

01-02-22

DATE _____
PAGE _____

Assignment 1

Q1. To find the decision boundary that minimizes the error in the given cases.

We can minimize the probability of error by the likelihood ratio.

$$\text{If } \frac{P(x|\omega_1)}{P(x|\omega_2)} > \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \frac{P(\omega_2)}{P(\omega_1)}$$

then decide ω_1 else decide ω_2 .

(i) Zero-one loss

$$\Rightarrow \lambda_{11} = 0; \lambda_{12} = 1; \lambda_{21} = 1; \lambda_{22} = 0$$

$$\text{RHS} = \frac{1}{1-0} \cdot \frac{3}{4} = * \cdot 3$$

$$\begin{aligned} \text{LHS} &= \frac{P(x|\omega_1)}{P(x|\omega_2)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2} \cdot \frac{\sqrt{2\pi}}{e^{-\frac{1}{2}(x-5)^2}} \\ &= e^{-\frac{1}{2}[(x-2)^2 - (x-5)^2]} \\ &= e^{-\frac{1}{2}[(x-2+x-5)(x-2-x+5)]} \\ &= e^{-\frac{1}{2}(2x-7)(3)} \\ &= \end{aligned}$$



DATE _____

PAGE _____

$$\rightarrow x < 3.038 \Rightarrow x < 3.1338$$

Ans. The decision boundary is as follows:

If $x < 3.1338$ decide ω_1 ,

else decide ω_2 . Decision boundary = 3.1338

(ii) Given: $\lambda_{12} = 2$; $\lambda_{21} = 3$; $\lambda_{11} = 0$; $\lambda_{22} = 0$

$$RHS = \frac{2-0}{3-0} 3 = \frac{2}{3} \cdot 3 = 2$$

$$LHS \Rightarrow e^{-\frac{3}{2}(2x-7)} > 2$$

$$\Rightarrow -\frac{3}{2}(2x-7) > \ln 2$$

$$\Rightarrow 2x-7 < -0.462$$

$$\Rightarrow x < 3.2689$$

Ans. The decision boundary is as follows:

If $x < 3.2689$ decide ω_1 ,

else decide ω_2 .

Decision boundary = 3.2689

In tasks like cancer prediction, predicting no cancer when the patient already has cancer is more dangerous than predicting cancer for a no-cancer patient. In zero-one loss, the penalties for false positives is same as false negatives.

Hence, zero-one loss for such tasks should not be preferred considering serious life implications.

Q2.

Given $y = A^T x + B$

$$A = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$y = A^T x + B$$

$$\therefore E[y] = E[A^T x + B]$$

$\therefore E[\cdot]$ is a linear operator

$$\Rightarrow E[y] = A^T E[x] + B$$

$$= [2 \ -1 \ 2] \begin{bmatrix} 5 \\ -5 \\ 6 \end{bmatrix} + 5$$

$$= 27 + 5$$

$$= 32$$

Ans.

32



Q3 Given: $P(x|\omega_i) = \frac{1}{\pi b} \cdot \frac{1}{1 + \left(\frac{x-a_i}{b}\right)^2}, i=1,2$

$$\text{If } \frac{P(x|\omega_1)}{P(x|\omega_2)} > \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \frac{P(\omega_2)}{P(\omega_1)}$$

then, $P(\omega)$ then decide ω_1 else decide ω_2 .

$$\therefore P(\omega_1) = P(\omega_2)$$

$$\Rightarrow \text{If } \frac{P(x|\omega_1)}{P(x|\omega_2)} > \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}}$$

then decide ω_1 else decide ω_2 .

For zero-one loss $\lambda_{12} = 1, \lambda_{22} = 0, \lambda_{21} = 1, \lambda_{11} = 0$

The likelihood ratio becomes

$$\text{If } P(x|\omega_1) > P(x|\omega_2)$$

decide ω_1 else ω_2 .

$$\Rightarrow \frac{1}{\pi b} \cdot \frac{1}{1 + \left(\frac{x-a_1}{b}\right)^2} > \frac{1}{\pi b} \cdot \frac{1}{1 + \left(\frac{x-a_2}{b}\right)^2}$$

$$\Rightarrow 1 + \left(\frac{x-a_2}{b}\right)^2 > 1 + \left(\frac{x-a_1}{b}\right)^2$$

$$\Rightarrow \left(\frac{x-a_2}{b}\right)^2 - \left(\frac{x-a_1}{b}\right)^2 > 0$$

$$\Rightarrow \left(\frac{x-a_2+x-a_1}{b}\right) \left(\frac{x-a_2-x+a_1}{b}\right) > 0$$

$$\Rightarrow \left(\frac{2x-(a_1+a_2)}{b}\right) \left(\frac{a_1-a_2}{b}\right) > 0$$

In cauchy pdf $b > 0$

Let $a_1 > a_2$

If $a_1 < a_2$

$$\Rightarrow 2x > a_1 + a_2$$

$$\Rightarrow x > \frac{a_1 + a_2}{2}$$

$$\Rightarrow 2x < a_1 + a_2$$

$$\Rightarrow x < \frac{a_1 + a_2}{2}$$

Thus, the decision boundary is $\frac{a_1 + a_2}{2}$.

