On the Ultimate Complexity of Factorials

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Abstract. It has long been observed that certain factorization algorithms provide a way to write product of a lot of integers succinctly. In this paper, we study the problem of representing the product of *all* integers from 1 to n ($n!$) by straight-line programs. Formally, we say that a sequence of integers a_n is ultimately $f(n)$-computable, if there exists a nonzero integer sequence m_n such that for any n, $a_n m_n$ can be computed by a straight-line program (using only additions, subtractions and multiplications) of length at most $f(n)$. Shub and Smale [12] showed that if $n!$ is ultimately hard to compute, then algebraic version of $NP \neq P$ is true. Assuming a widely believed number theory conjecture concerning smooth numbers in short interval, a subexponential upper bound ($exp(c\sqrt{\log n \log\log n})$) for the ultimate complexity of $n!$ is proved in this paper, and a random subexponential algorithm constructing such a short straight-line program is presented as well.

Classification of Topics: Computational and structural complexity.

1 Introduction

Computing the factorial function ($n!$) is an important problem in computational complexity research. Due to the size of $n!$, computing it certainly takes exponential time in the Turing model. One can instead study the modular factorial $n! \bmod m$. Given an integer m, the smallest α such that $\gcd(\alpha! \bmod m, m) > 1$ is the smallest prime factor of m. For every $n \geq \alpha$, $\gcd(n! \bmod m, m)$ is greater than 1, hence we can use binary search to find α if we know how to compute $n! \bmod m$ efficiently for any m and n. This shows that the integer factorization problem can be reduced to computing $n! \bmod m$. It is very interesting to compare modular exponentiation with modular factorial. In some sense, the reason that primality testing is easy while factoring is hard is because modular exponentiation is easy but modular factorial is hard. This statement may underestimate the complexity of modular factorial, as it is believed that computing $n! \bmod m$ is much harder than the integer factorization problem. We don't even know whether computing modular factorial is an NP-easy problem or not.

One approach we may take to compute $n! \bmod m$ is to find a short straight-line program for $n!$. This problem relates to the algebraic model of computation [2,1,3]. In the algebraic model, it makes sense to ask whether the factorial problem has a polynomial time algorithm, because in this context, we only count

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the number of ring operations used to compute $n!$ in the time complexity, regardless of the size of the operands. Sometimes, algebraic complexity is also called non-scalar complexity. If the function $n!$ has polynomial time algebraic complexity, or equivalently, every integer $n!$ has a short straight-line program uniformly, then by doing modulo m in every step, we obtain a polynomial time algorithm to compute $n! \bmod m$ in the Turing model, which is thought to be unlikely. Throughout this paper, we assume that a straight-line program only contains ring operations. Shamir [11] showed that if division (computing remainder and quotient) is allowed, then $n!$ can be computed by a straight-line program of polynomial length.

The ultimate complexity of a number was first studied in [12] by Shub and Smale. They found a surprising relation between the ultimate complexity of $n!$ and the algebraic version of NP vs. P problem. We say that $n!$ is ultimately hard to compute, if there does not exist a non-zero integer sequence m_n, such that $n!m_n$ can be computed by straight-line programs of length $(\log n)^c$ for an absolute constant c. It was proved in [12]:

> *If $n!$ is ultimately hard to compute, then the algebraic version of $NP \neq P$ is true.*

Note that in the Turing model, proving that the modular factorial problem is hard does not necessarily imply that $NP \neq P$. There is no corresponding concept of the ultimate complexity in the Turing model.

