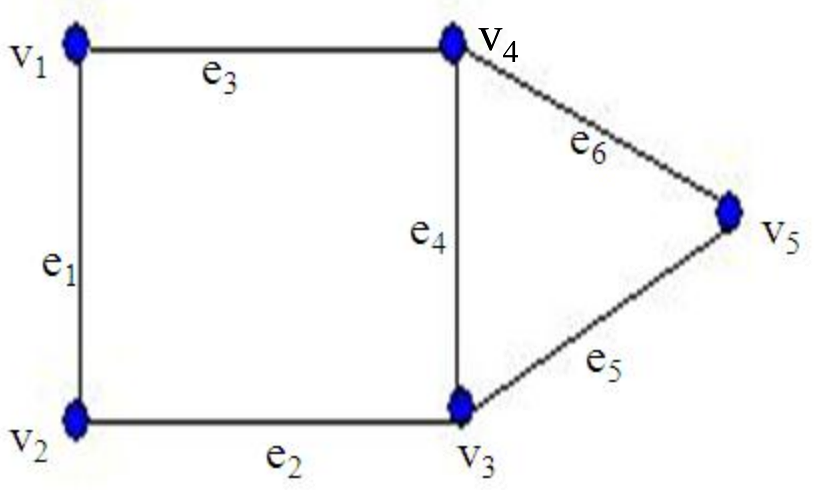
Unit 5

Graphs and Graph algorithms

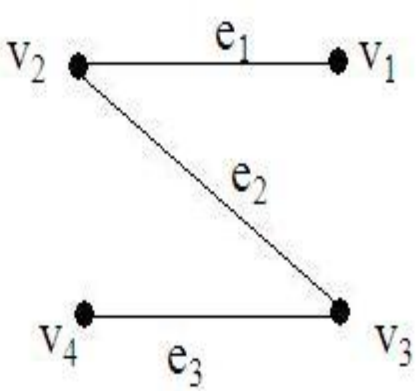
GRAPH

- Consider an ordered pair G=<V,E> or (V,E), where V is the non-empty set of element called vertices (also called node, point, dot) and E is the possibly empty set of element called edge.
- Vertex set $V=\{v_1,v_2,v_3,v_4,v_5\}$
- Edge set $E=\{e_1,e_2,e_3,e_4,e_5,e_6\}$



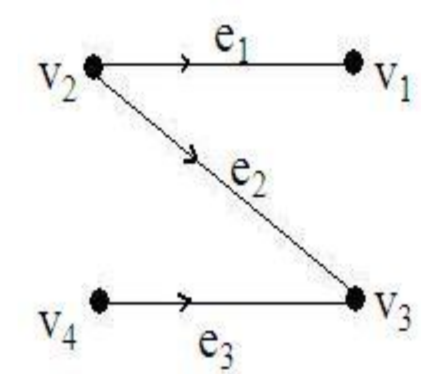
END POINT

- Let G=<V,E> be a graph and e belongs to E is any edge associated with the pair (u, v) then u and v are known as end point of an edge e.
- We say that e joins u and v. we write as e $\leftrightarrow$(u, v)



$e_1 \leftrightarrow (v_1, v_2)$

[Undirected graph]



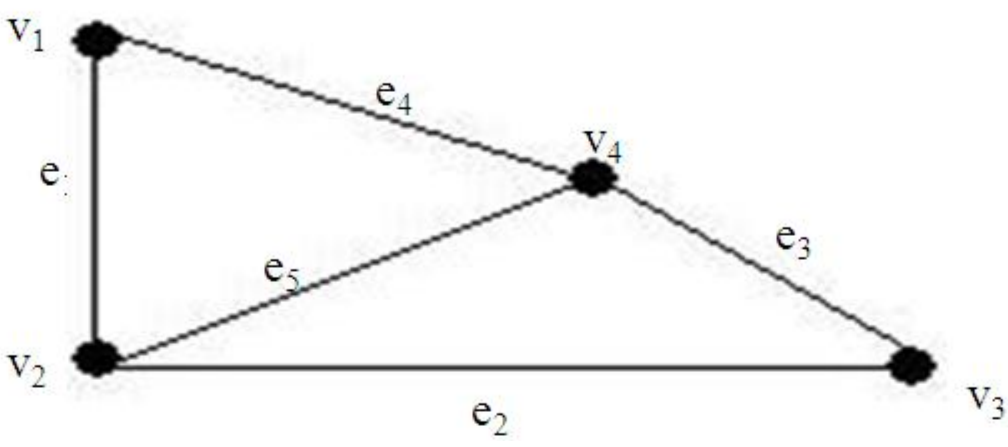
$e_1 \leftrightarrow (V_2, v_1)$

[Directed graph]

UNDIRECTED GRAPH

- A graph G=<V,E> is said to be undirected if every edge is associated with an unordered pair of vertices.

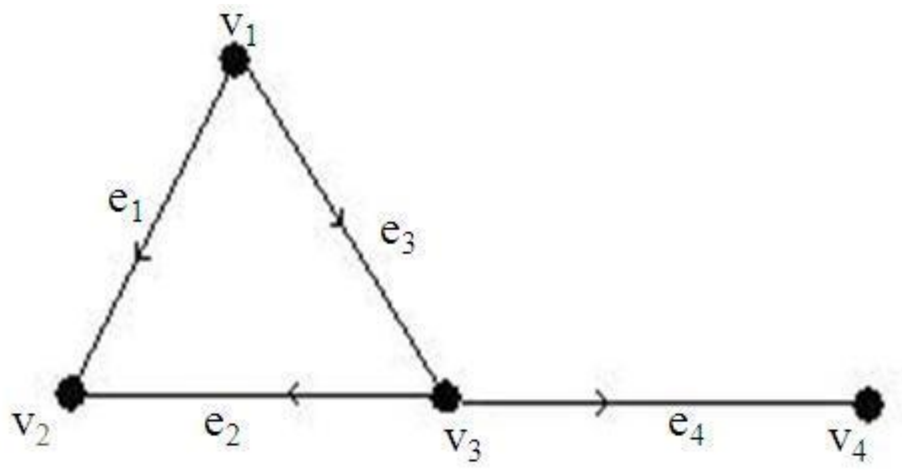
- In this case each edge is known as undirected



DIRECTED GRAPH (DIGRAPH)

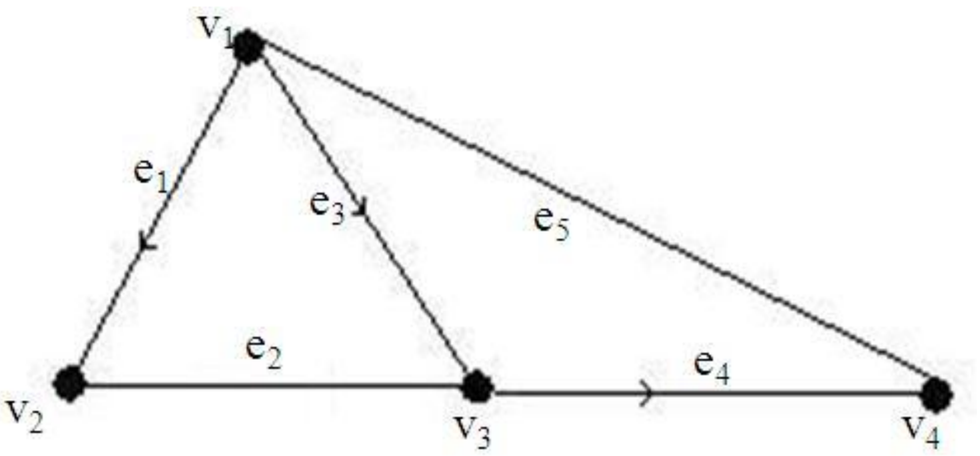
- A graph G=<V,E> is said to be directed if every edge is associated with an ordered pair of vertices.

- In this case each edge is known as directed



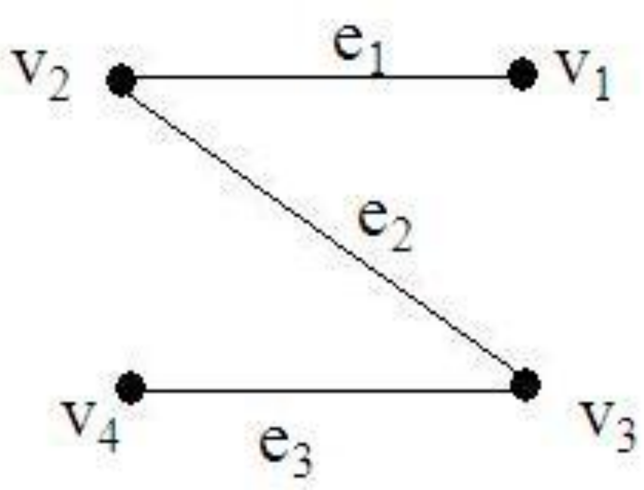
MIXED GRAPH

- A graph is known as mixed graph if some edges are directed and some edges are undirected



ADJACENT VERTICES

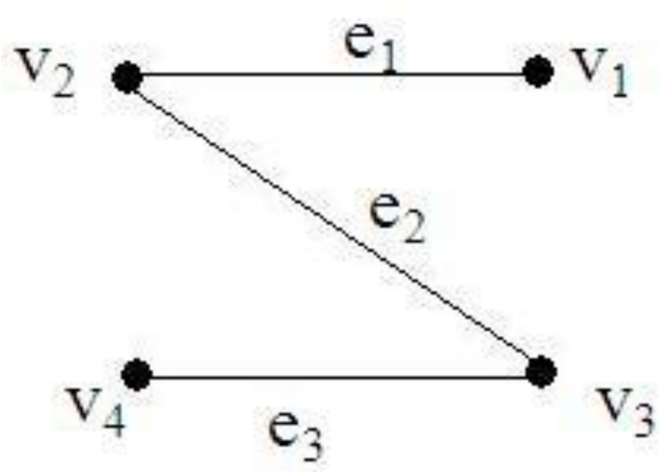
➢ Let G be a graph, then two vertices of a graph G are said to be adjacent if there is an edge associated with them.



• Here, v_1 and v_2 are adjacent vertices.

ADJACENT EDGES

- Let G be a graph then two or more edges of a graph are said to be adjacent if there is a common end vertex.

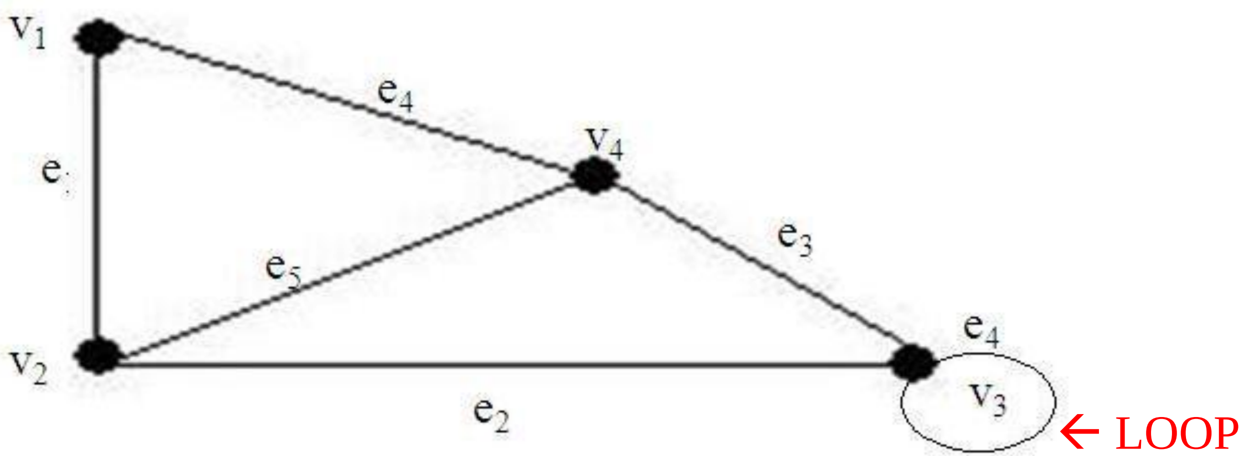


• Here, e_1 and e_2 are adjacent edges.

LOOP (SLING)

- Let G be a graph then an edge of a graph is said to be loop or sling if it joins a vertex to itself.

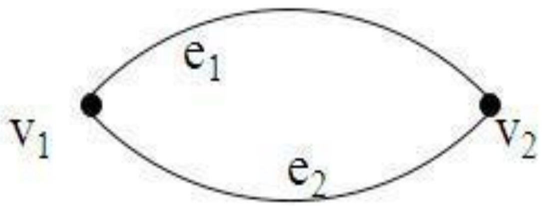
- i.e. origin and terminal are same.



PARALLEL EDGES

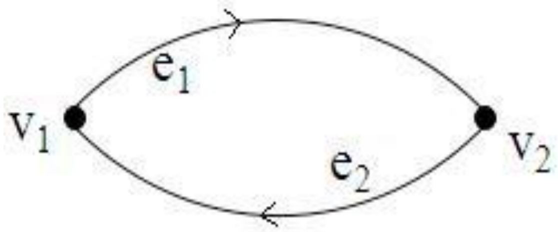
- Let G be a graph then two or more edges are said to be parallel if their end points are identical.

[Undirected graph]



← parallel edges

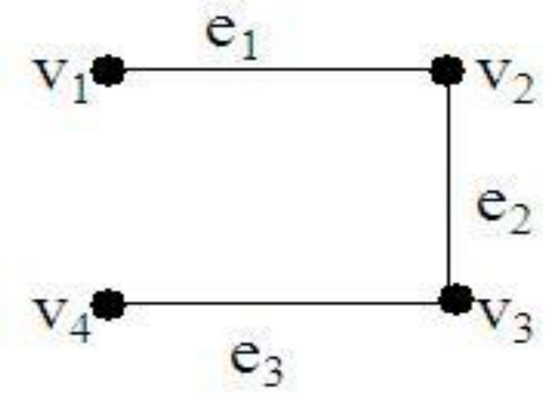
[Directed graph]



← Not parallel edges

PENDANT VERTAX

- Let G be a graph, then a vertex of a graph G is said to be pendant if it incident with only one edges of a graph.

