March 18, 2022 Exam 2 is on Tuesday March 29

10.7 Taylor Polynomials / $\underbrace{\text{Maclaurin Polynomials}}_{\text{special case when } a=0}$ → best polynomial to approximate a function near a point $x=a$.

Assume general function $f(x)$ can be represented by a polynomial of order n centered at $x=a$

$$f(x) \approx \underbrace{c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n}_{T_n(x) \text{ — Taylor polynomial of order } n}$$

n – represents the (possible) highest power of the polynomial

To be a good approximation:

want $f(a) = T_n(a)$

$$\boxed{f(a) = c_0} + c_1(\cancel{a-a})^0 + c_2(\cancel{a-a})^2 + c_3(\cancel{a-a})^3 + \cdots + c_n(\cancel{a-a})^n$$

want same slope at a: $f'(a) = T_n'(a)$

$$T_n' = 0 + c_1(1) + 2c_2(x-a)^1(1) + 3c_3(x-a)^2(1) + \cdots + nc_n(x-a)^{n-1}$$

need: $\boxed{f'(a) = c_1} + 2c_2(\cancel{a-a})^0 + 3c_3(\cancel{a-a})^2 + \cdots + nc_n(\cancel{a-a})^{n-1}$

want same concavity: $f''(a) = T_n''(a)$

$$T_n'' = 0 + 2c_2(1) + 3\cdot 2c_3(x-a)^1 + 4\cdot 3c_4(x-a)^2 + \cdots + n(n-1)c_n(x-a)^{n-2}$$

need: $\boxed{f''(a) = 2c_2} + 3\cdot 2c_3(\cancel{a-a})^0 + 4\cdot 3c_4(\cancel{a-a})^2 + \cdots + n(n-1)c_n(\cancel{a-a})^{n-2}$

want same $f'''(a) = T_n'''(a)$

$$T_n'''(x) = 0 + 3\cdot 2c_3(1) + 4\cdot 3\cdot 2c_4(x-a)^1 + \cdots + n(n-1)(n-2)c_n(x-a)^{n-3}$$

need: $\boxed{f'''(a) = 3\cdot 2c_3} + 4\cdot 3\cdot 2c_4(\cancel{a-a})^0 + \cdots + n(n-1)(n-2)c_n(\cancel{a-a})^{n-3}$

KNOW:

$c_0 = f(a) = \dfrac{f(a)}{0!}$

$c_1 = f'(a) = \dfrac{f'(a)}{1!}$

$c_2 = \dfrac{f''(a)}{2} = \dfrac{f''(a)}{2!}$

$c_3 = \dfrac{f'''(a)}{3\cdot 2} = \dfrac{f'''(a)}{3!}$

A Taylor Polynomial for $f(x)$ of order n centered at $x = a$ has the form

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

- Special case, when $a=0$ sometimes called a Maclaurin polynomials
- $f^{(n)}(a)$ — represents the n^{th} derivative of $f(x)$
- $n=0$ term $\quad \frac{f^{(0)}(a)}{0!}(x-a)^0 \quad f^{(0)}$ — 0 derivative $f(a)$

$0! \equiv 1 \quad (x-a)^0 = 1$

Ex 1: Find T_2 and T_3 for $f(x) = \cos(2x)$ centered at $a = \pi/2$.

ALWAYS create chart

* must start with $n=0$

n	$f^{(n)}(x)$	$f^{(n)}(a) = f^{(n)}(\pi/2)$	$\frac{f^{(n)}(a)}{n!}$
0	$\cos(2x)$	$\cos(2\cdot\pi/2) = \cos(\pi) = -1$	
1	$-2\sin(2x)$	$-2\sin(2\cdot\pi/2) = 0$	
2	$-2\cdot2\cos(2x)$	$-4\cos(2\cdot\pi/2) = 4$	
3	$2\cdot2\cdot2\sin(2x)$	$8\sin(2\cdot\pi/2) = 0$	

column included in book

$$T_n(x) = \underbrace{\overbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2}^{T_2} + \frac{f'''(a)}{3!}(x-a)^3}_{T_3} + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$$T_2 = -1 + 0(x-\pi/2) + \frac{4}{2!}(x-\pi/2)^2$$

simplified $\quad T_2 = -1 + 2(x-\pi/2)^2$

$$T_3 = T_2 + \frac{f'''(\pi/2)}{3!}(x-\pi/2)^3$$

$$= -1 + 2(x-\pi/2)^2 + \frac{0}{3!}(x-\pi/2)^3$$

so $\quad T_3 = -1 + 2(x-\pi/2)^2$

EX 2: Find Taylor polynomial of order 4 ($T_4(x)$) and Taylor polynomial of order n ($T_n(x)$ in summation notation) for $f(x) = \ln(1+x)$ centered at $a = 0$. [could ask for Maclaurin]

