Q: Let $\{-1, -\frac{1}{2}, 1, \frac{5}{2}, 3\}$ be a realization of a random sample of size 5 from a population having $N\left(\frac{1}{2}, \sigma^2\right)$ distribution, where $\sigma > 0$ is an unknown parameter. Let $T$ be an unbiased estimator of $\sigma^2$ whose variance attains the Cramer-Rao lower bound. Then, based on the above data, the realized value of $T$ (rounded off to two decimal places) equals

**Solution:**

**Definition 1.** *Unbiased Estimator is defined as*

$$E(\hat{\sigma^2}) = \sigma^2 \tag{1}$$

*where, $E(\hat{\sigma^2})$ represents the expected value of the estimator $\hat{\sigma^2}$ and $\sigma^2$ represents the true parameter*

**Definition 2.** *The Cramér-Rao bound can be defined as follows:*

$$Var(\sigma^2) \geq \frac{1}{I(\sigma^2)} \tag{2}$$

*where $I(\sigma^2)$ represents fisher information for the parameter $\sigma^2$. Mathematically,*

$$I(\sigma^2) = -E\left[\frac{\partial^2}{\partial(\sigma)^2} \log P_X(X|\sigma^2)\right]$$

*where, $E[\cdot]$ represents the expected value and $P_X(X|\sigma^2)$ is the p.d.f of random variable $X$ given the parameter $\sigma^2$.*

$P_X(X|\sigma^2)$ is given by:

$$P_X(X|\sigma^2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(X-\frac{1}{2})^2}{2\sigma^2}\right) \tag{3}$$

$$\log p_X(X|\sigma^2) = \log\left(\frac{1}{2\pi\sigma^2} \exp\left(-\frac{(X-\frac{1}{2})^2}{2\sigma^2}\right)\right) \tag{4}$$

$$= -\frac{1}{2}\log(2\pi\sigma^2) - \frac{(X-\frac{1}{2})^2}{2\sigma^2} \tag{5}$$

$$\frac{\partial^2}{\partial(\sigma^2)^2} \log P_X(X|\sigma^2) = \frac{1}{2\pi\sigma^2} - \frac{3(X-\frac{1}{2})^2}{\sigma^4} \tag{6}$$

$$I(\sigma^2) = \frac{3}{\sigma^4}E[X^2] - \frac{3}{\sigma^4}E[X] + \frac{3}{4\sigma^4} - \frac{1}{2\pi\sigma^4} \tag{7}$$

$$E[X^2] = \sigma^2 + \left(\frac{1}{2}\right)^2 \tag{8}$$

$$E[X] = \frac{1}{2} \tag{9}$$

$$\implies I(\sigma^2) = \left(3 - \frac{1}{2\pi}\right)\frac{1}{\sigma^2} \tag{10}$$

Hence, Cramér-Rao bound is given as $\frac{\sigma^2}{(3-\frac{1}{2\pi})}$

**Definition 3.** *Variance of $T$ attains Cramer-Rao lower bound*
*$\implies$ $T$ has attained minimum possible variance and $T$ is an efficient estimator*

Therefore,

$$T = \frac{\sum(X_i - \mu)^2}{n} \tag{11}$$

$$n = 5 \tag{12}$$

$$\mu = \frac{1}{2} \tag{13}$$

| $X_i$ | -1 | $-\frac{1}{2}$ | 1 | $\frac{5}{2}$ | 3 |
|---|---|---|---|---|---|
| $(X_i - \mu)^2$ | $\frac{9}{4}$ | 1 | $\frac{1}{4}$ | 4 | $\frac{25}{4}$ |

TABLE 0
TABLE 1

$$\sum (X_i - \mu)^2 = 13.75 \tag{14}$$

Hence,

$$T = 2.75 \tag{15}$$

Since, $T$ is an unbiased estimator of $\sigma^2$,

$$\text{Cramér-Rao bound} = \frac{T}{\left(3 - \frac{1}{2\pi}\right)} \tag{16}$$

$$= 0.968 \tag{17}$$