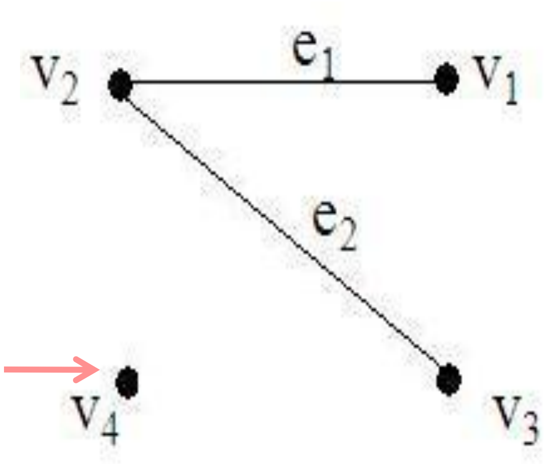


- v_1 and v_4 are pendant vertex

ISOLATED VERTAX

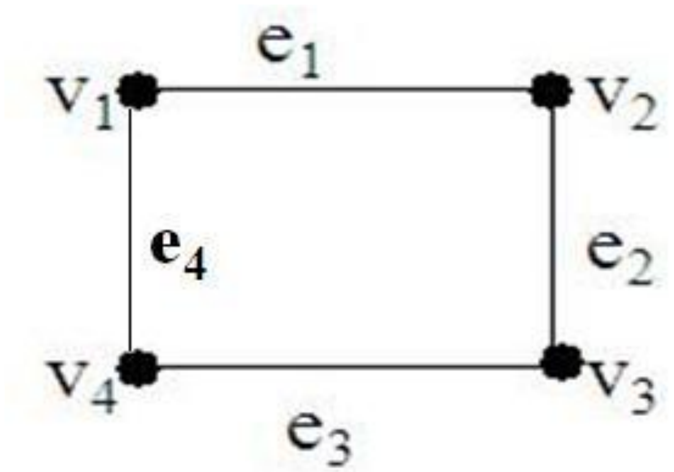
- Let G be a graph then a vertex of a graph G is said to be isolated if it doesn't incident with any edge of a graph.

Isolated vertex



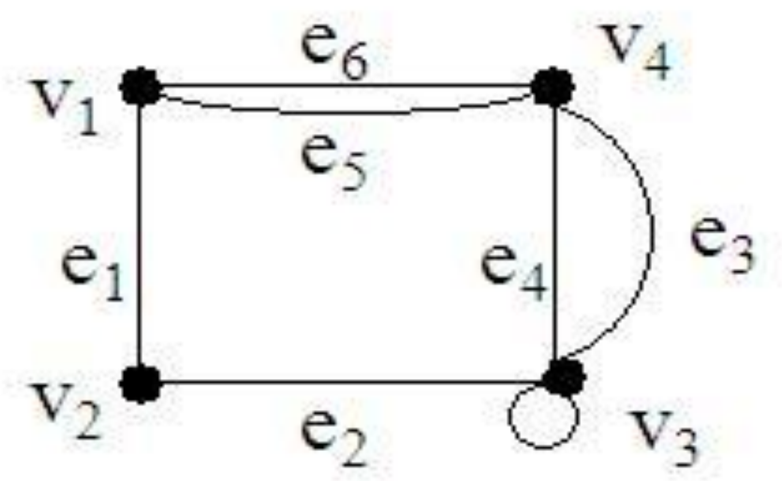
SIMPLE GRAPH

- A graph which neither contain loops nor parallel edges is known as simple graph.



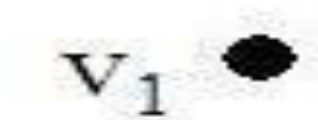
MULTI-GRAPH

- A graph is not simple is known as multi-graph.
- i.e. A graph which contains either loop or parallel edges or both.

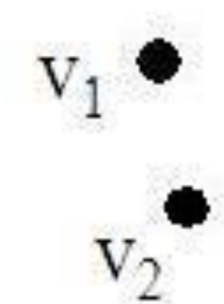


NULL GRAPH

- Let G=<V,E> be a graph, then it is known as null graph if E= Ø.
- i.e. there is no edges in graph or all the vertices of a graph are isolated.

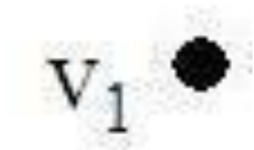
N_1 = 

[Also called trivial graph]

N_2 = 

TRIVIAL GRAPH

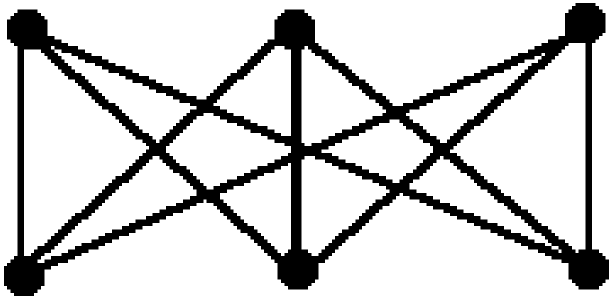
- A graph having one vertex and no edges is known as trivial graph.



← Trivial graph

FINITE GRAPH and INFINITE GRAPH

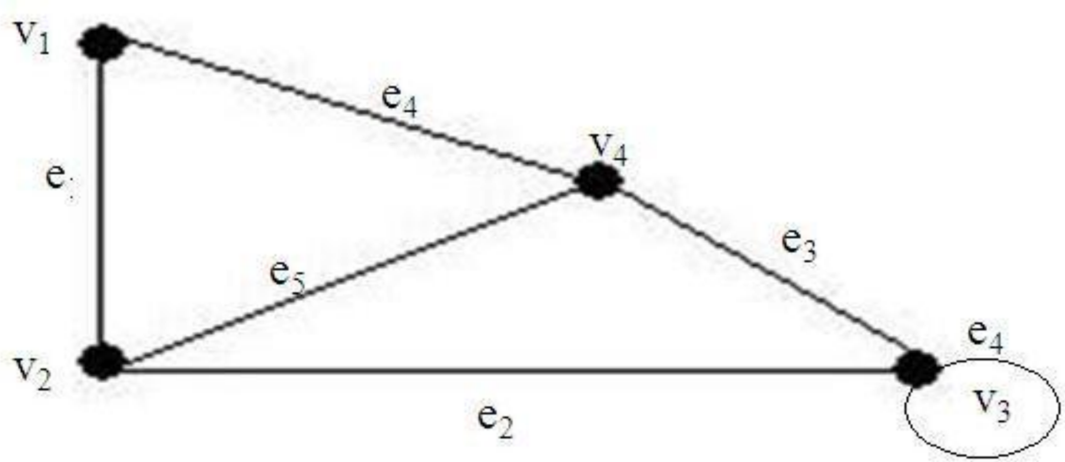
- Let G be a graph, then it said to be a finite graph if number of vertices and number of edges are finite.



- Let G be a graph, then it said to be a Infinite graph if number of vertices and number of edges are Infinite.

DEGREE (FOR UNDIRECTED GRAPH)

- Let G=<V,E> be an undirected graph and v belongs to V be any vertex, then degree of a vertex V is the total no. of edges incident with V and each loop is counted twice .
- Each loop counted twice.
- It is denoted by d(v) or deg(v) or $d_G(V)$



$d(v_1)=2$
$d(v_2)=3$
$d(v_3)=4$
$d(v_4)=3$

DEGREE (FOR DIRECTED GRAPH)

- Let G=<V,E> be a directed graph and v belongs to V be any vertex then there are three types of degree.

Indegree of a vertex:-

- Total no. of edges for which v is terminal vertex.
- It is denoted by $d_i(v)$.

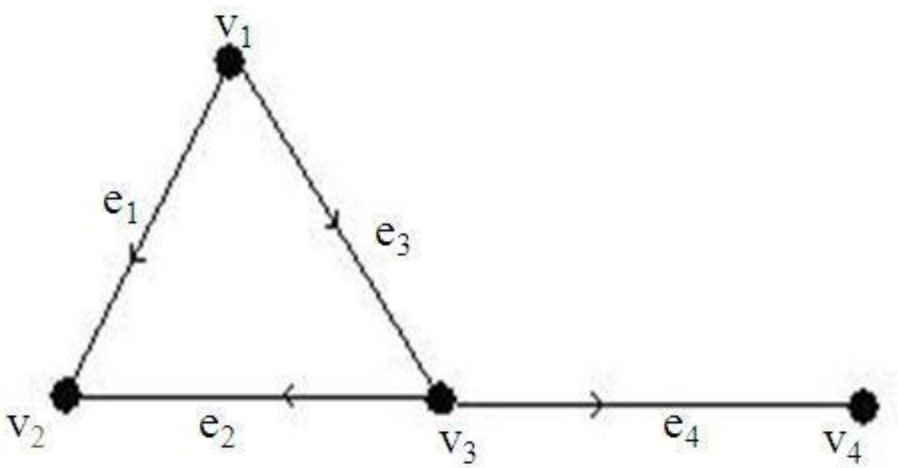
Outdegree of a vertex:-

- Total no. of edges for which v is initial vertex.
- It is denoted by $d_o(v)$.

Total degree of a vertex:-

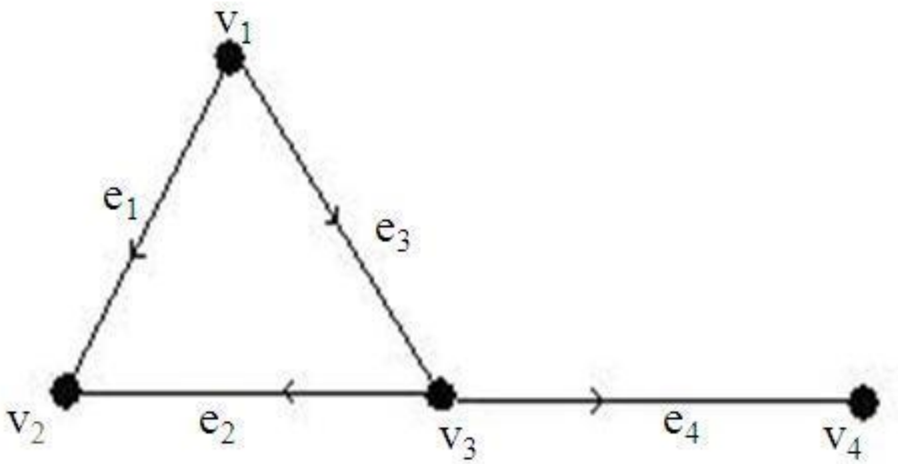
- Sum of indegree and outdegree of a vertex.
- It is denoted by d(v).

EXAMPLE:-



Indegree	Outdegree	total degree
$d_i(v_1)=$	$d_o(v_1)=$	$d(v_1)=$
$d_i(v_2)=$	$d_o(v_2)=$	$d(v_2)=$
$d_i(v_3)=$	$d_o(v_3)=$	$d(v_3)=$
$d_i(v_4)=$	$d_o(v_4)=$	$d(v_4)=$

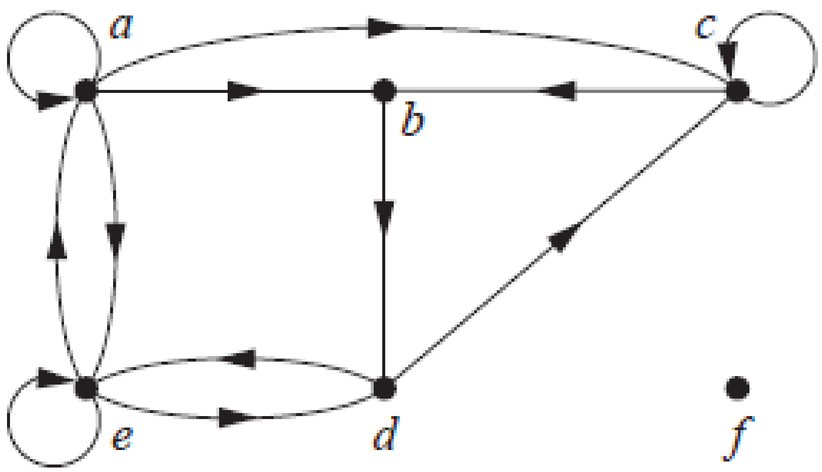
EXAMPLE:-



Indegree	Outdegree	total degree
$d_i(v_1)=0$	$d_o(v_1)=2$	$d(v_1)=2$
$d_i(v_2)=2$	$d_o(v_2)=0$	$d(v_2)=2$
$d_i(v_3)=1$	$d_o(v_3)=2$	$d(v_3)=3$
$d_i(v_4)=1$	$d_o(v_4)=0$	$d(v_4)=1$

Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in

Figure



G

Theorem

Let $G = (V, E)$ be a graph with directed edges. Then

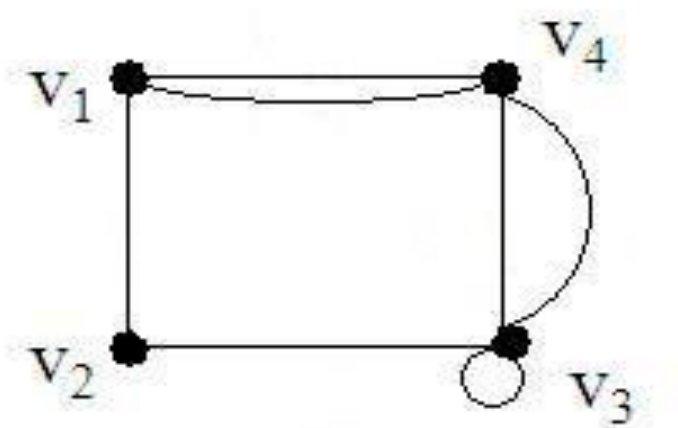
$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

Where $|E|$ is total no. of edges.

