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PROBABILITY, STATISTICS AND RANDOM PROCESSES

Second Edition

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Chapter 2

Random Variables

The outcomes of random experiments may be numerical or non-numerical in nature. For example, the number of telephone calls received in a board in 1 h is numerical in nature, while the result of a coin tossing experiment in which 2 coins are tossed at a time is non-numerical in nature. As it is often useful to describe the outcome of a random experiment by a number, we will assign a number to each non-numerical outcome of the experiment. For example, in the 2 coins tossing experiment we could assign the value 0 to the outcome of getting 2 tails, 1 to the outcome of getting 1 head and 1 tail and 2 to the outcome of getting 2 heads. Thus in any experimental situation we can assign a real number x to every element s of the sample space S. That is, the function $X(s) = x$ that maps the elements of the sample space into real numbers is called the random variable associated with the concerned experiment. A formal definition may be given as follows.

Definition: A random variable (abbreviatively RV) is a function that assigns a real number $X(s)$ to every element $s \in S$, where S is the sample space corresponding to a random experiment E.)

Note *Although we are expected to perform the random experiment E, we observe the outcome $s \in S$ and then evaluate $X(s)$ [i.e., assign a real number x to $X(s)$], the number $x = X(s)$ itself can be thought of as the outcome of the experiment and R_x as the sample space of the experiment. In this sense, we will hereafter talk about a random variable X taking the value x and $P(X = x)$. Actually, $P(X = x) = P\{s: X(s) = x\}$.*

Hereafter, R_x will be referred to as ***Range space***.

Similarly $\{X \le x\}$ represents the subset $\{s: X(s) \le x\}$ and hence an event associated with the experiment.

Discrete Random Variable

If X is a random variable (RV) which can take a finite number or countably infinite number of values, X is called a discrete RV. When the RV is discrete, the possible values of X may be assumed as $x_1, x_2, \ldots, x_n, \ldots$. In the finite case, the list of values terminates and in the countably infinite case, the list goes upto infinity.

For example, the number shown when a die is thrown and the number of alpha particles emitted by a radioactive source are discrete RVs.

Probability Function

If X is a discrete RV which can take the values $x_1, x_2, x_3, \ldots$ such that $P(X = x_i) = p_i$, then p_i is called the *probability function or probability mass function or point probability function*, provided p_i ($i = 1, 2, 3, \ldots$) satisfy the following conditions:

(i) $p_i \geq 0$, for all i, and

(ii) $\sum_i p_i = 1$

The collection of pairs $\{x_i, p_i\}$, $i = 1, 2, 3, \ldots$, is called *the probability distribution of the* RV X, which is sometimes displayed in the form of a table as given below:

$X = x_i$	$P(X = x_i)$
x_1	p_1
x_2	p_2
$\vdots$	$\vdots$
x_r	p_r
$\vdots$	$\vdots$

Continuous Random Variable

If X is an RV which can take all values (i.e., *infinite number* of values) in an interval, then X is called a *continuous* RV.

For example, the length of time during which a vacuum tube installed in a circuit functions is a continuous RV.

Probability Density Function

If X is a continuous RV such that

$$P\left\{x - \frac{1}{2}dx \leq X \leq x + \frac{1}{2}dx\right\} = f(x)dx$$

then $f(x)$ is called the *probability density function* (shortly denoted as pdf) of X, provided $f(x)$ satisfies the following conditions:

(i) $f(x) \geq 0$, for all $x \in R_x$, and

(ii) $\int_{R_x} f(x)dx = 1$

Moreover, $P(a \leq X \leq b)$ or $P(a < X < b)$ is defined as

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

The curve $y = f(x)$ is called the *probability curve of the* RV X.

Note *When X is a continuous RV*

$$P(X = a) = P(a \leq X \leq a) = \int_a^a f(x)\,dx = 0$$

This means that it is almost impossible that a continuous RV assumes a specific value. Hence $P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b)$.

Cumulative Distribution Function (cdf)

If X is an R V, discrete or continuous, then $P(X \leq x)$ is called the *cumulative distribution function* of X or *distribution function* of X and denoted as $F(x)$.
If X is discrete,

$$F(x) = \sum_{\substack{j \\ x_j \leq x}} p_j$$

If X is continuous,

$$F(x) = P(-\infty < X \leq x) = \int_{-\infty}^{x} f(x)dx$$

Properties of the *cdf* $F(x)$

1. $F(x)$ is a non-decreasing function of x, i.e., if $x_1 < x_2$, then $F(x_1) \leq F(x_2)$.
2. $F(-\infty) = 0$ and $F(\infty) = 1$.
3. If X is a discrete R V taking values $x_1, x_2, \ldots$, where $x_1 < x_2 < x_3 < \ldots < x_{i-1} < x_i < \ldots$, then $P(X = x_i) = F(x_i) - F(x_{i-1})$.
4. If X is a continuous R V, then $\frac{d}{dx}F(x) = f(x)$, at all points where $F(x)$ is differentiable.

Note *Although we may talk of probability distribution of a continuous RV, it cannot be represented by a table as in the case of a discrete RV. The probability distribution of a continuous RV is said to be known, if either its pdf or cdf is given.*

Special Distributions

The probability mass functions of some discrete RVs and the probability density functions of some continuous RVs, which are of frequent applications, are as follows:

Discrete Distributions

1. If the discrete RV X can take the values 0, 1, 2, ..., n, such that $P(X = i) = nC_i\, p^i\, q^{n-i}$, $i = 0, 1, \ldots, n$, where $p + q = 1$, then X is said to follow a *binomial distribution* with parameters n and p, which is denoted a $B(n, p)$.
2. If the discrete RV X can take the values 0, 1, 2, ..., such that $P(x = i) = \dfrac{e^{-\lambda}\lambda^i}{i!}$, $i = 0, 1, 2, \ldots$, then X is said to follow a Poisson distribution with parameter λ.
3. If the discrete RV X can take the values 0, 1, 2, ..., such that $P(X = i) = (n + i - 1)C_i\, p^n q^i$, $i = 0, 1, 2, \ldots$, where $p + q = 1$, then X is said to follow a *Pascal (or negative binomial) distribution* with parameter n.
4. A Pascal distribution with parameter 1 [i.e., $P(X = i) = pq^i$, $i = 0, 1, 2, \ldots$ and $p + q = 1$] is called a *geometric distribution*.

Continuous Distributions

5. If the pdf of a continuous RV X is $f(x) = \dfrac{1}{b-a}$ (a constant), $a \le x \le b$, then X follows a *uniform distribution* (*or rectangular distribution*).
6. If the pdf of a continuous RV X is $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}$, $-\infty < x < \infty$, then X is said to follow a *normal distribution* (or *Gaussian distribution*) with parameters μ and σ, which will be hereafter denoted as $N(\mu, \sigma)$.
7. If the pdf of a continuous RV X is $f(x) = \dfrac{1}{\Gamma(n)}\, e^{-x} x^{n-1}$, $0 < x < \infty$ and $n > 0$, then X follows a *gamma distribution* with parameter n. Gamma distribution is a particular case of *Erlang distribution*, the pdf of which is $f(x) = \dfrac{c^n}{\Gamma(n)}\, x^{n-1} e^{-cx}$, $0 < x < \infty$, $n > 0$, $c > 0$.
8. An Erlang distribution with $n = 1$ [i.e., $f(x) = ce^{-cx}$, $0 < x < \infty$, $c > 0$] is called *an exponential* (or *negative exponential*) *distribution* with parameter c.
9. If the pdf of a continuous RV X is $f(x) = \dfrac{x}{\alpha^2}\, e^{-x^2/2\alpha^2}$, $0 < x < \infty$, then X follows a *Rayleigh distribution* with parameter α.
10. If the pdf of a continuous RV X is $f(x) = \dfrac{\sqrt{2}}{\alpha^3\sqrt{\pi}}\, x^2 e^{-x^2/2\alpha^2}$, $0 < x < \infty$, then X follows a *Maxwell distribution* with parameter α.

11. If the pdf of a continuous RV X is $f(x) = \dfrac{1}{2\lambda}\, e^{-|x-\mu|/\lambda}$, $-\infty < x < \infty$, $\lambda > 0$, then X follows a *Laplace* (or *double exponential*) *distribution* with parameters λ and μ.
12. If the pdf of a continuous RV X is $f(x) = \dfrac{\alpha}{\pi} \times \dfrac{1}{x^2 + \alpha^2}$, $\alpha > 0$, $-\infty < x < \infty$, then X follows a *Cauchy distribution* with parameter α.

Worked Example 2(A)

Example 1

From a lot containing 25 items, 5 of which are defective, 4 items are chosen at random. If X is the number of defectives found, obtain the probability distribution of X, when the items are chosen (i) without replacement and (ii) with replacement.

Since only 4 items are chosen, X can take the values 0, 1, 2, 3 and 4. The lot contains 20 non-defective and 5 defective items.

Case (i): When the items are chosen without replacement, we can assume that all the 4 items are chosen simultaneously.

$$\therefore \quad P(X = r) = P(\text{choosing exactly } r \text{ defective items})$$
$$= P(\text{choosing } r \text{ defective and } (4 - r) \text{ good items})$$
$$= \frac{5C_r \times 20\,C_{4-r}}{25\,C_4} \quad (r = 0, 1, \ldots, 4).$$

Case (ii): When the items are chosen with replacement, we note that the probability of an item being defective remains the same in each draw.

i.e., $$p = \frac{5}{25} = \frac{1}{5},\; q = \frac{4}{5} \text{ and } n = 4$$

The problem is one of performing 4 Bernoulli's trials and finding the probability of exactly r successes.

$$\therefore \quad P(X = r) = 4C_r \left(\frac{1}{5}\right)^r \left(\frac{4}{5}\right)^{4-r} \quad (r = 0, 1, \ldots, 4)$$

Example 2

A shipment of 6 television sets contains 2 defective sets. A hotel makes a random purchase of 3 of the sets. If X is the number of defective sets purchased by the hotel, find the probability distribution of X. (MU — Apr. 96)

All the 3 sets are purchased simultaneously. Since there are only 2 defective sets in the lot, X can take the values 0, 1 and 2.

