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On the effective Nullstellensatz

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Dedicated to Professor Arkadiusz Płoski

Abstract. Let $\mathbb{K}$ be an algebraically closed field and let $X \subset \mathbb{K}^m$ be an n-dimensional affine variety. Assume that $f_1, \ldots, f_k$ are polynomials which have no common zeros on X. We estimate the degrees of polynomials $A_i \in \mathbb{K}[X]$ such that $1 = \sum_{i=1}^k A_i f_i$ on X. Our estimate is sharp for $k \le n$ and nearly sharp for $k > n$. Now assume that $f_1, \ldots, f_k$ are polynomials on X. Let $I = (f_1, \ldots, f_k) \subset \mathbb{K}[X]$ be the ideal generated by f_i. It is well-known that there is a number $e(I)$ (the Noether exponent) such that $\sqrt{I}^{e(I)} \subset I$. We give a sharp estimate of $e(I)$ in terms of n, deg X and deg f_i. We also give similar estimates in the projective case. Finally we obtain a result from the elimination theory: if $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$ is a system of polynomials with a finite number of common zeros, then we have the following optimal elimination:

$$\phi_i(x_i) = \sum_{j=1}^{n} f_j g_{ij}, \; i = 1, \ldots, n,$$

where deg $f_j g_{ij} \le \prod_{i=1}^n \deg f_i$.

1. Introduction

In recent years there has been a great deal of interest in the problem of finding effective versions of Hilbert's Nullstellensatz (see e.g. [1], [2], [3], [8], [9], [11]). Let $A = \mathbb{K}[x_1, \ldots, x_n]$ denote the ring of polynomials. The

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Mathematics Subject Classification (1991): 14D06, 14Q20 classical theorem states if polynomials $f_1, \ldots, f_k \in A$ have no common zeros in $\mathbb{K}^n$, then they generate the unit ideal, i.e., there exist $g_j \in A$ such that

$$\sum g_j f_j = 1. \tag{1.1}$$

The problem is to bound the degrees of the g_j in terms of degrees of the f_j. A more general version of the Nullstellensatz states that if a polynomial G vanishes on the zero set of the ideal $I = (f_1, \ldots, f_k) \subset A$, then $G^l \in I$ for some integer $l > 0$. A natural problem is to estimate l in terms of the degrees of the f_j. This leads directly to the problem of determining the so called *Noether exponent* $e(I)$ of the ideal I:

$$e(I) = \min\{\mu : \sqrt{I}^{\mu} \subset I\}$$

in terms of the degrees of generators of the ideal I. All these problems have natural generalization to the case of an arbitrary affine algebra $A = \mathbb{K}[X]$, where X is an affine (or a projective) variety (see e.g [3], [9] and [11]).

The first aim of this paper is to find an estimate of $\deg f_i g_i$ for $f_i, g_i \in A$ in (1.1) as well as to estimate the Noether exponent of the ideal $I \subset A$ for an arbitrary affine algebra $A = \mathbb{K}[X]$. Moreover, our proof gives a simple method of effective computation of the polynomials g_i in (1.1).

We always assume that X is of positive dimension and $f_i|_X \not\equiv 0$. For a sequence $d_1 \geq d_2 \geq \cdots \geq d_k$ let

$$N(d_1, \ldots, d_k; n) = \begin{cases} \prod_{i=1}^{k} d_i & \text{if } n \geq k \geq 1, \\ \left(\prod_{i=1}^{n-1} d_i\right) d_k & \text{if } k > n > 1 \\ d_k & \text{if } n = 1. \end{cases}$$

For technical reasons we also introduce a number $N'(d_1, \ldots, d_k; n) = N(d_1, \ldots, d_k; n)$ for $n > 1$ and $N'(d_1, \ldots, d_k; 1) = d_1$. Our first result is:

Theorem 1.1. *Let $\mathbb{K}$ be an algebraically closed field, and $X \subset \mathbb{K}^m$ be an affine n-dimensional variety of degree D. Let $f_1, \ldots, f_k \in \mathbb{K}[X]$ be non-constant polynomials without common zeros. Assume that* $\deg f_i = d_i$, *where $d_1 \geq \cdots \geq d_k$. There exist polynomials $g_i \in \mathbb{K}[X]$ with*

$$\deg f_i g_i \leq \begin{cases} DN'(d_1, \ldots, d_k; n) & if\ k \leq n, \\ 2DN'(d_1, \ldots, d_k; n) - 1 & if\ k > n, \end{cases}$$

such that $1 = \sum_{i=1}^{k} f_i g_i$.

This result is sharp for $k \leq n$ and nearly sharp in the case $k > n$. Moreover, if $k \leq n$ then our result is sharp even in the particular case $X = \mathbb{K}^n$ (see Example 1.5 b)).

Now we review some previous results on the subject. For $X = \mathbb{K}^n$ Kollár [8] obtained an optimal estimate in almost all cases. He showed that if $d_i \geq 3$ (and $n > 1$), then $\deg f_i g_i \leq N'(d_1, \dots, d_k; n)$, and this estimate is sharp. Moreover Sombra [11] obtained the estimate $\deg f_i g_i \leq 2N'(d_1, \dots, d_k; n)$, which holds for arbitrary d_i. Thus for $k \leq n$ we slightly improve Kollár's bound (we remove the restrictions on the degrees). We also improve Sombra's estimate (especially for $k \leq n$). However, in the case $k > n$ (and $X = \mathbb{K}^n$) our estimate is slightly weaker than the one in [8] (for $d_i \geq 3$).

Remark 1.2. Let $f_1 = \phi(x)$, $f_2 = \psi(x) + y^d$, $f_3 = y \in \mathbb{K}[x, y]$, where $\phi, \psi \in \mathbb{K}[x]$ are sufficiently general polynomials of degree d. If

$$1 = \sum_{i=1}^{3} f_i g_i,$$

then it is easy to see that

$$\max\{\deg f_i g_i\} = 2d - 1 = 2N'(d_1, d_2, d_3; 2) - 1.$$

This means that at least for $X = \mathbb{K}^2$ (and $X = \mathbb{K}$ of course) our estimation is optimal (in some cases) also for $k > n$. □

Now let $X \subset \mathbb{K}^m$ be an arbitrary affine variety of dimension n and of degree D. Let $f_1, \dots, f_k \in \mathbb{K}[X]$ be polynomials of degrees $d_1, \dots, d_k$. Kollár's results in [9] imply the estimate:

$$\deg f_i g_i \leq (m + 1) D N'(d_1, \dots, d_k; n + 2)$$

(where m is the dimension of the ambient space). On the other hand Sombra [11] proved the following estimate (where $d = \max d_i$):

$$\deg f_i g_i \leq (n + 1)^2 D d^{n+1},$$

under the additional assumption that the projective closure of X is a Cohen-Macaulay variety. Theorem 1.1 improves both these results.