ODD VERTEX AND EVEN VERTEX

- A vertex whose degree is odd number is known as odd vertex.

- A vertex whose degree is even number is known as even vertex.



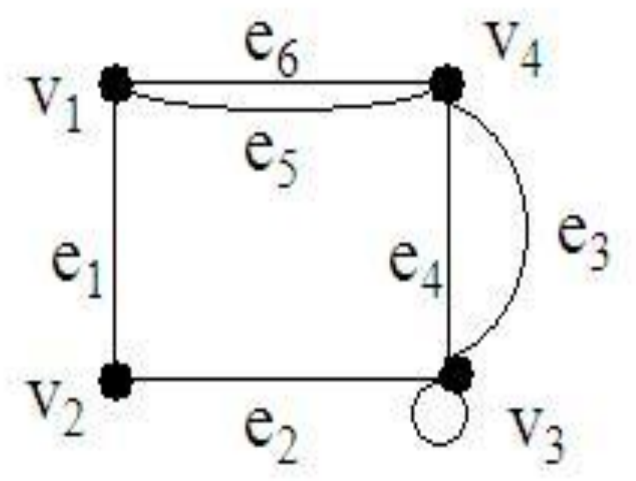
Odd vertex	Even vertex
v1,v3	v2,v4
d(v1)=3	d(v2)=2
d(v3)=5	d(v4)=4

HANDSHAKING THEORAM(1 st Theorem of Graph Theory)

- For any graph G=<V,E>, the sum of degree of all vertices is twice to the total no. of edges.

$$\sum_{v \in V} d(v) = 2e$$

EXAMPLE:-



$d(v_1) + d(v_2) + d(v_3) + d(v_4)$

$= 3+2+5+4$

Sum of degree=14

Here, total edges=7

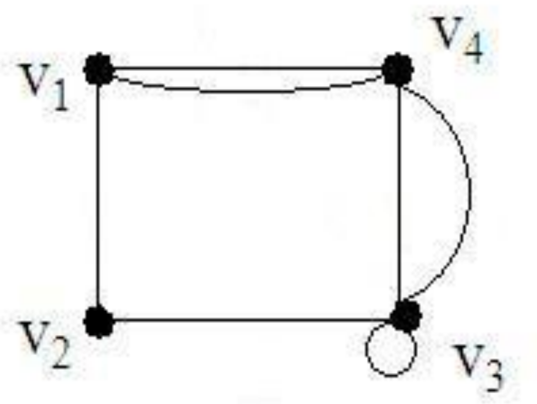
Now, $2e = 2*7$

$= 14$

COROLLARY

- For any graph G, total no. of odd vertices is always even.

EXAMPLE:-



- Here, odd vertex is v_1 and v_3.

$d(v_1) + d(v_3)$
$= 3+5$
$= 8$
$=$ even

EXAMPLE :

How many edges are there in a graph with 10 vertices each of degree six?

REGULAR GRAPH

- A graph is said to be regular if all the vertices having same total degree.

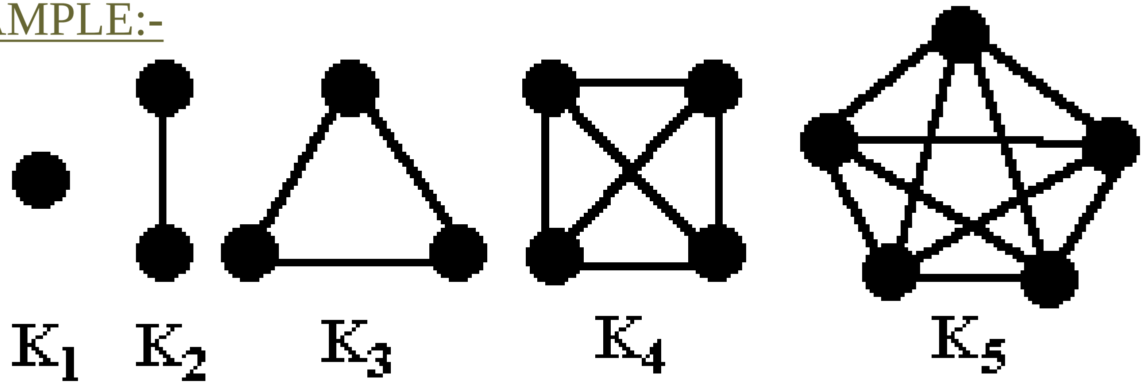
K-REGULAR GRAPH:-

- Regular graph having K degree than it's called K-regular graph.

COMPLETE GRAPH

- A simple graph is said to be complete if every pair of distinct vertices is adjacent .
- No loop ,no parallel edges.
- It is denoted by Kn.

EXAMPLE:-



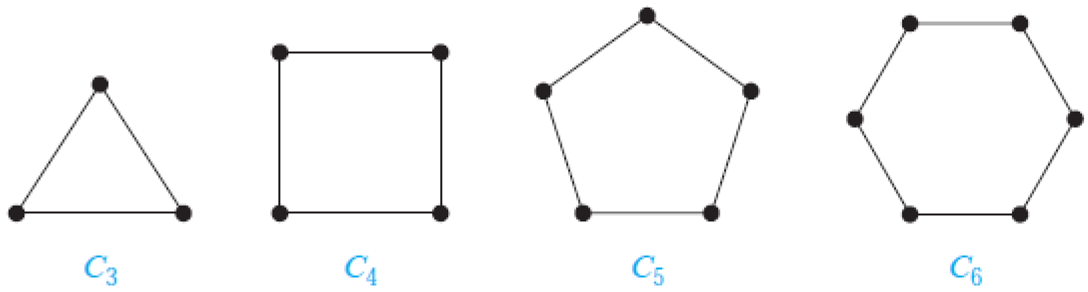
Result

- A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called **non complete**.

- Complete graph with 'n' vertices has precisely ${}_nC_2 \left(i.e. \frac{n(n-1)}{2} \right)$ edges.

Cycles:

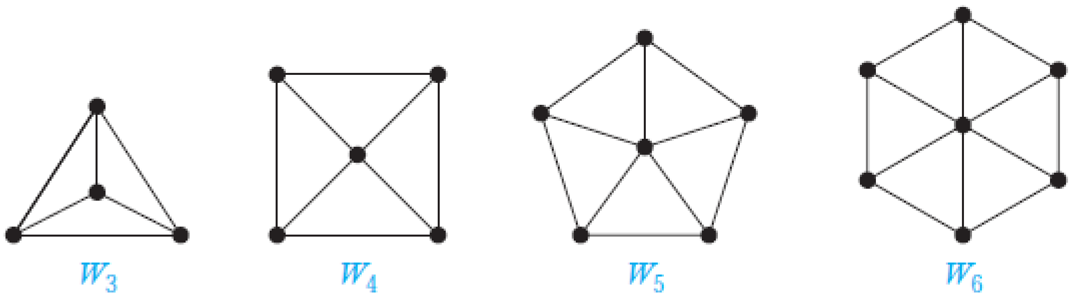
A **cycle** $\boldsymbol{C_n}$, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles C_3, C_4, C_5, and C_6 are displayed in Figure .



Wheels

We obtain a **wheel *Wn*** when we add an additional vertex to a cycle *Cn*, for $n \geq 3$,and connect this new vertex to each of the *n* vertices in *Cn*, by new edges.

The wheels *W*3, *W*4,*W*5, and *W*6 are displayed in Figure .



Bipartite Graph

A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets $V1$ and $V2$ such that every edge in the graph connects a vertex in $V1$ and a vertex in $V2$

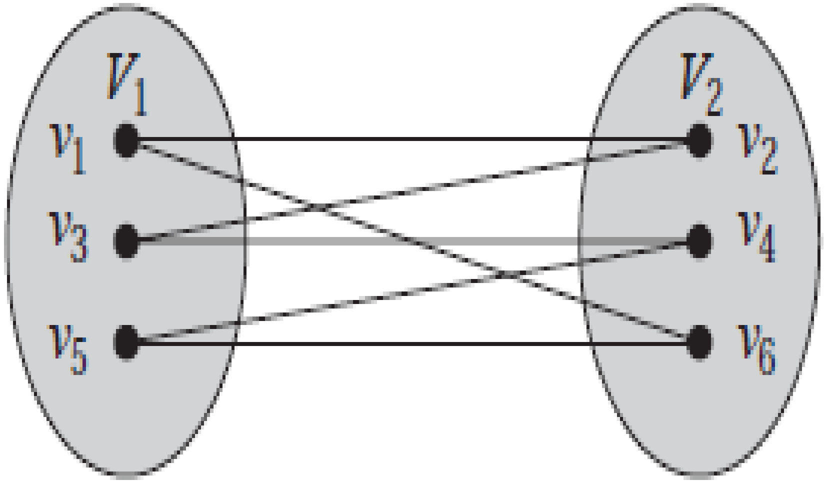
(so that no edge in G connects either two vertices in $V1$ or two vertices in $V2$).

When this condition holds, we call the pair $(V1, V2)$ a *bipartition* of the vertex set V of G.

C_6 is Bipartite or not.

C_6 is bipartite

because its vertex set can be partitioned into the two sets $V1 = \{v1, v3, v5\}$ and $V2 = \{v2, v4, v6\}$, and every edge of $C6$ connects a vertex in $V1$ and a vertex in $V2$.



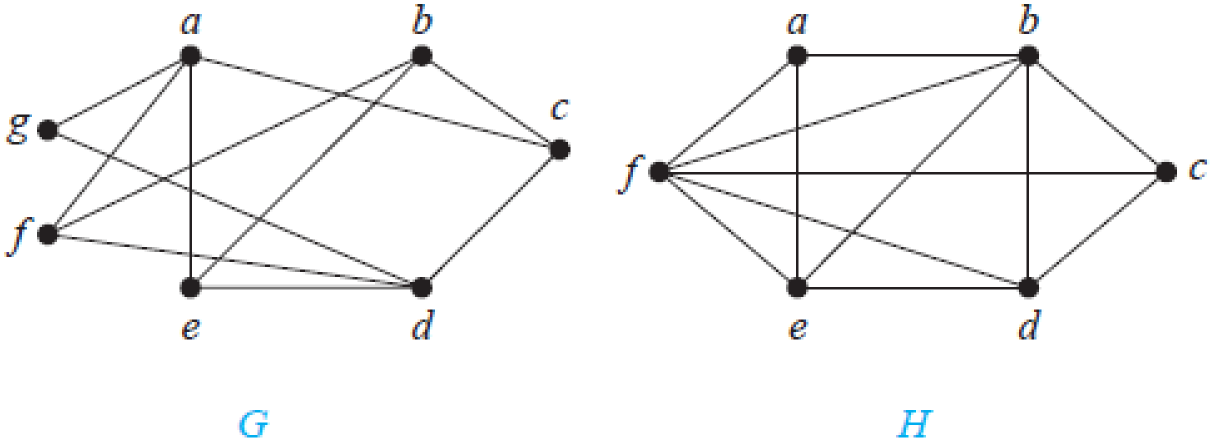
K_3 is Bipartite or not.

K_3 is not bipartite

If we divide the vertex set of K_3 into two disjoint sets, one of the two sets must contain two vertices.

If the graph were bipartite, these two vertices could not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge.

Are the graphs *G* and *H* displayed in Figure bipartite?



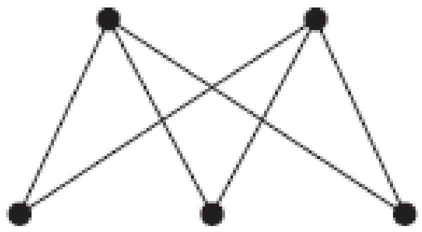
Theorem

A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

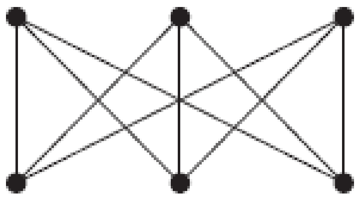
Complete Bipartite Graph

A **complete bipartite graph on m and n vertices,$\mathbf{K_{m,n}}$** is a graph that has its vertex set partitioned into two subsets of *m* and *n* vertices, respectively with an edge between two each pair of vertices.

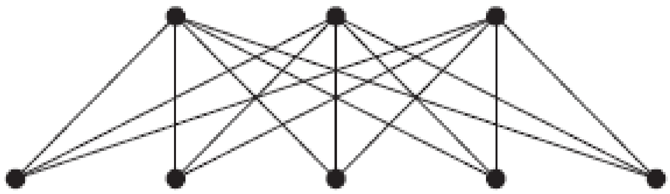
$K_{2,3}$, $K_{3,3}$, $K_{3,5}$, and $K_{2,6}$ are the complete bipartite graphs



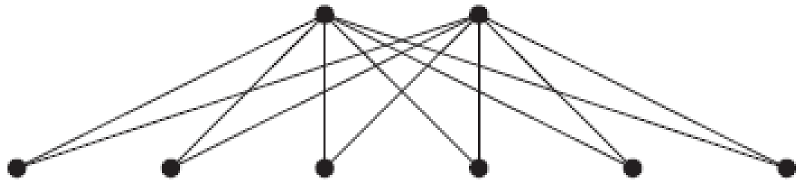
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



$K_{2,6}$

Note :

1. $K_{m,n}$ *has* $m+n$ *vertices and* mn *edges.*
2. *A complete bipartite graph* $K_{m,n}$ *is not a regular if* $m \neq n$.
3. *A graph which contains a triangle can not be bipartite.*

Isomorphism of Graphs

Introduction :

We often need to know whether it is possible to draw two graphs in the same way.

i.e. do the graphs have the same structure when we ignore the identities of their vertices?