$\therefore \quad P(X = r) = P(\text{choosing exactly } r \text{ defective sets})$

$= P[\text{choosing } r \text{ defective and } (3 - r) \text{ good sets}]$

$$= \frac{2\,C_r \times 4\,C_{3-r}}{6\,C_3} \quad (r = 0, 1, 2)$$

The required probability distribution is represented in the form of the following table.

$X = r$	p_r
0	1/5
1	3/5
2	1/5
Total	1

Example 3

A random variable X may assume 4 values with probabilities $(1 + 3x)/4$, $(1 - x)/4$, $(1 + 2x)/4$ and $(1 - 4x)/4$. Find the condition on x so that these values represent the probability function of X?

$$P(X = x_1) = p_1 = (1 + 3x)/4;\ p_2 = (1 - x)/4;$$
$$p_3 = (1 + 2x)/4;\ p_4 = (1 - 4x)/4$$

If the given probabilities represent a probability function, each $p_i \geq 0$ and $\sum_i p_i = 1$.

In this problem, $p_1 + p_2 + p_3 + p_4 = 1$, for any x.

But $p_1 \geq 0$, if $x \geq -1/3$; $p_2 \geq 0$, if $x \leq 1$; $p_3 \geq 0$, if $x \geq -1/2$ and $p_4 \geq 0$; if $x \leq 1/4$.

Therefore, the values of x for which a probability function is defined lie in the range $-1/3 \leq x \leq 1/4$.

Example 4

If the random variable X takes the values 1, 2, 3 and 4 such that $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$, find the probability distribution and cumulative distribution function of X.

Let $P(X = 3) = 30K$. Since $2P(X = 1) = 30K$, $P(X = 1) = 15K$.

Similarly $P(X = 2) = 10K$ and $P(X = 4) = 6K$.

Since $\Sigma\, p_i = 1$, $15K + 10K + 30K + 6K = 1$.

$$\therefore K = \frac{1}{61}$$

The probability distribution of X is given in the following table:

$X = i$	1	2	3	4
p_i	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

The cdf $F(x)$ is defined as $F(x) = P(X \leq x)$. Accordingly the cdf for the above distribution is found out as follows:

When $x < 1$, $F(x) = 0$

When $1 \leq x < 2$, $F(x) = P(X = 1) = \frac{15}{61}$

When $2 \leq x < 3$, $F(x) = P(X = 1) + P(X = 2) = \frac{25}{61}$

When $3 \leq x < 4$, $F(x) = P(X = 1) + P(X = 2) + P(X = 3) = \frac{55}{61}$

When $x \geq 4$, $F(x) = P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4) = 1$.

Example 5

A random variable X has the following probability distribution.

x:	-2	-1	0	1	2	3
$p(x)$:	0.1	K	0.2	$2K$	0.3	$3K$

(a) Find K, (b) Evaluate $P(X < 2)$ and $P(-2 < X < 2)$, (c) find the cdf of X and (d) evaluate the mean of X. (BU — Apr. 96)

(a) Since $\Sigma\, P(x) = 1$, $6K + 0.6 = 1$

$$\therefore \quad K = \frac{1}{15}$$

$\therefore$ the probability distribution becomes

x :	-2	-1	0	1	2	3
$p(x)$:	1/10	1/15	1/5	2/15	3/10	1/5

(b) $P(X < 2) \quad = P(X = -2, -1, 0 \text{ or } 1)$

$= P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1)$

[since the events $(X = -2)$, $(X = -1)$ etc. are mutually exclusive]

$$= \frac{1}{10} + \frac{1}{15} + \frac{1}{5} + \frac{2}{15} = \frac{1}{2}$$

$P(-2 < X < 2) = P(X = -1, 0 \text{ or } 1)$

$= P(X = -1) + P(X = 0) + P(X = 1)$

$$= \frac{1}{15} + \frac{1}{5} + \frac{2}{15} = \frac{2}{5}$$

(c) $F(x) = 0$, when $x < -2$

$= \frac{1}{10}$, when $-2 \leq x < -1$

$= \frac{1}{6}$, when $-1 \leq x < 0$

$$= \frac{11}{30}, \text{ when } 0 \le x < 1$$

$$= \frac{1}{2}, \text{ when } 1 \le x < 2$$

$$= \frac{4}{5}, \text{ when } 2 \le x < 3$$

$$= 1, \text{ when } 3 \le x$$

(d) The mean of X is defined as $E(X) = \Sigma\, xp(x)$ (refer to Chapter 4)

$$\therefore \text{Mean of } X = \left(-2\times\frac{1}{10}\right)+\left(-1\times\frac{1}{15}\right)+\left(0\times\frac{1}{5}\right)$$

$$+\left(1\times\frac{2}{15}\right)+\left(2\times\frac{3}{10}\right)+\left(3\times\frac{1}{5}\right)$$

$$= -\frac{1}{5}-\frac{1}{15}+\frac{2}{15}+\frac{3}{5}+\frac{3}{5}=\frac{16}{15}$$

Example 6

The probability function of an infinite discrete distribution is given by $P(X=j) = 1/2^j$ ($j = 1, 2, \ldots, \infty$). Verify that the total probability is 1 and find the mean and variance of the distribution. Find also $P(X$ is even$)$, $P(X \ge 5)$ and $P(X$ is divisible by 3$)$.

Let $P(X=j) = p_j$

$$\sum_{j=1}^{\infty} p_j = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\infty, \text{ that is a geometric series.}$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

The mean of X is defined as $E(X) = \sum_{j=1}^{\infty} jp_j$ (refer to Chapter 4).

$$\therefore \quad E(X) = a + 2a^2 + 3a^3 + \ldots \infty, \text{ where } a = \frac{1}{2}$$

$$= a(1 + 2a + 3a^2 + \ldots \infty)$$

$$= a(1-a)^{-2} = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 2$$

The variance of X is defined as $V(X) = E(X^2) - [E(X)]^2$,

where $E(X^2) = \sum_{j=1}^{\infty} j^2 p^j$ (refer to Chapter 4).

$$E(X^2) = \sum_{j=1}^{\infty} j^2 a^j, \text{ where } a = \frac{1}{2}$$

$$= \sum_{j=1}^{\infty} [j(j+1)-j]a^j = \sum_{j=1}^{\infty} j(j+1)a^j - \sum_{j=1}^{\infty} ja^j$$

$$= a\,(1.2 + 2.3a + 3.4a^2 + \ldots \infty) - a(1 + 2a + 3a^3 + \ldots \infty)$$

$$= a \times 2\,(1-a)^3 - a \times (1-a)^{-2}$$

$$= \frac{2a}{(1-a)^3} - \frac{a}{(1-a)^2} = 8 - 2 = 6$$

$$\therefore \quad V(X) = E(X^2) - \{E(X)\}^2 = 6 - 4 = 2$$

$P(X$ is even$) = P(X = 2$ or $X = 4$ or $X = 6$ or etc.$)$

$P(X = 2) + P(X = 4) + \ldots + \infty$

(since the events are mutually exclusive)

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \cdots + \infty$$

$$= \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$$

$$P(X \ge 5) = P(X = 5 \text{ or } X = 6 \text{ or } X = 7 \text{ or etc.})$$

$$= P(X = 5) + P(X = 6) + \ldots + \infty$$

$$= \frac{\frac{1}{2^5}}{1-\frac{1}{2}} = \frac{1}{16}$$

$P(X \text{ is divisible by } 3) = P(X = 3 \text{ or } X = 6 \text{ or } X = 9 \text{ etc.})$

$$= P(X = 3) + P(X = 6) + \ldots + \infty$$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^9 + \cdots + \infty$$

$$= \frac{\frac{1}{8}}{1 - \frac{1}{8}} = \frac{1}{7}$$

Example 7

A random variable X has the following probability distribution.

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2 + K$

Find (i) the value of K, (ii) $P(1.5 < X < 4.5/X > 2)$ and (iii) the smallest value of λ for which $P(X \le \lambda) > 1/2$.

$$\Sigma\, p(x) = 1$$

$\therefore$ $10K^2 + 9K = 1$

i.e., $(10K - 1)(K + 1) = 0$

$\therefore$ $K = \frac{1}{10}$ or -1.

The value $K = -1$ makes some values of $p(x)$ negative, which is meaningless.

$\therefore$ $K = \frac{1}{10}$

The actual distribution is given below:

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

(i) $P(1.5 < X < 4.5 / X > 2) = P(A/B)$, say

$$= \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P[(1 \cdot 5 < X < 4 \cdot 5) \cap (X > 2)]}{P(X > 2)}$$

$$= \frac{P(X = 3) + P(X = 4)}{\sum_{r=3}^{7} (X = r)} = \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}$$

(ii) By trials, $P(X \le 0) = 0$; $P(X \le 1) = \frac{1}{10}$; $P(X \le 2) = \frac{3}{10}$

$$P(X \le 3) = \frac{5}{10};\ P(X \le 4) = \frac{8}{10}$$

Therefore, the smallest value of λ satisfying the condition $P(X \le \lambda) > 1/2$ is 4.

Example 8

If $p(x) = \begin{cases} x\, e^{-x^2/2} & x \ge 0 \\ 0 & x < 0 \end{cases}$

(a) show that $p(x)$ is a pdf (of a continuous RV X.)

(b) find its distribution function $P(x)$. (BU — Nov. 96)

(a) If $p(x)$ is to be a pdf, $p(x) \ge 0$ and

$$\int_{R_X} p(x)\, dx = 1$$

Obviously, $p(x) = xe^{-x^2/2} \ge 0$, when $x \ge 0$

Now $\int_0^\infty p(x)dx = \int_0^\infty xe^{-x^2/2}\, dx = \int_0^\infty e^{-t}\, dt$ (putting $t = x^2/2$)

$$= 1$$

$\therefore$ $p(x)$ is a legitimate pdf of a RV X.

$$F(x) = P(X \le x) = \int_0^x f(x)dx$$

$\therefore$ $F(x) = 0$, when $x < 0$

and $F(x) = \int_0^x xe^{-x^2/2}\, dx = 1 - e^{-x^2/2}$, when $x \ge 0$.

Example 9

If the density function of a continuous RV X is given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

(i) find the value of a

(ii) find the cdf of X

(iii) If x_1, x_2 and x_3 are 3 independent observations of X, what is the probability that exactly one of these 3 is greater than 1.5?

