Exercises 1. Vectors

1. Create the vectors:

(a) $(1, 2, 3, \ldots, 19, 20)$

(b) $(20, 19, \ldots, 2, 1)$

(c) $(1, 2, 3, \ldots, 19, 20, 19, 18, \ldots, 2, 1)$

(d) $(4, 6, 3)$ and assign it to the name `tmp`.

For parts (e), (f) and (g) look at the help for the function `rep`.

(e) $(4, 6, 3,\ 4, 6, 3, \ldots, 4, 6, 3)$ where there are 10 occurrences of 4.

(f) $(4, 6, 3,\ 4, 6, 3, \ldots, 4, 6, 3, 4)$ where there are 11 occurrences of 4, 10 occurrences of 6 and 10 occurrences of 3.

(g) $(4, 4, \ldots, 4,\ 6, 6, \ldots, 6,\ 3, 3, \ldots, 3)$ where there are 10 occurrences of 4, 20 occurrences of 6 and 30 occurrences of 3.

2. Create a vector of the values of $e^x \cos(x)$ at $x = 3, 3.1, 3.2, \ldots, 6$.

3. Create the following vectors:

(a) $(0.1^3 0.2^1, 0.1^6 0.2^4, \ldots, 0.1^{36} 0.2^{34})$ **(b)** $\left(2, \frac{2^2}{2}, \frac{2^3}{3}, \ldots, \frac{2^{25}}{25}\right)$

4. Calculate the following:

(a) $\displaystyle\sum_{i=10}^{100} \left(i^3 + 4i^2\right)$. **(b)** $\displaystyle\sum_{i=1}^{25} \left(\frac{2^i}{i} + \frac{3^i}{i^2}\right)$

5. Use the function `paste` to create the following character vectors of length 30:

(a) `("label 1", "label 2", ....., "label 30")`.
Note that there is a single space between `label` and the number following.

(b) `("fn1", "fn2", ..., "fn30")`.
In this case, there is no space between `fn` and the number following.

6. Execute the following lines which create two vectors of random integers which are chosen with replacement from the integers 0, 1, ..., 999. Both vectors have length 250.

```
set.seed(50)
xVec <- sample(0:999, 250, replace=T)
yVec <- sample(0:999, 250, replace=T)
```

Suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ denotes the vector `xVec` and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ denotes the vector `yVec`.

(a) Create the vector $(y_2 - x_1, \ldots, y_n - x_{n-1})$.

(b) Create the vector $\left(\frac{\sin(y_1)}{\cos(x_2)}, \frac{\sin(y_2)}{\cos(x_3)}, \ldots, \frac{\sin(y_{n-1})}{\cos(x_n)}\right)$.

(c) Create the vector $(x_1 + 2x_2 - x_3, x_2 + 2x_3 - x_4, \ldots, x_{n-2} + 2x_{n-1} - x_n)$.

(d) Calculate $\displaystyle\sum_{i=1}^{n-1} \frac{e^{-x_{i+1}}}{x_i + 10}$.

7. This question uses the vectors `xVec` and `yVec` created in the previous question and the functions `sort`, `order`, `mean`, `sqrt`, `sum` and `abs`.

(a) Pick out the values in `yVec` which are > 600.

(b) What are the index positions in `yVec` of the values which are > 600?

(c) What are the values in `xVec` which correspond to the values in `yVec` which are > 600? (By correspond, we mean at the same index positions.)

(d) Create the vector $(|x_1 - \bar{\mathbf{x}}|^{1/2}, |x_2 - \bar{\mathbf{x}}|^{1/2}, \ldots, |x_n - \bar{\mathbf{x}}|^{1/2})$ where $\bar{\mathbf{x}}$ denotes the mean of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

(e) How many values in `yVec` are within 200 of the maximum value of the terms in `yVec`?

(f) How many numbers in `xVec` are divisible by 2? (Note that the modulo operator is denoted `%%`.)

(g) Sort the numbers in the vector `xVec` in the order of increasing values in `yVec`.

(h) Pick out the elements in `yVec` at index positions $1, 4, 7, 10, 13, \ldots$.

8. By using the function `cumprod` or otherwise, calculate

$$1 + \frac{2}{3} + \left(\frac{2}{3}\frac{4}{5}\right) + \left(\frac{2}{3}\frac{4}{5}\frac{6}{7}\right) + \cdots + \left(\frac{2}{3}\frac{4}{5}\cdots\frac{38}{39}\right)$$

Exercises 2. Matrices

1. Suppose

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

(a) Check that $\mathbf{A}^3 = \mathbf{0}$ where $\mathbf{0}$ is a 3×3 matrix with every entry equal to 0.

(b) Replace the third column of $\mathbf{A}$ by the sum of the second and third columns.