So far the best algorithm we know computes $n!$ in $O(\sqrt{n}\log^2 n)$ ring operations over $\mathbf{Z}$ [14,4]. No better upper bound has been reported for the ultimate complexity of $n!$. It has long been noticed that certain factorization algorithms provide a way to write product of a lot of primes succinctly. For instance, Lenstra's elliptic curve factorization method [10], performs algebraic computation modulo the integer to be factored, but the operations remain the same for all inputs of certain size. The algebraic computation essentially generates a number with a lot of prime factors, since it factors almost all integers of the size. However, these algorithms do not directly give us a straight-line program to compute a product of $n!$, because

1. Divisions are essential in these algorithms. For instance, in the elliptic curve factorization method, splitting of an integer n happens precisely when taking inverse of a integer modulo n cannot process.
2. More importantly, all the fast factorization algorithms are random in nature. The time complexity of a random algorithm is an *average* measurement. Practically the algorithms should work as expected, but theoretically it is possible that for any integer n, there are bad choices of random bits such that the algorithm will take exponential time to stop. Hence for any choice of random bits of certain length, there are integers which cannot be factored.

In this paper we give a formal proof of a subexponential upper bound for the ultimate complexity of $n!$ under a widely believed number theory conjecture. The essential part of the proof is still based on Lenstra's elliptic curve factorization

method. Our result is constructive in the sense that we can construct the straight-line program from n in subexponential time. More precisely, our paper presents a Monte Carlo random algorithm (certainly in the Turing model), given a natural number n as input, output a straight-line program which computes a non-zero multiple of $n!$ with probability better than a constant. The algorithm runs in subexponential time, hence the output straight-line program will have at most a subexponential length. Our result suggests that the ultimate complexity of $n!$ is not as high as the complexity of $n!$. This result also shows that the complexity of certain multiple of $n!$ is much closer to the integer factorization problem than the complexity of $n!$ itself.

It is interesting to note that we don't know whether there exists a subexponential straight line program for the polynomial $(x-1)\cdots(x-n)$, or any polynomial with a lot of distinct integral roots. If we apply the same technique in the paper to construct straight line program, we encounter obstacles from the Uniform Boundedness Theorem [6].

Let $\mathcal{E}$ be an elliptic curve $y^2 = x^3 + ax + b$ with $a, b \in \mathbf{Z}$ and $P_s(x)$ be the univariate s-th division polynomial of $\mathcal{E}$. Given n, $P_n(x)$ can be computed by a straight-line program of length $O(\log n)$ using 1, x, a and b as constants. If x, a and b are integers less than n, then $P_n(x)$ can be calculated by $O(\log n)$ arithmetic operations using 1 as the only constant. Let x be an integer which is not the x-coordinate of a torsion on $\mathcal{E}$, i.e. $P_i(x) \neq 0$ for any positive integer i. For any prime p, we have $p|P_s(x)$ if s is divisible by $|E(\mathbf{F}_p)|$, where E is the reduction of $\mathcal{E}$ at p and x is the x-coordinate of a point on $E(\mathbf{F}_p)$ (x may or may not be an x-coordinate of a point on $\mathcal{E}(\mathbf{Q})$).

If the reduction of $\mathcal{E}$ at a random prime p takes a random number between $p - 2\sqrt{p} + 1$ and $p + 2\sqrt{p} + 1$ as the order over $\mathbf{F}_q$, then with probability greater than 1 over a subexponential function on $\log p$, the reduction curve has a smooth order over $\mathbf{F}_p$. Furthermore, given an elliptic curve $E/\mathbf{F}_p$, a random integer x becomes an x-coordinate of a point on $E(\mathbf{F}_p)$ with a constant probability (about $1/2$). Hence if S is a large smooth number and x is an arbitrary integer, $P_S(x)$ contains a lot of distinct prime factors. In order to get a multiple of $n!$, we only need to collect subexponentially many elliptic curves and evaluate their S-th division polynomials at polynomially many integers. We will show that randomly chosen elliptic curves and integers suffice. The effects of the global torsions will be carefully controlled.

1.1 Main Results

We call a number smooth if all of its prime factors are small. More precisely, a number is said to be y-smooth, if all of its prime factors are less than y. Let

$$\Psi(x, y) = |\{n \leq x, n \; is \; y - smooth\}|.$$

Throughout this paper, let $L_x(\beta, c)$ denote $e^{c \log^{\beta} x \log\log^{1-\beta} x}$. A fundamental theorem about $\Psi(x, L_x(1/2, a))$ was proved in [5].