(i) Since $f(x)$ is a pdf, $\int_{R_X} f(x)\mathrm{d}x = 1$

i.e., $\int_0^3 f(x)\mathrm{d}x = 1$

i.e., $\int_0^1 ax\mathrm{d}x + \int_1^2 a\mathrm{d}x + \int_2^3 (3a - ax)\mathrm{d}x = 1$

i.e., $2a = 1$

$\therefore$ $a = \dfrac{1}{2}$

(ii) $F(x) = P(X \le x) = 0$, when $x < 0$

$$F(x) = \int_0^x \frac{x}{2}\,\mathrm{d}x = \frac{x^4}{4}, \text{ when } 0 \le x \le 1$$

$$= \int_0^1 \frac{x}{2}\,\mathrm{d}x + \int_1^x \frac{1}{2}\,\mathrm{d}x = \frac{x}{2} - \frac{1}{4} \text{ when } 1 \le x \le 2$$

$$= \int_0^1 \frac{x}{2}\,\mathrm{d}x + \int_1^2 \frac{1}{2}\,\mathrm{d}x + \int_2^x \left(\frac{3}{2} - \frac{x}{2}\right)\mathrm{d}x = \frac{3}{2}x - \frac{x^2}{4} - \frac{5}{4}, \text{ when } 2 \le x \le 3$$

$= 1$, when $x > 3$

(iii) $p(X > 1.5) = \int_{1.5}^3 f(x)\,\mathrm{d}x$

$$= \int_{1.5}^2 \frac{1}{2}\,\mathrm{d}x + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)\mathrm{d}x = \frac{1}{2}$$

Choosing an X and observing its value can be considered as a trial and $(X > 1.5)$ can be considered a success.

$\therefore$ $p = 1/2$, $q = 1/2$

As we choose 3 independent observations of X, $n = 3$.

By Bernoulli's theorem,

P(exactly one value > 1.5)

$$= P(1 \text{ success}) = 3C_1 \times (p)^1 \times (q)^2 = \frac{3}{8}$$

Example 10

A continuous RV X that can assume any value between $x = 2$ and $x = 5$ has a density function given by $f(x) = k(1 + x)$. Find $P(X < 4)$. (MU — Apr. 96)

By the property of pdf,

$\int_{R_x} f(x)\,\mathrm{d}x = 1$. X takes values between 2 and 5.

$\therefore$ $\int_2^5 k(1 + x)\mathrm{d}x = 1$

i.e., $\dfrac{27}{2}k = 1$

$\therefore$ $k = \dfrac{2}{27}$

Now $p(X < 4) = p(2 < X < 4) = \int_2^4 k(1 + x)\mathrm{d}x = \dfrac{16}{27}$

Example 11

A continuous RV X has a pdf $f(x) = kx^2e^{-x}$; x 1 0. Find k, mean and variance. (MKU — Apr. 97)

By the property of pdf,

$$\int_0^\infty kx^2e^{-x}\mathrm{d}x = 1$$

i.e., $2k = 1$

$\therefore$ $k = \dfrac{1}{2}$

Mean of X is defined as

$$E(X) = \int_{R_x} xf(x)\mathrm{d}x$$

(refer to Chapter 4)

Variance of X is defined as

$$V(X) = E(X^2) - \{E(X)\}^2,$$

where $E(X^2) = \int_{R_x} x^2f(x)\mathrm{d}x$ (refer to Chapter 4)

$$\therefore \quad E(X) = \frac{1}{2}\int_0^\infty x^3e^{-x}\mathrm{d}x$$

$$= \frac{1}{2}\left[x^3(-e^{-x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})\right]_0^\infty$$

$$= 3$$

$$E(X^2) = \frac{1}{2}\int_0^\infty x^4 e^{-x}dx$$

$$= \frac{1}{2}\,[x^4(-e^{-x}) - 4x^3(e^{-x}) + 12x^2(-e^{-x}) - 24x(e^{-x}) + 24(-e^{-x})]_0^\infty$$

$$= 12$$

$\therefore \quad V(X) = E(X^2) - \{E(X)\}^2 = 3$

Example 12

The probability that a person will die in the time interval (t_1, t_2) is given by

$$p(t_1 \le t \le t_2) = \int_{t_1}^{t_2} a(t)\,dt.$$

The function $a(t)$ is determined from long records and can be assumed to be

$$a(t) = \begin{cases} 3 \times 10^{-9} t^2 (100-t)^2 & 0 \le t \le 100 \\ 0 & \text{elsewhere} \end{cases}$$

Determine (i) the probability that a person will die between the ages 60 and 70 and (ii) the probability that he will die between those ages, assuming he lived upto 60. (MSU — Apr. 96)

(i) $P(60 < t < 70) = \int_{60}^{70} a(t)dt$

$$= 3 \times 10^{-9} \int_{60}^{70} t^2(100-t)^2 dt$$

$$= 0.1544$$

(ii) $P(60 < t < 70/t \ge 60) = P(60 < t < 70/60 \le t \le 100)$

$$= \frac{P(60 < t < 70)}{P(60 < t < 100)}$$

$$= \int_{60}^{70} a(t)dt \Big/ \int_{60}^{100} a(t)dt$$

$$= \frac{0{\cdot}15436}{0{\cdot}31744} = 0.4863$$

Example 13

A continuous RV has a pdf $f(x) = 3x^2$, $0 \le x \le 1$. Find a and b such that
(i) $P(X \le a) = P(X > a)$ and
(ii) $P(X > b) = 0.05$ (BDU — Nov. 96)

(i) $P(X \le a) = P(X > a)$

$$\therefore \int_0^a 3x^2 dx = \int_a^1 3x^2 dx$$

i.e., $a^3 = 1 - a^3$

i.e., $a^3 = \frac{1}{2}$

$\therefore a = 0.7937$

(ii) $P(X > b) = 0.05$

$$\int_b^1 3x^2 dx = 0.05$$

i.e., $b^3 = 95$

$\therefore \quad b = 0.9830$

Example 14

The distribution function of a RV X is given by $F(x) = 1 - (1+x)e^{-x}$, $x \ge 0$. Find the density function, mean and variance of X. (MKU — Nov. 96)

By the property of $F(x)$, the pdf $f(x)$ is given by $f(x) = F'(x)$ at points of continuity of $F(x)$.

The given cdf is continuous for $x \ge 0$.

$$\therefore \quad f(x) = (1+x)e^{-x} - e^{-x} = xe^{-x},\ x \ge 0$$

$$E(X) = \int_0^\infty x^2 e^{-x} dx = 2$$

$$E(X^2) = \int_0^\infty x^3 e^{-x} dx = 6$$

$$V(X) = E(X^2) - [E(X)]^2 = 2$$

Example 15

The cdf of a continuous RV X is given by

$$F(x) = 0,\ x < 0$$
$$= x^2,\ 0 \le x < \frac{1}{2}$$
$$= 1 - \frac{3}{25}(3-x)^2,\ \frac{1}{2} \le x < 3$$
$$= 1,\ x \ge 3$$

Find the pdf of X and evaluate $P(|X| \le 1)$ and $P(\frac{1}{3} \le X < 4)$ using both the pdf and cdf.

The points $x = 0$, 1/2 and 3 are points of continuity

$$\therefore \quad f(x) = 0, \; x < 0$$
$$= 2x, \; 0 \le x < \frac{1}{2}$$
$$= \frac{6}{25}(3 - x), \; \frac{1}{2} \le x < 3$$
$$= 0, \; x \ge 3$$

Although the points $x = 1/2$, 3 are points of discontinuity for $f(x)$, we may assume that $f\left(\frac{1}{2}\right) = \frac{3}{5}$ and $f(3) = 0$.

$$P(|X| \le 1) = p(-1 \le x \le 1)$$
$$= \int_{-1}^{1} f(x)dx = \int_{0}^{1/2} 2x dx + \int_{1/2}^{1} \frac{6}{25}(3 - x)\, dx \text{ (using property of pdf)}$$
$$= \frac{13}{25}$$

If we use property of cdf

$$P(|X| \le 1) = P(-1 \le x \le 1) = F(1) - F(-1) = \frac{13}{25}$$

If we use the property of pdf

$$P(1/3 \le X < 4) = \int_{1/3}^{1/2} 2x dx + \int_{1/2}^{3} \frac{6}{25}(3 - x)dx = \frac{8}{9}$$

If we use the property of cdf

$$P(1/3 \le X < 4) = F(4) - F\left(\frac{1}{3}\right)$$
$$= 1 - \frac{1}{9} = \frac{8}{9}$$

Example 16

If the RV k is uniformly distributed over (0, 5) what is the probability that the roots of the equation $4x^2 + 4kx + (k + 2) = 0$ are real?

The RV k is U (0, 5).

$$\therefore \quad \text{pdf of } k = \frac{1}{5}, \; 0 < k < 5$$

$P(\text{Roots of } 4x^2 + 4kx + k + 2 = 0 \text{ are real})$

$= P(\text{Discriminant of the equation} \ge 0)$

$= P(k^2 - k - 2 \ge 0) = P[(k - 2)(k + 1) \ge 0]$

$= P[(k \ge -1 \text{ and } k \ge 2) \text{ or } (k \le 2 \text{ and } k \le -1)]$

$= P(k \ge 2 \text{ or } k \le -1) = P(k \ge 2)$ [*since k takes values in* (0, 5)]

$$= \int_{2}^{5} f(k)dk = \frac{1}{5}(5 - 2)$$
$$= \frac{3}{5}$$

Example 17

A point P is taken at random on a line AB of length $2a$, all positions of the point being equally likely. Find the probability that the product $(AP \times PB) > \frac{a^2}{2}$.