Isomorphism of Graphs

The simple graphs $G_1 = (V_1,E_1)$ and

$G_2 = (V_2,E_2)$ are *isomorphic* if there exists a one to one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1.

Such a function f is called an *isomorphism*.

∗ Two simple graphs that are not isomorphic are called *non isomorphic*.

Isomorphic Graphs

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic graphs if there exists functions

$f : V_1 \to V_2$ & $g : E_1 \to E_2$ such that

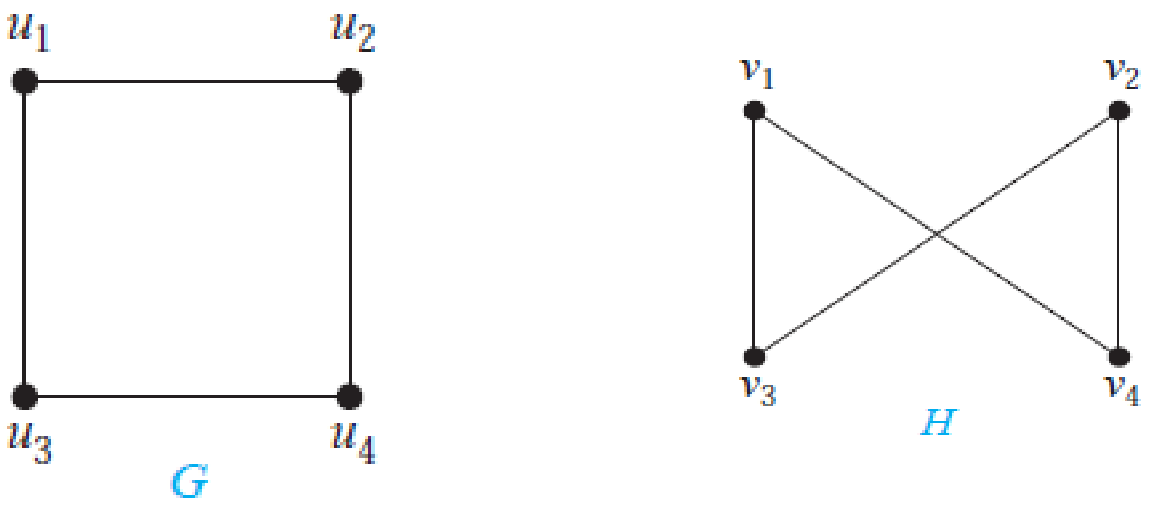
1. $f : V_1 \to V_2$ is one – one and onto function
2. $g : E_1 \to E_2$ is one – one and onto function
3. The incidence relations are preserved.(If there is an edge e between nodes u & v of V_1 then $g(e)$ should be an edge between $f(u)$ & $f(v)$ of V_2

Conclusion

We conclude that Isomorphic graphs has

1. Same no. of vertices
2. Same no. of edges
3. An equal no. of vertices with given degree
4. The incidence relation must be preserved (Preserves adjacency relationship)

The graphs $G = (V, E)$ and $H = (W, F)$, displayed in figure are isomorphic or not?



Solution:

1. Graphs have 4 vertices. $V=\{u_1,u_2,u_3,u_4\}$ & $W=\{v_1,v_2,v_3,v_4\}$
2. Graphs have 4 edges

   $$E=\{\{u_1,u_2\},\{u_1,u_3\},\{u_4,u_2\},\{u_4,u_3\}\},F=\{\{v_1,v_4\},\{v_1,v_3\},\{v_3,v_2\},\{v_4,v_2\}\}$$

3. 4 vertices having degree 2
4. Define a function $f:V\to W$ as $f(u_1)=v_1$, $f(u_2)=v_4$, $f(u_3)=v_3$, and $f(u_4)=v2$ is a one to one correspondence between V and W.

Continue …..

Further, Define a function $g : E \to F$ as

$\{u_1, u_2\} \in E$ and $\{f(u_1), f(u_2)\} = \{v_1, v_4\} \in F$

$\{u_1, u_3\} \in E$ and $\{f(u_1), f(u_3)\} = \{v_1, v_3\} \in F$

$\{u_4, u_2\} \in E$ and $\{f(u_4), f(u_2)\} = \{v_2, v_4\} \in F$

$\{u_4, u_3\} \in E$ and $\{f(u_4), f(u_3)\} = \{v_2, v_3\} \in F$

And $\{u_1, u_4\} \notin E$ and $\{f(u_1), f(u_4)\} = \{v_1, v_2\} \notin F$

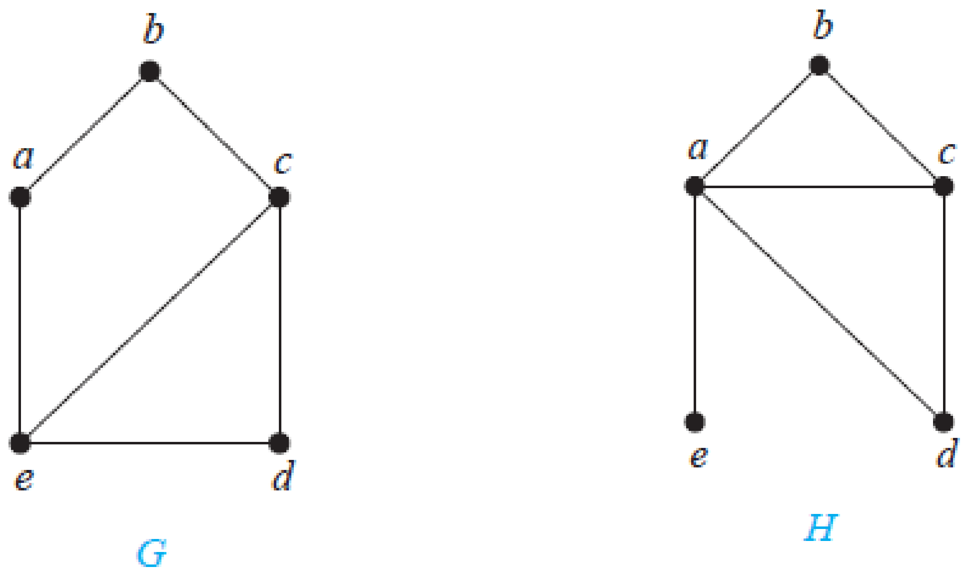
$\{u_2, u_3\} \notin E$ and $\{f(u_2), f(u_3)\} = \{v_4, v_3\} \notin F$

Continue….

Hence *f* preserves adjacency as well as non-adjacency of the vertices and edges also.

G and H are Isomorphic graphs.

The graphs in following figure are isomorphic or not?



Solution:

Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one.

It follows that G and H are not isomorphic.

Complement of a Graph

Complement of Graph G=(V,E) is denoted by $\overline{G}$ or G'. It is defined as $V(G)=V(\overline{G})$ and two vertices are adjacent in $\overline{G}$ iff they are not adjacent in G.

Example

What is the complement graph of complete bipartite graph $K_{2,3}$?

Is it regular graph.

Note:

A graph G is self-complementary if it is isomorphic its complement.

Path, Cycles and Connectivity

Path:

A path from ‘u’ to ‘v’ in the graph is a sequence of one or more edges in such a way that terminal of every edge will become initial of next edge of a graph. That is,

$$P = \{(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)\}$$

where v_0 is an initial vertex and v_n is a terminal vertex.

Path length:

Total number of edges involved in a path is known as path length.

Simple Path:

A path in which all edges are distinct is known as Simple path.

Elementary Path:

A path in which all vertices are distinct is known as Elementary path.

Trivial Path:

A path of length zero is known as Trivial Path.

Closed Path:

A path in which initial and terminal vertices are same is known as closed path.

Cycle (Circuit):

A non trivial path in which initial and terminal vertices are same is known as cycle.

Simple Cycle:

A cycle in which all the edges are distinct is known as simple cycle.

Elementary Cycle:

A cycle in which all the internal vertices are distinct is known as elementary cycle.

Cyclic Graph:

A graph which contains at least one cycle is known as cyclic graph.

Acyclic Graph:

A graph which does not contain any cycle is known as acyclic graph.

Reachability:

A vertex ‘v’ of a simple graph is said to be rechable from the vertex ‘u’ of the same graph, if there exists a path from ‘u’ to ‘v’.

Remark: Every vertex is reachable from vertex itself with a path of length zero (trivial path).

Reachable set:

The set of vertices which are reachable from vertex ‘v’ is said to be rechable set of ‘v’ and it is denoted by R(v).

$$R(v) = \{u \in V / u \text{ is reachable from } v\}$$

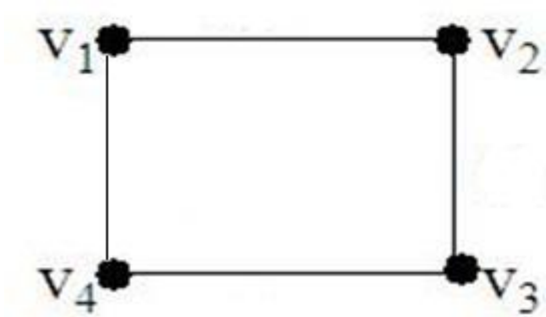
Node base / Vertex base :

In a Graph $G = (V, E)$, a subset X of V is said to be Node base of V if $R(X) = V$ and no proper subset of X has the same property.

SUBGRAPH

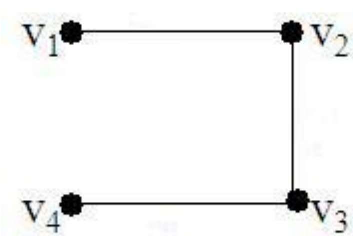
- Let G=<V$_1$,E$_1$> and H=<V$_2$,E$_2$> be any two graphs then H is said to be sub graph of graph G if,

1) $V_2(H) \subseteq V_1(G)$
2) $E_2(H) \subseteq E_1(G)$



[G]

[H$_1$]



[H$_2$]

H$_2$ is a sub graph of G.
H$_1$ is a sub graph of G.

A sub graph G_1 of G_2 is " proper sub graph of G_2" if $G_1 \neq G_2$.

PROPER SUBGRAPH

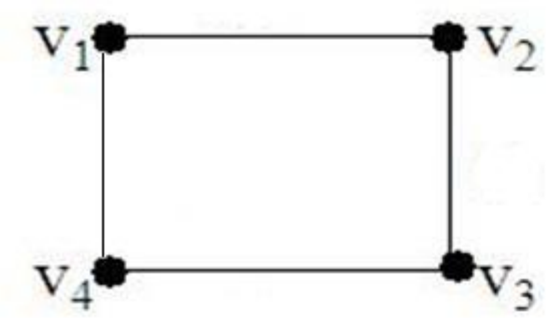
- Let $G=\langle V_1,E_1\rangle$ and $H=\langle V_2,E_2\rangle$ be any two graphs then H is said to be proper sub graph of graph G if,

1) $V_2(H) \subset V_1(G)$
2) $E_2(H) \subseteq E_1(G)$

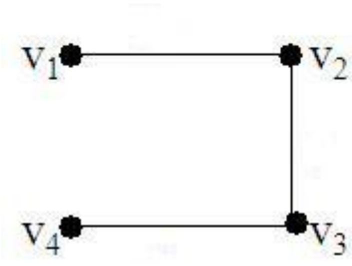
SPANNING SUBGRAPH

➢Let $G_1=<V_1,E_1>$ and $G_2=<V_2,E_2>$ be any two graph then G_1 is said to be a spanning graph of G_2 if,

1) $V_1=V_2$
2) $E_1 \subset E_2$



[G]



[H_1]

[H_2]

H_1 is spanning graph of G.
H_2 in not spanning graph of G.

EDGE DELETED SUBGRAPH

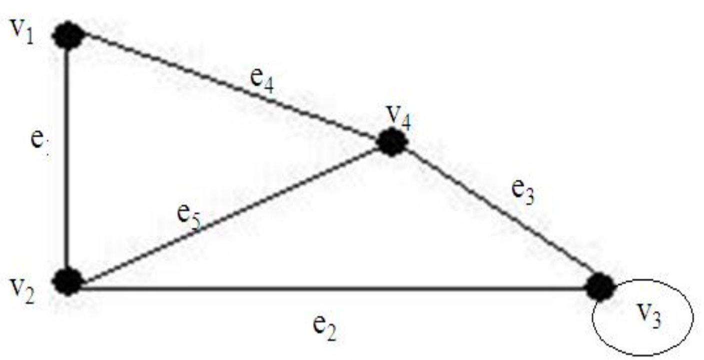
Given a graph $G = (V, E)$ and an edge $e \in E$, we can produce a sub graph of G by removing the edge e. The resulting sub graph, denoted by

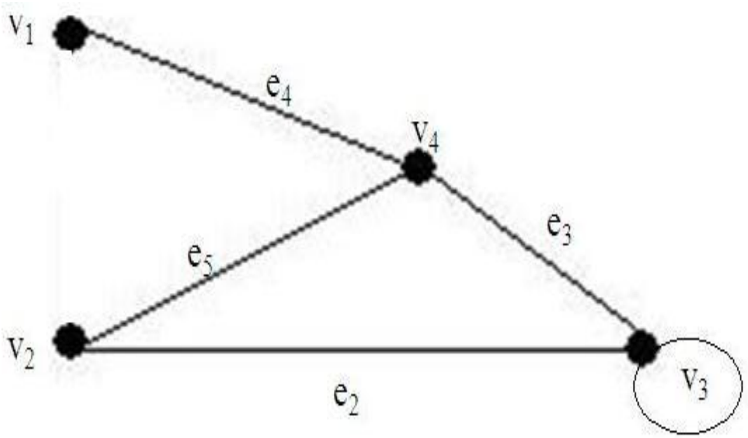
$G - e$, has the same vertex set V as G. Its edge set is $E - e$.