Our second result is the following sharp estimate of the Noether exponent in the affine case (for a slightly more exact version of this theorem see Corollary 4.6):

Theorem 1.3. *Let $\mathbb{K}$ be an algebraically closed field and let $X \subset \mathbb{K}^m$ be an affine n-dimensional variety of degree D. Let $f_1, \dots, f_k \in \mathbb{K}[X]$ be non-zero polynomials. Assume that* $\deg f_i = d_i$, *where $d_1 \geq \dots \geq d_k$. If $I = (f_1, \dots, f_k)$, then*

$$\sqrt{I}^{DN(d_1, \dots, d_k; n)} \subset I.$$

For $X = \mathbb{K}^n$ this result was obtained by Kollár in [8] under the additional assumption that $d_i \geq 3$. For an arbitrary affine n-dimensional variety $X \subset \mathbb{K}^m$ Kollár's results from [9] imply the estimate

$$e(I) \leq mDN(d_1, \ldots, d_k; n+1),$$

where m is the dimension of the ambient space. Moreover Sombra [11] obtained the estimate (where $d = \max d_i$)

$$e(I) \leq (n+1)^2 Dd^{n+1},$$

under the additional assumption that the projective closure of X is a Cohen-Macaulay variety. Theorem 1.3 improves both these results.

Theorem 1.3 implies the following (asymptotically sharp) homogeneous version:

Corollary 1.4. *Let $X \subset \mathbb{P}^m$ be a projective n-dimensional variety of degree D. Let $f_1, \ldots, f_k \in \mathbb{K}[X]$ be non-zero homogeneous polynomials and set $I = (f_1, \ldots, f_k)$. Assume that* $\deg f_i = d_i$, *where $d_1 \geq \cdots \geq d_k$. Let I_t denote the degree t part of the homogeneous ideal I (with grading induced from $\mathbb{K}[X]$). Then*

(a) $((\sqrt{I})^{DN(d_1,\ldots,d_k;n)})_t \subset I_t$ *for sufficiently large $t \geq t_0$,*
(b) $(\sqrt{I})^{DN(d_1,\ldots,d_k;n+1)} \subset I$.

Statement (a) of this Corollary seems to be optimal. Assertion (b) is sharp for $k \leq n$.

For $X = \mathbb{P}^n$ Kollár [8] obtained for $d_i \geq 3$ (and $n > 1$) the sharp estimate $e(I) \leq N(d_1, \ldots, d_k; n)$. Thus for $k \leq n$ Corollary 1.4 slightly improves Kollár's result (we remove the assumption that $d_i \geq 3$). However, for $k > n$, $d_i \geq 3$ and $X = \mathbb{P}^n$ Kollár's result is better.

Now let $X \subset \mathbb{P}^m$ be an arbitrary projective variety of dimension n. Ein and Lazarsfeld [3] proved the following estimate (where $d = \max d_i$):

$$e(I) \leq (n+1)Dd^n,$$

under the assumption that X is a smooth complex projective variety and additionally that X is embedded in $\mathbb{P}^m$ in a special way. Moreover Sombra [11] obtained the estimate

$$e(I) \leq (n+1)^2 Dd^{n+1},$$

under the additional assumption that X is a Cohen-Macaulay variety. Corollary 1.4 improves both these results in the case $k \leq n$. If $k > n$ our result improves the estimate of Sombra. However in the case $k > n$ our bound is in some instances slightly weaker than the estimate of Ein-Lazarsfeld (but our assumptions are much weaker).

Example 1.5. (a) We show that Theorem 1.3 is sharp, i.e., for every $D, d_1, \dots, d_k$ there exists an n-dimensional affine variety $X \subset \mathbb{K}^m$ of degree D and polynomials $f_i \in \mathbb{K}[X]$ of degrees $d_1, \dots, d_k$ such that $e((f_1, \dots, f_k)) = DN(d_1, \dots, d_k; n)$. In particular our result is also sharp in the case $X = \mathbb{K}^n$.

First we assume that $X = \mathbb{K}^n$. We use Kollár's Example 2.3 from [8]. Let $k = n$ and set $f_1 = x_1^{d_1}$, $f_2 = x_1 - x_2^{d_2}, \dots, f_{n-1} = x_{n-2} - x_{n-1}^{d_{n-1}}$, $f_n = x_{n-1} - x_n^{d_n}$. Clearly $\deg f_i = d_i$. Moreover we have

$$\mathbb{K}[x_1, \dots, x_n]/(f_1, \dots, f_n) \cong \mathbb{K}[x_n]/\left(x_n^{d_1 d_2 \dots d_n}\right).$$

Therefore we see that x_n is in the radical of $(f_1, \dots, f_n)$, but $x_n^{N(d_1,\dots,d_n;n)-1}$ is not in $(f_1, \dots, f_n)$. If $k \geq n$ then we arrange the d_i in such a way that d_n is the smallest (i.e., we exchange d_n with d_k) and take f_i for $i \leq n$ as before and f_i = appropriate multiple of f_n for $i > n$. If $k < n$, then we can consider the above example with k variables and consider this as polynomial in n variables.

Now, in the general case, for given data n, k, d_i and D we first arrange $f_1, \dots, f_k$ in $\mathbb{K}^n$ as above and then add the polynomial $g = x_n - x_{n+1}^D$. Now we can consider the polynomials $f_1, \dots, f_k$ and g in $\mathbb{K}^{n+1}$. If we take $X = \{x : g(x) = 0\}$ and we consider the f_i and x_{n+1} as polynomials on X, then we see that x_{n+1} is in the radical of $(f_1, \dots, f_k)$, but $x_{n+1}^{DN(d_1,\dots,d_k;n)-1}$ is not in $(f_1, \dots, f_k)$.

(b) We show that Corollary 1.4 and Theorem 1.1 are sharp for $k \leq n$. Indeed, for every $D, d_1, \dots, d_k$ (with $k \leq n$) we construct an n-dimensional projective variety $X \subset \mathbb{P}^m$ of degree D and homogeneous polynomials $f_i \in \mathbb{K}[X]$ of degrees $d_1, \dots, d_k$ such that

$$e(f_1, \dots, f_k) = DN(d_1, \dots, d_k).$$

For simplicity assume that $k = n$. Set $f_1 = x_1^{d_1}$, $f_2 = x_1 x_{n+1}^{d_2-1} - x_2^{d_2}, \dots, f_{n-1} = x_{n-2} x_{n+1}^{d_{n-1}-1} - x_{n-1}^{d_{n-1}}$, $f_n = x_{n-1} x_{n+1}^{d_n-1} - x_n^{d_n}$ and $g = x_{n-1} x_{n+1}^{D-1} - x_0^D$. Now it is enough to take $X = \{x \in \mathbb{P}^{n+1} : g(x) = 0\}$ and regard f_i as elements of $\mathbb{K}[X]$. Putting $x_{n+1} = 1$ we can show (similarly as in (a)) that x_0 is in the radical of $(f_1, \dots, f_k)$, but $x_0^{DN(d_1,\dots,d_k;n)-1}$ is not in $(f_1, \dots, f_k)$. Consequently $e(f_1, \dots, f_k) = DN(d_1, \dots, d_k)$.