Chart:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	$\ln(1+0) = \ln(1) = 0$
1	$\frac{1}{1+x}(1) = (1+x)^{-1}$	$(1+0)^{-1} = 1^{-1} = 1$
2	$-1(1+x)^{-2}$	$-1(1)^{-2} = -1$
3	$(-1)(-2)(1+x)^{-3}$	$2(1)^{-3} = 2$
4	$(-1)(-2)(-3)(1+x)^{-4}$	$-6(1)^{-4} = -6$

$$T_n(x) = \cancel{f(a)} + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$$T_4 = 0 + 1(x-0)^1 - \frac{1}{2!}(x-0)^2 + \underbrace{\frac{2}{3!}}_{\frac{\cancel{2}}{1\cdot\cancel{2}\cdot 3} = \frac{1}{3}}(x-0)^3 - \underbrace{\frac{6}{4!}}_{\frac{\cancel{6}}{1\cdot\cancel{2}\cdot\cancel{3}\cdot 4} = \frac{1}{4}}(x-0)^4$$

Simplified: $T_4 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$

1 term is positive

$T_n(x) \longrightarrow$ find pattern of coefficient $\frac{f^{(n)}(a)}{n!} \longrightarrow$ above we see coefficient for n^{th} term is $\frac{1}{n}$

so $T_n(x) = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k} x^k$

for alternating will be $(-1)^k$ or $(-1)^{k+1}$ whichever matches your terms

Error Bound: If $f^{(n+1)}$ exists and is continuous and one can find K such that $\left|f^{(n+1)}(u)\right| \leq K$ for all u between a and x, then the error bound for using $T_n(x)$ to approximate $f(x)$ at a value x near a is given by

$$\left|f(x) - T_n(x)\right| \leq K \frac{|x-a|^{n+1}}{(n+1)!}$$

Ex 3: Find T_2, T_3 and T_n for $f(x)=e^{3x}$ centered at $a=1$ AND use the error bound formula to find the maximum possible error for using T_3 to approximate e^{3x} at $x=1.1$

chart

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	e^{3x}	e^3
1	$3e^{3x}$	$3e^3$
2	3^2e^{3x}	3^2e^3
3	3^3e^{3x}	3^3e^3
$\vdots$		$\vdots$
4	3^4e^{3x}	
n	3^ne^{3x}	3^ne^3

$$T_n(x) = \underbrace{\overbrace{f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2}^{T_2}+\frac{f'''(a)}{3!}(x-a)^3}_{T_3}+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$$T_2 = e^3 + 3e^3(x-1) + \frac{3^2e^3}{2!}(x-1)^2$$

$$T_3 = e^3 + 3e^3(x-1) + \frac{3^2e^3}{2!}(x-1)^2 + \frac{3^3e^3}{3!}(x-1)^3$$

note $\frac{3^3e^3}{3!} = \frac{3^2e^3}{2}$

$T_n \longrightarrow$ need pattern (see chart ... $f^{(n)}(1)=3^ne^3$)

$$T_n(x) = \sum_{k=0}^{n} \frac{3^ke^3}{k!}(x-1)^k$$

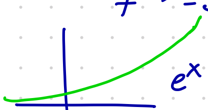
Error bound for $|f(1.1)-T_3(1.1)| \longrightarrow$ error of using $T_3(1.1)$ to approximate $e^{3.3}$

$$\boxed{\left|f(x)-T_n(x)\right| \leq K\,\frac{|x-a|^{n+1}}{(n+1)!}}$$

$K \geq |f^{(4)}(u)|$ for u between $\underset{x}{1.1}$ and $\underset{a}{1}$

$f^{(4)} = 3^4e^{3x}$ need max of $|3^4e^{3u}|$ on $1 \leq u \leq 1.1$

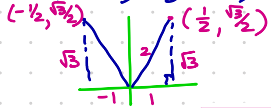
we get $K = 3^4e^{3.3}$



e^x increasing $\Rightarrow$ max at R endpoint of interval

$$|f(1.1)-T_3(1.1)| \leq 3^4e^{3.3}\,\frac{|1.1-1|^4}{4!}$$

Ex 4: Find Taylor polynomial of order 3, $T_3(x)$, for $f(x)=\cos(2x)$ centered at $a=\pi/3$.