Let $AP = X$.

$$\therefore \quad PB = (2a - X)$$

Since all positions of the point P are equally likely, $X(= AP)$ is uniformly distributed over $(0, 2a)$.

$$\therefore \quad \text{pdf of } X = \frac{1}{2a}, \; 0 < x < 2a$$

$$P[AP \times PB > \frac{a^2}{2}] = P[X(2a - X) > \frac{a^2}{2}]$$
$$= P(2X^2 - 4aX + a^2 < 0)$$
$$= P\left[\left\{X - \left(1 - \frac{1}{\sqrt{2}}\right)a\right\}\left\{X - \left(1 + \frac{1}{\sqrt{2}}\right)a\right\} < 0\right]$$

[since the factors of $(2x^2 - 4ax + a^2)$ are $x - \left(1 - \frac{1}{\sqrt{2}}\right)a$ and $x - \left(1 + \frac{1}{\sqrt{2}}\right)a$]

$$= P\left[\left(1 - \frac{1}{\sqrt{2}}\right)a < X < \left(1 + \frac{1}{\sqrt{2}}\right)a\right]$$
$$= \int_{\left(1 - \frac{1}{\sqrt{2}}\right)a}^{\left(1 + \frac{1}{\sqrt{2}}\right)a} f(x)\, dx = \frac{1}{2a}[x]_{\left(1 - \frac{1}{\sqrt{2}}\right)a}^{\left(1 + \frac{1}{\sqrt{2}}\right)a} = \frac{\sqrt{2}\, a}{2a} = \frac{1}{\sqrt{2}}$$

Example 18

If the continuous RV. X represents the time of failure of a system, that has been put into operation at $t = 0$, find the conditional density function of X, given that the system has survived upto time t. Deduce the same when X follows an exponential distribution with parameter λ.

The conditional distribution function of X, subject to the given condition, is given by

$$F(x/X > t) = \frac{P[X \le x \text{ and } X > t]}{P(X > t)} \text{ [since unconditional } F(x) = P(X \le x)]$$

$$= \frac{P[t < X \le x]}{P[t < X < \infty]}$$

$$= \frac{F(x) - F(t)}{1 - F(t)} \text{ for } x > t$$

$$= 0 \text{ for } x < t$$

Therefore, the conditional density function $f(x/X > t)$ is given by

$$f(x/X > t) = \frac{d}{dx} F(x/X > t)$$

$$= \frac{f(x)}{1 - F(t)}, x > t$$

For the exponential distribution with parameter λ,

$$f(x) = \lambda e^{-\lambda x}, x > 0, \text{ and } F(x) = \int_0^X \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}.$$

$$\therefore \quad f(x/X > t) = \frac{\lambda e^{-\lambda X}}{e^{-\lambda t}} = \lambda e^{-\lambda(x-t)} = f(x - t)$$

Example 19

If $f(t)$ is the unconditional density of the time of failure of a system and $h(t)$ is the hazard rate (or conditional failure rate or conditional density of T, given $T > t$) find $f(t)$ in terms of $h(t)$. Deduce that T follows a Rayleigh distribution, when $h(t) = t$.

The conditional density of T, given $T > t$ or the hazard rate, is given by $h(t) = f(t/T > t)$

$$= \frac{f(t)}{1 - F(t)} = \frac{F'(t)}{1 - F(t)}$$

$$\therefore \int_0^t h(t)dt = \int_0^t \frac{F'(t)}{1 - F(t)} dt$$

$$= [-\log\{1 - F(t)\}]_0^t$$

$= -\log\{1 - F(t)\}$ [since $F(0) = P(T \le 0) = 0$ as the system was put into operation at $t = 0$]

$$\therefore \quad F(t) = 1 - e^{-\int_0^t h(t)\,dt}$$

$$\therefore \quad f(t) = h(t) \times e^{-\int_0^t h(t)dt}$$

When $h(t) = t$,

$$f(t) = t \times e^{-\int_0^t t\,dt}$$

$= te^{-t^2/2}$, which is the pdf of a Raleigh distribution

Example 20

If a continuous RV X follows N(0, 2), find

$P\{1 \le X \le 2\}$ and $P\{1 \le X \le 2/X \ge 1\}$.

X follows $N(0, 2)$, the density function of which is $f(x) = \frac{1}{2\sqrt{2\pi}} e^{-x^2/8}$, $-\infty < x < \infty$.

$$\therefore \quad P\{1 \le X \le 2\} = \int_1^2 f(x)dx$$

$$= \int_{0.5}^1 \phi(t)dt, \text{ putting } t = \frac{x}{2}$$

where $\phi(t)$ is the standard normal density.

$$= \int_0^1 \phi(t)dt - \int_0^{0.5} \phi(t)dt$$

$= 0.3413 - 0.1915$ (from the normal tables)

$= 0.1498$

$$P\{1 \le X \le 2/X \ge 1\} = \frac{P\{(1 \le X \le 2) \text{ and } X \ge 1\}}{P\{X \ge 1\}}$$

$$= \frac{P\{1 \le X \le 2\}}{P\{1 \le X < \infty\}}$$

$$= \frac{0.1498}{\int_0^\infty \phi(t)\,dt - \int_0^{0.5} \phi(t)\,dt}$$

$$= \frac{0.1498}{0.5 - 0.1915} = 0.4856$$

Exercise 2(A)

Part A (Short answer questions)

1. Define a RV with an example.
2. Define a discrete RV with an example.
3. Define a continuous RV and give an example for the same.
4. Distinguish between a discrete RV and a continuous RV.
5. Define the probability mass function of a discrete RV.
6. Write down the probability distribution of the outcome when 2 fair dice are tossed.
7. Define the pdf of a continuous RV.
8. State the properties of the pdf of a continuous RV.
9. What is the probability curve of a continuous RV? Give an example.
10. Prove that it is almost impossible that a continuous RV assumes a specific value. (OR) If X is a continuous RV prove that $P(X = a) = 0$.
11. If X represents the total number of heads obtained, when a fair coin is tossed 5 times, find the probability distribution of X.
12. If the probability distribution of X is given as:

x:	1	2	3	4
p_x:	0.4	0.3	0.2	0.1

find $P\left(\frac{1}{2} < X < \frac{7}{2} / X > 1\right)$

13. Define the cdf of a RV. Explain how to find it for both kinds of RV.
14. Differentiate between the pdf and cdf of a RV.
15. State the properties of the cdf of a RV.
16. Verify whether $f(x) = \begin{cases} |x| & \text{in} -1 \le X \le 1 \\ 0, & \text{elsewhere} \end{cases}$

can be the pdf of a continuous RV.

17. If $f(x) = kx^2$, $0 < x < 3$, is to be a density function, find the value of k.
18. If the pdf of a RV X is given by

$$f(x) = \begin{cases} 1/4 & \text{in } -2 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

find $P\{|X| > 1\}$.

19. Find the value of k, if $f(x) = \begin{cases} k x e^{-x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$

is the pdf of a RV X.

20. If the pdf of a RV X is $f(x) = \frac{x}{2}$ in $0 \le x \le 2$, find $P\{X > 1.5/X > 1\}$.
21. The RV X has the following probability distribution:

x:	−2	−1	0	1
p_x:	0.4	k	0.2	0.3

find k and the mean value of X.

22. If X represents the outcome of the toss of a 6 faced dice, find $P(X \le x)$ as a function of x.
23. If the pdf of a RV X is $f(x) = 2x$, $0 < x < 1$, find the cdf of X.
24. If the cdf of a RV X is given by $F(x) = 1 - e^{-\lambda x}$, when $x \ge 0$ and $= 0$, when $x < 0$, find the pdf of X.
25. If the cdf of a RV is given by $F(x) = 0$, for $x < 0$; $= x^2/16$ for $0 \le x < 4$ and $= 1$, for $4 \le x$, find $P(X > 1/X < 3)$.
26. Define binomial distribution, What are its mean and variance?
27. Give the probability law of Poisson distribution and also its mean and variance.
28. Define the exponential distribution.
29. If X follows an exponential distribution with parameter 1, find $P(|X| < 1)$.
30. Define Pascal distribution and define geometric distribution as a particular case of Pascal distribution.
31. Write down the pdf's of general normal distribution and standard normal distribution.
32. Define Erlang distribution. Deduce Gamma distribution as a particular case of Erlang distribution.
33. Deduce the pdf of an exponential distribution as a particular case of that of Erlang distribution.
34. Give the pdf of Raleigh distribution.
35. Define Maxwell distribution.
36. Write down the pdf of Laplace distribution.
37. Define Cauchy distribution.

Part B

38. Find the formula for the probability distribution of the number of heads, when a fair coin is tossed 4 times. (MU — Nov. 96)
39. A coin is known to come up heads 3 times as often as tails. This coin is tossed 3 times. Write down the probability distribution of the number of heads that appear and also the cdf. Make a sketch of both.
40. Consider the experiment of tossing a coin, the 2 events of the space being occurrence of head of tail. Assign probabilities p and q for head and tail respectively and define a random variable X by $X(h) = 1$ and $X(t) = 0$. Determine and plot the probability function $f(x)$ and the distribution function $F(x)$. (MSU — Apr. 96)
41. Two dice are tossed. If X is the sum of the numbers shown up, find the probability mass function of X.
42. Consider the experiment of tossing a fair coin 4 times. Define $X = 0$, if 0 or 1 head appears; $X = 1$, if 2 heads appear; $X = 2$, if 3 or 4 heads appear. Find the probability function, mean and variance of X.

43. A discrete RV X has the following probability distribution.

x:	0	1	2	3	4	5	6	7	8
$p(x)$:	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

Find the value of a, $P(X < 3)$, variance and distribution function of X.
(MKU — Apr. 97)

44. The probability distribution of a R ' is given below:

x:	0	1	2	3
$p(x)$:	0.1	0.3	0.5	0.1

If $Y = X^2 + 2X$, find the proba' ility distribution, mean and varian of Y.

45. The probability mass function of a RV X is defined as $P(X = 0) = 3C^2$, $P(X = 1) = 4C - 10C^2$ and $P(X = 2) = 5C - 1$, where $C > 0$ and $P(X = r) = 0$, if $r \neq 0, 1, 2$.
(i) Find the value of C, (ii) Find $P\{0 < X < 2/X > 0\}$, (iii) the distribution function of X, (iv) the largest value for which $F(x) < \frac{1}{2}$ and (v) the smallest value of X for which $F(x) > \frac{1}{2}$.

46. If the probability mass function of a RV X is given by $P(X = r) = kr^3$; $r = 1, 2, 3, 4$, find (i) the value of k (ii) $P(1/2 < X < 5/2 / X > 1)$, (iii) the mean and variance of X and (iv) the distribution function of X.