Finally we can dehomogenize x_0 to show that Theorem 1.1 is sharp for $k \leq n$. □

The second aim of this paper is to solve the following problem from elimination theory. Let $f_1, \dots, f_n \in \mathbb{K}[x_1, \dots, x_n]$ be polynomials with a finite number of common zeros. A classical method to solve the system of equations

$$f_i = 0, \quad i = 1, \dots, n,$$

is to eliminate variables, i.e., to find non-zero polynomials $\phi_i \in \mathbb{K}[x_i]$ and polynomials $g_{ij} \in \mathbb{K}[x_1, \ldots, x_n]$, such that

$$\phi_i(x_i) = \sum_{j=1}^{n} g_{ij} f_j \text{ for } j = 1, \ldots, n.$$

If the degrees of ϕ_j, g_{ji} are fixed beforehand, we can actually find a solution, because the problem then reduces to solving a $\mathbb{K}$-linear system of equations. From this point of view a fundamental problem here is to find an optimal bound for the degrees of ϕ_j, g_{ji}. In this paper we give such an optimal bound:

Theorem 1.6. *Let $\mathbb{K}$ be an algebraically closed field. Let $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$ be polynomials. Assume that the zeros set of $f_1, \ldots, f_n$ is finite. Then there exist polynomials $g_{ij} \in \mathbb{K}[x_1, \ldots, x_n]$ and non-zero polynomials $\phi_i(x_i) \in \mathbb{K}[x_i]$, such that*

(a) $\deg g_{ij} f_j \leq \prod_{i=1}^{n} \deg f_i$,
(b) $\phi_i(x_i) = \sum_{j=1}^{n} g_{ij} f_j$ *for every* $i = 1, \ldots, n$.

Moreover, in a sufficiently general system of coordinates,

(c) if $\phi_j(a) = 0$ for some $a \in \mathbb{K}$, then there exist $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \in \mathbb{K}$ such that $f_l(a_1, \ldots, a_{j-1}, a, a_{j+1}, \ldots, a_n) = 0$ for $l = 1, \ldots, n$.

A simple application of the Bézout Theorem shows that this result is optimal. For further generalizations of Theorem 1.6 see Theorems 3.10 and 4.3.

Most of our results hold for an arbitrary infinite field $\mathbb{K}$. All our methods are elementary and the paper is nearly self-contained. For further applications of our method see [7].

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2. Terminology

We assume that $\mathbb{K}$ is an algebraically closed field. If $X \subset \mathbb{K}^m$ is an affine variety of codimension k, then by $\deg X$ we mean the number of common points of X and a sufficiently general linear subspace M of dimension k. In particular if $X = \mathbb{K}^m$, then $\deg X = 1$.

If $X \subset \mathbb{K}^m$ is an affine variety then by $\mathbb{K}[X]$ we mean the ring of polynomial functions on X. For a projective variety $X \subset \mathbb{P}^m$ by $\mathbb{K}[X]$ we mean the homogeneous coordinate ring of X (for details see e.g. [4]). If $g \in \mathbb{K}[X]$ is a polynomial, then we put

$$\deg g = \min\{\deg G : G \in \mathbb{K}[x_1, \ldots, x_m] \text{ and } G|_X = g\}.$$

In an analogous way, we define the degree of a homogeneous polynomial on a projective variety. Let $I \subset \mathbb{K}[X]$ be the ideal. We define the zero set of I as $V(I) = \{x : g(x) = 0 \textit{ for every } g \in I\}$.

3. Effective Nullstellensatz

We start with a simple lemma:

Lemma 3.1. *Let $f \in \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial and let $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$. The polynomial f is reduced if and only if ∇f does not vanish identically on any irreducible component of the set $V(f) = \{x : f(x) = 0\}$.*

Proof. If f is reduced then ∇f does not vanish identically on irreducible components of $V(f)$, because the set of smooth points is dense in $V(f)$. On the other hand, if f is not reduced, then $f = g^2 h$ and $\nabla f = g(2\nabla g h + g\nabla h) = 0$ on $V(g)$. □

Corollary 3.2. *Let $f \in \mathbb{K}[x_1, \ldots, x_n]$ be a reduced polynomial. Assume that $V(f)$ does not contain hyperplanes $\{x_i = a_i\}$ (where $a_i \in k$) as irreducible components. Then for any positive integers $m_1, \ldots, m_n$ the polynomial $F = f(T_1^{m_1} + T_1, \ldots, T_n^{m_n} + T_n)$ is also reduced.*

Proof. Indeed,

$$\nabla F = \nabla f \cdot \langle m_1 T_1^{m_1 - 1} + 1, \ldots, m_n T_n^{m_n - 1} + 1\rangle,$$

(where $\cdot$ denotes the scalar product). By the assumption and Lemma 3.1 this product does not vanish on irreducible components of $V(F)$. □

Now we are ready to prove the following generalization of the Perron Theorem (see [10], Satz 57, p. 129, for the classical version):

Theorem 3.3 (Generalized Perron Theorem). *Let $\mathbb{L}$ be a field and let $Q_1, \ldots, Q_{n+1} \in \mathbb{L}[x_1, \ldots, x_m]$ be non-constant polynomials with $\deg Q_i = d_i$. Assume that $X \subset \mathbb{L}^m$ is an affine variety of dimension n and of degree D. If the mapping $Q = (Q_1, \ldots, Q_{n+1}) : X \to \mathbb{L}^{n+1}$ is generically finite, then there exists a non-zero polynomial $W(T_1, \ldots, T_{n+1}) \in \mathbb{L}[T_1, \ldots, T_{n+1}]$ such that*

(a) $W(Q_1, \ldots, Q_{n+1}) = 0$ *on X,*
(b) $\deg W(T_1^{d_1}, T_2^{d_2}, \ldots, T_{n+1}^{d_{n+1}}) \leq D \prod_{j=1}^{n+1} d_j$.