(B) $\because p(\omega_1) = p(\omega_2)$

$$\Rightarrow p(\omega_1) = p(\omega_2) = 0.5$$

$$p(\omega_1|x) = \frac{p(x|\omega_1)p(\omega_1)}{p(x)}$$

$$p(x) = \sum_{j=1}^2 p(x|\omega_j)p(\omega_j)$$

$$= p(x|\omega_1)p(\omega_1) + p(x|\omega_2)p(\omega_2)$$

$$= 0.5(p(x|\omega_1) + p(x|\omega_2))$$

$$= 0.5 \left(\frac{1}{\pi b} \frac{1}{1+(x-3)^2} + \frac{1}{\pi b} \frac{1}{1+(x-5)^2} \right)$$

$$= 0.5 \left(\frac{1}{\pi b} ((x-3)^{-2} + (x-5)^{-2}) \right)$$



$$\Rightarrow p(\omega_1|x) = \frac{0.5}{\pi b} \cdot \frac{1}{1+(x-3)^2}$$

~~$$0.5 \cdot \frac{1}{\pi b} \left(\frac{1}{1+(x-3)^2} + \frac{1}{1+(x-5)^2} \right)$$~~

$$\Rightarrow p(\omega_1|x) = \frac{1}{1+(x-3)^2} + \frac{1}{1+(x-5)^2}$$

$$(C) P(\text{error}) = \int_{-\infty}^{\infty} \min [p(\omega_1|x), p(\omega_2|x)] p(x) dx$$

$$= \int_{-\infty}^{\infty} \min [p(x|\omega_1)p(\omega_1), p(x|\omega_2)p(\omega_2)] dx$$

Now, for the decision boundary point where $p(x|\omega_1)$ and $p(x|\omega_2)$ intersect

$$\Rightarrow \frac{1}{\pi b} \frac{1}{1+(x-3)^2} = \frac{1}{\pi b} \frac{1}{1+(x-5)^2}$$

$$\Rightarrow (x-3)^2 = (x-5)^2$$

$$\Rightarrow (x-3 - x+5)(x-3 + x-5) = 0$$

$$\Rightarrow 2x = 8$$

$$\Rightarrow x = 4$$

For $x < 4$

$$P(x|\omega_1) \text{ let } x=0 : = \frac{1}{\pi} \frac{1}{10} = \frac{1}{10\pi}$$

$$P(x|\omega_2) \quad " \quad : = \frac{1}{\pi} \frac{1}{26} = \frac{1}{26\pi}$$



$$\Rightarrow P(x|\omega_1) > P(x|\omega_2) \quad \forall x \in (-\infty, 4)$$

$$\Rightarrow P(x|\omega_1) < P(x|\omega_2) \quad \forall x \in (4, \infty)$$

$$\Rightarrow P(\text{error}) = 0.5 \int_{-\infty}^4 \frac{1}{\pi} \cdot \frac{1}{1+(x-5)^2} dx + 0.5 \int_4^{\infty} \frac{1}{\pi} \cdot \frac{1}{1+(x-3)^2} dx$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^4 \frac{1}{1+(x-5)^2} dx + \int_4^{\infty} \frac{1}{1+(x-3)^2} dx \right]$$

$$= \frac{1}{2\pi} \left[\text{Let } u = x-5 \text{ and } v = x-3 \\ \Rightarrow du = dx \text{ and } dv = dx \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\infty}^{-1} \frac{1}{1+u^2} du + \int_1^{\infty} \frac{1}{1+v^2} dv \right]$$

$$= \frac{1}{2\pi} \left[\left(\tan^{-1} u \right)^{-1} + \left(\tan^{-1} v \right)^\infty \right]$$

$$= \frac{1}{2\pi} \left(-\frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{4} \right)$$

$$= \frac{1}{2\pi} \left[\frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{4} \right]$$

$$= \frac{1}{4}$$

$$= 0.25$$

Ans. The overall error rate 0.25



4(a)

$$f(a) = \theta^a (1-\theta)^{1-a}$$

$$f(b) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{b-m}{\sigma} \right)^2}$$

