# Transmission line temperature modeling

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**Abstract**

This document derives an approximate line temperature equation that is second-order in angle difference. This makes it compatible with the temporal instanton optimization formulation. Numerical experimentation indicates that the derived equation matches the IEEE 738 standard closely, and where the two models differ, the approximate equation is conservative.

# 1 Approximate line temperature equation

## 1.1 The heat balance equation

The change in temperature of any object may be expressed as a differential equation. This equation, called the heat balance equation, relates change in temperature to a sum of various sources of heating. For a transmission line, these sources are:

1. Resistive heating, also known as $I^2R$ losses. This heat source is a function of power flow on the line. In general, the line's resistance varies with temperature, making this calculation more complicated than it appears.
2. Convective heating, which is directly proportional to the temperature difference between the line and the surrounding air. If the line is hotter than the surrounding air, it will "give off heat" via convection – particles moving away from the line will carry energy with them.
3. Radiative heating. This heating is related to line and ambient temperatures raised to the fourth power. Physically, thermal radiation is the process by which thermal energy is converted to electromagnetic energy. It takes place in all objects whose absolute temperature is greater than zero.
4. Solar irradiation. The sun imparts energy to the transmission line according. The magnitude of this energy transfer is a function of clouds, geometry, and the line's insulation material. Reflective insulation admits very little solar energy, while black insulation results in significant solar thermal heating.

These heat sources are summed together in the heat balance equation, which tells us how quickly (and in what direction) the transmission line's temperature will change:

$$\frac{dT}{dt} = \frac{1}{m \cdot C_p} \left[ I^2 \cdot R(T) - q_c - q_r + q_s \right] \tag{1}$$

In this equation, $T$ is the conductor average temperature, $I^2 \cdot R(T)$ represents line losses, $q_c$ is the convective heat rate, $q_r$ is the radiative heat rate, and $q_s$ is the solar heat rate.

1. In the temporal instanton setting, $I^2 \cdot R(T)$ is replaced by $f_{ij}^{\text{loss}}$, the DC approximate line losses.
2. The convection heat rate is expressed as

$$\eta_c \cdot (T - T_{\text{amb}}) \ , \tag{2}$$

where $T_{\text{amb}}$ is the ambient temperature (of surrounding air).
3. The radiation heat rate is given by

$$\eta_r \cdot \left[ (T + 273)^4 - (T_{\text{amb}} + 273)^4 \right] \tag{3}$$

4. The solar heat rate is fixed to some conservative constant (representing full direct sun, for example).

## 1.2 Linearization of radiation heat rate

When (1) is combined with a given initial temperature $T_0$, the resulting initial value problem makes it possible to determine the temperature at some later time. Let's substitute the DC loss approximation, (2), and (3) into (1) and attempt to solve for temperature:

$$\frac{dT}{dt} = \frac{1}{mC_p} \left[ f_{ij}^{\text{loss}} - \eta_c \left( T(t) - T_{\text{amb}} \right) - \eta_r \left( (T(t) + 273)^4 - (T_{\text{amb}} + 273)^4 \right) + q_s \right] \tag{4}$$

Suppose that power flow, ambient temperature, and solar heat rate are constant during the temperature change calculation. Then there are only two variable terms on the right-hand side: one first-order, one fourth-order. The fourth-order term (which corresponds to radiation) makes this equation difficult to solve. Fortunately, this term is approximately linear over the temperature range we are interested in. The figure below shows how we might linearize the radiation heat rate conservatively[1].

The green trace is the linearization of $q_r$ about $T_{\text{mid}}$, the midpoint between the ambient and limit temperatures. Let's replace $q_r$ by this linear approximation $\tilde{q}_r$:

$$\frac{dT}{dt} = \frac{1}{mC_p} \left[ f_{ij}^{\text{loss}} - \eta_c \left( T(t) - T_{\text{amb}} \right) - \tilde{q}_r + q_s \right] \tag{5}$$

where

$$\tilde{q}_r = \eta_r \left( (T_{\text{mid}} + 273)^4 - (T_{\text{amb}} + 273)^4 \right) + 4\eta_r (T_{\text{mid}} + 273)^3 \cdot (T(t) - T_{\text{mid}}) \tag{6}$$

[1] Because a transmission line is typically at a higher temperature than surrounding air, radiation tends to decrease line temperature. Thus, a conservative approach will underestimate the radiation heat rate, leading to slightly higher temperatures.