Proof. Without loss of generality we can assume that the field $\mathbb{L}$ is algebraically closed. Let $\tilde{X} = \{(x, w) \in X \times \mathbb{L}^{n+1} : Q_i(x) = w_i^{d_i} + w_i\}$ (if $d_i = 1$ we take $Q_i(x) = w_i$). Let W be an irreducible polynomial such that $W(Q_1, \dots, Q_{n+1}) = 0$ on X and take $P(T_1, \dots, T_{n+1}) = W(T_1^{d_1} + T_1, \dots, T_{n+1}^{d_{n+1}} + T_{n+1})$. Let $Y = \{w \in \mathbb{L}^{n+1} : P(w) = 0\}$. By Corollary 3.2 we have $\deg Y = \deg P$. The sets $\tilde{X}$, Y are affine of pure dimension n. Now consider the mapping

$$\pi : \tilde{X} \ni (x, w) \to w \in Y.$$

It is easy to see that π is a dominant generically finite mapping. Consequently,

$$\deg \pi \deg Y \leq \deg \tilde{X}.$$

By the Bézout Theorem we have $\deg \tilde{X} \leq D \prod_{j=1}^{n+1} d_j$. This finishes the proof. □

Remark 3.4 (due to the referee). Let us note that Theorem 3.3 can also be obtained as the elementary degree estimate one gets by looking at the image of the map given by the Q_i to $\mathbb{L}^{n+1}$, viewed as an affine chart of the weighted projective space $\mathbb{WP}(d_1, \dots, d_{n+1}, 1)$.

The following lemma is obvious:

Lemma 3.5. *Let $\mathbb{K}$ be an infinite field. Let $X \subset \mathbb{K}^m$ be an affine algebraic variety of dimension n. For sufficiently general numbers $a_{ij} \in \mathbb{K}$ the mapping*

$$\pi : X \ni (x_1, \dots, x_m) \to \left(\sum_{j=1}^{m} a_{1j}x_j, \sum_{j=2}^{m} a_{2j}x_j, \dots, \sum_{j=n}^{m} a_{1j}x_j \right) \in \mathbb{K}^n$$

is finite. □

First we prove:

Theorem 3.6. *Let $\mathbb{K}$ be an algebraically closed field and let $f_1, \dots, f_k \in \mathbb{K}[x_1, \dots, x_m]$ be non-zero polynomials. Let $X \subset \mathbb{K}^m$ be an affine algebraic variety of dimension n and of degree D. Assume that $k \leq n$. Let* $\deg f_i = d_i$, *where $d_1 \geq \dots \geq d_k$. If $V(f_1, \dots, f_k) \cap X = \emptyset$, then there exist polynomials g_i, such that*

(a) $\deg f_i g_i \leq DN(d_1, \dots, d_k; n)$,
(b) $1 = \sum_{i=1}^{k} f_i g_i$ *on X.*

Proof. By the assumption there are polynomials A_j such that $\sum_{i=1}^{k} A_i f_i = 1$. Let us consider the mapping

$$\Phi : X \times \mathbb{K} \ni (x, z) \to (x, f_1(x)z, \dots, f_k(x)z) \in \mathbb{K}^m \times \mathbb{K}^k.$$

Then Φ is an embedding. Indeed, denote variables in the space $\mathbb{K}^m \times \mathbb{K}^k$ in such a way that $X_i = x_i$, $W_j = f_j(x)z$. We have $z = \sum_{i=1}^{k} A_i(X) W_i$ and $x_i = X_i$ (on X). This implies that the mapping $\Phi^{-1} : \mathrm{cl}(\Phi(X \times \mathbb{K})) \to X \times \mathbb{K}$ is a well-defined morphism and consequently Φ is an embedding.

In particular the mapping Φ is finite and the set $\Gamma = \Phi(X \times \mathbb{K})$ is closed. Let $\pi : \Gamma \to \mathbb{K}^{n+1}$ be a generic projection. Then π is a finite mapping. Define $\Psi := \pi \circ \Phi(x, z)$. By Lemma 3.5, we can assume that

$$\Psi = \Big(\sum_{j=1}^{n} \gamma_{1j} f_j z + l_1(x), \sum_{j=2}^{n} \gamma_{2j} f_j z + l_2(x), \\ \dots, \sum_{j=n}^{n} \gamma_{nj} f_j z + l_n(x), l_{n+1}(x) \Big),$$

where l_j are linear forms (and $f_j := 0$ for $j > k$). Set $\Psi = (\Psi_1, \dots, \Psi_{n+1})$.

Apply Theorem 3.3 to $\mathbb{L} = \mathbb{K}(z)$, the polynomials $\Psi_1, \dots, \Psi_{n+1} \in \mathbb{L}[x_1, ..., x_m]$ and the variety X considered over $\mathbb{L}$. Thus there exists a non-zero polynomial $W(T_1, \dots, T_{n+1}) \in \mathbb{L}[T_1, \dots, T_{n+1}]$ such that

$$W(\Psi_1, \dots, \Psi_{n+1}) = 0 \quad \text{and}$$

$$\deg W\big(T_1^{d_1}, T_2^{d_2}, \dots, T_k^{d_k}, T_{k+1}, \dots, T_{n+1}\big) \leq D \prod_{j=1}^{k} d_j.$$

Since the coefficients of W are in $\mathbb{K}(z)$, there is a non-zero polynomial $\tilde{W} \in \mathbb{K}[T_1, \dots, T_{n+1}, Y]$ such that

(a) $\tilde{W}(\Psi_1(x, z), \dots, \Psi_{n+1}(x, z), z) = 0$,
(b) $\deg_T \tilde{W}(T_1^{d_1}, T_2^{d_2}, \dots, T_k^{d_k}, T_{k+1}, \dots, T_{n+1}, Y) \leq D \prod_{j=1}^{k} d_j$, where $\deg_T$ denotes the degree with respect to the variables $T = (T_1, \dots, T_{n+1})$.

Since the mapping $\Psi = (\Psi_1, \dots, \Psi_{n+1}) : X \times \mathbb{K} \to \mathbb{K}^{n+1}$ is finite, for every polynomial $H \in \mathbb{K}[X][z]$ there is a minimal (monic with respect to Y) polynomial $P_H(T, Y) \in \mathbb{K}[T_1, \dots, T_{n+1}][Y]$ such that $P_H(\Psi_1, \dots, \Psi_{n+1}, H) = H^r + \sum_{i=1}^{r} b_i(\Psi_1, \dots, \Psi_{n+1}) H^{r-i} = 0$. Set $H = z$. By the minimality of P_z, we have

$$P_z(T, Y) | \tilde{W}(T, Y),$$

in particular

$$\deg_T P_z\big(T_1^{d_1}, T_2^{d_2}, \dots, T_k^{d_k}, T_{k+1}, \dots, T_{n+1}, z\big) \leq D \prod_{j=1}^{k} d_j.$$

Let N be the degree of P_z with respect to Y. Add all terms of the form $z^N Q(x)$ which occur in the expression $P_z(\Psi_1, \dots, \Psi_{n+1}, z)$. It is easy to see that Q is either 1 or of the form $f_1^{s_1} \dots f_k^{s_k} P(x)$, where $\sum s_i > 0$ and $\deg f_1^{s_1} \dots f_k^{s_k} P(x) \le D \prod_{j=1}^k d_j$. Thus we have obtained polynomials g_i such that $1 + \sum f_i g_i = 0$, where $\deg f_i g_i \le D \prod_{j=1}^k d_j$. □

Remark 3.7. In the same way we can show that in the general case ($k > n$) we have the estimate $\deg f_i g_i \le DN(d_1, \dots, d_k; n+1)$. However in that case we will find a better estimate (see Theorem 3.12).

