G)$. Pilśniak and Woźniak conjectured that for any graph G with at least two vertices, $\chi''_{\rm nsd}(G) \leqslant \Delta(G) + 3$. In this paper, by using the famous Combinatorial Nullstellensatz, we show that $\chi''_{\rm nsd}(G) \leqslant 2\Delta(G) + \mathrm{col}(G) - 1$, where $\mathrm{col}(G)$ is the coloring number of G. Moreover, we prove this assertion in its list version.

Keywords neighbor sum distinguishing total coloring, coloring number, Combinatorial Nullstellensatz, list total coloring

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1 Introduction

The terminology and notation used but undefined in this paper can be found in [2]. Let $G = (V, E)$ be a graph and denote the maximum degree of G by $\Delta(G)$. Let $d_G(v)$ or simply $d(v)$ denote the degree of a vertex v in G.

Given a graph $G = (V, E)$ and a positive integer k, a *total k-coloring* of G is a mapping $\phi : V \cup E \to \{1, 2, \dots, k\}$ such that

(a) $\phi(u) \neq \phi(v)$ for every pair u, v of adjacent vertices;
(b) $\phi(v) \neq \phi(e)$ for every vertex v and every edge e incident with v;
(c) $\phi(e) \neq \phi(e')$ for every pair e, e' of adjacent edges.

Given a total k-coloring ϕ of G, let $C_\phi(v)$ denote the set of colors of the edges incident with v and the color of v. A total k-coloring is called *neighbor set distinguishing* or *adjacent vertex distinguishing* if for each edge uv, $C_\phi(u)$ is different from $C_\phi(v)$. The smallest k is called the *neighbor set distinguishing total chromatic number* or *adjacent vertex distinguishing total chromatic number*, denoted by $\chi''_{\rm nd}(G)$. Zhang [25] proposed that the following conjecture.

Conjecture 1 (See [25])**.** *For any graph G with at least two vertices, $\chi''_{\rm nd}(G) \leqslant \Delta(G) + 3$.*

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Conjecture 1 has been confirmed for subcubic graphs, K_4-minor free graphs and planar graphs with large maximum degree, see [3, 6, 20–22].

Colorings and labellings related to sums of colors have been studied widely. The family of such problems includes e.g., vertex-coloring k-edge-weightings [8], total weight choosability [16,23], magic and antimagic labellings [7, 24] and the irregularity strength [13, 14]. Among them there are the 1-2-3 Conjecture due to Karoński et al. [9] and 1-2 Conjecture due to Przybyło and Woźniak [17]. For more information, see the survey paper [19]. In a total k-coloring of G, let $f(v)$ denote the total sum of colors of the edges incident with v and the color of v. If for each edge uv, $f(u) \neq f(v)$, we call such total k-coloring a *total k-neighbor sum distinguishing coloring*. The smallest number k is called the *neighbor sum distinguishing total chromatic number*, denoted by $\chi''_{\mathrm{nsd}}(G)$. For neighbor sum distinguishing total colorings, we have the following conjecture due to Pilśniak and Woźniak [12].

Conjecture 2 (See [12]). *For any graph G with at least two vertices, $\chi''_{\mathrm{nsd}}(G) \leqslant \Delta(G) + 3$.*

Clearly Conjecture 2 implies Conjecture 1 since it is easy to check that $\chi''_{\mathrm{nd}}(G) \leqslant \chi''_{\mathrm{nsd}}(G)$. Pilśniak and Woźniak [12] proved that Conjecture 2 holds for complete graphs, cycles, bipartite graphs and subcubic graphs. Dong and Wang [5] showed that Conjecture 2 holds for sparse graphs. Li et al. [11] proved that Conjecture 2 holds for K_4-minor free graphs. Later, Li et al. [10] proved that Conjecture 2 holds for planar graphs with maximum degree at least 13. Cheng et al. [4] proved that $\chi''_{\mathrm{nsd}}(G) \leqslant \Delta(G) + 2$ for planar graphs with maximum degree at least 14.