Proposition 1. *For any constant a, $\Psi(x, L_x(1/2, a)) = xL_x(1/2, -1/(2a) + o(1))$.*

It was conjectured that the smooth number in some short interval is as dense as in a large interval. In particular,

Conjecture 1. For any constant $a > 0$,

$$\Psi(p+1+2\sqrt{p}, L_p(1/2, a)) - \Psi(p+1-2\sqrt{p}, L_p(1/2, a)) = \sqrt{p}L_p(1/2, -1/(2a) + o(1)).$$

Though this conjecture has not been proved yet, it is widely believed to be true. See [10,9] for details. In fact, Lenstra's elliptic curve factorization algorithm relies on this conjecture to achieve the subexponential time complexity.

Theorem 1. *Assume that Conjecture 1 is true. There exist absolute constants c_1 and c_2 such that for any natural number n, a non-zero multiple of $n!$ can be computed by a straight-line program of length at most $L_n(1/2, c_1)$. Furthermore, the straight-line program can be constructed in time $L_n(1/2, c_2)$ by a probabilistic Turing machine.*

In this paper, we focus on the theoretical aspects of the problem and do not consider the practical performance. This paper is organized as follows. In Section 2, we define the straight-line program and the ultimate complexity, and prove a lemma about bipartite graphs. In Section 3, we review some facts about elliptic curves. In Section 4, we prove the main theorem. We conclude this paper by a discussion section.

2 Preliminaries

A straight-line program of an integer is a sequence of ring operations, which outputs the integer in the last operation. Formally

Definition 1. *A straight-line program of an integer m is a sequence of instructions*

$$z \leftarrow x \alpha y$$

where $\alpha \in \{+, -, *\}$, x, y are two previously appeared symbols or 1 and z is a new symbol, such that after we execute the instructions sequentially, the last symbol will represent the value of m. The length of the program is the number of instructions. The length of the shortest straight-line program of m is called the straight-line complexity of m.

An integer n has a straight-line complexity at most $2 \log n$. In some cases, a straight-line program is a very compact description of an integer. It can represent a huge number in small length. For example, the number n^m can be computed using the repeated squaring technique and hence has a straight-line complexity at most $2 \log n + 2 \log m$.

Definition 2. *A sequence of integer a_n is ultimately $f(n)$-computable, if there exists a nonzero integer sequence m_n such that $a_n m_n$ can be computed by a straight-line program of length at most $f(n)$. The smallest $f(n)$ is called the ultimate complexity of a_n.*

In this paper, we study the ultimate complexity of $n!$. First we show that this problem can be reduced to studying the ultimate complexity of the product of primes up to n.

Lemma 1. *Let p_n be the n-th prime number. If the sequence $\alpha_n = p_1 p_2 \cdots p_m$, where p_m is the largest prime less than or equal to n, can be ultimately computed by a straight-line program of length $f(n)$, then $n!$ can be ultimately computed by a straight-line program of length $f(n) + 2\log n$.*

Proof. This follows from a simple fact that $n! | \alpha_n^n$. Note that the exponent n is minimum possible.

2.1 Lemma about Bipartite Graphs

Given a bipartite graph $G = X \times Y$, we say that a subset $A \subseteq X$ dominates a subset $B \subseteq Y$, if every vertex in B is adjacent to at least one vertex in A.

Lemma 2. *For a simple undirected bipartite graph $X \times Y$, let $m = |X|$ and $n = |Y|$. If every vertex in X has degree greater than $d = \lceil n/r \rceil$, $r > 2$, then there exists a subset $S \subseteq Y$, with cardinality $g = \lceil 2r \log m \rceil$, which dominates X. Moreover if randomly choose a subset of Y with cardinality g, it dominates X with probability greater than $1 - \frac{1}{m}$.*

Proof. From $X \times Y$, we construct a new bipartite graph $X \times \mathcal{Y}$ as follows. $\mathcal{Y}$ is the set of all the subsets of Y with g elements. For any $u \in X$ and $v \in \mathcal{Y}$, u and v are joined by an edge iff in $X \times Y$, u is adjacent to at least one vertex in $v \subseteq Y$.