2. Create the following matrix $\mathbf{B}$ with 15 rows:

$$\mathbf{B} = \begin{bmatrix} 10 & -10 & 10 \\ 10 & -10 & 10 \\ \cdots & \cdots & \cdots \\ 10 & -10 & 10 \end{bmatrix}$$

Calculate the 3×3 matrix $\mathbf{B}^{\mathrm{T}}\mathbf{B}$. (Look at the help for `crossprod`.)

3. Create a 6×6 matrix `matE` with every entry equal to 0. Check what the functions `row` and `col` return when applied to `matE`. Hence create the 6×6 matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

4. Look at the help for the function `outer`. Hence create the following patterned matrix:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

5. Create the following patterned matrices. In each case, your solution should make use of the special form of the matrix—this means that the solution should easily generalise to creating a larger matrix with the same structure and should not involve typing in all the entries in the matrix.

(a)
$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix}$$

(b)
$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

(c)
$$\begin{pmatrix} 0 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 0 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\ 2 & 1 & 0 & 8 & 7 & 6 & 5 & 4 & 3 \\ 3 & 2 & 1 & 0 & 8 & 7 & 6 & 5 & 4 \\ 4 & 3 & 2 & 1 & 0 & 8 & 7 & 6 & 5 \\ 5 & 4 & 3 & 2 & 1 & 0 & 8 & 7 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 & 8 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

6. Solve the following system of linear equations in five unknowns

$$\begin{aligned}
x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 7\\
2x_1 + x_2 + 2x_3 + 3x_4 + 4x_5 &= -1\\
3x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 &= -3\\
4x_1 + 3x_2 + 2x_3 + x_4 + 2x_5 &= 5\\
5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 &= 17
\end{aligned}$$

by considering an appropriate matrix equation $\mathbf{A}\mathbf{x} = \mathbf{y}$.

Make use of the special form of the matrix **A**. The method used for the solution should easily generalise to a larger set of equations where the matrix **A** has the same structure; hence the solution should not involve typing in every number of **A**.

7. Create a 6×10 matrix of random integers chosen from 1, 2,. . . , 10 by executing the following two lines of code:

```
set.seed(75)
aMat <- matrix( sample(10, size=60, replace=T), nr=6)
```

(a) Find the number of entries in each row which are greater than 4.
(b) Which rows contain exactly two occurrences of the number seven?
(c) Find those pairs of columns whose total (over both columns) is greater than 75. The answer should be a matrix with two columns; so, for example, the row $(1, 2)$ in the output matrix means that the sum of columns 1 and 2 in the original matrix is greater than 75. Repeating a column is permitted; so, for example, the final output matrix could contain the rows $(1, 2)$, $(2, 1)$ and $(2, 2)$.
What if repetitions are not permitted? Then, only $(1, 2)$ from $(1, 2)$, $(2, 1)$ and $(2, 2)$ would be permitted.

8. Calculate

(a) $\displaystyle\sum_{i=1}^{20}\sum_{j=1}^{5}\frac{i^4}{(3+j)}$ **(b)** (Hard) $\displaystyle\sum_{i=1}^{20}\sum_{j=1}^{5}\frac{i^4}{(3+ij)}$ **(c)** (Even harder!) $\displaystyle\sum_{i=1}^{10}\sum_{j=1}^{i}\frac{i^4}{(3+ij)}$

Exercises 3. Simple Functions

1. **(a)** Write functions `tmpFn1` and `tmpFn2` such that if `xVec` is the vector $(x_1, x_2, \ldots, x_n)$, then `tmpFn1(xVec)` returns the vector $\left(x_1, x_2^2, \ldots, x_n^n\right)$ and `tmpFn2(xVec)` returns the vector $\left(x_1, \dfrac{x_2^2}{2}, \ldots, \dfrac{x_n^n}{n}\right)$.

   (b) Now write a function `tmpFn3` which takes 2 arguments `x` and `n` where `x` is a single number and `n` is a strictly positive integer. The function should return the value of
$$1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}$$

2. Write a function `tmpFn(xVec)` such that if `xVec` is the vector $\mathbf{x} = (x_1, \ldots, x_n)$ then `tmpFn(xVec)` returns the vector of moving averages:
$$\frac{x_1 + x_2 + x_3}{3}, \quad \frac{x_2 + x_3 + x_4}{3}, \quad \ldots, \quad \frac{x_{n-2} + x_{n-1} + x_n}{3}$$
Try out your function; for example, try `tmpFn( c(1:5,6:1) )`.

3. Consider the continuous function
$$f(x) = \begin{cases} x^2 + 2x + 3 & \text{if } x < 0 \\ x + 3 & \text{if } 0 \le x < 2 \\ x^2 + 4x - 7 & \text{if } 2 \le x. \end{cases}$$
Write a function `tmpFn` which takes a single argument `xVec`. The function should return the vector of values of the function $f(x)$ evaluated at the values in `xVec`.
Hence plot the function $f(x)$ for $-3 < x < 3$.