To prove the next result, we have to recall some our former results (see [5], [6]):

Definition 3.8. Let $f : X \to Y$ be a generically finite dominant polynomial mapping of affine varieties. We say that f *is finite at a point* $y \in Y$ if there exists a Zariski open neighborhood U of y such that the mapping $\mathrm{res}_{f^{-1}(U)} f : f^{-1}(U) \to U$ is finite.

Let S_f denote the set of all points $y \in Y$ at which f is not finite. We say that S_f is *the set of non-properness* of the mapping f. We recall our result about this set (see [6] for a short and elementary proof):

Theorem 3.9. *Let X be an affine n-dimensional variety and let $f = (f_1, \dots, f_n) : X \to \mathbb{K}^n$ be a generically finite mapping. Then the set S_f is either empty or is a hypersurface. Moreover, for every polynomial $G \in \mathbb{K}[X]$, if $W_G(T_1, \dots, T_n, t) = \sum_{i=0}^s a_i(T) t^{s-i} \in \mathbb{K}[T, t]$ is an irreducible polynomial and*

$$W_G(f_1, \dots, f_n, G) = 0,$$

then

$$\left\{T \in \mathbb{K}^n : a_0(T) = 0\right\} \subset S_f.$$

□

Now we are in a position to prove the following important:

Theorem 3.10 (Elimination Theorem). *Let $\mathbb{K}$ be an algebraically closed field and $k \le n$. Let $f_1, \dots, f_k \in \mathbb{K}[x_1, \dots, x_m]$ be polynomials with $\deg f_i = d_i$ and let $X \subset \mathbb{K}^m$ be an affine algebraic n-dimensional variety of degree D. Assume that the set $V(f_1, \dots, f_k) \cap X$ is finite. If we take a sufficiently general system of coordinates $(x_1, \dots, x_m)$, then there exist polynomials $g_{ij} \in \mathbb{K}[x_1, \dots, x_m]$ and non-zero polynomials $\phi_i(x_i) \in \mathbb{K}[x_i]$ such that*

(a) $\deg g_{ij} f_j \le DN(d_1, \dots, d_k; n)$,
(b) $\phi_i(x_i) = \sum_{j=1}^k g_{ij} f_j$ *for every* $i = 1, \dots, m$ *(on X),*
(c) *if* $\phi_j(a) = 0$ *for some* $a \in \mathbb{K}$, *then there exist* $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_m \in \mathbb{K}$ *such that* $(a_1, \dots, a_{j-1}, a, a_{j+1}, \dots, a_m) \in X$ *and* $f_l(a_1, \dots, a_{j-1}, a, a_{j+1}, \dots, a_m) = 0$ *for* $l = 1, \dots, k$.

Proof. By Theorem 3.6 we can assume that $k = n$ and $V(f_1, \dots, f_n) \cap X = \{a_1, \dots, a_r\}$ is a finite non-empty set. The mapping

$$\Phi : X \times \mathbb{K} \ni (x, z) \to (x, f_1(x)z, \dots, f_n(x)z) \in \mathbb{K}^m \times \mathbb{K}^n$$

is a (non-closed) embedding outside the set $\{a_1, \dots, a_r\} \times \mathbb{K}$. Take $\Gamma = \mathrm{cl}(\Phi(X \times \mathbb{K}))$. Let $\pi : \Gamma \to \mathbb{K}^{n+1}$ be a generic projection. Define $\Psi := \pi \circ \Phi(x, z)$. By Lemma 3.5 we can assume that

$$\Psi = \Big(\sum_{j=1}^{n} \gamma_{1j} f_j z + l_1(x), \dots, \sum_{j=n}^{n} \gamma_{nj} f_j z + l_n(x), l_{n+1}(x) \Big),$$

where $l_1, \dots, l_{n+1}$ are generic linear form. In particular we can assume that l_{n+1} is the variable x_1 in a generic system of coordinates.

Note that the mapping $\Psi = (\Psi_1, \dots, \Psi_{n+1}) : X \times \mathbb{K} \to \mathbb{K}^{n+1}$ is finite outside the set $\bigcup_{j=1}^{s} \{T \in \mathbb{K}^{n+1} : T_{n+1} = a_{1j}\}$, where a_{1j} is the first coordinate of a_j (recall that we consider a generic system of coordinates in which $x_1 = l_{n+1}$!). In particular the set of non-properness of the mapping Ψ is contained in the hypersurface $S = \{T \in \mathbb{K}^{n+1} : \prod_{j=1}^{s} (T_{n+1} - a_{1j}) = 0\}$. Since the mapping Ψ is finite outside S, for every $H \in \mathbb{K}[x_1, \dots, x_n, z]$ there is a minimal polynomial $P_H(T, Y) \in \mathbb{K}[T_1, \dots, T_{n+1}][Y]$ such that $P_H(\Psi_1, \dots, \Psi_{n+1}, H) = \sum_{i=0}^{r} b_i(\Psi_1, \dots, \Psi_{n+1}) H^{r-i} = 0$ and the coefficient b_0 satisfies $\{T \in \mathbb{K}^{n+1} : b_0(T) = 0\} \subset S$ (see Theorem 3.9). In particular b_0 depends only on T_{n+1}.

Now set $H = z$. Arguing as in the proof of Theorem 3.6 we deduce

$$\deg_T P_z\big(T_1^{d_1}, T_2^{d_2}, \dots, T_n^{d_n}, T_{n+1}, Y\big) \le D \prod_{j=1}^{n} d_j$$

and consequently we obtain the equality $b_0(x_1) + \sum f_i g_i = 0$, where $\deg f_i g_i \le D \prod_{j=1}^{n} d_j$. Set $\phi_1 = b_0$. By the construction the polynomial ϕ_1 has zeros only at $a_{11}, \dots, a_{1s}$.