 $x = [a \ b]$ a 2-d vectorThe covariance of x

$$\begin{bmatrix} \theta(1-\theta) & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

~~$$\text{Now, cov of } x = \begin{bmatrix} \sigma_x^2 & \sigma_{x_1 x_2} \\ \sigma_{x_1 x_2} & \sigma_x^2 \end{bmatrix}$$~~

~~$$\text{Now, cov of } x = \begin{bmatrix} \sigma_x^2 & \sigma_{x_1 x_2} \\ \sigma_{x_1 x_2} & \sigma_x^2 \end{bmatrix}$$~~

where, $\sigma_{x_i}^2$ the the variance of x_i in \bar{x} . $\sigma_{x_i x_j}$ is the covariance of x_i and x_j in \bar{x} .

$$\therefore \sigma_{x_1 x_2}^2 = 0$$

 $\Rightarrow x_1$ and x_2 are independent (assumed as sir told in class)

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$f(x) = f(a, b) = f(a) f(b)$$

Ans.

$$\Rightarrow f(x) = \frac{\theta^a (1-\theta)^{1-a}}{\sigma \sqrt{2\pi}} \cdot e^{-\frac{1}{2} \left(\frac{b-m}{\sigma} \right)^2}$$



$$\Rightarrow p(x) = \frac{\theta^a (1-\theta)^{1-a} - \frac{1}{2} \left(\frac{b-m}{\sigma} \right)^2}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{b-m}{\sigma} \right)^2}$$

$q(x)$ = N iid distribution of $p(x)$.

$$q(x) = \prod_{i=1}^N p(x_i)$$

$$= \prod_{i=1}^N \frac{\theta^{a_i} (1-\theta)^{1-a_i} - \frac{1}{2} \left(\frac{b_i-m}{\sigma} \right)^2}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{b_i-m}{\sigma} \right)^2}$$

$$\ln q(x) = \sum_{i=1}^N \ln \left[\frac{\theta^{a_i} (1-\theta)^{1-a_i} - \frac{1}{2} \left(\frac{b_i-m}{\sigma} \right)^2}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{b_i-m}{\sigma} \right)^2} \right]$$

$$= \sum_{i=1}^N \left[-\ln [\sigma \sqrt{2\pi}] + \ln \theta^{a_i} + \ln (1-\theta)^{1-a_i} - \frac{1}{2} \left(\frac{b_i-m}{\sigma} \right)^2 \right]$$

Now, differentiating $\ln q(x)$ wrt θ

$$\Rightarrow \frac{1}{q(x)} \frac{d q(x)}{d\theta} \cdot \frac{dx}{d\theta} = \sum_{i=1}^N \left[\frac{a_i}{\theta} - \frac{(1-a_i)}{1-\theta} \right]$$

Putting $\frac{d q(x)}{d\theta} = 0$ to maximize $q(x)$

$$\Rightarrow \left\{ \sum_{i=1}^N \left[\frac{a_i}{\theta} - \frac{(1-a_i)}{1-\theta} \right] \right\} \frac{q(x)}{d\theta} = 0$$

$$\Rightarrow \sum_{i=1}^N \frac{a_i}{\theta} = \sum_{i=1}^N \frac{1-a_i}{1-\theta}$$

$$\text{Let } \sum_{i=1}^n a_i = Y$$

$$\Rightarrow \frac{Y}{\theta} = \frac{1-Y}{1-\theta}$$

$$Y(1-\theta) = \theta(N-Y)$$

$$\Rightarrow Y - Y\theta = N\theta - \theta Y$$

$$\Rightarrow \theta = \frac{Y}{N}$$

Ans. $\Rightarrow \boxed{\theta = \frac{1}{N} \sum_{i=1}^N a_i}$

A few (good) partitions fit with

$$(D-1) = \sum_{i=1}^{d-1} a_i \text{ subject to } a_i \geq 1$$

(the question is, $\theta = \text{(good) partition}$)

$$\theta = \frac{1}{N} \left((D-1) + \sum_{i=1}^{d-1} a_i \right)$$

$$\theta = \frac{1}{N} \left(D - 1 + \sum_{i=1}^{d-1} a_i \right)$$

$$Y = \frac{1}{N} \sum_{i=1}^{d-1} a_i$$

$$\frac{1}{N} \sum_{i=1}^{d-1} a_i = Y - \frac{1}{N}$$