The *coloring number* of a graph G, $\mathrm{col}(G)$, is defined as the least integer k such that G has a vertex enumeration in which each vertex is preceded by fewer than k of its neighbors, hence $\mathrm{col}(G) - 1 \leqslant \Delta(G)$. Sometimes, $\mathrm{col}(G)$ may be much smaller than $\Delta(G)$. For example, it is obvious that $\mathrm{col}(G) \leqslant 6$ for any planar graph G. For a given graph G, let $L_w (w \in V \cup E)$ be a set of lists of real numbers, each of size k. The smallest k for which for any specified collection of such lists there exists a neighbor sum (set) distinguishing total coloring using colors from L_w for each $w \in V \cup E$ is called the neighbor sum (set) distinguishing total choosability of G, and denoted by $ch''_{\mathrm{nsd}}(G)(ch''_{\mathrm{nd}}(G))$. In this paper, we prove the following theorem.

Theorem 1.1. *If G is a graph with at least 2 vertices, then $ch''_{\mathrm{nsd}}(G) \leqslant 2\Delta(G) + \mathrm{col}(G) - 1$. Moreover, if G does not contain a component which is a regular subgraph of degree $\Delta(G)$, then $ch''_{\mathrm{nsd}}(G) \leqslant 2\Delta(G) + \mathrm{col}(G) - 2$.*

Clearly, $ch''_{\mathrm{nd}}(G) \leqslant ch''_{\mathrm{nsd}}(G)$ and $\chi''_{\mathrm{nsd}}(G) \leqslant ch''_{\mathrm{nsd}}(G)$, so the result is also true for $ch''_{\mathrm{nd}}(G)$ and $\chi''_{\mathrm{nsd}}(G)$.

2 Sketch of proof

Our proof is based on iterative applications of the Combinatorial Nullstellensatz, which was inspired by [15, 24]. The following two theorems are the main tools we shall use to prove our result.

Theorem 2.1 (See [1], Combinatorial Nullstellensatz). *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(x_1, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \ldots, x_n]$. Suppose the degree $\deg(P)$ of P equals $\sum_{i=1}^{n} k_i$, where each k_i is a non-negative integer, and suppose the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in P is non-zero. Then if $S_1, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > k_i$, there are $s_1 \in S_1, \ldots, s_n \in S_n$ so that $P(s_1, \ldots, s_n) \neq 0$.*

Theorem 2.2 (See [18]). *If $P(x_1, x_2, \ldots, x_n) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ is of degree $\leqslant s_1 + s_2 + \cdots + s_n$, where $s_1, s_2, \ldots, s_n$ are non-negative integers, then*

$$\begin{aligned}&\left(\frac{\partial}{\partial x_1}\right)^{s_1} \left(\frac{\partial}{\partial x_2}\right)^{s_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{s_n} P(x_1, x_2, \ldots, x_n)\\ &\quad = \sum_{x_1=0}^{s_1} \cdots \sum_{x_n=0}^{s_n} (-1)^{s_1+x_1} \binom{s_1}{x_1} \cdots (-1)^{s_n+x_n} \binom{s_n}{x_n} P(x_1, x_2, \ldots, x_n).\end{aligned}$$

The proof of Theorem 1.1 is inductive. Firstly, we choose a vertex v with $d_G(v) < \mathrm{col}(G)$. Assume that we have been able to provide a satisfactory coloring for each of the components of $G - v$, and we want to extend this partial coloring to whole graph G. Let $e_1 = vv_1, \ldots, e_{k-1} = vv_{k-1}$ be the remaining uncolored edges of G. Undoubtedly, some of the colors cannot be used to paint these edges and vertex v. It is easy to know that the number of colors forbidden for each of these edges is at most $2\Delta - 1$ and the number of colors forbidden for the vertex v is $k-1$ (since the coloring is supposed to be proper and the sum at v_i is supposed to be distinct from the sum of any its neighbors in $V\backslash\{v, v_1, \ldots, v_{k-1}\}$). Let $S_1, \ldots, S_k$ be the lists of the remaining colors available for $e_1, \ldots, e_{k-1}, v$, respectively. All we need to do now is to choose colors from these lists so that they are pairwise distinct, the obtained final sums of all neighbors of v are also pairwise distinct (some of these may be joined by edges in G), and distinct from the obtained sum at v. We associate with each e_i a variable x_i and v a variable x_k. Then it is sufficient to find a color for each x_i from S_i respectively to make the following polynomial non-zero:

$$P(x_1, \ldots, x_k) = \prod_{1 \leqslant i < j \leqslant k} (x_i - x_j) \prod_{1 \leqslant i < j \leqslant k-1} (x_i + w_i - x_j - w_j) \prod_{j=1}^{k-1} \left(\left(\sum_{t=1}^{k} x_t \right) - x_j - w_j \right),$$

where $w_1, \ldots, w_{k-1}$ are partial sums of the vertices, resp., $v_1, \ldots, v_{k-1}$ provided by our initial coloring of the graph $G - v$.

To prove the first part of our theorem, we consider the monomial $x_1^{k-1} \cdots x_k^{k-1}$ in P. We can prove the coefficient of this monomial $x_1^{k-1} \cdots x_k^{k-1}$ in P is non-zero and $|S_i| > k-1$ for $i = 1, 2, \ldots, k$. To prove the second part of our theorem, we consider the monomial $x_1^{k-2} \cdots x_{k-1}^{k-2} x_k^{2k-2}$ in P. We will prove its coefficient is non-zero. We know that $|S_i| \geqslant \mathrm{col}(G) - 1$ $(i = 1, 2, \ldots, k-1)$ and $|S_k| \geqslant 2\Delta - 1$. It is easily verified that if G does not contain a component which is a regular subgraph of degree $\Delta(G)$ then $\mathrm{col}(G) \leqslant \Delta(G)$. Hence, $|S_i| > k-2$ for $i = 1, 2, \ldots, k-1$ and $|S_k| > 2k-2$. Here, we give the following key lemma.

Lemma 2.3. *Let*

$$Q_n(x_1, \ldots, x_n) = \prod_{1 \leqslant i < j \leqslant n-1} (x_i - x_j)^2 \prod_{j=1}^{n-1} (x_j - x_n) \left(\left(\sum_{t=1}^{n} x_t \right) - x_j \right)$$

be a polynomial in n variables, $n \geqslant 2$, and let a and b be the coefficients of the monomial $x_1^{n-1} \cdots x_n^{n-1}$ and $x_1^{n-2} \cdots x_{n-1}^{n-2} x_n^{2n-2}$ in Q_n, respectively. Then $a = (-1)^{\frac{(n-1)(n+2)}{2}} (n-1)!$ and $b = (-1)^{\frac{n(n-1)}{2}} (n-1)!$. Consequently, $a \neq 0$ and $b \neq 0$.

3 The proof of Lemma 2.3

Firstly, we prove $a \neq 0$. Note that the degree of Q_n equals $n(n-1)$. By Theorem 2.2, we have

$$\begin{aligned}
[(n-1)!]^n a &= \left(\frac{\partial}{\partial x_1} \right)^{n-1} \left(\frac{\partial}{\partial x_2} \right)^{n-1} \cdots \left(\frac{\partial}{\partial x_n} \right)^{n-1} Q_n(x_1, \ldots, x_n) \\
&= \sum_{x_1=0}^{n-1} \cdots \sum_{x_n=0}^{n-1} (-1)^{n(n-1)+x_1+\cdots+x_n} \binom{n-1}{x_1} \cdots \binom{n-1}{x_n} Q_n(x_1, \ldots, x_n) \\
&= \sum_{\sigma} (-1)^{\sigma(1)+\cdots+\sigma(n)} \binom{n-1}{\sigma(1)} \cdots \binom{n-1}{\sigma(n)} Q_n(\sigma(1), \ldots, \sigma(n)),
\end{aligned}$$