For every $u \in X$, its degree in $X \times \mathcal{Y}$ is greater than $\binom{n}{g} - \binom{n-d}{g}$. The total number of edges in $X \times \mathcal{Y}$ is thus greater than $m(\binom{n}{g} - \binom{n-d}{g})$. The average degree of elements in $\mathcal{Y}$ is greater than

$$\frac{m(\binom{n}{g} - \binom{n-d}{g})}{\binom{n}{g}} = m(1 - \binom{n-d}{g} / \binom{n}{g}).$$

We have

$$\begin{aligned}
\binom{n-d}{g} / \binom{n}{g} &= \frac{(n-d)!/(n-d-g)!}{n!/(n-g)!} \\
&= \frac{(n-d)(n-d-1)\cdots(n-d-g+1)}{n(n-1)\cdots(n-g+1)} \\
&< (1 - \frac{d}{n})^g < (1 - \frac{1}{r})^{2r\log m} \\
&< \frac{1}{m^2}.
\end{aligned}$$

Suppose that $x|\mathcal{Y}|$ vertices in $\mathcal{Y}$ have degree less than m. The average degree of vertices in $\mathcal{Y}$ is less than $m(1-x)+(m-1)x = m-x$. Hence $m-x > m(1-\frac{1}{m^2})$. This implies that $x < \frac{1}{m}$.

This lemma will be used several times in the paper. First we present a simple consequence of the lemma.

Corollary 1. *Let p be a prime. If we randomly pick $n = \lceil 6 \log p \rceil$ integers $a_1, a_2, \cdots, a_n$ between 2 and p inclusive, then with probability at least $1 - \frac{2 \log p}{p}$, for every prime q, $2 < q \leq p$, at least one of integers in $\{a_1, a_2, \cdots, a_n\}$ is a quadratic nonresidue modulo q.*

Proof. For every prime q, $2 < q \leq p$, at most $p/3$ of the integers between 2 and p inclusive have prime factor q. For the rest of integers, half of them are quadratic nonresidues modulo q. Hence at least $p/3$ of the integers in the same range are quadratic nonresidues modulo q. By replacing r with 3 in Lemma 2 we get the corollary. Note that there are about $\frac{p}{\log p}$ primes less than p.

3 Elliptic Curves

An elliptic curve is a smooth cubic curve. Let k be a field. If the characteristic of k is not 2 or 3, we may assume that the elliptic curve is given by an equation of the form

$$y^2 = x^3 + ax + b, \qquad a, b \in k.$$

The discriminant of this curve is defined as $-16(4a^3 + 27b^2)$, whose essential part is the discriminant of the polynomial $x^3 + ax + b$. It should be non-zero as the curve is smooth. For detailed information about elliptic curves, we refer to Silverman's book [13].

The set of points on an elliptic curve consists of the solution set of the definition equation plus a point at infinity. These points form an abelian group with the infinity point as the identity. We call a point *torsion* if it has a finite order in the group. The x-coordinates of the torsions of order $n > 3$ are the solutions of $P_n^{\mathcal{E}}(x)$, the n-th division polynomial of E. These polynomials can be computed recursively. Sometimes we omit the superscription $\mathcal{E}$ if there is no confusion about the curve. We can verify that the recursion for $P_n(x)$ does not involve any division. By applying the technique similar to the repeated squaring, we get

Proposition 2. *For any integers $n(> 0)$ and x, the integer $P_n^{\mathcal{E}}(x)$ can be computed by a straight-line program of length $O(\log n + \log(|x| + 1) + \log(|a| + 1) + \log(|b| + 1))$, where $\mathcal{E}$ is the elliptic curve $y^2 = x^3 + ax + b$ with $a, b \in \mathbf{Z}$.*

See [6] for the proof of (a stronger version of) the proposition.