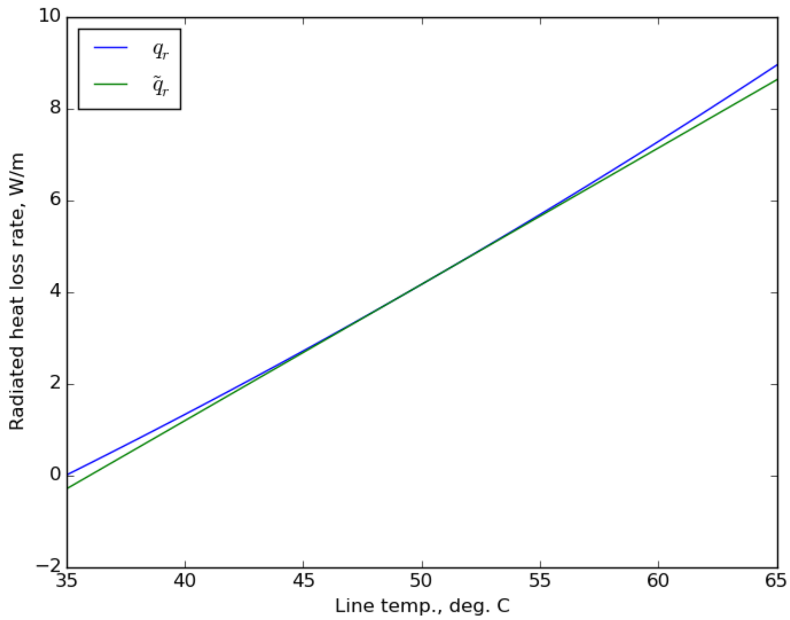


Figure 1: Comparison of fourth-order radiation heat rate (3) with a conservative linearization across a range of temperatures. Note that the ambient temperature is 35°C.

## 1.3 Line temperature as IVP solution

After simplification, (5) becomes

$$\frac{dT}{dt} = aT(t) + b \tag{7}$$

where constants $a$ and $b$ are defined as

$$a = \frac{1}{mC_p}\left[-\eta_c - 4\eta_r(T_{\text{mid}} + 273)^3\right] \tag{8a}$$

$$b = \frac{1}{mC_p}\left[f_{ij}^{\text{loss}} + \eta_c T_{\text{amb}} - \eta_r\left((T_{\text{mid}} + 273)^4 - (T_{\text{amb}} + 273)^4\right) + 4\eta_r T_{\text{mid}}(T_{\text{mid}} + 273)^3 + q_s\right] \tag{8b}$$

The differential equation (7) has a straightforward solution:

$$T(t) = ke^{at} - \frac{b}{a} \tag{9}$$

where $k = T(0) + b/a$. Note that $b$ is influenced by power flow (via $f_{ij}^{\text{loss}}$), but $a$ is not.

## 2 From temperature equation to optimization constraint

The important thing to note about the temperature equation (9) is that it is influenced only by initial temperature and the angle difference during each time interval. There is therefore a recursive relationship between final temperature and initial temperature that involves only angle differences.

### 2.1 Recursive relationship between final and initial temperatures

Suppose there are three time intervals: $t_1$, $t_2$, and $t_3$. Then the line temperature at the end of the third interval is given by

$$\begin{aligned} T(t_3) &= k_3 e^{(t_3-t_2)a} - \frac{b_3}{a} \\ &= \left[ k_2 e^{(t_2-t_1)a} - \frac{b_2}{a} + \frac{b_3}{a} \right] e^{(t_3-t_2)a} - \frac{b_3}{a} \\ &= \left[ \left( T(t_1) + \frac{b_2}{a} \right) e^{(t_2-t_1)a} - \frac{b_2}{a} + \frac{b_3}{a} \right] e^{(t_3-t_2)a} - \frac{b_3}{a} \\ &= \left\{ \left[ \left( T(t_0) + \frac{b_1}{a} \right) e^{(t_1-t_0)a} - \frac{b_1}{a} + \frac{b_2}{a} \right] e^{(t_2-t_1)a} - \frac{b_2}{a} + \frac{b_3}{a} \right\} e^{(t_3-t_2)a} - \frac{b_3}{a} \end{aligned} \tag{10}$$

The recursive pattern is apparent. Because all time intervals are the same length, we have

$$e^{(t_3-t_2)a} = e^{(t_2-t_1)a} = e^{(t_1-t_0)a} \ ,$$

and we can distribute in (10) to obtain

$$T(t_3) = (e^{(t_1-t_0)a})^3 \left( T(t_0) + \frac{b_1}{a} \right) + (e^{(t_1-t_0)a})^2 \left( -\frac{b_1}{a} + \frac{b_2}{a} \right) + (e^{(t_1-t_0)a}) \left( -\frac{b_2}{a} + \frac{b_3}{a} \right) - \frac{b_3}{a} \tag{11}$$