where the summation runs through all n-permutations of the set $\{0, 1, \ldots, n-1\}$, i.e., all injective mapping $\sigma : \{1, 2, \ldots, n\} \to \{0, 1, \ldots, n-1\}$ (for non-injective ones, the value of Q_n is by its definition equal to 0). Denote the set of all such n-permutations by A, and let $A_i = \{\sigma \in A : \sigma(n) = i\}$ for $i \in \{0, 1, \ldots, n-1\}$. Let $p = \frac{n(n-1)}{2}$. Note that $|A_i| = (n-1)!$. For a given $\sigma \in A_i$, we have

$$\begin{aligned}
&(-1)^{\sigma(1)+\cdots+\sigma(n)} = (-1)^p, \\
&\binom{n-1}{\sigma(1)} \cdots \binom{n-1}{\sigma(n)} = \frac{[(n-1)!]^{n-2}}{[1! \cdots (n-2)!]^2},
\end{aligned}$$

$$\prod_{1\leqslant t<j\leqslant n-1}(\sigma(t)-\sigma(j))^2\prod_{j=1}^{n-1}(\sigma(j)-\sigma(n))=(-1)^i\frac{[1!\cdots(n-1)!]^2}{i!(n-1-i)!},$$

$$\binom{n-1}{\sigma(1)}\cdots\binom{n-1}{\sigma(n)}\prod_{1\leqslant t<j\leqslant n-1}(\sigma(t)-\sigma(j))^2\prod_{j=1}^{n-1}(\sigma(j)-\sigma(n))=(-1)^i[(n-1)!]^{n-1}\binom{n-1}{i},$$

$$\prod_{j=1}^{n-1}\left(\left(\sum_{t=1}^{n}\sigma(t)\right)-\sigma(j)\right)=\frac{\prod_{j=0}^{n-1}(p-j)}{p-i}.$$

For a given real number x, denote $x^{\underline{0}}=1$, and let $x^{\underline{j}}=x(x-1)^{\underline{j-1}}$ for $j=1,2,\ldots$ We may then write

$$[(n-1)!]^n a=[(n-1)!]^n\sum_{i=0}^{n-1}(-1)^{p+i}\binom{n-1}{i}\frac{p^{\underline{n}}}{p-i},$$

i.e.,

$$a=\sum_{i=0}^{n-1}(-1)^{p+i}\binom{n-1}{i}\frac{p^{\underline{n}}}{p-i}.$$

Lemma 3.1. *For all integers n,t with $0\leqslant t\leqslant n-1$, and for each real number x,*

$$\sum_{j=t}^{n-1}(-1)^j j^{\underline{t}}\binom{n-1}{j}x^{\underline{n-1-t}}=0.$$

Proof. For $t=0$, we have

$$\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}x^{\underline{n-1}}=x^{\underline{n-1}}(-1+1)^{n-1}=0.$$

It follows that

$$\begin{aligned}\sum_{j=t}^{n-1}(-1)^j j^{\underline{t}}\binom{n-1}{j}x^{\underline{n-1-t}}&=\sum_{j=t}^{n-1}(-1)^t(-1)^{j-t}(n-1)^{\underline{t}}\binom{n-1-t}{j-t}x^{\underline{n-1-t}}\\&=(-1)^t(n-1)^{\underline{t}}\sum_{r=0}^{n-1-t}(-1)^r\binom{n-1-t}{r}x^{\underline{n-1-t}}\\&=0.\end{aligned}$$ □

Lemma 3.2. *For all integers n,p,t with $0\leqslant t\leqslant n-1$,*

$$\sum_{j=t}^{n-1}(-1)^j j^{\underline{t}}\binom{n-1}{j}\frac{(p-t)^{\underline{n-t}}}{p-j}=\sum_{j=t+1}^{n-1}(-1)^j j^{\underline{t+1}}\binom{n-1}{j}\frac{[p-(t+1)]^{\underline{n-(t+1)}}}{p-j}.$$

Proof.