Proposition 3. *Let $\mathcal{E} : y^2 = x^3 + ax + b$ be an elliptic curve defined over $\mathbf{Z}$. Assume that p doesn't divide the discriminant. If x is an integer and*

1. *it is the x-coordinate of a point on $E(\mathbf{F}_p)$,*
2. *the point $(x, \sqrt{x^3+ax+b})$ is not a torsion on $\mathcal{E}$,*

then $P_l(x) \neq 0$ and $p|P_l(x)$, where l is any non-zero multiple of $|E(\mathbf{F}_p)|$.

Let $\mathcal{E} : y^2 = x^3+ax+b$ be an elliptic curve defined over $\mathbf{Z}$. The torsion points on $\mathcal{E}$ with integral x-coordinates (thus y-coordinates are integers or quadratic algebraic numbers) have order at most 18, as shown in the celebrated Uniform Boundedness Theorem in the quadratic number fields [7,8]. Hence such integers must be the roots of $P_n(x)$ with $n \leq 18$, or of x^3+ax+b. The maximal possible roots of those equations are bounded by the sum of the degrees of the equations, which is an absolute constant. Let B denote this constant. Define

$$R_{\mathcal{E}}(p) = \{x | x \in \mathbf{Z}, 1 \leq x \leq p, (x, \sqrt{x^3+ax+b}) \text{ is not a torsion on } \mathcal{E}\}.$$

Then $|R_{\mathcal{E}}(p)| \geq p - B$. Given an integer, we can decide whether the integer is in $R_{\mathcal{E}}(p)$ in polynomial time. From Lemma 2, we conclude

Corollary 2. *Let p be a prime and $\mathcal{E} : y^2 = x^3+ax+b$ be an elliptic curve defined over $\mathbf{Z}$ with $1 \leq a \leq p-1$ and $1 \leq b \leq p-1$. If $n = \lceil 6 \log p \rceil$ integers $x_1, x_2, \cdots, x_n$ are randomly chosen from $R_{\mathcal{E}}(p)$, then with probability greater than $1 - \frac{2 \log p}{p}$, for any prime q satisfying $7B < q \leq p$ and $q \not| 4a^3+27b^2$, one of x_i is the x-coordinate of a point on the reduction of $\mathcal{E}$ at q.*

Proof. We construct a bipartite graph $P \times Y$ as follows. The set P consists of all the prime numbers from $7B$ to p which are not the prime factors of $4a^3+27b^2$. Let $Y = R_{\mathcal{E}}(p)$. For any $q \in P$ and $x \in Y$, draw an edge between q and x iff x^3+ax+b is a quadratic residue modulo q.

The degree of every element in P is greater than $(\frac{q-2\sqrt{q}}{2} - B) \times \frac{p}{q} > \frac{q}{3} \times \frac{p}{q} = \frac{p}{3}$ for $q > 7B$. The theorem now follows from Lemma 2.

The j-invariant of the curve $y^2 = x^3+ax+b$ is defined as $j = 1728 \frac{4a^3}{4a^3+27b^2}$. Two elliptic curves with a same j-invariant are isomorphic over the algebraic closed field. For elliptic curves defined over a prime finite field $\mathbf{F}_p$, $p > 3$, two curves with a same j-invariant may not be isomorphic. If $j \neq 0$ or 1728, there are exactly two isomorphic classes which have the same j-invariant, one can be represented by $y^2 = x^3+kx+k$ and the other by $y^2 = x^3+c^2kx+c^3k$, where $k = \frac{27j}{4(1728-j)}$ and c is a quadratic nonresidue modulo p. There are different number of points over the two classes of curves. There are at most 6 isomorphic classes with $j = 0$, and at most 4 isomorphic classes with $j = 1728$.

We are interested in counting the number of non-isomorphic elliptic curves with the number of points coming from a given set. In [10] the following proposition was proved.

