Q: Let $\{-1, -\frac{1}{2}, 1, \frac{5}{2}, 3\}$ be a realization of a random sample of size 5 from a population having $N\left(\frac{1}{2}, \sigma^2\right)$ distribution, where $\sigma > 0$ is an unknown parameter. Let T be an unbiased estimator of σ^2 whose variance attains the Cramer-Rao lower bound. Then, based on the above data, the realized value of T (rounded off to two decimal places) equals

Solution:

Definition 1. *Unbiased Estimator is defined as*

$$E(\hat{\sigma^2}) = \sigma^2 \tag{1}$$

where, $E(\hat{\sigma^2})$ represents the expected value of the estimator $\hat{\sigma^2}$ and σ^2 represents the true parameter

Definition 2. *The Cramér-Rao bound can be defined as follows:*

$$Var(\sigma^2) \geq \frac{1}{I(\sigma^2)} \tag{2}$$

where $I(\sigma^2)$ represents fisher information for the parameter σ^2. Mathematically,

$$I(\sigma^2) = -E\left[\frac{\partial^2}{\partial(\sigma)^2} \log P_X(X|\sigma^2)\right]$$

where, $E[\cdot]$ represents the expected value and $P_X(X|\sigma^2)$ is the p.d.f of random variable X given the parameter σ^2.

$P_X(X|\sigma^2)$ is given by:

$$P_X(X|\sigma^2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(X - \frac{1}{2})^2}{2\sigma^2}\right) \tag{3}$$

$$\log p_X(X|\sigma^2) = \log\left(\frac{1}{2\pi\sigma^2} \exp\left(-\frac{(X - \frac{1}{2})^2}{2\sigma^2}\right)\right) \tag{4}$$

$$= -\frac{1}{2}\log(2\pi\sigma^2) - \frac{(X - \frac{1}{2})^2}{2\sigma^2} \tag{5}$$

$$\frac{\partial^2}{\partial(\sigma^2)^2} \log P_X(X|\sigma^2) = \frac{1}{2\pi\sigma^2} - \frac{3(X - \frac{1}{2})^2}{\sigma^4} \tag{6}$$

$$I(\sigma^2) = \frac{3}{\sigma^4}E[X^2] - \frac{3}{\sigma^4}E[X] + \frac{3}{4\sigma^4} - \frac{1}{2\pi\sigma^4} \tag{7}$$

$$E[X^2] = \sigma^2 + \left(\frac{1}{2}\right)^2 \tag{8}$$

$$E[X] = \frac{1}{2} \tag{9}$$

$$\implies I(\sigma^2) = \left(3 - \frac{1}{2\pi}\right)\frac{1}{\sigma^2} \tag{10}$$

Hence, Cramér-Rao bound is given as $\frac{\sigma^2}{(3 - \frac{1}{2\pi})}$

Definition 3. *Variance of T attains Cramer-Rao lower bound*
$\implies$ *T has attained minimum possible variance and T is an efficient estimator*

Therefore,

$$T = \frac{\sum(X_i - \mu)^2}{n} \tag{11}$$

$$n = 5 \tag{12}$$

$$\mu = \frac{1}{2} \tag{13}$$

X_i	-1	$-\frac{1}{2}$	1	$\frac{5}{2}$	3
$(X_i - \mu)^2$	$\frac{9}{4}$	1	$\frac{1}{4}$	4	$\frac{25}{4}$

TABLE 0

TABLE 1

$$\sum (X_i - \mu)^2 = 13.75 \tag{14}$$

Hence,

$$T = 2.75 \tag{15}$$