4. Write a function which takes a single argument which is a matrix. The function should return a matrix which is the same as the function argument but every odd number is doubled.

   Hence the result of using the function on the matrix
$$\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$
should be:
$$\begin{bmatrix} 2 & 2 & 6 \\ 10 & 2 & 6 \\ -2 & -2 & -6 \end{bmatrix}$$
Hint: First try this for a specific matrix on the Command Line.

5. Write a function which takes 2 arguments `n` and `k` which are positive integers. It should return the $n \times n$ matrix:
$$\begin{bmatrix} k & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & k & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & k & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & k & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & k & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & k \end{bmatrix}$$
Hint: First try to do it for a specific case such as $n = 5$ and $k = 2$ on the Command Line.

6. Suppose an angle α is given as a positive real number of degrees.
If $0 \le \alpha < 90$ then it is quadrant 1. If $90 \le \alpha < 180$ then it is quadrant 2.
If $180 \le \alpha < 270$ then it is quadrant 3. If $270 \le \alpha < 360$ then it is quadrant 4.
If $360 \le \alpha < 450$ then it is quadrant 1. And so on.

   Write a function `quadrant(alpha)` which returns the quadrant of the angle α.

7. **(a)** Zeller's congruence is the formula:

$$f = \big([2.6m - 0.2] + k + y + [y/4] + [c/4] - 2c\big) \bmod 7$$

where $[x]$ denotes the integer part of x; for example [7.5] = 7.
Zeller's congruence returns the day of the week f given:
k = the day of the month,
y = the year in the century
c = the first 2 digits of the year (the century number)
m = the month number (where January is month 11 of the preceding year, February is month 12 of the preceding year, March is month 1, etc.)
For example, the date 21/07/1963 has $m = 5$, $k = 21$, $c = 19$, $y = 63$; whilst the date 21/2/1963 has $m = 12$, $k = 21$, $c = 19$ and $y = 62$.

Write a function `weekday(day,month,year)` which returns the day of the week when given the numerical inputs of the day, month and year.
Note that the value of 1 for f denotes Sunday, 2 denotes Monday, etc.

(b) Does your function work if the input parameters `day`, `month` and `year` are vectors with the same length and with valid entries?

8. **(a)** Suppose $x_0 = 1$ and $x_1 = 2$ and

$$x_j = x_{j-1} + \frac{2}{x_{j-1}} \qquad \text{for } j = 1, 2, \ldots.$$

Write a function `testLoop` which takes the single argument n and returns the first $n-1$ values of the sequence $\{x_j\}_{j\geq 0}$: that means the values of $x_0, x_1, x_2, \ldots, x_{n-2}$.

(b) Now write a function `testLoop2` which takes a single argument `yVec` which is a vector. The function should return

$$\sum_{j=1}^{n} e^j$$

where n is the length of `yVec`.

9. *Solution of the difference equation* $x_n = rx_{n-1}(1 - x_{n-1})$*, with starting value* x_1.

(a) Write a function `quadmap( start, rho, niter )` which returns the vector $(x_1, \ldots, x_n)$ where $x_k = rx_{k-1}(1 - x_{k-1})$ and

`niter` denotes n,
`start` denotes x_1, and
`rho` denotes r.

Try out the function you have written:

- for $r = 2$ and $0 < x_1 < 1$ you should get $x_n \to 0.5$ as $n \to \infty$.
- try `tmp <- quadmap(start=0.95, rho=2.99, niter=500)`

Now switch back to the Commands window and type:

```
plot(tmp, type="l")
```

Also try the plot `plot(tmp[300:500], type="l")`

(b) Now write a function which determines the number of iterations needed to get $| x_n - x_{n-1} | < 0.02$. So this function has only 2 arguments: `start` and `rho`. (For `start=0.95` and `rho=2.99`, the answer is 84.)

10. **(a)** Given a vector $(x_1, \ldots, x_n)$, the sample autocorrelation of lag k is defined to be

$$r_k = \frac{\sum_{i=k+1}^{n}(x_i - \bar{x})(x_{i-k} - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Thus

$$r_1 = \frac{\sum_{i=2}^{n}(x_i - \bar{x})(x_{i-1} - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{(x_2 - \bar{x})(x_1 - \bar{x}) + \cdots + (x_n - \bar{x})(x_{n-1} - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

Write a function `tmpFn(xVec)` which takes a single argument `xVec` which is a vector and returns a

`list` of two values: r_1 and r_2.
In particular, find r_1 and r_2 for the vector $(2, 5, 8, \ldots, 53, 56)$.

(b) (Harder.) Generalise the function so that it takes two arguments: the vector `xVec` and an integer `k` which lies between 1 and $n-1$ where n is the length of `xVec`.
The function should return a vector of the values $(r_0 = 1, r_1, \ldots, r_k)$.

If you used a loop to answer part (b), then you need to be aware that much, much better solutions are possible—see exercises 4. (Hint: `sapply`.)