Hence $G - e = (V, E - \{e\})$

EDGE DELETED SUBGRAPH

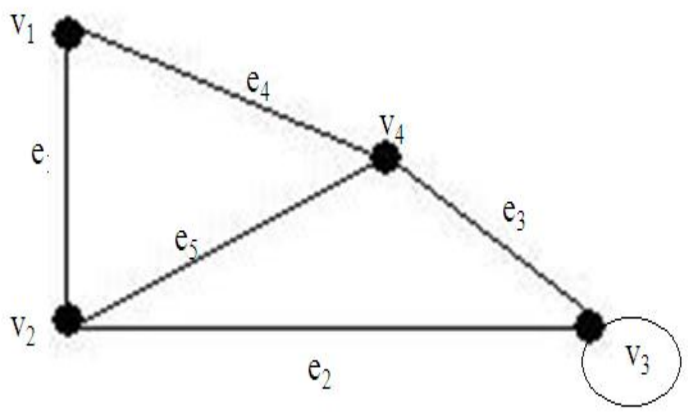
- We can delete only one edge from the given graph.
  - Suppose, we deleted e_1 edge.

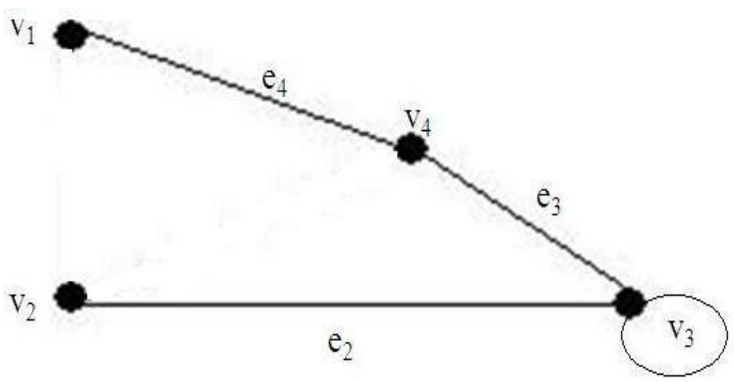




EDGE SET DELETED SUBGRAPH

- We can delete more then one edges from the given graph.
  - Suppose, we deleted e_1 and e_5 both edges.





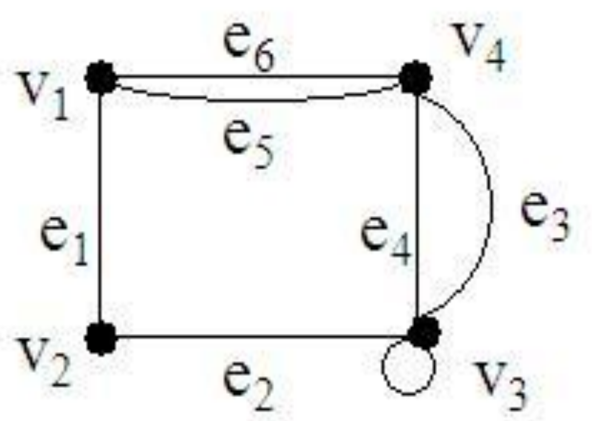
VERTEX DELETED SUBGRAPH

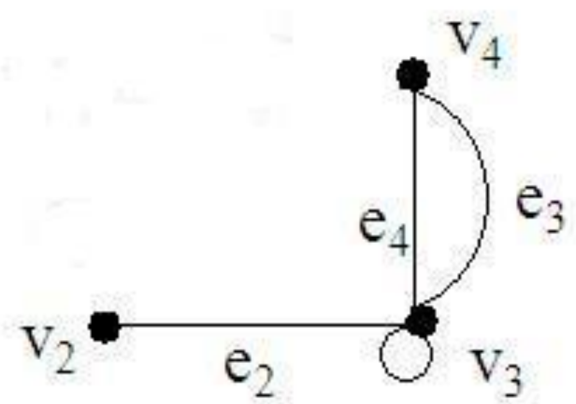
When we remove a vertex v and all edges incident to it from $G = (V, E)$, we produce a subgraph, denoted by $G - v$. Observe that

$G - v = (V - v, E)$, where E is the set of edges of G not incident to v.

VERTEX DELETED SUBGRAPH

- We can delete only one vertex from the given graph.
  - Suppose, we deleted only v_1 vertex.

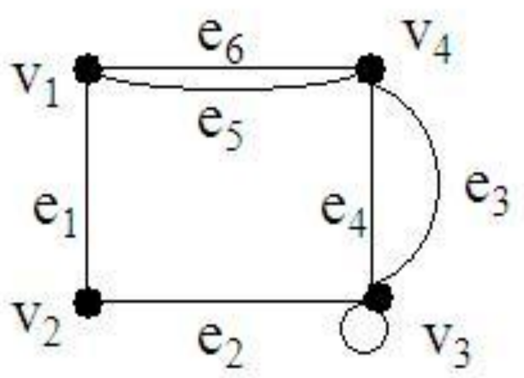


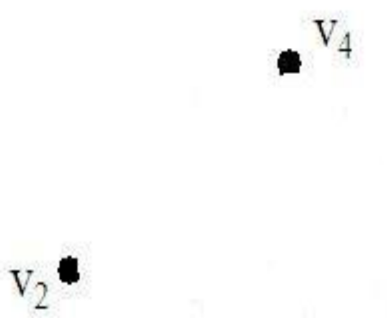


VERTEXSET DELETED SUBGRAPH

- We can delete more then one vertex from the given graph.

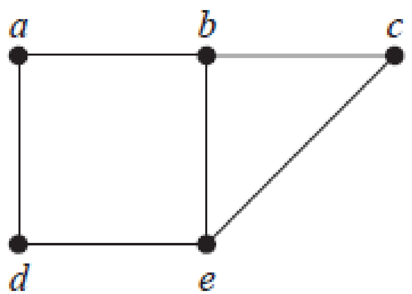
  - Suppose, we deleted v_1 and v_3 vertex.



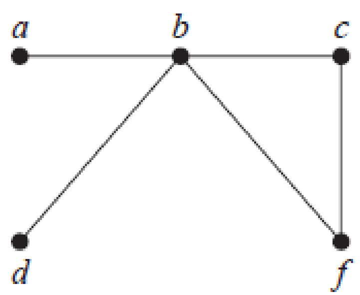


Graph Unions

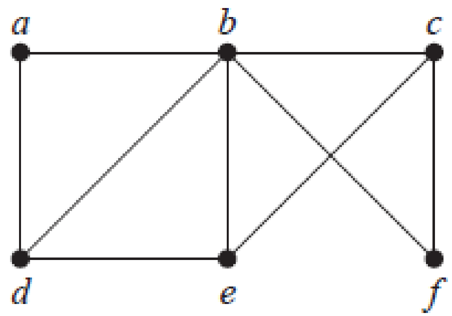
The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.



G_1



G_2



$G_1 \cup G_2$

Connectedness in Undirected Graphs

- An undirected graph is connected if every distinct pair of vertices are joined by a path.

Otherwise, it is known as disconnected.

- Every disconnected graph can be split into number of connected sub graphs, called **connected components**.

Thus, any two computers in the network can communicate if and only if the graph of this network is connected.

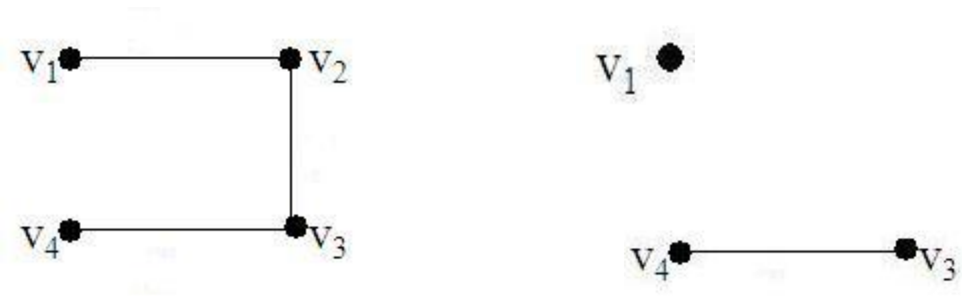
CONNECTED COMPONENTS

A **connected component** of a graph *G* is a connected subgraph of *G* that is not a proper subgraph of another connected subgraph of *G*.

A graph *G* that is not connected has two or more connected components that are disjoint and have *G* as their union.

Cut vertex

➢Removal of vertex will becomes disconnected graph.



Here, V_2 is cut vertex.

Cut Vertices

- The removal from a graph of a vertex and all incident edges produces a sub graph with more connected components. Such vertices are called **cut vertices**.
- The removal of a cut vertex from a connected graph produces a sub graph that is not connected.

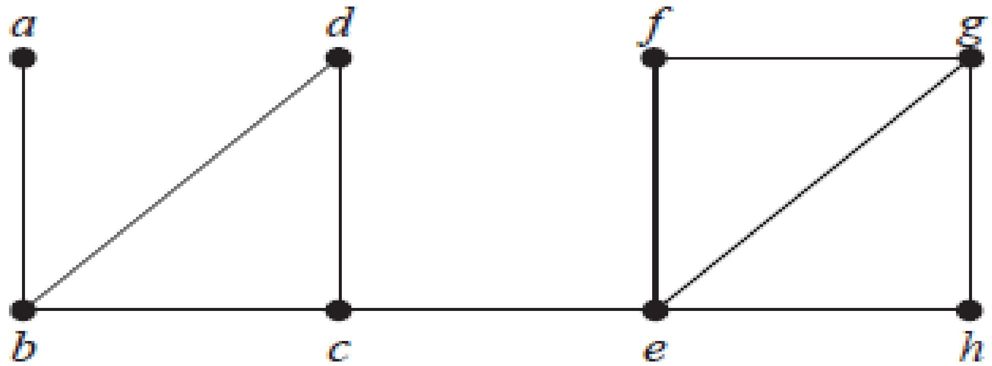
Cut Edge / Bridge

An edge whose removal produces a graph with more connected components than in the original graph is called a **cut edge** or **bridge**.

Note that in a graph representing a computer network, a cut vertex and a cut edge represent an essential router and an essential link that cannot fail for all computers to be able to communicate.

Example

Find the cut vertices and cut edges in the graph $G1$ shown in Figure



G_1

Solution:

The cut vertices of $G1$ are b, c, and e.

The removal of one of these vertices (and its adjacent edges) disconnects the graph.

The cut edges are $\{a, b\}$ and $\{c, e\}$.

Removing either one of these edges disconnects $G1$.

For Directed Graphs:

- **Weakly connected**: A digraph is called weakly connected if it is connected as an undirected graph in which each edge directed edge is converted to an undirected edge.

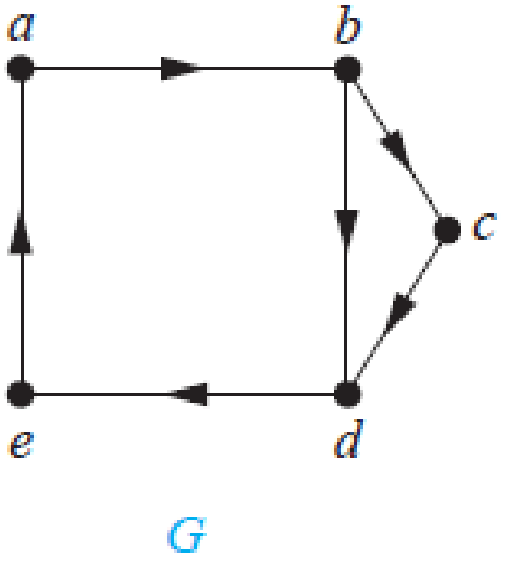
- **Unilaterally connected**: A digraph is said to be unilaterally connected if for every pair of vertices of the graph at least one of the vertices of the pair is rechable from other vertex.

- **Strongly connected**: A digraph is said to be strongly connected if for every pair of vertices of the graph both vertices must be reachable from one another.

Note: Any strongly connected directed graph is also weakly connected.

Example

Are the directed graphs *G* and *H* shown in Figure strongly connected? Are they weakly connected?



Solution:

G is strongly connected because there is a path between any two vertices in this directed graph .

Hence, G is also weakly connected.

The graph H is not strongly connected. There is no directed path from a to b in this graph.

However, H is weakly connected, because there is a path between any two vertices in the underlying Undirected graph of H.

Connected Components(DIGRAPH)

- Weak Component

- Unilateral Component

- Strong Component

Connectivity

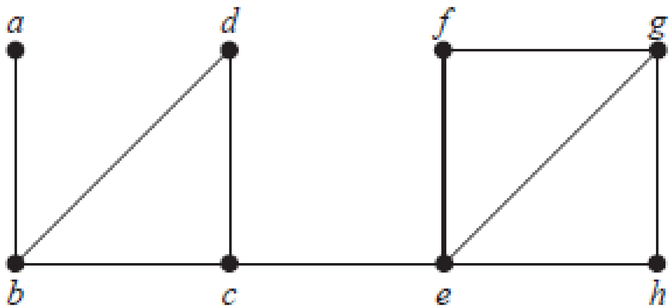
Edge Connectivity:

Let G be a connected graph. The edge connectivity of G is the minimum number of edges whose removal results in a disconnected or trivial graph. The edge connectivity of a connected graph G is denoted by G is denoted by $\lambda(G)$.