Further, since the form l_{n+1} is generic, we can find n forms of this type which are linearly independent. Hence in a similar way we can construct polynomials $\phi_i(x_i)$, $i = 2, \dots, m$, as in (b). Moreover, it is easy to check that the ϕ_i also satisfy (c). $\square$

Remark 3.11. If we do not change coordinates, then still there exist polynomials $g_{ij} \in \mathbb{K}[x_1, \dots, x_m]$ and non-zero polynomials $\phi_i(x_i) \in \mathbb{K}[x_i]$ such that

(a) $\deg g_{ij} f_j \le DN(d_1, \dots, d_k; n)$,
(b) $\phi_i(x_i) = \sum_{j=1}^{k} g_{ij} f_j$ for every $i = 1, \dots, m$ (on X)

(see Theorem 4.3 in Sect. 4). However in this case we can lose (c).

Now we are in a position to prove the main result of this section:

Theorem 3.12. *Let $\mathbb{K}$ be an algebraically closed field, and $X \subset \mathbb{K}^m$ be an affine n-dimensional variety of degree D. Let $f_1, \ldots, f_k \in \mathbb{K}[X]$ be non-constant polynomials without common zeros. Assume that* $\deg f_i = d_i$, *where $d_1 \geq \cdots \geq d_k$. There exist polynomials $g_i \in \mathbb{K}[X]$ with*

$$\deg f_i g_i \leq \begin{cases} DN'(d_1, \ldots, d_k; n) & \text{if } k \leq n, \\ 2DN'(d_1, \ldots, d_k; n) - 1 & \text{if } k > n, \end{cases}$$

such that $1 = \sum_{i=1}^{k} f_i g_i$.

Proof. If $f_1, f_2 \in \mathbb{K}[x]$ are two polynomials of one variable of degree at most d without common zeros, then there exist polynomials $g_1, g_2 \in \mathbb{K}[x]$ such that

$$\deg g_i \leq d - 1 \ \text{ and } \ f_1 g_1 + f_2 g_2 = 1. \tag{3.1}$$

This can be easily derived from the properties of the resultant of f, g. Now take polynomials $f_1, \ldots, f_k \in \mathbb{K}[x_1, \ldots, x_m]$ which have no common zeros on X. We can rearrange the f_i in such a way that $d_2 \geq d_3 \geq \cdots \geq d_k \geq d_1$. Using general linear combinations

(3.2)

$$F_1 = f_1, \quad F_i = \sum_{j=i}^{k} \gamma_{ij} f_j, \quad i = 2, \ldots, k \quad (\text{or } F_1 = \sum_{j=i}^{k} \gamma_j f_j \text{ for } n = 1),$$

we can assume that $k \leq n + 1$. If $V(F_1, \ldots, F_n) = \emptyset$, then we are done by Theorem 3.6. Let $k = n + 1$. Thus $V(F_1, \ldots, F_n)$ is a finite non-empty set. If $F'_1, \ldots, F'_n$ is another sufficiently general linear combination (3.2), then $V(F_1, \ldots, F_n) \cap V(F'_1, \ldots, F'_n) = \emptyset$. By the Elimination Theorem (Theorem 3.10) there exist polynomials $g_i, g'_i \in \mathbb{K}[x_1, \ldots, x_n]$ and polynomials $\phi(x_1), \phi'(x_1) \in \mathbb{K}[x_1]$ such that

(a) $\deg F_i g_i \leq DN'(d_1, \ldots, d_k; n)$ and $\deg F'_i g'_i \leq DN'(d_1, \ldots, d_k; n)$,
(b) $\phi(x_1) = \sum_{i=1}^{n} F_i g_i$ and $\phi'(x_1) = \sum_{i=1}^{n} F'_i g'_i$ on X,
(c) the polynomials ϕ and ϕ' have no common zeros (we assume that the system of coordinates is sufficiently general).

Now Theorem 3.12 follows directly from (3.1). □

4. On the Noether exponent

We start with a simple but basic lemma:

Lemma 4.1. *Let $X \subset \mathbb{K}^m$ be an affine variety of dimension $n + 1$. Let G be any polynomial which is non-constant on X. Let $\Pi = (G, L_1, \ldots, L_n) : X \to \mathbb{K}^{n+1}$, where $L_1, \ldots, L_n$ are sufficiently general linear forms, and let*

S_Π denote the set of non-properness of the mapping Π (see Sect. 3). Then there is a polynomial $\rho \in \mathbb{K}[t_1]$ such that

$$S_\Pi = \big\{(t_1, \ldots, t_{n+1}) \in \mathbb{K}^{n+1} : \rho(t_1) = 0\big\}.$$

Proof. Let $l_1, \ldots, l_{n+1}$ be general linear forms. Since G and l_i are algebraically dependent on X, there exists a non-zero polynomial $W(T, T_1, \ldots, T_{n+1}) \in \mathbb{K}[T, T_1, \ldots, T_{n+1}]$ such that we have $W(G, l_1, \ldots, l_{n+1}) = 0$ on X. Take $L_i = l_i - \alpha_i l_{n+1}$ where α_i are sufficiently general. Thus we have a non-trivial relation (where the polynomial ρ is non-zero):

$$l_{n+1}^N \rho(G) + \sum_{j=1}^{N} l_{n+1}^{N-j} A_j(G, L_1, \ldots, L_n) = 0. \tag{4.1}$$

Now let $L = (G, L_1, \ldots, L_n, l_{n+1})$. The mapping $L : X \to \mathbb{K}^{n+2}$ is finite (because $(L_1, \ldots, L_n, l_{n+1})$ is). Let $X' = L(X)$ and consider the projection

$$\pi : X' \ni (x_1, \ldots, x_{n+2}) \to (x_1, \ldots, x_{n+1}) \in \mathbb{K}^{n+1}.$$

By (4.1) there is a polynomial $\rho \in \mathbb{K}[t_1]$ such that

$$S_\pi = \big\{(t_1, \ldots, t_{n+1}) \in \mathbb{K}^{n+1} : \rho(t_1) = 0\big\}.$$

Now it is enough to note that $\Pi = \pi \circ L$ and the mapping L is proper. $\square$

We now state a result we shall frequently use (we omit a simple proof):

Lemma 4.2. *Let X be an affine algebraic n-dimensional variety. Let $f_1, \ldots, f_k \in \mathbb{K}[X]$ be non-zero polynomials and assume that $\deg f_1 \geq \deg f_2 \geq \ldots \geq \deg f_k$. Let $I = (f_1, \ldots, f_k)$. Let $V(I) = \bigcup_{i=1}^m Z_i$ and define $r_i :=$ codim Z_i. Take sufficiently general $\gamma_{ij} \in \mathbb{K}$ and define polynomials $F_1 := f_k$, $F_i := \gamma_{ii} f_i + \cdots + \gamma_{ik} f_k$ for $i \geq 2$. Then for each $i = 1, \ldots, m$ the intersection $V(F_1) \cap \cdots \cap V(F_{r_i})$ along Z_i is proper and $V(I) = V(F_1, \ldots, F_{n+1})$.*

We are thus led to the following strengthening of Theorem 3.10:

Theorem 4.3 (Generalized Elimination Theorem). *Let $X \subset \mathbb{K}^m$ be an affine algebraic n-dimensional variety of degree D. Let $G, f_1, \ldots, f_k \in \mathbb{K}[X]$ be polynomials with $\deg f_i = d_i$. Assume that G is constant on every connected component of the set $V(f_1, \ldots, f_k)$. Then there exist polynomials $g_i \in \mathbb{K}[X]$ and a non-zero polynomial $\phi \in \mathbb{K}[t]$ such that*

(a) $\deg g_j f_j \leq (\deg G) D N(d_1, \ldots, d_k; n)$,
(b) $\phi(G) = \sum_{j=1}^k g_j f_j$.