47. Find the values of a for which $P(x = j) = (1 - a)a^j, j = 0, 1, 2, \ldots$ represents a probability mass function. Show also that for any 2 positive integers m and n

$$P(X > m + n/X > m) = P(X \geq n).$$

48. If a discrete probability distribution is given by $P(X = r) = k(1 - a)^{r-1}$, $0 < a < 1$, for $r = 1, 2, \ldots, \infty$, find the value of k and also the mean and variance of X.

49. If the probability distribution of a discrete RV X is given by $P(X = x) = ke^{-t}(1 - e^{-t})^{x-1}$, $x = 1, 2, \ldots, \infty$, find the value of k and also the mean and variance of X.

50. In a continuous distribution, the probability density is given by $f(x) = kx(2 - x)$, $0 < x < 2$. Find k, mean, variance and the distribution function.
(MKU — Nov. 96)

51. The diameter of an electric cable X is a continuous RV with pdf $f(x) = kx(1 - x)$, $0 \leq x \leq 1$. Find (i) the value of k, (ii) cdf of X, (iii) the value of a such that $P(X < a) = 2P(X > a)$ and (iv) $P(X \leq 1/2 / 1/3 < X < 2/3)$.

52. X is a continuous RV with pdf given by $f(x) = kx$, in $0 \leq x \leq 2$; $= 2k$, in $2 \leq x \leq 4$, and $= 6k - kx$, in $4 \leq x \leq 6$. Find the value of k and $F(x)$.

53. The continuous RV X has pdf $f(x) = \frac{x}{2}$, $0 \leq x \leq 2$. Two independent determinations of X are made. What is the probability that both these determinations will be greater than 1? If 3 independent determinations had been made, what is the probability that exactly 2 of these are larger than 1?

54. A continuous RVX that can assume values between $x = 2$ and $x = 5$ has density function given by $f(x) = 2(1 + x)/27$. Find $P(3 < X < 4)$.
(MU — Nov. 9

55. A continuous RV has the pdf $f(x) = kx^4$, $-1 < x < 0$. Find the value of k an also $P(X > -1/2 / X < -1/4)$.

56. Suppose that the life length of a certain radio tube (in hours) is a contin ous RV X with pdf $f(x) = \frac{100}{x^2}$ $x > 100$ and $= 0$, elsewhere.

(a) What is the probability that a tube will last less than 200 h, if it is know that the tube is still functioning after 150 h of service?

(b) What is the probability that if 3 such tubes are installed in a set, exactly will have to be replaced after 150 h of service?

(c) What is the maximum number of tubes that may be inserted into a set s that there is a probability of 0.1 that after 150 h of service all of them ar still functioning?

57. If the cdf of a continuous RV X is given by $F(x) = \frac{1}{2}e^{kx}$, $x \leq 0$, and $F(x)$ $1 - \frac{1}{2}e^{-kx}$, $x > 0$, find $P(|x| \leq 1/k)$. Prove that the density function of X i

$$f(x) = \frac{k}{2}e^{-k|x|} \quad -\infty < x < \infty, \text{ given that } k > 0.$$

58. If the distribution function of a continuous RV X is given by $F(x) =$ (when $x < 0$; $= x$, when $0 \leq x < 1$ and $= 1$, when $1 \leq x$, find the pdf of X. Als find $P(1/3 < X < 1/2)$ and $P(1/2 < X < 2)$ using the cdf of X.

59. A point is chosen on a line of length a at random. What is the probabilit that the ratio of the shorter to the longer segment is less than 1/4?

60. If the RV k is uniformly distributed over (1, 7) what is the probability tha the roots of the equation $x^2 + 2kx + (2k + 3) = 0$ are real?

61. If $f(t)$ is the unconditional density of time to failure T of a system and $h(t$ is the conditional density of T, given $T > t$, find $h(t)$ when (i) $f(t) = \lambda e^{-\lambda}$ (ii) $f(t) = \lambda^2 te^{-\lambda t}$, $t > 0$. Prove also that $h(t)$ is not a density function.

62. If the continuous RV X follows $N(1000, 20)$, find
(i) $P(X < 1024)$, (ii) $P(X < 1024 / X > 961)$ and
(iii) $P(31 < \sqrt{X} \leq 32)$.

Two-Dimensional Random Variables

So far we have considered only the one-dimensional RV, i.e., we have considere such random experiments, the outcome of which had only one characteristic an hence was assigned a single real value. In many situations, we will be intereste in recording 2 or more characteristics (numerically) of the outcome of a randon experiment. For example, both voltage and current might be of interest in certain experiment.

Definitions: Let S be the sample space associated with a random experiment E Let $X = X(s)$ and $Y = Y(s)$ be two functions each assigning a real number to eacl outcomes $s \in S$. Then (X, Y) is called a *two-dimensional random variable*.

If the possible values of (X, Y) are finite or countably infinite, (X, Y) is called *a two-dimensional discrete* RV. When (X, Y) is a two-dimensional discrete RV the possible values of (X, Y) may be represented as (x_i, y_j), $i = 1, 2, \ldots, m, \ldots; j = 1, 2, \ldots, n, \ldots$.

If (X, Y) can assume all values in a specified region R in the xy-plane, (X, Y) is called a *two-dimensional continuous* RV.

Probability Function of (X, Y)

If (X, Y) is a two-dimensional discrete RV such that $P(x = x_i, y = y_j) = p_{ij}$, then p_{ij} is called the *probability mass function* or simply the *probability function* of (X, Y) provided the following conditions are satisfied.

(i) $p_{ij} \geq 0$, for all i and j

(ii) $\sum\limits_j \sum\limits_i p_{ij} = 1$

The set of triples $\{x_i, y_j, p_{ij}\}$, $i = 1, 2, \ldots, m, \ldots, j = 1, 2, \ldots, n, \ldots$, is called *the joint probability distribution of (X, Y).*

Joint Probability Density Function

If (X, Y) is a two-dimensional continuous RV such that.

$P\left\{x - \frac{dx}{2} \leq X \leq x + \frac{dx}{2} \text{ and } y - \frac{dy}{2} \leq Y \leq y + \frac{dy}{2}\right\} = f(x, y)\,dx\,dy$, then $f(x, y)$ is called *the joint pdf* of (X, Y), provided $f(x, y)$ satisfies the following conditions.

(i) $f(x, y) \geq 0$, for all $(x, y) \in R$, where R is the range space.

(ii) $\iint\limits_R f(x, y)\,dx\,dy = 1$.

Moreover if D is a subspace of the range space R, $P\{(X, Y) \in D\}$ is defined as

$$P\{(X, Y) \in D\} = \iint\limits_D f(x, y)\,dx\,dy. \text{ In particular}$$

$$P\{a \leq X \leq b, c \leq Y \leq d\} = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

Cumulative Distribution Function

If (X, Y) is a two-dimensional RV (discrete or continuous), then $F(x, y) = P\{X \leq x \text{ and } Y \leq y\}$ is called *the cdf of (X, Y).*

In the discrete case,

$$F(x, y) = \sum_{\substack{j \\ y_j \leq y}} \sum_{\substack{i \\ x_i \leq x}} p_{ij}$$

In the continuous case,

$$F(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y)\,dx\,dy$$

Properties of F(x, y)

(i) $F(-\infty, y) = 0 = F(x, -\infty)$ and $F(\infty, \infty) = 1$

(ii) $P\{a < X < b, Y \leq y\} = F(b, y) - F(a, y)$

(iii) $P\{X \leq x, c < Y < d\} = F(x, d) - F(x, c)$

(iv) $P\{a < X < b, c < Y < d\} = F(b, d) - F(a, d) - F(b, c) + F(a, c)$

(v) At points of continuity of $f(x, y)$

$$\frac{\partial^2 F}{\partial x \partial y} = f(x, y)$$

Marginal Probability Distribution

$$P(X = x_i) = P\{(X = x_i \text{ and } Y = y_1) \text{ or } (X = x_i \text{ and } Y = y_2) \text{ or etc.}\}$$

$$= p_{i1} + p_{i2} + \ldots = \sum_j p_{ij}$$

$P(X = x_i) = \sum\limits_j p_{ij}$ is called *the marginal probability function of X*. It is defined for $X = x_1, x_2, \ldots$ and denoted as P_{i*}. The collection of pairs $\{x_i, p_{i*}\}$, $i = 1, 2, 3, \ldots$ is called the *marginal probability distribution of X.*

Similarly the collection of pairs $\{y_j, p_{*j}\}$, $j = 1, 2, 3, \ldots$ is called *the marginal probability distribution of Y*, where $p_{*j} = \sum\limits_i P_{ij} = P(Y = y_j)$.

In the continuous case,

$$P\{x - \frac{1}{2}dx \leq X \leq x + \frac{1}{2}dx, -\infty < Y < \infty\}$$

$$= \int_{-\infty}^{\infty} \int_{x - \frac{1}{2}dx}^{x + \frac{1}{2}dx} f(x, y)\,dx\,dy$$

$$= \left[\int_{-\infty}^{\infty} f(x, y)\,dy\right] dx \text{ [since } f(x, y) \text{ may be treated a constant in}$$

$$(x - 1/2dx, x + 1/2\,dx)]$$

$$= f_X(x)dx, \text{ say}$$

$f_X(x) = \int\limits_{-\infty}^{\infty} f(x, y)dy$ is called *the marginal density of X.*

Similarly, $f_Y(y) = \int\limits_{-\infty}^{\infty} f(x, y)\,dx$ is called *the marginal density of Y.*

Note $P(a \le X \le b) = P(a \le X \le b, -\infty < Y < \infty)$

$$= \int_{-\infty}^{\infty} \int_{a}^{b} f(x, y)\,dx\,dy$$

$$= \int_{a}^{b} \left[\int_{-\infty}^{\infty} f(x, y)\,dy \right] dx = \int_{a}^{b} f_X(x)\,dx$$

Similarly, $P(c \le Y \le d) = \int_{c}^{d} f_Y(y)\,dy$

Conditional Probability Distribution

$P\{X = x_i / Y = y_j\} = \dfrac{P\{X = x_i, Y = y_j\}}{P\{Y = y_j\}} = \dfrac{p_{ij}}{p_{*j}}$ is called *the conditional probability function of* X, given that $Y = y_j$.