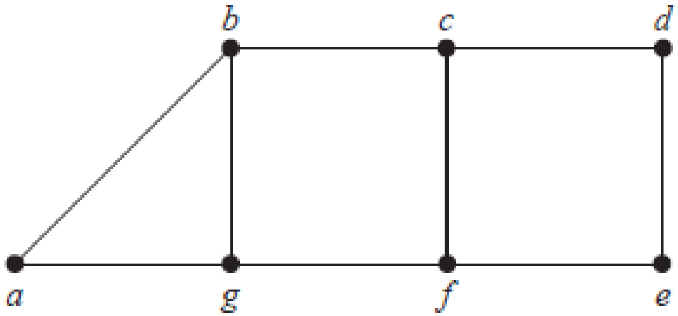
- If G is disconnected graph, then $\lambda(G) = 0$
- Edge connectivity of a connected graph G with a bridge is 1.

Example:

Find the edge connectivity for each of the graphs in following figure



G_1



G_3

Solution:

G_1 has a cut edge, so $\lambda(G_1) = 1$.

$\lambda(G3) = 2$, because $G3$ has no cut edges, but the removal of the two edges $\{b, c\}$ and $\{f, g\}$ disconnects it.

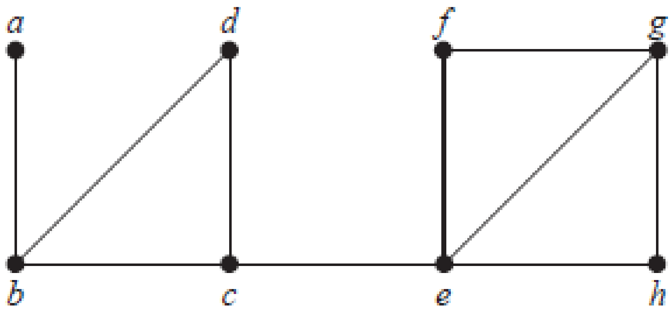
Vertex Connectivity

Let G be a connected graph. The vertex connectivity of G is the minimum number of vertices whose removal results in a disconnected or a trivial graph. The vertex connectivity of a connected graph is denoted by $\kappa(G)$.

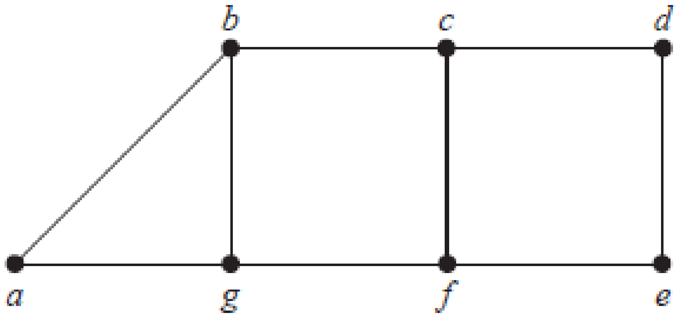
- If G is a disconnected graph then $\kappa(G)=0$.
- The vertex connectivity of a graph having a bridge is always one.

Example:

Find the vertex connectivity for each of the graphs in following figure



G_1



G_3

Solution:

G_1 is a connected graph with a cut vertex. we know that $\kappa(G_1) = 1$.

$G3$ has no cut vertices. but that $\{b, g\}$ is a vertex cut. Hence, $\kappa(G3) = 2$.

Matrix Representation of Graphs

Adjacency Matrices (Simple Undirected graph) :

Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$.

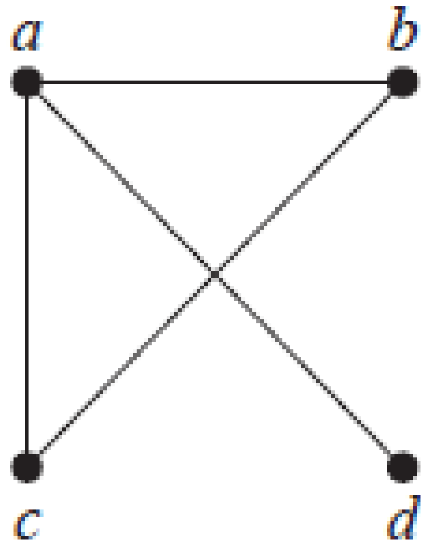
The **adjacency matrix A=[**a_{ij} **]**(*or* **A**$_G$) of G, with respect to this listing of the vertices, is

$a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G,

0 otherwise.

Example:

Use an adjacency matrix to represent the graph shown in Figure



Solution:

We order the vertices as *a*, *b*, *c*, *d*. The matrix representing this graph is

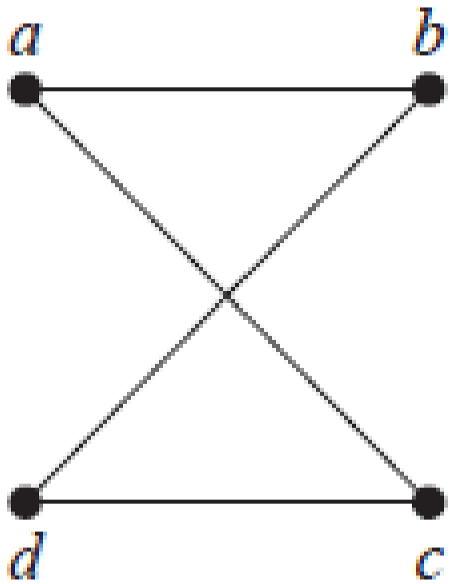
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Example:

Draw a graph with the adjacency matrix with respect to ordering a,b,c,d

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Solution:



Adjacency matrices can also be used to represent undirected graphs with loops and with multiple edges.

A loop at the vertex v_i is represented by a 1 at the *(i, i)*th position of the adjacency matrix.

When multiple edges connecting the same pair of vertices v_i and v_j, or multiple loops at the same vertex

- Here the *(i, j)*th entry of this matrix equals the number of edges that are associated to $\{vi , vj \}$.

Adjacency Matrices (Multi Undirected graph) :

Suppose that $G = (V, E)$ is a multi graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$.

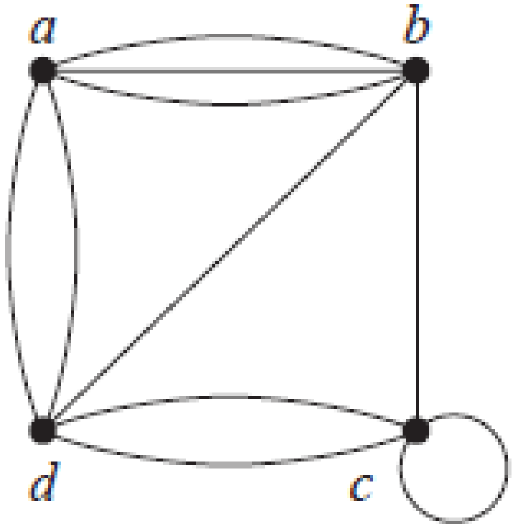
The **adjacency matrix A=[**a_{ij} **]**(*or* $\mathbf{A}_G$) of G, with respect to this listing of the vertices, is

a_{ij} = total no. of edges between v_i *and* v_j of G,

0 otherwise.

Example:

Use an adjacency matrix to represent graph



Solution:

The adjacency matrix using the ordering of vertices *a*, *b*, *c*, *d* is

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

Adjacency Matrices (Directed graph) :

Suppose that $G = (V, E)$ is a directed graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$.

The **adjacency matrix A=[**a_{ij} **]**(*or* $\mathbf{A}_G$) of G, with respect to this listing of the vertices, is

a_{ij} = Total no. of edges from v_i to v_j of G,

0 otherwise.

Incidence Matrices (Undirected graph)

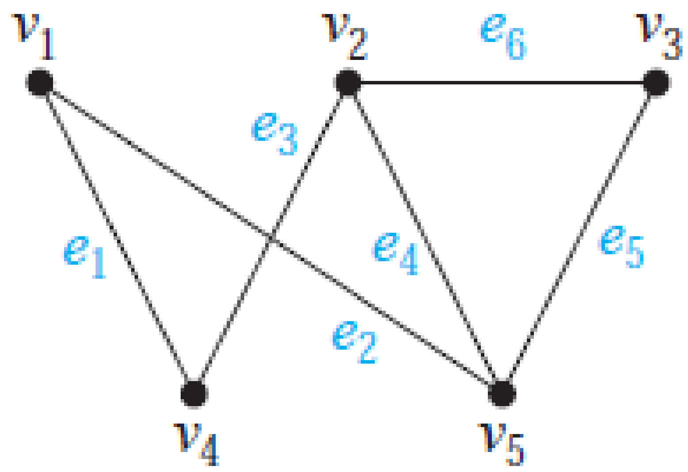
Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and

$e_1, e_2, \ldots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Example:

Represent the graph shown in Figure with an incidence matrix.



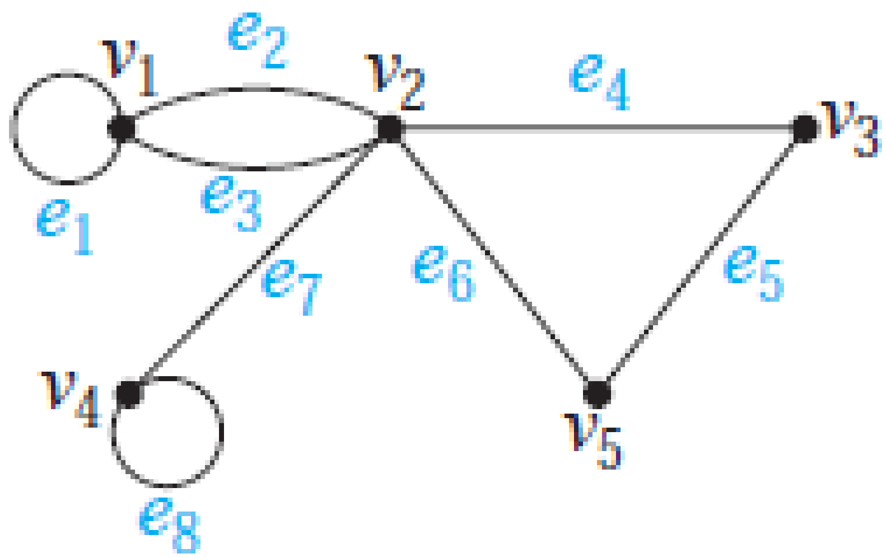
Solution

The incidence matrix is

$$\begin{array}{c c} & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right]. \end{array}$$

Example

Represent the graph shown in Figure with an incidence matrix.



Solution

The incidence matrix is

$$
\begin{array}{c}
\\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}.
\end{array}
$$

Incidence Matrices (Directed graph)

Let $G = (V, E)$ be an directed graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and

$e_1, e_2, \ldots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]$, where

$m_{ij} =$ 1 when v_i is the initial vertex of edge e_j,

-1 when v_i is terminal vertex of edge e_j,

0 otherwise.

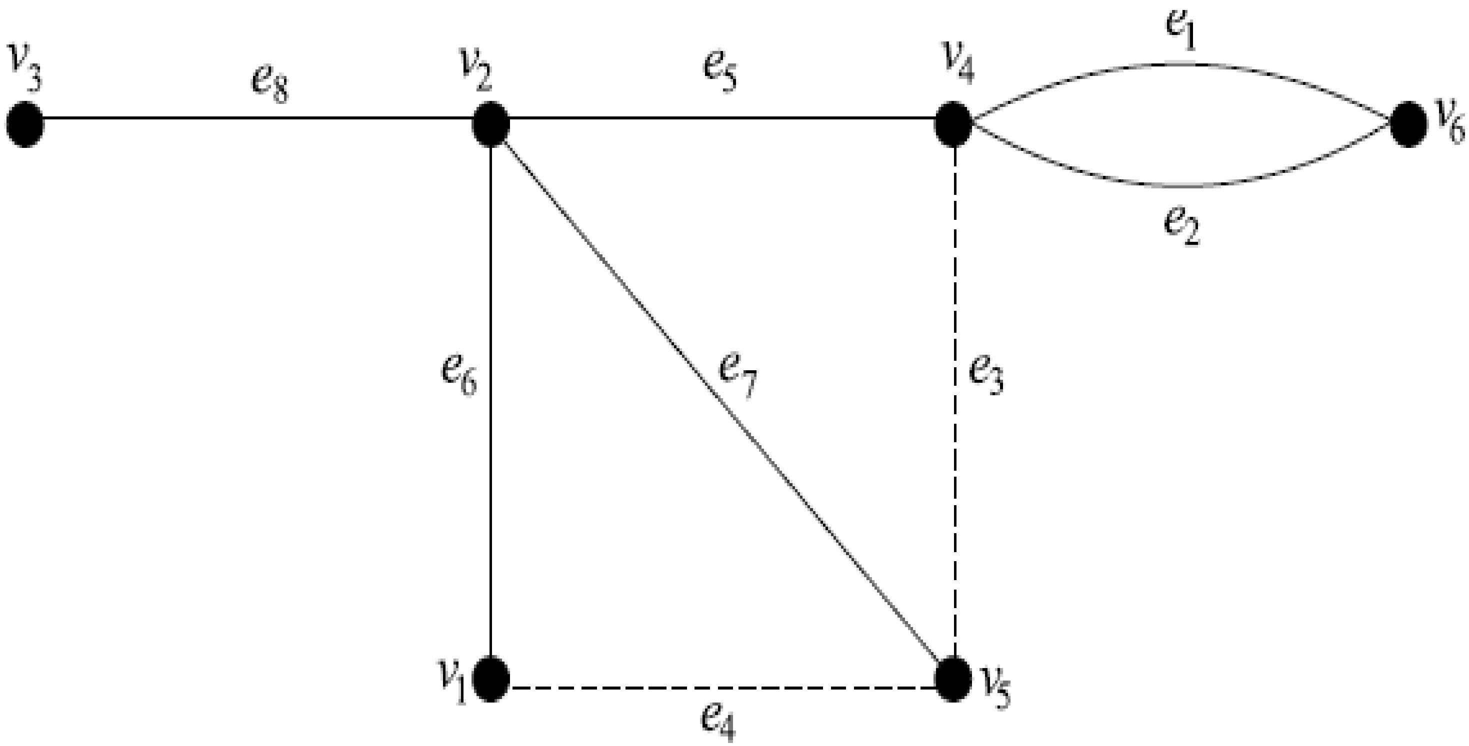
Path Matrix

Let (u,v) be a pair of specific vertices in a graph. A path matrix is denoted by $P(u,v)$. The rows in $P(u,v)$ corresponds to different paths between vertices u & v; and the columns corresponds to the edges in G. The path matrix for u & v vertices is defined as follows;

$P(u,v) = \left[P_{ij}\right]$; where $P_{ij} = 1$; if j^{th} edge lies in i^{th} path

= 0; otherwise

Example



Solution

The different paths between the vertices v_3 and v_4 are

$$p_1 = \{e_8, e_5\},\ p_2 = \{e_8, e_7, e_3\} \text{ and } p_3 = \{e_8, e_6, e_4, e_3\}.$$

The path matrix for v_3, v_4 is given by

$$P(v_3, v_4) = \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

Eulerian & Hamiltonian graph

The town of Königsberg, Prussia (now called Kaliningrad and part of the Russian republic),was divided into four sections by the branches of the Pregel River. These four sections included the two regions on the banks of the Pregel, Kneiphof Island, and the region between the two branches of the Pregel. In the eighteenth century seven bridges connected these regions.