Because power flow data enters through $b_1$, $b_2$, and $b_3$, it makes sense to group terms accordingly:

$$\begin{aligned} T(t_3) = ((e^{(t_1-t_0)a})^3 T(t_0) + \left( \frac{(e^{(t_1-t_0)a})^3}{a} - \frac{(e^{(t_1-t_0)a})^2}{a} \right) b_1 + \\ + \left( \frac{(e^{(t_1-t_0)a})^2}{a} - \frac{(e^{(t_1-t_0)a})^1}{a} \right) b_2 + \left( \frac{e^{(t_1-t_0)a}}{a} - \frac{1}{a} \right) b_3 \end{aligned} \tag{12}$$

The pattern in the above expression makes it easy to extend $T(t_3)$ to cover the general case $T(t_n)$:

$$T(t_n) = (e^{(t_1-t_0)a})^n T(t_0) + \frac{1}{a} \sum_{i=1}^{n} \left( (e^{(t_1-t_0)a})^i - (e^{(t_1-t_0)a})^{i-1} \right) b_{n-i+1} \tag{13}$$

Ultimately (13) will be used to constrain a line's final temperature to some limiting value. The remainder of this section will relate (13) back to power flow angles so it can be "plugged in" to the optimization framework.

## 2.2 Relating temperature to angle differences

Recall that $b_n$ depends on the angle difference at time $t_n$:

$$b_n = \frac{1}{mC_p}\left[\frac{r_{ij}}{x_{ij}^2} \cdot \frac{S_b}{3L_{ij}}\theta_{ij}(t_n)^2 + \eta_c T_{\text{amb}} - \eta_r \left((T_{\text{mid}}+273)^4 - (T_{\text{amb}}+273)^4\right) + 4\eta_r T_{\text{mid}}(T_{\text{mid}}+273)^3 + q_s\right]$$

$$b_n = c\theta_{ij}(t_n)^2 + d \tag{14}$$

where constants $c$ and $d$ are defined as:

$$c = \frac{r_{ij}S_b}{3mC_p x_{ij}^2 L_{ij}}$$

$$d = \frac{1}{mC_p}\left[\eta_c T_{\text{amb}} - \eta_r \left((T_{\text{mid}}+273)^4 - (T_{\text{amb}}+273)^4\right) + 4\eta_r T_{\text{mid}}(T_{\text{mid}}+273)^3 + q_s\right]$$

Substitute (14) into (13):

$$T(t_n) = (e^{(t_1-t_0)a})^n T(t_0) + \frac{1}{a}\sum_{i=1}^{n}\left((e^{(t_1-t_0)a})^i - (e^{(t_1-t_0)a})^{i-1}\right)(c\theta_{ij}(t_{n-i+1})^2 + d)$$

Expand the sum term:

$$\frac{1}{a}\sum_{i=1}^{n}\left((e^{(t_1-t_0)a})^i - (e^{(t_1-t_0)a})^{i-1}\right)(c\theta_{ij}(t_{n-i+1})^2 + d) = \frac{c}{a}\left[\sum_{i=1}^{n}\left((e^{(t_1-t_0)a})^i - (e^{(t_1-t_0)a})^{i-1}\right)\theta_{ij}(t_{n-i+1})^2\right] + \\ + \frac{d}{a}\left[\sum_{i=1}^{n}\left((e^{(t_1-t_0)a})^i - (e^{(t_1-t_0)a})^{i-1}\right)\right] \tag{15}$$

Substitute (15) into the line temperature equation, introducing the constant $f$ to keep things a bit neater:

$$T(t_n) = f + \frac{c}{a}\left[\sum_{i=1}^{n}\left((e^{(t_1-t_0)a})^i - (e^{(t_1-t_0)a})^{i-1}\right)\theta_{ij}(t_{n-i+1})^2\right] \tag{16}$$

where

$$f = (e^{(t_1-t_0)a})^n T(t_0) + \frac{d}{a}\left[\sum_{i=1}^{n}\left((e^{(t_1-t_0)a})^i - (e^{(t_1-t_0)a})^{i-1}\right)\right] \tag{17}$$

