

Q: Suppose that X_1, X_2, \dots, X_n are independent and identically distributed random variables, each having probability density function $f(\cdot)$ and median θ . We want to test

$H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$

Consider a test that rejects H_0 if $S > c$ for some c depending on the size of the test, where S is the cardinality of the set $\{i : X_i > \theta_0, 1 \leq i \leq n\}$. Then which one of the following statements is true?

- (A) Under H_0 , the distribution of S depends on $f(\cdot)$.
- (B) Under H_1 , the distribution of S does not depend on $f(\cdot)$.
- (C) The power function depends on θ .
- (D) The power function does not depend on θ .

Solution:

Definition 1. Median θ is defined as

$$\Pr(X_i \leq \theta) = 0.5 \text{ for all } i \text{ from } 1 \text{ to } n.$$

$$E(S) = E\left(\sum_{i=1}^n I(X_i > \theta_0)\right) \quad (1)$$

$$= \sum_{i=1}^n E(I(X_i > \theta_0)) \quad (2)$$

$I(X_i > \theta_0)$ represents an indicator function.

$$I(X_i > \theta_0) = \begin{cases} 1, & \text{if } X_i > \theta_0 \\ 0, & \text{if } X_i \leq \theta_0 \end{cases} \quad (3)$$

Therefore,

$$E(I(X_i > \theta_0)) = P(X_i > \theta_0) = \int_{\theta_0}^{\infty} f(x) dx \quad (4)$$

$$E(S) = \sum_{i=1}^n \int_{\theta_0}^{\infty} f(x) dx \quad (5)$$

1) From (5), under H_0 , the distribution of S depends on $f(\cdot)$.

2) The power function can be expressed as:

$$\pi(\theta) = \Pr(\text{Reject } H_0 \mid H_1 \text{ is true}) \quad (6)$$

$$= \Pr(S > c \mid \theta) \quad (7)$$

Therefore, power function depends on value of θ .