$$\begin{aligned}\sum_{j=t}^{n-1}(-1)^j j^{\underline{t}}\binom{n-1}{j}\frac{(p-t)^{\underline{n-t}}}{p-j}&=\sum_{j=t}^{n-1}(-1)^j j^{\underline{t}}\binom{n-1}{j}\frac{(p-t)(p-t-1)^{\underline{n-t-1}}}{p-j}\\&=\sum_{j=t}^{n-1}(-1)^j j^{\underline{t}}\binom{n-1}{j}\frac{(p-j+j-t)(p-t-1)^{\underline{n-t-1}}}{p-j}\\&=\sum_{j=t}^{n-1}(-1)^j j^{\underline{t}}\binom{n-1}{j}\frac{(p-j)(p-t-1)^{\underline{n-t-1}}}{p-j}\\&\quad+\sum_{j=t}^{n-1}(-1)^j j^{\underline{t}}\binom{n-1}{j}\frac{(j-t)(p-t-1)^{\underline{n-t-1}}}{p-j}\end{aligned}$$

$$= \sum_{j=t+1}^{n-1} (-1)^j j^{\underline{t+1}} \binom{n-1}{j} \frac{[p-(t+1)]^{\underline{n-(t+1)}}}{p-j},$$

where the last equality follows by the above lemma. □

Lemma 3.3. *For all integers n, p with $n \geqslant 1$,*

$$\sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \frac{p^{\underline{n}}}{p-j} = (-1)^{n-1}(n-1)!.$$

The proof of this lemma follows directly from $n-1$ repeated applications of the above lemma.

Hence, we have $a = (-1)^{\frac{(n-1)(n+2)}{2}}(n-1)!$ and $a \neq 0$.

Now we will prove that $b \neq 0$. For $n = 2$, it is easily verified that $b = -1$. Now we assume $n \geqslant 3$, by Theorem 2.2, we have

$$\begin{aligned}
&(2n-2)![(n-2)!]^{n-1} b \\
&= \left(\frac{\partial}{\partial x_1}\right)^{n-2} \left(\frac{\partial}{\partial x_2}\right)^{n-2} \cdots \left(\frac{\partial}{\partial x_{n-1}}\right)^{n-2} \left(\frac{\partial}{\partial x_n}\right)^{2n-2} Q_n(x_1, x_2, \ldots, x_n) \\
&= \sum_{x_1=0}^{n-2} \cdots \sum_{x_{n-1}=0}^{n-2} \sum_{x_n=0}^{2n-2} (-1)^{n(n-1)+x_1+\cdots+x_n} \binom{n-2}{x_1} \cdots \binom{n-2}{x_{n-1}} \binom{2n-2}{x_n} Q_n(x_1, x_2, \ldots, x_n) \\
&= \sum_{x_n=n-1}^{2n-2} (-1)^{x_n} \binom{2n-2}{x_n} \sum_{\sigma} (-1)^{\sum_{i=1}^{n-1} \sigma(i)} \binom{n-2}{\sigma(1)} \cdots \binom{n-2}{\sigma(n-1)} Q_n(\sigma(1), \sigma(2), \ldots, \sigma(n-1), x_n),
\end{aligned}$$

where the second summation runs through all $(n-1)$-permutations of the set $\{0, 1, \ldots, n-2\}$, i.e., all injective mapping $\sigma : \{1, 2, \ldots, n-1\} \to \{0, 1, \ldots, n-2\}$ (for non-injective ones, the value of Q_n is by its definition equal to 0). Since the second summation runs through all $(n-1)$-permutations of the set $\{0, 1, \ldots, n-2\}$, the first summation can only runs from $x_n = n-1$ to $x_n = 2n-2$ (for $x_n \in \{0, 1, \ldots, n-2\}$, the value of Q_n is by its definition equal to 0). For each σ, we have

$$\begin{aligned}
&\binom{n-2}{\sigma(1)} \cdots \binom{n-2}{\sigma(n-1)} = \binom{n-2}{0} \cdots \binom{n-2}{n-2} = \frac{[(n-2)!]^{n-3}}{[1!2!\cdots(n-3)!]^2}, \\
&Q_n(x_1, \ldots, x_n) = [1!\cdots(n-2)!]^2 \prod_{j=0}^{n-2} (j - x_n) \left(\left(\sum_{i=0}^{n-2} i \right) + x_n - j \right), \\
&(-1)^{\sigma(1)+\cdots+\sigma(n-1)} = (-1)^{\frac{(n-1)(n-2)}{2}}.
\end{aligned}$$