The collection of pairs, $\left\{x_i, \dfrac{p_{ij}}{p_{*j}}\right\}$ $i = 1, 2, 3, \ldots,$

is called *the conditional probability distribution of* X, *given* $Y = y_j$.

Similarly, the collection of pairs, $\left\{Y_j, \dfrac{p_{ij}}{p_{i*}}\right\}$, $j = 1, 2, 3, \ldots$, is called the *conditional probability distribution of* Y *given* $X = x_i$. In the continuous case,

$$P\left\{x - \frac{1}{2}dx \le X < x + \frac{1}{2}dx / Y = y\right\}$$

$$= P\left\{x - \frac{1}{2}dx \le X \le x + \frac{1}{2}dx / y - \frac{1}{2}dy \le Y \le y + \frac{1}{2}dy\right\}$$

$$= \frac{f(x, y)\,dx\,dy}{f_Y(y)\,dy} = \left\{\frac{f(x, y)}{f_Y(y)}\right\} dx.$$

$\dfrac{f(x, y)}{f_Y(y)}$ is called *the conditional density of* X, *given* Y, and is denoted by $f(x/y)$.

Similarly, $\dfrac{f(x, y)}{f_X(x)}$ is called *the conditional density of* Y, given X, and is denoted by $f(y/x)$.

Independent RVs

If (X, Y) is a two-dimensional discrete RV such that $P\{X = x_i / Y = y_j\} = P(X = x_i)$

i.e., $\dfrac{p_{ij}}{p_{*j}} = p_{i*}$, i.e., $p_{ij} = p_{i*} \times p_{*j}$ for all i, j then X and Y are said to be independent RVs.

Similarly if (X, Y) is a two-dimensional continuous RV such that $f(x, y) = f_X(x) \times f_Y(y)$, then X and Y are said to be independent RVs.

Random Vectors

Sometimes we may have to be concerned with Random experiments whose outcomes will have 3 or more simultaneous numerical characteristics. To study the outcomes of such random experiments we require knowledge of *n-dimensional random variables* or *random vectors*. For example, the location of a space vehicle in a cartesian co-ordinate system is a three-dimensional random vector.

Most of the concepts introduced above for the two-dimensional case can be extended to the n-dimensional one.

Definitions: A vector X: $[X_1, X_2, \ldots, X_n]$ whose components X_i are RVs is called a *random vector*. $(X_1, X_2, \ldots, X_n)$ can assume all values in some region R_n of the n-dimensional space. R_n is called the *range space*.

The joint distribution function of $(X_1, X_2, \ldots, X_n)$ is defined as $F(x_1, x_2, \ldots, x_n)$ $= P[X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n]$

The joint pdf of $(X_1, X_2, \ldots, X_n)$ is defined as $f(x_1, x_2, \ldots, x_n)$

$$= \frac{\partial^n F(x_1, x_2 \cdots x_n)}{\partial x_1 . \partial x_2 \cdots \partial x_n}$$ and satisfies the following conditions.

(i) $f(x_1, x_2, \ldots, x_n) \ge 0$, for all $(x_1, x_2, \ldots, x_n)$

(ii) $\displaystyle\iint_{R_n}\int \cdots \int f(x_1, x_2, \ldots, x_n)\,dx_1 dx_2 \ldots dx_n = 1$

(iii) $P[(X_1, X_2, \ldots, X_n) \in D] = \displaystyle\iint_{D} \cdots \int f(x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n$ where D is a subset of the range space R_n.

The marginal pdf of any subset of the n RVs $X_1, X_2, \ldots, X_n$ is obtained by "integrating out" the variables not in the subset. For example, if $n = 3$, then

$f_{X_1}(x_1) = \displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_2 dx_3$ is the marginal pdf of the one-dimensional RV X_1 and $f_{X_1, X_2}(x_1, x_2) = \displaystyle\int_{-\infty}^{\infty} f(x_1\, x_2\, x_3)\,dx_3$ is the marginal joint pdf of the two-dimensional RV (X_1, X_2). The concept of independent RVs is also extended in a natural way. The RVs $(X_1, X_2, \ldots, X_n)$ are said to be independent, if

$f(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdot \ldots \cdot f_{X_n}(x_n)$

The conditional density functions are defined as in the following examples.

If $n = 3$,

$$f(x_1, x_2 / x_3) = \frac{f(x_1, x_2, x_3)}{f_{X_3}(x_3)}$$ and

$$f(x_1 / x_2, x_3) = \frac{f(x_1, x_2, x_3)}{f_{X_2, X_3}(x_2, x_3)}$$

Worked Example 2(B)

Example 1

Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn, find the joint probability distribution of (X, Y).

As there are only 2 white balls in the box, X can take the values 0, 1 and 2 and Y can take the values 0, 1, 2 and 3.

$P(X = 0, Y = 0)$ = P(drawing 3 balls none of which is white or red)
= P(all the 3 balls drawn are black)

$$= 4C_3/9C_3 = \frac{1}{21}$$

$P(X = 0, Y = 1)$ = P(drawing 1 red and 2 black balls)

$$= \frac{3C_1 \times 4C_2}{9C_3} = \frac{3}{14}$$

Similarly, $P(X = 0, Y = 2) = \frac{3C_2 \times 4C_1}{9C_3} = \frac{1}{7}$; $P(X = 0, Y = 3) = \frac{1}{84}$

$P(X = 1, Y = 0) = \frac{1}{7}$; $P(X = 1, Y = 1) = \frac{2}{7}$; $P(X = 1, Y = 2) = \frac{1}{14}$;

$P(X = 1, Y = 3) = 0$ (since only 3 balls are drawn)

$P(X = 2, Y = 0) = \frac{1}{21}$; $P(X = 2, Y = 1) = \frac{1}{28}$; $P(X = 2, Y = 2) = 0$;

$P(X = 2, Y = 3) = 0$

The joint probability distribution of (X, Y) may be represented in the form of a table as given below:

X	Y			
	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

Note *Sum of all the cell probabilities = 1.*

Example 2

For the bivariate probability distribution of (X, Y) given below, find $P(X \le 1)$, $P(Y \le 3)$, $P(X \le 1, Y \le 3)$, $P(X \le 1/Y \le 3)$, $P(Y \le 3/X \le 1)$ and $P(X + Y \le 4)$.

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

$P(X \le 1) = P(X = 0) + P(X = 1)$

$$= \sum_{j=1}^{6} P(X = 0, Y = j) + \sum_{j=1}^{6} P(X = 1, Y = j)$$

$$= \left(0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{2}{32} + \frac{3}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$= \frac{1}{4} + \frac{5}{8} = \frac{7}{8}$$

$P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$

$$= \sum_{i=0}^{2} P(X = i, Y = 1) + \sum_{i=0}^{2} P(X = i, Y = 2) + \sum_{i=0}^{2} P(X = i, Y = 3)$$

$$= \left(0 + \frac{1}{16} + \frac{1}{32}\right) + \left(0 + \frac{1}{16} + \frac{1}{32}\right) + \left(\frac{1}{32} + \frac{1}{8} + \frac{1}{64}\right)$$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$$

$$P(X \le 1, Y \le 3) = \sum_{j=1}^{3} P(X = 0, Y = j) + \sum_{j=1}^{3} P(X = 1, Y = j)$$

$$= \left(0+0+\frac{1}{32}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\right) = \frac{9}{32}$$

$$P(X \le 1/Y \le 3) = \frac{P(X \le 1, Y \le 3)}{P(Y \le 3)} = \frac{9/32}{23/64} = \frac{18}{23}$$

$$P(Y \le 3/X \le 1) = \frac{P(X \le 1, Y \le 3)}{P(X \le 1)} = \frac{9/32}{7/8} = \frac{9}{28}$$

$$P(X+Y \le 4) = \sum_{j=1}^{4} P(X=0, Y=j) + \sum_{j=1}^{3} P(X=1, Y=j) + \sum_{j=1}^{2} P(X=2, Y=j)$$

$$= \frac{3}{32}+\frac{1}{4}+\frac{1}{16} = \frac{13}{32}$$

Example 3

The joint probability mass function of (X, Y) is given by $p(x, y) = k(2x + 3y)$, $x = 0, 1, 2$; $y = 1, 2, 3$. Find all the marginal and conditional probability distributions. Also find the probability distribution of $(X + Y)$.

The joint probability distribution of (X, Y) is given below. The relevant probabilities have been computed by using the given law.

X	Y		
	1	2	3
0	$3k$	$6k$	$9k$
1	$5k$	$8k$	$11k$
2	$7k$	$10k$	$13k$

$$\sum_{j=1}^{3} \sum_{i=0}^{2} p(x_i, y_j) = 1$$

i.e., the sum of all the probabilities in the table is equal to 1.

i.e., $72k = 1$.

$$\therefore \quad k = \frac{1}{72}$$

Marginal Probability Distribution of X: $\{i, p_{i*}\}$

$X = i$	$p_{i*} = \sum_{j=1}^{3} p_{ij}$
0	$p_{01} + p_{02} + p_{03} = \frac{18}{72}$
1	$p_{11} + p_{12} + p_{13} = \frac{24}{72}$
2	$p_{21} + p_{22} + p_{23} = \frac{30}{72}$
	Total $= 1$

Marginal Probability Distribution of Y: $\{j, p_{*j}\}$

$Y = j$	$p_{*j} = \sum_{i=0}^{2} p_{ij}$
1	15/27
2	24/72
3	33/72
	Total = 1

Conditional distribution of X, given $Y = 1$, is given by $\{i, P(X = i/Y = 1)\} = \{i, P(X = i, Y = 1)/P(Y = 1)\}$
$= \{i, p_{i1}/p_{*1}\}$, $i = 0, 1, 2$.
The tabular representation is given below:

$X = i$	p_{i1}/p_{*1}
0	$3k/15k = \frac{1}{5}$
1	$5k/15k = \frac{1}{3}$
2	$7k/15k = \frac{7}{15}$
	Total = 1

The other conditional distributions are given below:

C.P.D. of X, given $Y = 2$	
$X = i$	p_{i2}/p_{*2}
0	$\frac{6k}{24k} = \frac{1}{4}$
1	$\frac{8k}{24k} = \frac{1}{3}$
2	$\frac{10k}{24k} = \frac{5}{12}$
	Total = 1