The Seven Bridges of Königsberg.

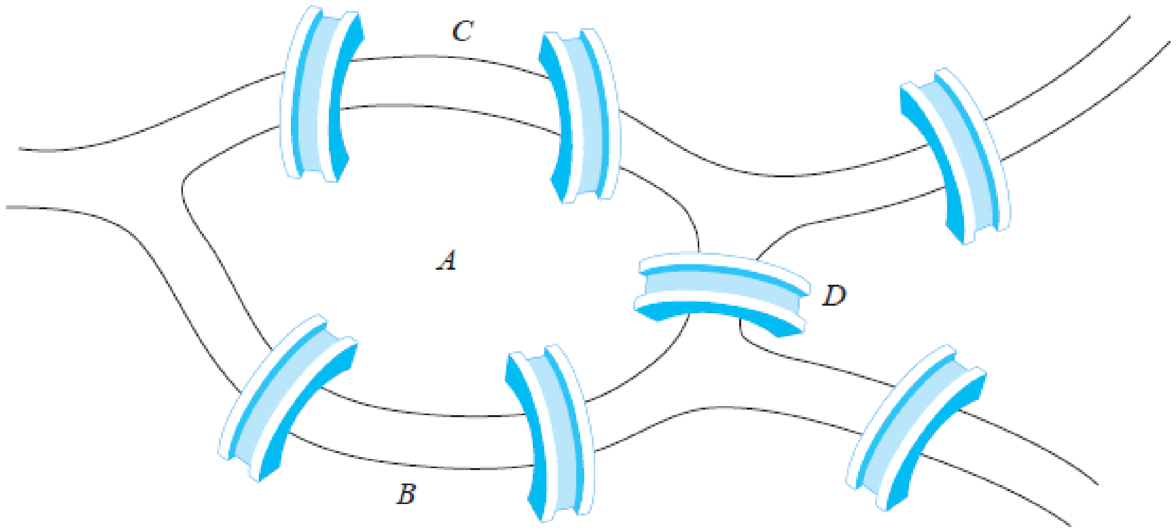


Figure depicts these regions and bridges.

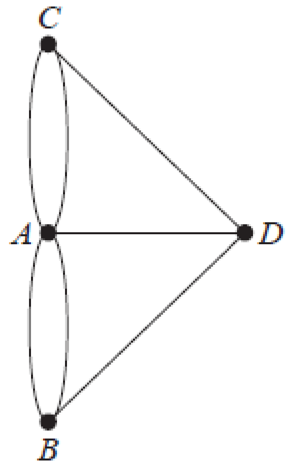
Problem

The townspeople took long walks through town on Sundays. They wondered whether it was possible to start at some location in the town, **travel across all the bridges once without crossing any bridge twice, and return to the starting point.**

The Swiss mathematician Leonhard Euler solved this problem. His solution, published in 1736, may be the first use of graph theory.

Euler studied this problem using the multigraph obtained when the four regions are represented by vertices and the bridges by edges. This multigraph is shown in Figure

Multigraph Model of the Town of Königsberg.



The question becomes: Is there a simple circuit in this multigraph that contains every edge?

Euler Paths and Circuits

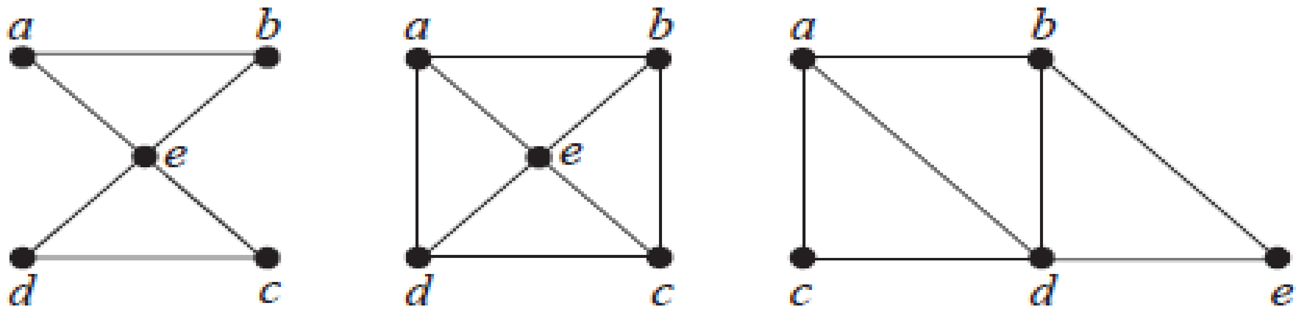
An ***Euler circuit*** in a graph G is a simple circuit containing every edge of G.

An ***Euler path*** in G is a simple path containing every edge of G.

Example:

Which of the undirected graphs in following Figure have an Euler circuit?

Of those that do not, which have an Euler path?



G_1 G_2 G_3

Solution:

The graph $G1$ has an Euler circuit,

for example, a, e, c, d, e, b, a.

Neither of the graphs $G2$ or $G3$ has an Euler circuit.

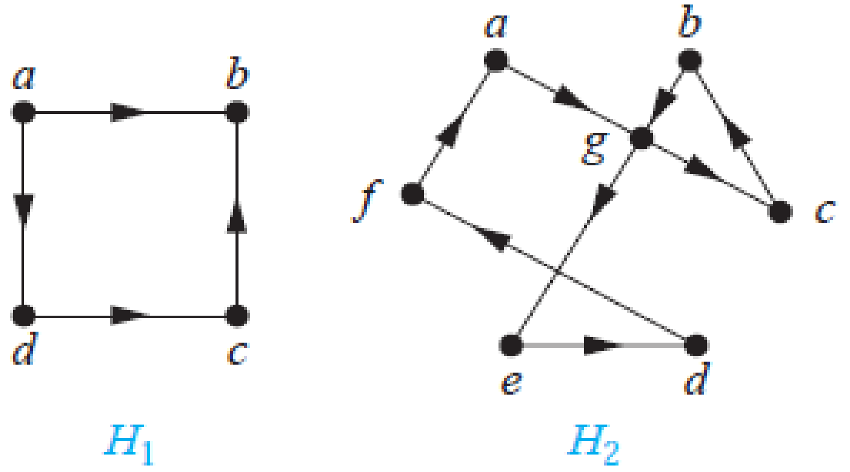
However,

$G3$ has an Euler path, namely, a, c, d, e, b, d, a, b.

$G2$ does not have an Euler path.

Example:

Which of the directed graphs in following figure have an Euler circuit? Of those that do not, which have an Euler path?



Solution:

The graph H_2 has an Euler circuit,

for example, $a, g, c, b, g, e, d, f, a$.

Neither $H1$ nor $H3$ has an Euler circuit .

$H3$ has an Euler path, namely,c, a, b, c, d, b, but $H1$ does not

Necessary And Sufficient Conditions For Euler Circuits And Paths

THEOREM 1:

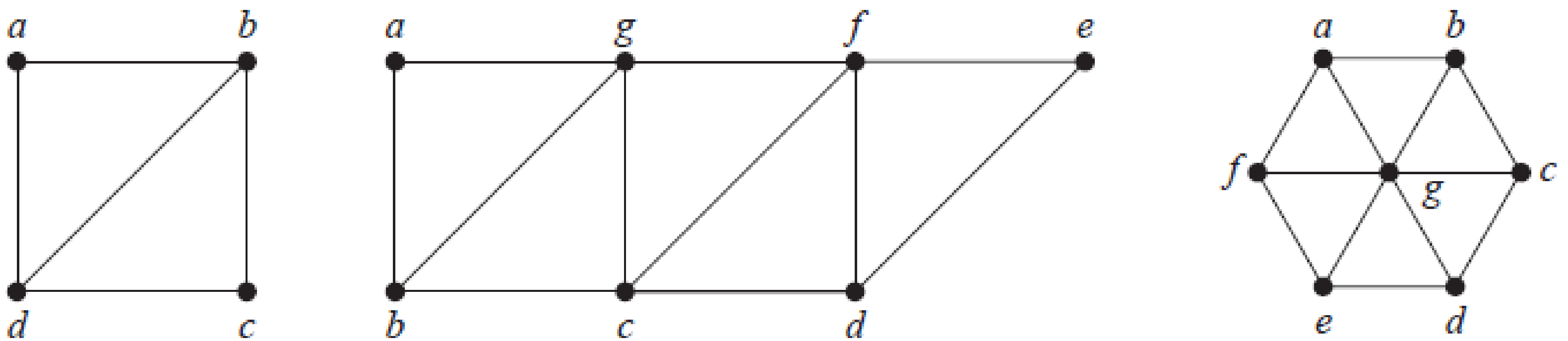
A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

THEOREM 2 :

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Example:

Which graphs shown in following figure have an Euler path?



G_1 G_2 G_3

Solution:

G_1 contains exactly two vertices of odd degree, namely, *b* and *d*. Hence, it has an Euler path that must have *b* and *d* as its endpoints. One such Euler path is *d*, *a*, *b*, *c*, *d*, *b*.

Similarly, G_2 has exactly two vertices of odd degree, namely, *b* and *d*. So it has an Euler path that must have *b* and *d* as endpoints. One such Euler path is *b*, *a*, *g*, *f*, *e*, *d*, *c*, *g*, *b*, *c*, *f*, *d*.

G_3 has no Euler path because it has six vertices of odd degree.

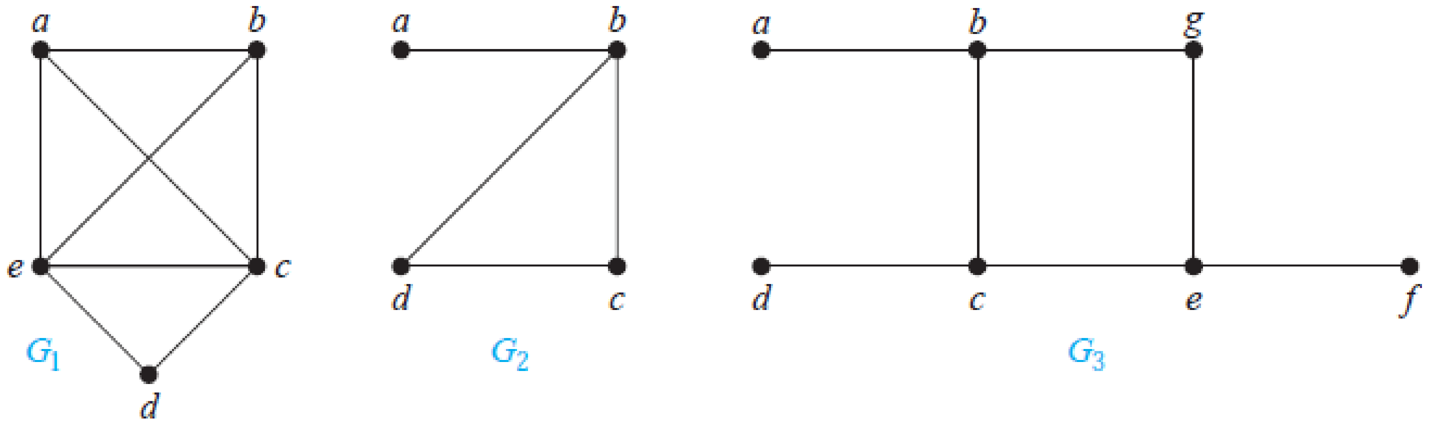
Hamilton Paths and Circuits

A simple path in a graph G that passes through every vertex exactly once is called a ***Hamilton path.***

A simple circuit in a graph G that passes through every vertex exactly once is called a ***Hamilton circuit***.

Example:

Which of the simple graphs in following figure have a Hamilton circuit or, if not, a Hamilton path?