Let $d = \frac{(n-1)(n-2)}{2}$. Hence,

$$(2n-2)![(n-2)!]^{n-1} b = (n-1)![(n-2)!]^{n-1} \sum_{x_n=n-1}^{2n-2} (-1)^{x_n+n-1+d} \binom{2n-2}{x_n} x_n^{\underline{n-1}} (d+x_n)^{\underline{n-1}}.$$

Let $q = d + n - 1$. Then we have

$$(-1)^q (2n-2)! b = (n-1)! \sum_{x_n=n-1}^{2n-2} (-1)^{x_n} \binom{2n-2}{x_n} x_n^{\underline{n-1}} (d+x_n)^{\underline{n-1}}.$$

It can be verified easily that the following equality holds:

$$(n-1)! \binom{2n-2}{x_n} x_n^{\underline{n-1}} = (2n-2)! \binom{n-1}{x_n - n + 1}.$$

Therefore,

$$(-1)^q b = \sum_{x_n=n-1}^{2n-2} (-1)^{x_n} \binom{n-1}{x_n - n + 1} (d+x_n)^{\underline{n-1}}.$$

Let $j = x_n - n + 1$. We have

$$(-1)^{q+n-1}b = \sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}(q+j)^{\underline{n-1}}.$$

Let $j + i = n - 1$. It follows that

$$(-1)^q b = \sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}(q+n-1-i)^{\underline{n-1}} = (n-1)!.$$

The above equality follows by the following lemma.

Lemma 3.4 (See [23]). *For any integers n, t with $n \geqslant t \geqslant 0$, for any real number x*

$$\sum_{j=t}^{n}(-1)^j j^{\underline{t}}\binom{n}{j}(x-j)^{\underline{n-t}} = (-1)^t n!.$$

Hence, we have $b = (-1)^{\frac{n(n-1)}{2}}(n-1)!$ and $b \neq 0$. This completes the whole proof of Lemma 2.3.

4 The proof of Theorem 1.1

Let $G = (V, E)$ be a graph. We prove our assertions by induction with respect to $m = |E|$, the size of G. For the first assertion, we may assume that G is a connected graph. The result is obvious for $m = 1$. We assume that $m \geqslant 2$ and that first assertion holds for graphs with smaller sizes.

Let L_w $(w \in (E \cup V))$ be any given set of lists of reals, each of size $2\Delta(G) + \mathrm{col}(G) - 1$. Among the enumerations $u_1, \ldots, u_n$ of the vertices of V in which every vertex is preceded by fewer than $\mathrm{col}(G)$ neighbors, choose one that maximizes $d_G(u_n)$. Denote $v = u_n$ and $d_G(v) = k - 1$, where $2 \leqslant k \leqslant \mathrm{col}(G)$.