Proof. We can assume that $G \neq$ const. Taking f_i as in Lemma 4.2 we can assume that $k \leq n$ and $G(V(f_1, \dots, f_k)) = \{a_1, \dots, a_s\}$ is a finite set. The mapping

$$\Phi : X \times \mathbb{K} \ni (x, z) \to (x, f_1(x)z, \dots, f_k(x)z) \in \mathbb{K}^m \times \mathbb{K}^n$$

is an embedding outside the set $\{(x, z) \in X \times \mathbb{K} : \prod_{i=1}^s (G - a_i) = 0\}$. Set $\Gamma = \mathrm{cl}(\Phi(X \times \mathbb{K}))$. Let $\Pi = (G, L_1, \dots, L_n) : \Gamma \to \mathbb{K}^{n+1}$ be as in Lemma 4.1. Define $\Psi := \Pi \circ \Phi(x, z)$. We can assume that

$$\Psi = \Big(G, \sum_{j=1}^{n} \gamma_{1j} f_j z + l_1(x), \dots, \sum_{j=n}^{n} \gamma_{nj} f_j z + l_n(x)\Big),$$

where l_i are generic linear forms (and $f_j := 0$ for $j > k$). Set $\Psi = (\Psi_1, \dots, \Psi_{n+1})$.

By Lemma 4.1 and the remarks above, the mapping $\Psi = (\Psi_1, \dots, \Psi_{n+1}) : X \times \mathbb{K} \to \mathbb{K}^{n+1}$ is finite outside the set $\{T \in \mathbb{K}^{n+1} : \prod_{i=1}^s (T_1 - a_j)\rho(T_1) = 0\}$. In particular the set of non-properness of Ψ is contained in the hypersurface $S = \{T \in \mathbb{K}^{n+1} : \prod_{i=1}^s (T_1 - a_j)\rho(T_1) = 0\}$. Since Ψ is finite outside S, for every $H \in \mathbb{K}[x_1, \dots, x_n, z]$ there is a minimal polynomial $P_H(T, Y) \in \mathbb{K}[T_1, \dots, T_{n+1}][Y]$ such that $P_H(\Psi_1, \dots, \Psi_{n+1}, H) = \sum_{i=0}^r b_i(\Psi_1, \dots, \Psi_{n+1}) H^{r-i} = 0$ and the coefficient b_0 satisfies $\{T \in \mathbb{K}^{n+1} : b_0(T) = 0\} \subset S$. In particular b_0 depends only on T_1.

Now set $H = z$. Let $d = \deg G$. Arguing as in the proof of Theorem 3.6 we deduce

$$\deg_T P_z\big(T_1^d, T_2^{d_1}, \dots, T_k^{d_{k-1}}, T_{k+1}^{d_k}, T_{k+2}, \dots, T_{n+1}, Y\big) \leq (\deg G) D \prod_{j=1}^{k} d_j$$

and consequently we obtain an equality $b_0(G) + \sum f_i g_i = 0$, where $\deg f_i g_i \leq D(\deg G) \prod_{j=1}^k d_j$. Set $\phi = b_0$. □

Definition 4.4. Let $X \subset \mathbb{K}^m$ be an affine variety of dimension n and let $I = (f_1, \dots, f_k) \subset \mathbb{K}[X]$. Further suppose that $G \in \mathbb{K}[X]$ vanishes on $V(f_1, \dots, f_k)$. Take a general linear combination: $F_1 = f_k$, $F_i = \sum_{j=i}^k \gamma_{ij} f_j$, $i = 2, \dots, n$. We put

$$\sigma(X, f_1, \dots, f_k) = \#(V(F_1, \dots, F_n) \setminus V(I)),$$
$$\sigma(X, f_1, \dots, f_k; G) = \#G(V(F_1, \dots, F_n) \setminus V(G)).$$

Additionally, we define

$$\kappa(X, G) = \begin{cases} \#(\mathbb{K} \setminus G(X)) & \text{if } G \neq \text{const on } X \\ 0 & \text{if } G = \text{const on } X. \end{cases}$$

□

Now we are in a position to prove Theorem 1.3. In fact we prove a slightly stronger result:

Theorem 4.5. *Let $X \subset \mathbb{K}^m$ be an affine variety of dimension n and of degree D. Let $G, f_1, \dots, f_k \in \mathbb{K}[X]$ and suppose a polynomial G vanishes on $V(f_1, \dots, f_k)$. Then one can find a natural number*

$$\mu \le DN(d_1, \dots, d_k; n) - \kappa(X, G) - \sigma(X, f_1, \dots, f_k; G)$$

and polynomials $h_i \in \mathbb{K}[X]$ such that

$$G^\mu = \sum_{i=1}^{k} h_i f_i.$$

Proof. Rearranging f_i in the usual way we find that $V(f_1, \dots, f_n) \setminus V(G)$ is a finite set. We can assume that G is non-constant. Thus by Theorem 4.3 there exist polynomials $g_i \in \mathbb{K}[x_1, \dots, x_m]$ and a non-zero polynomial $\phi \in \mathbb{K}[t]$ such that

(a) $\deg g_j f_j \le (\deg G) DN(d_1, \dots, d_k; n)$,
(b) $\phi(G) = \sum_{j=1}^n g_j f_j$.