Proposition 4. *There exist two constants c_1, c_2 such that if A is a set of integers between $p+1-\sqrt{p}$ and $p+1+\sqrt{p}$, the number of non-isomorphic classes of elliptic curves defined over $\mathbf{F}_p$ whose number of points over $\mathbf{F}_p$ are in A is*

$$c_1 \sqrt{p}(|A|-2)/\log p \leq N \leq c_2 \sqrt{p}|A| \log p (\log \log p)^2.$$

4 Proof of the Main Theorem

Our goal is to construct a straight-line program of some multiple of $\alpha_p = 2 \times 3 \times 5 \times \cdots \times p$ in $L_p(1/2, c_1)$ time for some constant c_1. Firstly, we compute a number $S = 2^{e_1} \times 3^{e_2} \cdots \times p_s^{e_s}$, where p_s is the maximal prime less than or equal to $L_p(1/2, 1)$ and for every $1 \leq i \leq s$, $p_i^{e_i}$ is the least p_i-power greater than $p + 1 + 2\sqrt{p}$. Certainly we can compute S in time $L_p(1/2, 2 + o(1))$.

Secondly, we randomly choose $l = \lceil 6 \log p \rceil$ integers $c_1, c_2, \cdots, c_l$ between 2 and p inclusive. The step is successful if for every prime $2 < q \leq p$, at least one of the integers is a quadratic nonresidue mod q. The step succeeds with probability greater than $1 - \frac{2 \log p}{p}$ according to Lemma 1.

Denote by $\mathbf{D}$ the set of elliptic curve $\{y^2 = x^3 + ax + a | 1 \leq a \leq p\} \cup \{y^2 = x^3 + ac_i^2 x + ac_i^3 | 1 \leq i \leq l, 1 \leq a \leq p\}$. Construct a bipartite graph $X \times \mathbf{D}$ as follows. X consists of all the primes between $7B + 1$ and p inclusive. For any prime $q \in X$ and any elliptic curve $\mathcal{E} \in \mathbf{D}$, connect q and $\mathcal{E}$ by an edge iff the reduction curve E of $\mathcal{E}$ at q is non-singular, and the order of $E(\mathbf{F}_q)$ is $L_p(1/2, 1)$-smooth.

Lemma 3. *The degree of every element in X is greater than $pL_p(1/2, -1/2 + o(1))$ under Conjecture 1.*

Proof. For any prime $7B < q \leq p$, consider the subset of $\mathbf{D}$:

$$\mathbf{D}_q = \{y^2 = x^3 + ax + a | 1 \leq a \leq q\} \cup \{y^2 = x^3 + ac_i^2 x + ac_i^3 | 1 \leq i \leq l, 1 \leq a < q\}.$$

The j-invariants of $y^2 = x^3 + ax + a$ and $y^2 = x^3 + ac_i^2 x + ac_i^3$ are $1728 \frac{4a}{4a+27}$. If one of integers in $\{c_1, c_2, \cdots, c_n\}$ is a quadratic nonresidue modulo q, then there exist representations of all the isomorphic classes of elliptic curves over $\mathbf{F}_q$ in $\mathbf{D}_q$, except for the curves with j-invariants 0 or 1728. There are at least $\frac{\sqrt{q}}{L_q(1/2,1/2+o(1))}$ $L_q(1/2, 1)$-smooth integers between $q - 2\sqrt{q} + 1$ and $q + 2\sqrt{q} + 1$. Hence there are at least $\sqrt{q} \frac{\sqrt{q}}{L_q(1/2,1/2+o(1))} = \frac{q}{L_q(1/2,1/2+o(1))}$ curves in $\mathbf{D}_q$ which have $L_q(1/2, 1)$-smooth orders over $\mathbf{F}_q$ according to Proposition 4. In the set $\mathbf{D}$, at least $\frac{q}{L_q(1/2,1/2+o(1))} \frac{p}{q} > \frac{p}{L_p(1/2,1/2+o(1))}$ curves have $L_p(1/2, 1)$-smooth order over $\mathbf{F}_q$. Hence the degree of q in $X \times \mathbf{D}$ is greater than $\frac{p}{L_p(1/2,1/2+o(1))}$.