C.P.D. of X, given $Y = 3$	
$X = i$	p_{i3}/p_{*3}
0	$\frac{9k}{33k} = \frac{3}{11}$
1	$\frac{11k}{33k} = \frac{1}{3}$
2	$\frac{13k}{33k} = \frac{13}{33}$
	Total = 1

C.P.D. of Y, given $X = 0$	
$Y = j$	p_{oj}/p_{o*}
1	$\frac{3k}{18k} = \frac{1}{6}$
2	$\frac{6k}{18k} = \frac{1}{3}$
3	$\frac{9k}{18k} = \frac{1}{2}$
	Total = 1

C.P.D. of Y, given $X = 1$	
$Y = j$	p_{1j}/p_{1*}
1	$\frac{5k}{24k} = \frac{5}{24}$
2	$\frac{8k}{24k} = \frac{1}{3}$
3	$\frac{11k}{24k} = \frac{11}{24}$
	Total = 1

C.P.D. of Y, given $X = 2$	
$Y = j$	p_{2j}/p_{2*}
1	$\frac{7k}{30k} = \frac{7}{30}$
2	$\frac{10k}{30k} = \frac{1}{3}$
3	$\frac{13k}{30k} = \frac{13}{30}$
	Total = 1

Probability distribution of $(X + Y)$	
$(X + Y)$	P
1	$p_{01} = \frac{3}{72}$
2	$p_{02} + p_{11} = \frac{11}{72}$
3	$p_{03} + p_{12} + p_{21} = \frac{24}{72}$
4	$p_{13} + p_{22} = \frac{21}{72}$
5	$p_{23} = \frac{13}{72}$
	Total = 1

Example 4

A machine is used for a particular job in the forenoon and for a different job in the afternoon. The joint probability distribution of (X, Y), where X and Y represent the number of times the machine breaks down in the forenoon and in the afternoon respectively, is given in the following table. Examine if X and Y are independent RVs.

X	Y		
	0	1	2
0	0.1	0.04	0.06
1	0.2	0.08	0.12
2	0.2	0.08	0.12

X and Y are independent, if $P_{i*} \times P_{*j} = P_{ij}$ for all i and j. So, let us find P_{i*} P_{*j} for all i and j.

$P_{0*} = 0.1 + 0.04 + 0.06 = 0.2$; $P_{1*} = 0.4$; $P_{2*} = 0.4$

$P_{*0} = 0.5$; $P_{*1} = 0.2$; $P_{*2} = 0.3$

Now $P_{0*} \times P_{*0} = 0.2 \times 0.5 = 0.1 = P_{00}$

$P_{0*} \times P_{*1} = 0.2 \times 0.2 = 0.04 = P_{01}$

$P_{0*} \times P_{*2} = 0.2 \times 0.3 = 0.06 = P_{02}$

Similarly we can verify that

$P_{1*} \times P_{*0} = P_{10}$; $P_{1*} \times P_{*1} = P_{11}$; $P_{1*} \times P_{*2} = P_{12}$;

$P_{2*} \times P_{*0} = P_{20}$; $P_{2*} \times P_{*1} = P_{21}$; $P_{2*} \times P_{*2} = P_{22}$

Hence the RVs X and Y are independent.

Example 5

The joint pdf of a two-dimensional RV (X, Y) is given by $f(x, y) = xy^2 + \frac{x^2}{8}$, $0 \le x \le 2, 0 \le y \le 1$.

Compute $P(X > 1)$, $P\left(Y < \frac{1}{2}\right)$, $P(X > 1/Y < 1/2)$

$P\left(Y < \frac{1}{2}/X > 1\right)$, $P(X < Y)$ and $P(X + Y \le 1)$.

Here the rectangle defined by $0 \le x \le 2, 0 \le y \le 1$ is the range space R. $R_1, R_2, \ldots$,are event spaces.

(i) $P(X > 1) = \int\int_{\substack{R_1 \\ (x>1)}} f(x, y)\, dx\, dy$

$$= \int_0^1 \int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dxdy = \frac{19}{24}$$

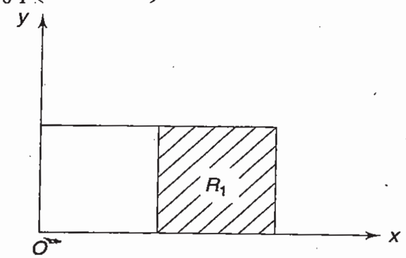

Fig. 2.1

(ii) $P(Y < 1/2) = \int\int_{\substack{R_2 \\ \left(y<\frac{1}{2}\right)}} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$

$$= \int_0^{1/2} \int_0^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \frac{1}{4}$$

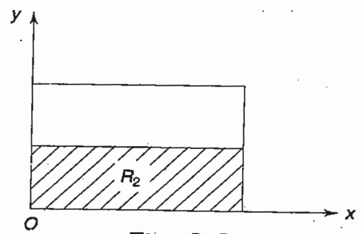

Fig. 2.2

(iii) $P(X > 1, Y < 1/2) = \int\int_{\substack{R_3 \\ \left(x>1\, \&\, y<\frac{1}{2}\right)}} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$

$$= \int_0^{1/2} \int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$$

$$= \frac{5}{24}$$

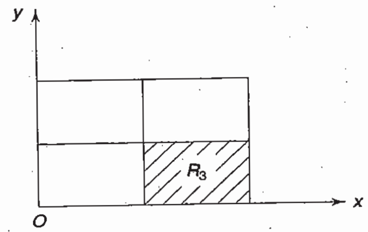

Fig. 2.3

(iv) $P\left(X > 1/Y < \frac{1}{2}\right) = \frac{P\left(X > 1, Y < \frac{1}{2}\right)}{P\left(Y < \frac{1}{2}\right)} = \frac{5/24}{1/4} = \frac{5}{6}$

(v) $P\left(Y < \frac{1}{2}/X > 1\right) = \frac{P\left(X > 1, Y < \frac{1}{2}\right)}{P(X > 1)} = \frac{5/24}{19/24} = \frac{5}{19}$

(vi) $P(X < Y) = \int\int_{\substack{R_4 \\ (x<y)}} \left(xy^2 + \frac{x^2}{8}\right) dx\, dy$

$$= \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\, dy = \frac{53}{480}$$

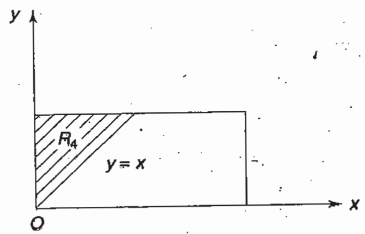

Fig. 2.4

(vii) $P(X+Y\le 1) = \underset{\substack{R_5\\(x+y\le 1)}}{\int\int} \left(xy^2 + \frac{x^2}{8}\right) dx\,dy$

$$= \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dx\,dy = \frac{13}{480}$$

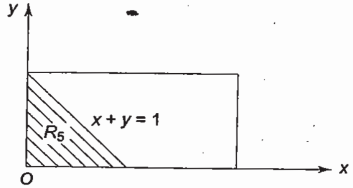

Fig. 2.5

Example 6

If the joint pdf of the RV (X, Y) is given by $f(x, y)$

$= \frac{1}{2\pi\sigma^2} \exp\{-(x^2+y^2)/2\sigma^2\}, -\infty < x, y < \infty$, find $P(X^2 + Y^2 \le a^2)$.

Here the entire xy-plane is the range space R and the event-space D is the interior of the circle $x^2 + y^2 = a^2$.

$$P(X^2+Y^2 \le a^2) = \iint_{x^2+y^2\le a^2} f(x, y)\, dx\, dy$$

Transform from cartesian system to polar system, i.e., put $x = r\cos\theta$ and $y = r\sin\theta$.

Then $dx\,dy = r\,dr\,d\theta$.

The domain of integration becomes $r \le a$.

Then $P(X^2+Y^2\le a^2) = \int_0^{2\pi}\int_0^a \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2}\, r\,dr\,d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(-e^{-r^2/2\sigma^2}\right)_0^a d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(1 - e^{-a^2/2\sigma^2}\right) d\theta$$

$$= 1 - e^{-a^2/2\sigma^2}$$

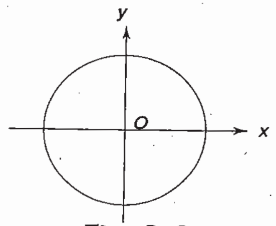

Fig. 2.6

Example 7

A gun is aimed at a certain point (origin of the co-ordinate system). Because of the random factors, the actual hit point can be any point (X, Y) in a circle of radius R about the origin. Assume that the joint density of X and Y is constant in this circle given by

$$f_{XY}(x, y) = c, \quad \text{for } x^2+y^2 \le R^2$$
$$= 0, \quad \text{otherwise}$$

(i) Compute c and (ii) show that

$$f_X(x) = \frac{2}{\pi R}\sqrt{1-\left(\frac{x}{R}\right)^2}, \text{ for } -R \le x \le R$$
$$= 0, \text{ otherwise}$$

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Here the range space is the interior of the circle $x^2 + y^2 = R^2$. By the property of joint pdf,

$$\iint_{x^2+y^2\le R^2} f(x, y)dx\,dy = 1$$

i.e., $\iint_{x^2+y^2\le R^2} c\,dx\,dy = 1$

Changing over to polar co-ordinates, we have

$$\int_0^{2\pi}\int_0^R c\,r\,dr\,d\theta = 1$$

$$\therefore \quad c = \frac{1}{\pi R^2}$$

Note *We have defined earlier that $f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$. This definition holds good if the range space is the entire xy-plane. If the range space is different from the entire xy-plane $f_X(x)$ is given by $\int f(x, y)dy$, for which the limits are fixed as follows: Draw an arbitrary line parallel to y-axis (since x is to be treated as a constant). The y co-ordinates of the ends of the segment of such a line that lies within the range space are the required limits. These limits will be either constants or functions of x.*

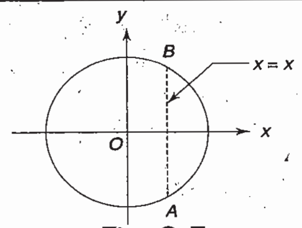


Fig. 2.7

The point $A = \left\{x, -\sqrt{R^2 - x^2}\right\}$

and the point $B = \left\{x, \sqrt{R^2 - x^2}\right\}$

Now, $f_X(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2} dy$

$$= \frac{2}{\pi R^2}\sqrt{R^2 - x^2} = \frac{2}{\pi R}\sqrt{1-\left(\frac{x}{R}\right)^2} \quad -R \le x \le R$$

Note *Whenever we are required to find the marginal and conditional density functions, the ranges of the concerned variables should also be specified.*

Example 8

The joint pdf of the RV (X, Y) is given by $f(x, y) = k\,x\,y\,e^{-(x^2+y^2)}$ $x > 0, y > 0$. Find the value of k and prove also that X and Y are independent.