Rearrange (16) to isolate angle differences:

$$\sum_{i=1}^{n}\left((e^{(t_1-t_0)a})^i - (e^{(t_1-t_0)a})^{i-1}\right)\theta_{ij}(t_{n-i+1})^2 = \frac{a}{c}(T(t_n) - f) \tag{18}$$

Now define

$$\hat{\theta}_{ij}(t_k) = \theta_{ij}(t_k)\sqrt{(e^{(t_1-t_0)a})^{n-k+1} - (e^{(t_1-t_0)a})^{n-k}} \tag{19}$$

to obtain

$$\sum_{k=1}^{n} \hat{\theta}_{ij}(t_k)^2 = \frac{a}{c}\left(T(t_n) - f\right) \tag{20}$$

The expression (20) may be used to constrain line temperature to be equal to some limiting value by the end of the last time interval.

This derivation has been somewhat messy. The line temperature constraint is summarized in the next section for convenience.

## 2.3 Summary of line temperature constraint

Suppose our time horizon consists of $n$ intervals, each on the order of ten minutes long. Power flow data is updated after each interval, but all other parameters remain constant. Choose a single transmission line in the network, and suppose it lies between nodes $i$ and $j$. This line has a thermal limit of $T_{\text{lim}}$ (°C). We can constrain the line's temperature to be equal to this limiting value by enforcing the second-order constraint:

$$\sum_{k=1}^{n} \hat{\theta}_{ij}(t_k)^2 = \frac{a}{c}\left(T_{\text{lim}} - f\right) \tag{21}$$

where

- $\hat{\theta}_{ij}(t_k) = \theta_{ij}(t_k)\sqrt{(e^{(t_1-t_0)a})^{n-k+1} - (e^{(t_1-t_0)a})^{n-k}}$
  - $\theta_{ij}(t_k)$ is the angle difference across line $ij$ at time interval $t_k$.
  - $(t_1 - t_0)$ is the length of each time interval (in seconds)
- $a = \frac{1}{mC_p}\left[-\eta_c - 4\eta_r(T_{\text{mid}} + 273)^3\right]$ is constant with units of $s^{-1}$
  - $mC_p$ is the heat capacity, with units of J/m-°C
  - $\eta_c$ is the conductive heat loss rate coefficient, with units of W/m-°C
  - $\eta_r$ is the conductive heat loss rate coefficient, with units of W/m-°C$^4$
  - $T_{\text{mid}}$ is the average of ambient temperature $T_{\text{amb}}$ and limit temperature $T_{\text{lim}}$, in Celsius
- $c = \frac{r_{ij}S_b}{3mC_p x_{ij}^2 L_{ij}}$ is a constant with units of W/m
  - $r_{ij}$ is resistance of line $ij$ in per unit
  - $x_{ij}$ is reactance of line $ij$ in per unit
  - $S_b$ is the system base (e.g. 100 MVA)
  - $L_{ij}$ is the length of one phase of line $ij$, in m
- $f = (e^{(t_1-t_0)a})^n T(t_0) + \frac{d}{a}\left[\sum_{i=1}^{n}\left((e^{(t_1-t_0)a})^i - (e^{(t_1-t_0)a})^{i-1}\right)\right]$ is a constant with units of degrees Celsius
  - $T(t_0)$ is the line's initial temperature (steady state temperature under base case power flow condition)
  - $d = \frac{1}{mC_p}\left[\eta_c T_{\text{amb}} - \eta_r\left((T_{\text{mid}} + 273)^4 - (T_{\text{amb}} + 273)^4\right) + 4\eta_r T_{\text{mid}}(T_{\text{mid}} + 273)^3 + q_s\right]$ is a constant with units of W/m
    - $q_s$ is the solar heat gain rate in W/m

# 3 Numerical comparison to IEEE 738 standard model

To validate the approximate line temperature model derived here, I compared it to the IEEE 738 standard model using RTS-96 and Waxwing conductor parameters.

## 3.1 Summary of IEEE 738 temperature calculation

IEEE recommends numerically integrating (1) to compute temperature changes. The temporal instanton framework uses approximate DC losses in place of $I^2R(T_{\text{avg}})$, so we will be integrating the following heat balance equation:

$$\frac{dT_{\text{avg}}}{dt} = \frac{1}{mC_p}\left(r_{ij}\frac{\theta_{ij}^2}{x_{ij}^2}\frac{S_b}{3L_{ij}} - q_c - q_r + q_s\right) \tag{22}$$

Heat rates $q_c$ and $q_r$ are calculated according to (2) and (3), respectively (copied here for convenience):

$$\begin{aligned} q_c &= \eta_c \cdot (T - T_{\text{amb}}) \\ q_r &= \eta_r \cdot ((T + 273)^4 - (T_{\text{amb}} + 273)^4) \end{aligned}$$

All other parameters are constant during temperature calculation. The important thing to keep in mind about IEEE 738 temperature calculation is that it requires numerical integration; there is no analytic temperature solution for (22). This means one must select an integration time step $\Delta t$. For each step, one computes the change in temperature by multiplying $\Delta t$ by the value of (22) computed at that step. IEEE recommends a step size smaller than 10% of the conductor thermal time constant[2]. A smaller integration step size yields more accurate results.