Now we proceed to the third step. We randomly choose $w = \lceil L_p(1/2, 1) \rceil$ curves $\mathcal{E}_1, \cdots, \mathcal{E}_w$ from D. The step is successful if for any prime $7B \leq q \leq p$, q doesn't divide discriminant of at least one of the curves in $\{\mathcal{E}_1, \cdots, \mathcal{E}_w\}$ and the reduction of this curve at q has a $L_p(1/2, 1)$-smooth order over $\mathbf{F}_q$. In the other words, in graph $X \times \mathbf{D}$, $\{\mathcal{E}_1, \cdots, \mathcal{E}_w\} \subseteq \mathbf{D}$ dominates X. Since $L_p(1/2, 1) > 2 \log p L_p(1/2, 1/2 + o(1))$, the step succeeds with probability at least $1 - \frac{2 \log p}{p}$ according to Lemma 2 and Lemma 3.

In the fourth step, for each $1 \leq i \leq w$, we pick $h = \lceil 6 \log p \rceil$ random integers $x_{i,1}, x_{i,2}, \cdots, x_{i,h}$ in $R_{\mathcal{E}_i}(p)$. The i-th sub-step is successful, if for any prime $7B < q \leq p$, at least one integer in $\{x_{i,1}, x_{i,2}, \cdots, x_{i,h}\}$ is the x-coordinate of a $\mathbf{F}_q$-point in the reduction curve of $\mathcal{E}_i$ at q. The successful probability for each

sub-step is greater than $1-\frac{2\log p}{p}$ according to Corollary 2. Hence the successfully probability for this step is greater than $(1-\frac{2\log p}{p})^w$.

Lemma 4. *All these four steps are successful with probability*

$$(1-\frac{2\log p}{p})^{L_p(1/2,1/2+o(1))} > 1/3.$$

If all the four steps are successful, then we can get a multiple of α_p by evaluating the S-th division polynomials of $\mathcal{E}_1, \cdots, \mathcal{E}_w$ on $x_{1,1}, x_{1,2}, \cdots, x_{1,h}; \cdots; x_{w,1}, \cdots, x_{w,h}$ respectively and multiplying the results together. Now we are ready to write the straight-line program for a multiple of $2 \times 3 \times 5 \times \cdots \times p$.

1. Start by computing the product of all the primes less than $7B$. Let the result be T_1.
2. Add instructions to compute
$$P_S^{\mathcal{E}_1}(x_{1,1}), \cdots, P_S^{\mathcal{E}_1}(x_{1,h}); \cdots; P_S^{\mathcal{E}_w}(x_{w,1}), \cdots, P_S^{\mathcal{E}_w}(x_{w,h}).$$
3. Add instructions to compute
$$T_2 \leftarrow \prod_{1\le i\le w, 1\le k\le h} P_S^{\mathcal{E}_i}(x_{i,k}).$$
4. Add $T \leftarrow T_1 \times T_2$ into the straight-line program.

Based on the analysis in this paper, it can be verified that the above straight-line program computes α_p ultimately and it has subexponential length.

5 Discussion

The relation between ultimate complexity and integer factorization can be further explored.

Firstly, can we derive a factorization algorithm from a straight-line program for a multiple of $n!$? The only problem here is that the multiple of $n!$, i.e. $n!m_n$, may contain primes greater that n. We must try to restrict the integer m_n such that it only has primes less than n. It seems hard to do this with the algorithm in this paper.

Secondly, is the lower bound of the ultimate complexity of $n!$ also subexponential? Since this problem is closely related to the integer factorization problem, which is believed not having a polynomial time algorithm, we suspect that the answer to this question is positive.

The existence of a short straight-line program for a large number does not imply that we can construct the short straight-line program in reasonable time. Given two integers m, n and a prime p, if m is the generator of $\mathbf{F}_p^*$ and $p \not| n$, then there exists a short straight-line program for a power of m which is congruent to n modulo p. But we don't know how to construct such a straight-line program from m, n and p, as the problem is equivalent to computing the discrete logarithm problem over $\mathbf{F}_p$. We believe that it may be possible that for some n, $n!$ or a multiple of $n!$ have very short straight-line programs, however constructing the program would be very hard.

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