Q: Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables, each having probability density function $f(\cdot)$ and median θ. We want to test
$H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$
Consider a test that rejects H_0 if $S > c$ for some c depending on the size of the test, where S is the cardinality of the set $\{i : X_i > \theta_0, 1 \leq i \leq n\}$. Then which one of the following statements is true?
(A) Under H_0, the distribution of S depends on $f(\cdot)$.
(B) Under H_1, the distribution of S does not depend on $f(\cdot)$.
(C) The power function depends on θ.
(D) The power function does not depend on θ.
Solution: UnderH_0, we have:

$$X_1, X_2, \ldots, X_n \sim f(\theta) \tag{1}$$

Now, let's consider the expected value (mean) of S under H_0:

$$E(S) = E\left(\sum_{i=1}^{n} I(X_i > \theta_0)\right) \tag{2}$$

$$= \sum_{i=1}^{n} E(I(X_i > \theta_0)) \tag{3}$$

$$E(I(X_i > \theta_0)) = P(X_i > \theta_0) = \int_{\theta_0}^{\infty} f(x)\, dx \tag{4}$$

So, the expected value of S depends on the probability density function $f(\theta)$.
Under H_1, we assume that $\theta > \theta_0$, which implies that we expect more of the X_i to be greater than θ_0.
It is assumed that the distribution of each random variable, X_i, and their independence from each other are the same under both hypotheses H_0 and H_1.
Therefore, the distribution of S under both H_0 and H_1 depends on the specific probability density function $f(\cdot)$.
The power function can be expressed as:

$$\pi(\theta) = P(\text{Reject } H_0 \,|\, H_1 \text{ is true}) \tag{5}$$

We are testing:

$$H_0 : \theta = \theta_0 \quad \text{(null hypothesis)} \tag{6}$$

$$H_1 : \theta > \theta_0 \quad \text{(alternative hypothesis)} \tag{7}$$

The power depends on the true value of θ because we are comparing the observed set $\{i \,|\, X_i > \theta_0\}$ to the critical value c, which depends on the size of the test and, indirectly, on θ_0.

Let $c(\theta_{\text{true}})$ be the critical value for a given true value of θ (θ_{true}). Then, the power can be expressed as:

$$\pi(\theta_{\text{true}}) = P(\text{Reject } H_0 \,|\, \theta = \theta_{\text{true}})$$

This probability depends on whether θ_{true} is greater than θ_0, which determines whether the alternative hypothesis is true or not. If $\theta_{\text{true}} > \theta_0$ (i.e., the alternative hypothesis is true), then the power depends on how likely the observed set $\{i \,|\, X_i > \theta_0\}$ is to be larger than the critical value $c(\theta_{\text{true}})$, given the true value of θ_{true}. This dependence on θ_{true} demonstrates that the power function depends on the true value of θ.