## 3.2 Comparison

I used RTS-96 network data and Waxwing conductor parameters to compare IEEE 738 standard temperature calculation to the model derived in Section 1. Figure 2 shows line temperatures calculated across three ten-minute time intervals, where each interval has a different angle difference (power flow). The angle differences are

| Interval | Angle difference (rad) |
|---|---|
| 1 | 0.09 |
| 2 | 0.04 |
| 3 | 0.15 |

The blue trace in Figure 2 results from numerical integration of (22) with a 1s step size. The green trace comes from the approximate model derived in Section 1. The green dot is the final temperature computed by (16).

Notes:

[2] A typical transmission line thermal time constant is ten minutes, which means IEEE recommends an integration step size of one minute

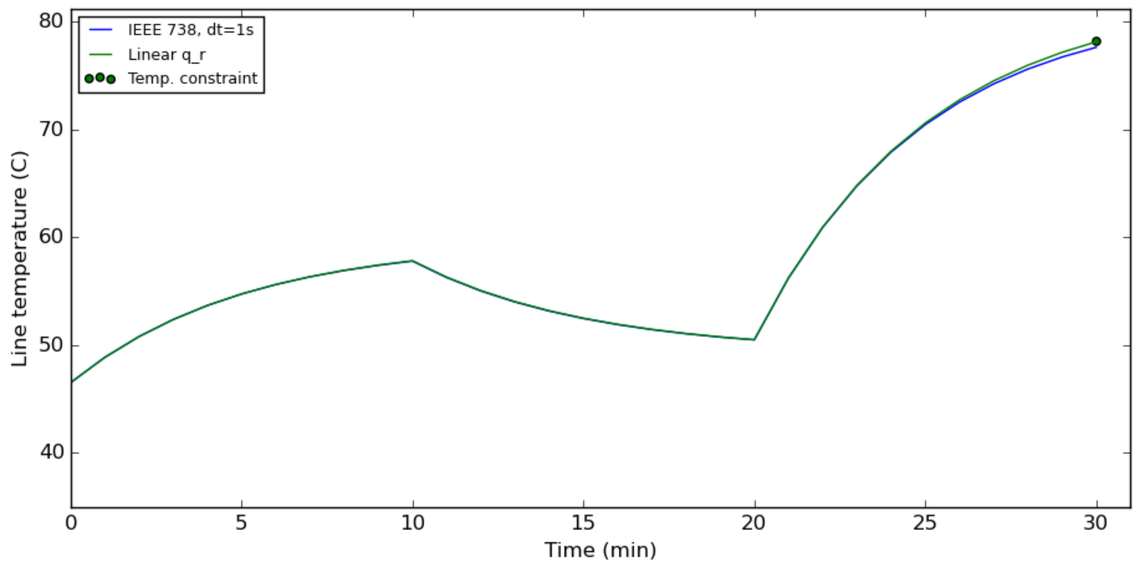


Figure 2: Comparison of IEEE 738 and approximate temperature calculation methods

- Because the approximate line temperature model is analytic (Equation (9) is continuous) while the 738 model requires numerical integration, I chose a very small integration step size of one second to facilitate comparison.
- Because the approximate model underestimates the radiative heat loss rate (see Figure 1), the green trace should lie slightly above the blue one. This makes the approximate model conservative, which is desirable in the temporal instanton setting. Figure 2 illustrates this conservative behavior: the green trace lies on or above the blue trace throughout the time horizon.
- The green dot, computed by (16), lies on top of the green curve. This validates the temperature constraint formulation (21).

# 4 Conclusion

An approximate line temperature equation was derived. This equation assumes radiation is linear (not fourth-order) in temperature. There is close numerical agreement between this approximate equation and IEEE 738 standard temperature calculation. The greatest benefit of the approximate temperature equation is that it may be rearranged into a second-order angle difference constraint, which is a requirement of the temporal instanton optimization framework.