We have $\phi(G) = G^\mu \prod_{j=1}^s (G - b_j) \in (f_1, \dots, f_n) \subset I = (f_1, \dots, f_k)$ (where $b_j \neq 0$). Moreover, by basic properties of the set of non-properness of a mapping Ψ (see the proof of Theorem 4.3), among b_j there are all values from $\mathbb{K} \setminus G(X)$ and from $G(V(f_1, \dots, f_n) \setminus V(G))$. Since $G - b_j$ is not in any associated prime ideal of I we see (by basic properties of primary decomposition of I) that in fact $G^\mu \in I$. But

$$\begin{aligned}(\mu + \kappa(X, G) + \sigma(X, f_1, \dots, f_k; G))(\deg G)\\ \le (\deg G) DN(d_1, \dots, d_k; n),\end{aligned}$$

hence $\mu + \kappa(X, G) + \sigma(X, f_1, \dots, f_k; G) \le DN(d_1, \dots, d_k; n)$. □

Corollary 4.6. *Let $X \subset \mathbb{K}^m$ be an affine variety of dimension n and of degree D. Let $f_1, \dots, f_k \in \mathbb{K}[X]$ and $I = (f_1, \dots, f_k) \subset \mathbb{K}[X]$. Assume that* $\deg f_i = d_i$. *Then*

$$(\sqrt{I})^\mu \subset I, \text{ where } \mu \le DN(d_1, \dots, d_k; n) - \sigma(X, f_1, \dots, f_k).$$

Proof. Let $\mathbf{a}, \mathbf{b} \subset R$ be two ideals of an affine $\mathbb{K}$-algebra R. Let $\{h_1, \dots, h_s\} \subset \mathbf{a}$ be a finite set of polynomials which contains a system of generators of $\mathbf{a}$. Consider a general linear form $l(c) = c_1 h_1 + \cdots + c_s h_s$, where $c_i \in \mathbb{K}$. We show first that if $l(c)^\mu \in \mathbf{b}$ for generic $(c_1, \dots, c_s)$, then $\mathbf{a}^\mu \subset \mathbf{b}$.

Indeed, if $l(c)^\mu \in \mathbf{b}$ for generic $c = (c_1, \dots, c_s)$, then using the Cramer rules we can compute every expression $h_{i_1} h_{i_2} \dots h_{i_\mu}$ as a $\mathbb{K}$-linear combination of elements of $\mathbf{b}$. This means that $\mathbf{a}^\mu \subset \mathbf{b}$.

Now take a general linear combination: $F_1 = f_k$, $F_i = \sum_{j=i}^k \gamma_{ij} f_j$, $i = 2, \dots, n$. Let $V(F_1, \dots, F_n) \setminus V(I) = \{a_1, \dots, a_r\}$. Since the a_i are not in

$V(I)$ there exist polynomials $H_i \in I$ such that $H_i(a_i) \neq 0$. Multiplying H_i by a suitable polynomial we can assume that $H_i(a_j) = 0$ for $j \neq i$.

Let $h_1, \dots, h_s$ be generators of $\sqrt{I}$ and consider a linear form $l(c) = c_1h_1 + \cdots + c_sh_s + d_1H_1 + \cdots + d_rH_r$, where $c_i, d_j \in \mathbb{K}$. Observe that for a generic $c = (c_1, \dots, c_s, d_1, \dots, d_r)$ we have $\sigma(X, f_1, \dots, f_k; l(c)) = \sigma(X, f_1, \dots, f_k)$. Now Corollary 4.6 follows directly from Theorem 4.5. □

Moreover, we have:

Corollary 4.7. *Let $X \subset \mathbb{K}^m$ be an affine n-dimensional variety of degree D. Let $f_1, \dots, f_k \in \mathbb{K}[X]$ be non-zero polynomials and set $I = (f_1, \dots, f_k)$. Assume that* $\deg f_i = d_i$, *where $d_2 \geq d_3 \geq \dots \geq d_k \geq d_1$. Let $I = \bigcap_{j=1}^s q_j$ be a primary decomposition of I. Then for every $i = 1, \dots, n$ we have:*

$$\text{if} \bigcap_{\mathrm{ht}(q_j)\leq i} q_j \neq \emptyset, \ \text{then } (\sqrt{I})^{Dd_1\dots d_i} \subset \bigcap_{\mathrm{ht}(q_j)\leq i} q_j.$$

Proof. By Lemma 4.2 we can assume that $\bigcup_{\mathrm{codim}\ Z_j \leq i} Z_j \subset V(f_1, \dots, f_i)$. Moreover, there exists a polynomial H, which does not vanish on any component or embedded component of codimension $\leq i$ of $V(I)$, such that for any $G \in \sqrt{I}$ the polynomial GH vanishes on $V(f_1, \dots, f_i)$. By Theorem 4.5 we have $(GH)^{Dd_1\dots d_i} \in (f_1, \dots, f_i) \subset I = \bigcap_{j=1}^s q_j$. Furthermore, if $\mathrm{ht}(q_j) \leq i$, then $H \notin \sqrt{q_i}$. Consequently, $G^{Dd_1\dots d_i} \in \bigcap_{\mathrm{ht}(q_j)\leq i} q_j$. Now arguing as in the proof of Corollary 4.6 we obtain $(\sqrt{I})^{Dd_1\dots d_i} \subset \bigcap_{\mathrm{ht}(q_j)\leq i} q_j$. □

Our next aim is to prove a projective version of Corollary 4.6:

Corollary 4.8. *Let $X \subset \mathbb{P}^m$ be a projective n-dimensional variety of degree D. Let $f_1, \dots, f_k \in \mathbb{K}[X]$ be non-zero homogeneous polynomials and set $I = (f_1, \dots, f_k)$. Assume that* $\deg f_i = d_i$. *Let $I = \bigcap_{i=1}^s q_i \cap J$ be a primary decomposition of I, and $\sqrt{q_i} \neq (x_0, \dots, x_m)$, $\sqrt{J} = (x_0, \dots, x_m)$. If $e(J)$ is an integer such that $(x_0, \dots, x_m)^{e(J)} \subset J$, then for every $t \geq e(J)$ we have*

$$((\sqrt{I})^{DN(d_1,\dots,d_k;n)})_t \subset I_t,$$

where I_t denotes the degree t part of the homogeneous ideal I (with grading induced from $\mathbb{K}[X]$). Moreover, for $I \neq J$ we have $(\sqrt{I})^{DN(d_1,\dots,d_k;n)} \subset \bigcap_{i=1}^s q_i$. Furthermore, we always have

$$(\sqrt{I})^{DN(d_1,\dots,d_k;n+1)} \subset I.$$

Proof. We can treat X as an affine $n+1$-dimensional cone in $\mathbb{K}^{m+1}$. Moreover, we can assume that $I \neq J$. Note that among primary ideals $q_1, \dots, q_s, J$ only the ideal J has height $n+1$. By Corollary 4.7 this means that $(\sqrt{I})^{DN(d_1,\dots,d_k;n)} \subset \bigcap_{i=1}^{s} q_i$. This gives $((\sqrt{I})^{DN(d_1,\dots,d_k;n)})_t \subset I_t$ for $t \geq e(J)$. Finally, the last statement follows directly from Corollary 4.6. □

From the proof above we have:

Corollary 4.9. *With the preceding notation, if* $\mu = \max(e(J), DN(d_1, \dots, d_k; n))$, *then*

$$(\sqrt{I})^{\mu} \subset I.$$

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