Denote the edges incident with v by $e_1 = vv_1, e_2 = vv_2, \ldots, e_{k-1} = vv_{k-1}$. Let $H_1, \ldots, H_l$ be the components of $G - v$. By induction, for each of these components which are of size at least 1 we can fix a neighbor sum distinguishing total coloring from the given lists. For components consisting of a single vertex, we choose any numbers from the appropriate lists. We shall now extend the obtained coloring to G. For this purpose, associate to each edges e_i incident with v a variable x_i and the vertex v a variable x_k. We shall choose a value for x_i $(i = 1, \ldots, k-1)$ and x_k from L_{e_i} $(i = 1, \ldots, k-1)$ and L_v, respectively, so that the obtained coloring is neighbor sum total distinguishing. To guarantee the coloring is proper, we cannot use the colors of other edges incident with v_i for e_i. For the same reason, we cannot use the color of v_i for e_i and v. Moreover, the sum obtained at v_i must be distinct from the other already fixed sums of the neighbors of v_i. Hence, we may have to forbid $2\Delta - 1$ colors in L_{e_i} and $\mathrm{col}(G) - 1$ colors in L_v. Let $S_i \subseteq L_{e_i}$ and $S_k \subseteq L_v$ be the set of the remaining colors from L_{e_i} and L_v, respectively. Then $|S_i| \geqslant (2\Delta(G) + \mathrm{col}(G) - 1) - (2\Delta(G) - 1) = \mathrm{col}(G) \geqslant k$ for $i = 1, \ldots, k-1$ and $|S_k| \geqslant (2\Delta(G) + \mathrm{col}(G) - 1) - (\mathrm{col}(G) - 1) = 2\Delta(G) \geqslant 2(k-1)$. All we have to do now is to choose distinct values of the variables x_i $(i = 1, \ldots, k)$, $x_i \in S_i$, so that the sums at $v_i's$ are pairwise distinct (in case of adjacency of some of these vertices), and distinct from the sum at v. Let w_i denote the contemporary sum at v_i (not counting the color of e_i) for $i = 1, \ldots, k-1$. Then it is enough to find appropriate substitutions for the variables x_i so that the value of the following polynomial is non-zero:

$$Q(x_1, \ldots, x_k) = \prod_{1 \leqslant i < j \leqslant k}(x_i - x_j)\prod_{1 \leqslant i < j \leqslant k-1}(x_i + w_i - x_j - w_j)\prod_{j=1}^{k-1}\left(\left(\sum_{t=1}^{k}x_t\right) - x_j - w_j\right).$$

By Theorem 2.1, it is enough to show that the coefficient of the monomial $x_1^{k-1}\cdots x_k^{k-1}$ in Q is non-zero. However, this coefficient is the same as the coefficient of the same monomial in

$$Q_k(x_1, \ldots, x_k) = \prod_{1 \leqslant i < j \leqslant k-1}(x_i - x_j)^2\prod_{j=1}^{k-1}(x_j - x_k)\left(\left(\sum_{t=1}^{k}x_t\right) - x_j\right),$$

hence the first assertion follows by Lemma 2.3.

For the second assertion, we may fix a neighbor sum distinguishing coloring from the given lists for each of the regular components by the first assertion (since the maximum degree of each of these components is smaller than $\Delta(G)$). So we may assume that G is a connected non-regular graph. The result is obvious for $m = 2$. Then we assume that $m \geqslant 3$, and that our assertion holds for non-regular graphs with smaller sizes.

Let $G_1, \ldots, G_r$ be the components of $G - v$. For each of the non-regular components, we may fix by induction a neighbor sum distinguishing total coloring from the given lists. For each of the regular components which are of size at least 1, each of their maximum degrees is smaller than $\Delta(G)$. Then we may fix a neighbor sum distinguishing total colorings for each of them from the given lists. We shall choose any number from the given lists for components consisting of a single vertex.

Since G is a non-regular graph, $\mathrm{col}(G) \leqslant \Delta(G)$. So we have that

$$|S_i| \geqslant (2\Delta(G) + \mathrm{col}(G) - 2) - (2\Delta(G) - 1) = \mathrm{col}(G) - 1 \geqslant k - 1,$$
$$|S_k| \geqslant (2\Delta(G) + \mathrm{col}(G) - 2) - (\mathrm{col}(G) - 1) = 2\Delta(G) - 1 \geqslant 2k - 1.$$

Since the coefficient of the monomial $x_1^{k-2} \cdots x_{k-1}^{k-2} x_k^{2k-2}$ in Q is the same as the coefficient of the same monomial in Q_k, the second assertion follows by Lemma 2.3. This completes the whole proof of Theorem 1.1.

Remark 4.1. In this paper, we give an upper bound for the neighbor sum distinguishing total chromatic number (in fact, we consider the neighbor sum distinguishing total choosability). To attack Conjecture 2, we propose the following question.

Question. *Is there a constant C such that $\chi''_{\mathrm{nsd}} \leqslant \Delta(G) + C$?*

It is also very interesting to consider the neighbor sum distinguishing total colorings of random graphs.

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