Solution:

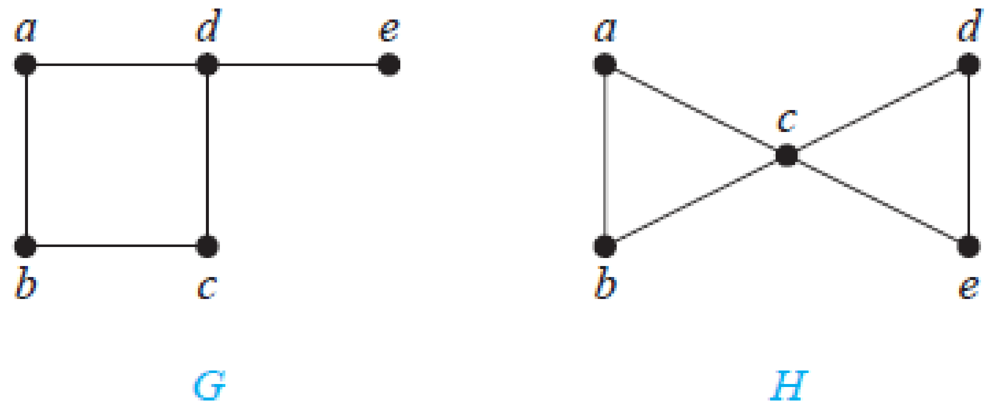
G_1 has a Hamilton circuit: a, b, c, d, e, a.

There is no Hamilton circuit in G_2 (this can be seen by noting that any circuit containing every vertex must contain the edge $\{a, b\}$ twice),but G_2 does have a Hamilton path, namely, a, b, c, d.

G_3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges $\{a, b\}$,$\{e, f\}$, and $\{c, d\}$ more than once.

Example:

Show that neither graph displayed in following figure has a Hamilton circuit.



Solution:

There is no Hamilton circuit in G because G has a vertex of degree one, namely, e.

Now consider H. Because the degrees of the vertices a, b, d, and e are all two, every edge incident with these vertices must be part of any Hamilton circuit.

It is now easy to see that no Hamilton circuit can exist in H, for any Hamilton circuit would have to contain four edges incident with c, which is impossible.

Sufficient conditions for a connected simple graph to have a Hamilton circuit.

DIRAC'S THEOREM :

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

ORE'S THEOREM :

If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Both Ore's theorem and Dirac's theorem provide **sufficient conditions** for a connected simple graph to have a Hamilton circuit.

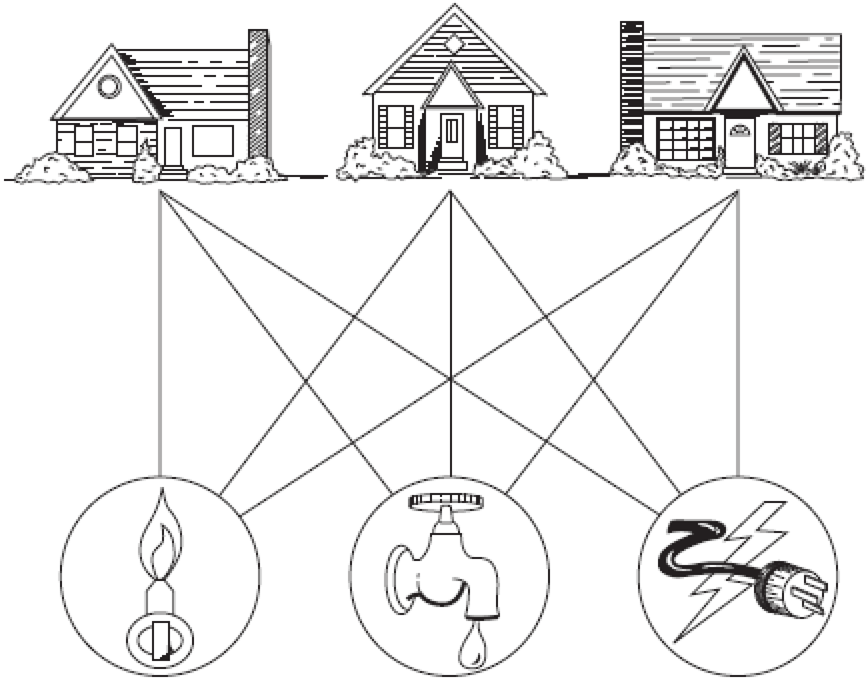
However, these theorems **do not** provide **necessary conditions** for the existence of a Hamilton circuit.

For example,

The graph C_5 has a Hamilton circuit but does not satisfy the hypotheses of either Ore's theorem or Dirac's theorem

Planar Graphs

Consider the problem of joining three houses to each of three separate utilities. Is it possible to join these houses and utilities so that none of the connections cross?



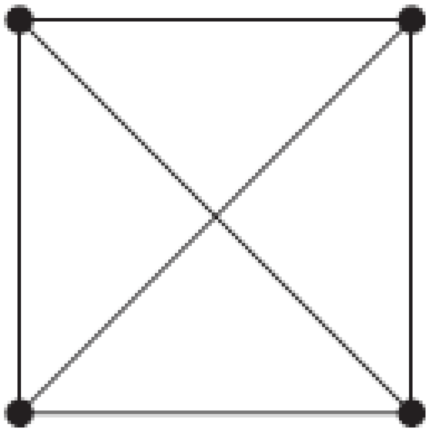
Definition:

A graph is called ***planar*** if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint).

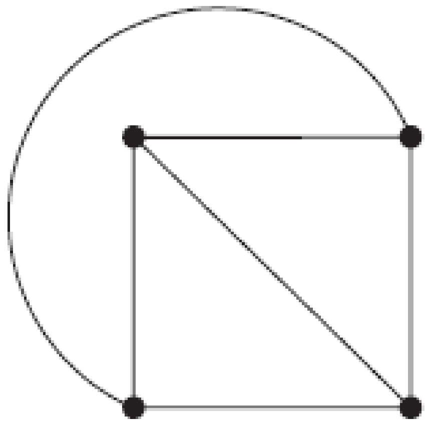
Such a drawing is called a ***planar representation*** of the graph.

Example:

Is K_4 (shown in Figure with two edges crossing) planar?



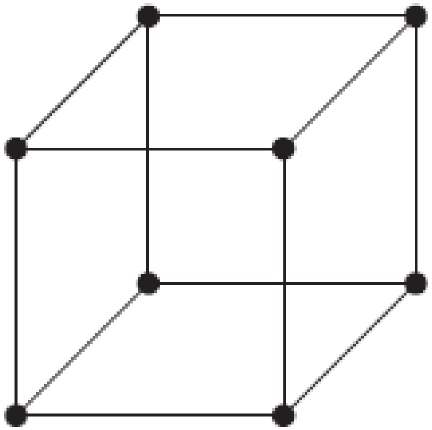
Solution:



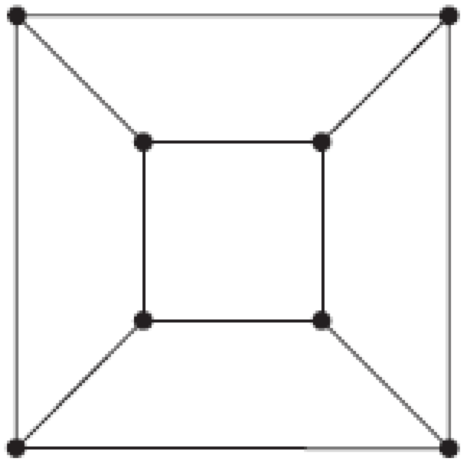
K_4 is planar because it can be drawn without crossings

Example:

Is *Q*3, shown in Figure planar?



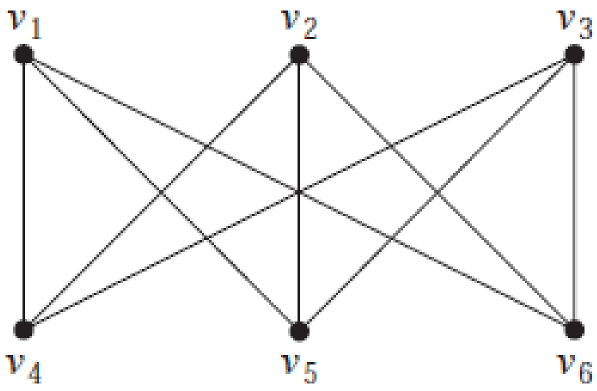
Solution:



Q_3 is planar, because it can be drawn without any edges crossing

Example:

Is $K_{3,3}$, shown in figure, planar?

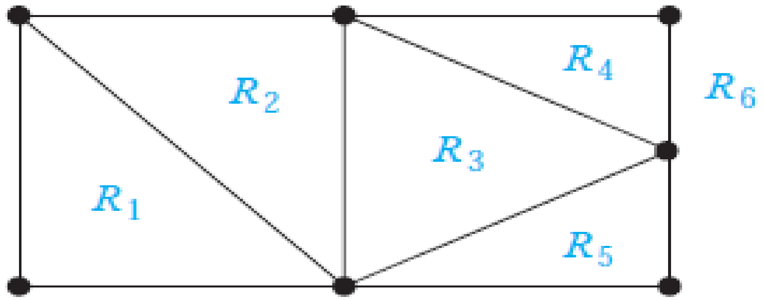


Solution:

$K_{3,3}$ is not planar.

Region of a Graph

A region of a planar graph is defined to be an area of the plane that is bounded by edges and it is not further divided into subareas.



Note:

In any graph 'G' at least one region always exists, which is unbounded region.

Euler's Formula

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G.

Then $r = e - v + 2$.

Example:

Suppose that a connected planar simple graph has 20 vertices, each of degree 3. Into how many regions does a representation of this planar graph split the plane?

Solution:

This graph has 20 vertices, each of degree 3, so $v = 20$.

Because the sum of the degrees of the vertices, $3v = 3 \cdot 20 = 60$, is equal to twice the number of edges, $2e$, we have $2e = 60$,or $e = 30$. Consequently, from Euler's formula, the number of regions is

$r = e - v + 2 = 30 - 20 + 2 = 12.$

Euler's formula can be used to establish some inequalities that must be satisfied by planar graphs.

Corollary 1:

If G is a connected planar simple graph with

e edges and v vertices, where $v \geq 3$, then

$e \leq 3v - 6$.

Corollary 2:

If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

If the inequality $e \leq 3v - 6$ is satisfied does *not* imply that a graph is planar.

K_5 is nonplanar using Corollary 1.

The graph K_5 has five vertices and 10 edges. However, the inequality $e \leq 3v - 6$ is not satisfied for this graph because $e = 10$ and $3v - 6 = 9$. Therefore, $K5$ is not planar

Corollary 3

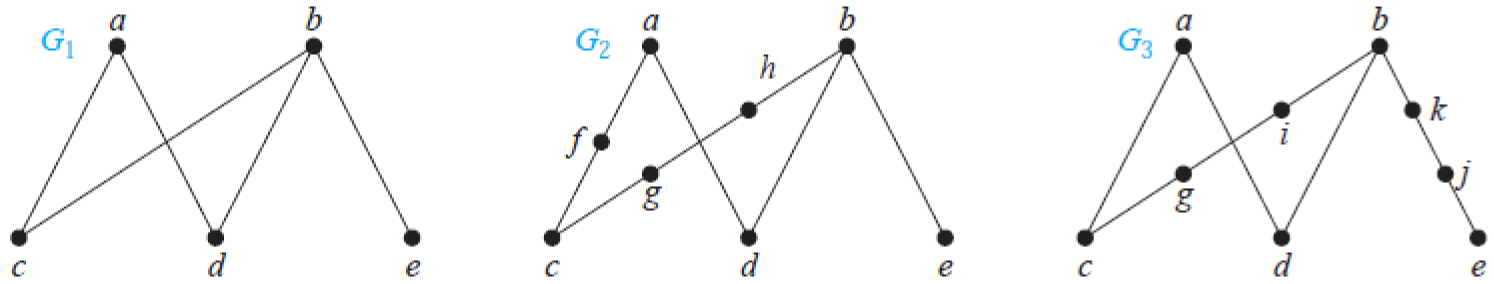
If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an **elementary subdivision**.

The graphs $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ are called **homeomorphic** if they can be obtained from the same graph by a sequence of elementary subdivisions.

Example:

Show that the graphs G_1, G_2, and G_3 displayed in following figure are all homeomorphic.



Solution:

These three graphs are homeomorphic because all three can be obtained from $G1$ by elementary subdivisions.

$G1$ can be obtained from itself by an empty sequence of elementary subdivisions.

To obtain $G2$ from $G1$ we can use this sequence of elementary subdivisions:

(i) remove the edge $\{a, c\}$, add the vertex f, and add the edges $\{a, f\}$ and $\{f, c\}$; (ii) remove the edge $\{b, c\}$, add the vertex g, and add the edges $\{b, g\}$ and $\{g, c\}$; and

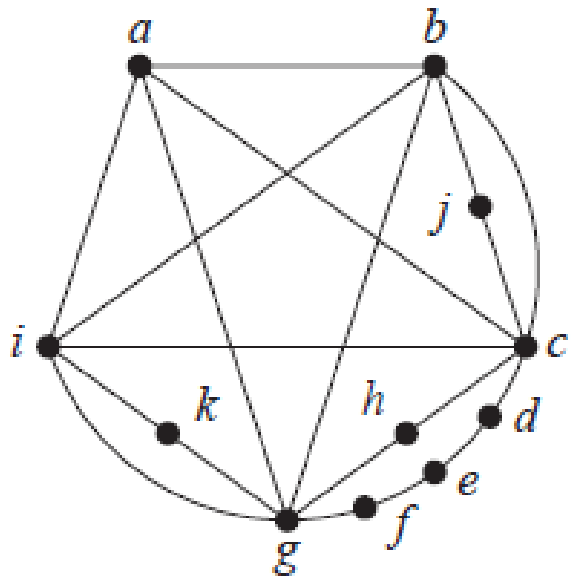
(iii) remove the edge $\{b, g\}$, add the vertex h, and add the edges $\{g, h\}$ and $\{b, h\}$.

Kuratowski's Theorem

A graph is nonplanar if and only if it contains a sub graph homeomorphic to $K_{3,3}$ or K_5.

Example:

Determine whether the graph *G* shown in Figure is planar.



G

Solution:

A subgraph H is homeomorphic to K_5