Chart

n	$f^{(n)}(x)$	$f^{(n)}(\pi/3)$
0	$\cos(2x)$	$\cos(2\cdot\pi/3) = -1/2$
1	$-2\sin(2x)$	$-2\sin(2\pi/3) = -2(\sqrt{3}/2) = -\sqrt{3}$
2	$-2^2\cos(2x)$	$-4\cos(2\pi/3) = -4(-1/2) = 2$
3	$2^3\sin(2x)$	$8\sin(2\pi/3) = 8(\sqrt{3}/2) = 4\sqrt{3}$
4	$2^4\cos(2x)$	

$$\boxed{T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3} + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$$T_3(x) = -\frac{1}{2} - \sqrt{3}(x-\pi/3) + \frac{2}{2!}(x-\pi/3)^2 + \frac{4\sqrt{3}}{3!}(x-\pi/3)^3$$

Use the Error bound formula to determine maximum possible error for using $T_3(x)$ to approximate $\cos(2\pi/5)$.

find an upper bound for $|\cos(2\pi/5) - T_3(\pi/5)|$

$$\boxed{|f(x) - T_n(x)| \le K\frac{|x-a|^{n+1}}{(n+1)!}}$$

$2\cdot x$ where $x = \pi/5$

$a = \pi/3$

$|x-a| = |\pi/5 - \pi/3| = \left|\frac{-2\pi}{15}\right| = \frac{2\pi}{15}$

$\cos(2x)$

$n=3$ then $n+1=4$

$$|\cos(2\pi/5) - T_3(\pi/5)| \le \frac{2^4\left(\frac{2\pi}{15}\right)^4}{4!}$$

K number where $|f^{(n+1)}(u)| \le K$ for u between a & x

$|f^{(4)}(u)| = |2^4\cos(2u)| \le K$ $\quad u$ between $\pi/5$ and $\pi/3$

$|2^4\cos(2u)| = 2^4|\cos(2u)| \le 2^4(1)$ so $K = 2^4$

KNOW $|\cos(2u)| \le 1$ ← use this!

Ex5: Find $T_4(x)$ for $f(x)=x\ln(x)$ centered at $a=1$ and use the error bound to find an upper bound on error for $\left|T_4(\tfrac{1}{2}) - \underbrace{\tfrac{1}{2}\ln(\tfrac{1}{2})}_{f(1/2)}\right|$ $\quad x = \tfrac{1}{2}$

Chart

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x\ln(x)$	$1\ln(1) = 0$
1	$\ln(x) + x(\frac{1}{x}) = \ln(x)+1$	$\ln(1)+1 = 0+1 = 1$
2	$1/x = x^{-1}$	$1^{-1} = 1$
3	$-1(x^{-2})$	$-1(1)^{-2} = -1$
4	$2x^{-3}$	$2(1)^{-3} = 2$
5	$-6x^{-4}$	

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

$$T_4 = 0 + 1(x-1) + \frac{1}{2!}(x-1)^2 - \frac{1}{3!}(x-1)^3 + \frac{2}{4!}(x-1)^4$$

Error Bound for $\left|f(\tfrac{1}{2}) - T_4(\tfrac{1}{2})\right|$

$$\boxed{\left|f(x) - T_n(x)\right| \le K \frac{|x-a|^{n+1}}{(n+1)!}}$$

$x = \tfrac{1}{2}$, $a = 1$; $n = 4$ so $n+1 = 5$

$|x-a| = |\tfrac{1}{2} - 1| = |-\tfrac{1}{2}| = \tfrac{1}{2}$

$f^{(5)}(x) = -6x^{-4} = \dfrac{-6}{x^4}$

$K = \max$ of $|f^{(5)}(u)|$ on $\tfrac{1}{2} \le u \le 1$

$= \max$ of $\left|\dfrac{-6}{u^4}\right|$

since u in denominator ...
largest when u is smallest
so at $u = \tfrac{1}{2}$

so $K = \dfrac{6}{(\frac{1}{2})^4} = 6 \cdot 2^4$

Upper Bound on Error:

$$\left|T_4(\tfrac{1}{2}) - \underbrace{\tfrac{1}{2}\ln(\tfrac{1}{2})}_{f(1/2)}\right| = \left|f(\tfrac{1}{2}) - T_4(\tfrac{1}{2})\right| \le 6 \cdot 2^4 \frac{(\frac{1}{2})^5}{5!} = \frac{\cancel{2}\cdot\cancel{3}\cdot\cancel{2^4}\cdot\frac{1}{\cancel{2^5}}}{1\cdot 2\cdot\cancel{3}\cdot 4\cdot 5} = \frac{1}{40}$$