Q: Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables, each having probability density function $f(\cdot)$ and median θ. We want to test
$H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$
Consider a test that rejects H_0 if $S > c$ for some c depending on the size of the test, where S is the cardinality of the set $\{i : X_i > \theta_0, 1 \leq i \leq n\}$. Then which one of the following statements is true?

(A) Under H_0, the distribution of S depends on $f(\cdot)$.
(B) Under H_1, the distribution of S does not depend on $f(\cdot)$.
(C) The power function depends on θ.
(D) The power function does not depend on θ.

Solution:

$$\Pr(X_i \leq \theta) = 0.5 \text{ for all i from 1 to n.} \tag{1}$$

$$H_0 : X_1, X_2, \ldots, X_n \sim f(\alpha) \tag{2}$$
$$H_1 : X_1, X_2, \ldots, X_n \sim f(\beta) \tag{3}$$

where, $f(\alpha)$ and $f(\beta)$ represents the probability density function (PDF) of the random variable X under H_0 and H_1 respectively.

$$\begin{aligned} E(S) &= E\left(\sum_{i=1}^{n} I(X_i > \theta_0)\right) && (4) \\ &= \sum_{i=1}^{n} E(I(X_i > \theta_0)) && (5) \end{aligned}$$

$I(X_i > \theta_0)$ represents an indicator function.

$$I(X_i > \theta_0) = \begin{cases} 1, & \text{if } X_i > \theta_0 \\ 0, & \text{if } X_i \leq \theta_0 \end{cases} \tag{6}$$

Therefore,

$$E(I(X_i > \theta_0)) = P(X_i > \theta_0) = \int_{\theta_0}^{\infty} f(x)\, dx \tag{7}$$

Therefore, distribution of S under H_0 can be taken as

$$E(S) = \sum_{i=1}^{n} \int_{\theta_0}^{\infty} f(\alpha)\, dx \tag{8}$$

And distribution of S under H_0 can be taken as

$$E(S) = \sum_{i=1}^{n} \int_{\theta_0}^{\infty} f(\beta)\, dx \tag{9}$$

Therefore, the distribution of S under both H_0 and H_1 depends on the probability density function $f(\cdot)$. The power function can be expressed as:

$$\begin{aligned} \pi(\theta) &= \Pr(\text{Reject } H_0 \,|\, H_1 \text{ is true}) && (10) \\ &= \Pr(S > c|\theta) && (11) \end{aligned}$$

Therefore, power function depends on value of θ.