

Q: Let $\{-1, -\frac{1}{2}, 1, \frac{5}{2}, 3\}$ be a realization of a random sample of size 5 from a population having $N(\frac{1}{2}, \sigma^2)$ distribution, where $\sigma > 0$ is an unknown parameter. Let T be an unbiased estimator of σ^2 whose variance attains the Cramer-Rao lower bound. Then, based on the above data, the realized value of T (rounded off to two decimal places) equals

Solution:

Definition 1. Unbiased Estimator is defined as

$$E(\hat{\sigma}^2) = \sigma^2 \quad (1)$$

where, $E(\hat{\sigma}^2)$ represents the expected value of the estimator $\hat{\sigma}^2$ and σ^2 represents the true parameter

Definition 2. The Cramér-Rao bound can be defined as follows:

$$\text{Var}(\sigma^2) \geq \frac{1}{I(\sigma^2)} \quad (2)$$

where $I(\sigma^2)$ represents fisher information for the parameter σ^2 . Mathematically,

$$I(\sigma^2) = -E \left[\frac{\partial^2}{\partial(\sigma^2)^2} \log P_X(X|\sigma^2) \right]$$

where, $E[\cdot]$ represents the expected value and $P_X(X|\sigma^2)$ is the p.d.f of random variable X given the parameter σ^2 .

$P_X(X|\sigma^2)$ is given by:

$$P_X(X|\sigma^2) = \frac{1}{2\pi\sigma^2} \exp \left(-\frac{(X - \frac{1}{2})^2}{2\sigma^2} \right) \quad (3)$$

$$\log p_X(X|\sigma^2) = \log \left(\frac{1}{2\pi\sigma^2} \exp \left(-\frac{(X - \frac{1}{2})^2}{2\sigma^2} \right) \right) \quad (4)$$

$$= -\frac{1}{2} \log(2\pi\sigma^2) - \frac{(X - \frac{1}{2})^2}{2\sigma^2} \quad (5)$$

$$\frac{\partial^2}{\partial(\sigma^2)^2} \log P_X(X|\sigma^2) = \frac{1}{2\pi\sigma^2} - \frac{3(X - \frac{1}{2})^2}{\sigma^4} \quad (6)$$

$$I(\sigma^2) = \frac{3}{\sigma^4} E[X^2] - \frac{3}{\sigma^4} E[X] + \frac{3}{4\sigma^4} - \frac{1}{2\pi\sigma^2} \quad (7)$$

$$E[X^2] = \sigma^2 + \left(\frac{1}{2} \right)^2 \quad (8)$$

$$E[X] = \frac{1}{2} \quad (9)$$

$$\Rightarrow I(\sigma^2) = \left(3 - \frac{1}{2\pi} \right) \frac{1}{\sigma^2} \quad (10)$$

Hence, Cramér-Rao bound is given as $\frac{\sigma^2}{(3 - \frac{1}{2\pi})}$

Definition 3. Variance of T attains Cramer-Rao lower bound

$\Rightarrow T$ has attained minimum possible variance and T is an efficient estimator

Therefore,

$$T = \frac{\sum (X_i - \mu)^2}{n} \quad (11)$$

$$n = 5 \quad (12)$$

$$\mu = \frac{1}{2} \quad (13)$$

X_i	-1	$-\frac{1}{2}$	1	$\frac{5}{2}$	3
$(X_i - \mu)^2$	$\frac{9}{4}$	1	$\frac{1}{4}$	4	$\frac{25}{4}$

TABLE 0

TABLE 1

$$\sum (X_i - \mu)^2 = 13.75 \quad (14)$$

Hence,

$$T = 2.75 \quad (15)$$

Since, T is an unbiased estimator of σ^2 ,

$$\text{Cramér-Rao bound} = \frac{T}{\left(3 - \frac{1}{2\pi}\right)} \quad (16)$$

$$= 0.968 \quad (17)$$