Here the range space is the entire first quadrant of the xy-plane.

By the property of the joint pdf

$$\iint_{x>0,\,y>0} kxy\, e^{-(x^2+y^2)}\, dx\, dy = 1$$

i.e., $k\int_0^\infty ye^{-y^2} dy \int_0^\infty xe^{-x^2} dx = 1$

i.e., $\frac{k}{4} = 1$

$\therefore$ $k = 4$

Now $f_X(x) = \int_0^\infty 4x\, e^{-x^2} \times ye^{-y^2} dy = 2x\, e^{-x^2}, x > 0$

Similarly, $f_Y(y) = 2ye^{-y^2}, y > 0.$

Now $f_X(x) \times f_Y(y) = 4\,x\,y\,e^{-(x^2+y^2)} = f(x, y)$

$\therefore$ The RVs x and y are independent.

Note *If $f(x, y)$ can be factorised as $f_1(x) \times f_2(y)$ then X and Y will be independent.*

Example 9

Given $f_{XY}(x, y) = cx\,(x - y), 0 < x < 2, -x <, y < x$, and 0 elsewhere, (a) evaluate c, (b) find $f_X(x)$, (c) $f_{Y/X}(y/x)$ and (d) $f_Y(y)$. (BDU — Apr. 96)

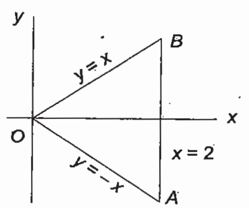


Fig. 2.8

Here the range space is the area within the triangle OAB (shown in the figure), defined by $0 < x < 2$ and $-x < y < x$.

(a) By the property of *jpdf*

$$\iint_{\Delta OAB} cx(x-y)dx\,dy = 1$$

$$\int_0^2\int_{-x}^{x} cx(x-y)dy\,dx = 1$$

i.e., $8c = 1$

$\therefore$ $c = \frac{1}{8}$

(b) $f_X(x) = \int_{-x}^{x} \frac{1}{8}x\,(x-y)dy$

$$= \frac{x^3}{4}, \text{ in } 0 < x < 2$$

(c) $f(y/x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{2x^2}(x - y), -x < y < x$

(d) $f_Y(y) = \int_{-y}^{2} \frac{1}{8}x\,(x-y)dx, \text{ in } -2 \le y \le 0$

$$= \int_{y}^{2} \frac{1}{8}x\,(x-y)dx, \text{ in } 0 \le y \le 2$$

i.e., $f_y(y) = \begin{cases} \frac{1}{3} - \frac{y}{4} + \frac{5}{48}y^3, & \text{in } -2 \le y \le 0 \\ \frac{1}{3} - \frac{y}{4} + \frac{1}{48}y^3, & \text{in } 0 \le y \le 2 \end{cases}$

Example 10

Train X arrives at a station at random in the time interval $(0, T)$ and stops for 'a' min. Train Y arrives independently in the same interval and stops for 'b' min.

(i) Find the probability P_1 that X will arrive before Y.
(ii) Find the probability P_2 that the two trains meet.
(iii) Assuming that they meet, find the probability P_3 that X arrived before Y. (MSU — Nov. 96)

Let the trains X and Y arrive at the station at time instances X and Y respectively.

Then the lengths of the intervals $(0, X)$ and $(0, Y)$, namely X and Y are continuous RVs. Each of X and Y is uniformly distributed in $(0, T)$ (since the times of arrival are equally likely) with pdf $\frac{1}{T}$.

Since the 2 trains arrive independently, X and Y are independent RVs.
$\therefore$ The joint pdf of (X, Y) is given by

$$f(x, y) = f_X(x) \times f_Y(y) = \frac{1}{T^2};\ 0 \le x, y \le T$$

The range space is the square defined by $0 \le x \le T$ and $0 \le y \le T$.

(i) $P_1 = P(X < Y) = \iint\limits_{\substack{x<y \\ (\Delta OBC)}} f(x, y)\,dx\,dy$

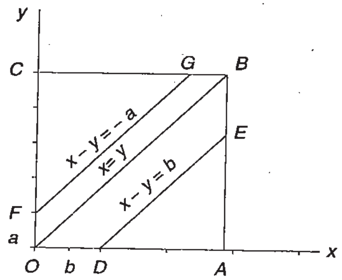


Fig. 2.9

$$= \frac{1}{T^2} \iint\limits_{\Delta OBC} dx\,dy = \frac{1}{T^2} \times \text{Area of } \Delta OBC$$

$$= \frac{1}{2}$$

(ii) If train X arrives first, the 2 trains will meet if $Y \le X + a$.
If train Y arrives first, the 2 trains will meet if $X \le Y + b$.

$\therefore$ For the 2 trains to meet, $-a \le X - Y \le b$.

$\therefore$ $P_2 = P(-a \le X - Y \le b) = \iint\limits_{\substack{-a \le x-y \le b \\ (ODEBGFO)}} f(x, y)\,dx\,dy$

$$= \frac{1}{T^2} \times \text{Area of the figure } ODEBGFO$$

$$= \frac{1}{T^2} \times (\text{Area of trapezium } ODEB + \text{that of } OBGF)$$

$$= \frac{1}{T^2} \times \left[\frac{1}{2} \times \frac{b}{\sqrt{2}}\left\{(T-b)\sqrt{2} + T\sqrt{2}\right\} + \frac{1}{2} \times \frac{a}{\sqrt{2}}\left\{(T-a)\sqrt{2} + T\sqrt{2}\right\}\right]$$

$$= \frac{1}{2T^2}\{a(2T-a) + b(2T-b)\}$$

$$= \frac{1}{2T^2}\{2(a+b)T - (a^2+b^2)\}$$

(iii) $P_3 = P\{X < Y / -a \le X - Y \le b\}$

$$= \frac{P[X < Y \text{ and } -a \le X - Y \le b]}{P(-a \le X - Y \le b)}$$

$$= \frac{\frac{1}{T^2} \times \text{area of trapezium } OBGF}{P_2}$$

$$= \frac{a(2T-a)}{2(a+b)T - (a^2+b^2)}$$

Example 11

Two trains arrive at a station at random between 7 A.M. and 7:30 A.M. One train stops for 5 min and the other for x min. For what value of x, will the probability that the 2 trains meet be equal to $\frac{1}{3}$?

In the notation of the previous problem,

$$T = 30,\ a = 5,\ b = x \text{ and } P_2 = \frac{1}{3}$$

$\therefore$ $\frac{1}{2T^2}\{2(a+b)T - (a^2+b^2)\} = \frac{1}{3}$

i.e., $\frac{1}{1800}\{60(x+5) - (x^2+25)\} = \frac{1}{3}$

i.e., $x^2 - 60x + 325 = 0.$

Solving, $x = 53.98$ (or) 6.02
As $x = 53.98$ is meaningless, $x = 6$ min (nearly).

Example 12

The two-dimensional RV (X, Y) follows a bivariate normal distribution $N(0, 0; \sigma_x, \sigma_y; r)$. Find the marginal density function of X and the conditional density function of Y, given X.

The notation $N(0, 0; \sigma_x, \sigma_y; r)$ refers to a bivariate normal distribution with mean of X = mean of $Y = 0$, variance of $X = \sigma^2_x$, variance of $Y = \sigma^2_y$ and the coefficient of correlation between X and $Y = r$.
The joint pdf of such a brivariate normal distribution is given by

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right\}$$

$$-\infty < x, y < \infty$$

The marginal density function of X is given

by $f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy$

$$= A \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-r^2)}\left(\frac{y}{\sigma_y} - \frac{rx}{\sigma_x}\right)^2\right\} \times$$

$$\exp\left(\frac{-x^2}{2\sigma_x^2}\right) dy, \text{ where } A = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}}$$

$$= A \exp\left(\frac{-x^2}{2\sigma_x^2}\right) \times \sqrt{2}\,\sigma_y\sqrt{1-r^2} \int_{-\infty}^{\infty} e^{-t^2}dt,$$

by putting $\frac{1}{\sqrt{2(1-r^2)}}\left(\frac{y}{\sigma_y} - \frac{rx}{\sigma_x}\right) = t$

$$= A \exp\left(\frac{-x^2}{2\sigma_x^2}\right)\sqrt{2}\cdot\sigma y\sqrt{1-r^2}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left(\frac{-x^2}{2\sigma_x^2}\right)\sqrt{2}\cdot\sigma y\sqrt{1-r^2}\sqrt{\pi}$$

$$= \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma_x^2}\right), -\infty < x < \infty$$

which is the density function of a normal distribution $N(0, \sigma_x)$.

The conditional density function of Y given X is given by $f\left(\frac{y}{x}\right) = \frac{f(x, y)}{f_X(x)}$

$$\therefore \quad f\left(\frac{y}{x}\right) = \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left\{\frac{-1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right\}}{\frac{1}{\sigma_x\sqrt{2\pi}} \exp\frac{-x^2}{2\sigma_x^2}}$$

$$= \frac{1}{\sqrt{2\pi}\left(\sigma_y\sqrt{1-r^2}\right)} \exp\left\{-\frac{1}{2\sigma_y^2(1-r^2)}\left(y^2 - 2r\frac{\sigma_y}{\sigma_x}xy + \frac{r^2\sigma_y^2}{\sigma_x^2}\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi}\left(\sigma_y\sqrt{1-r^2}\right)} \exp\left\{-\frac{1}{2\left(\sigma_y\sqrt{1-r^2}\right)^2}\left(y - \frac{r\sigma_y x}{\sigma_x}\right)^2\right\}$$

which is the density function of a normal distribution

$$N\left(\frac{r\sigma_y}{\sigma_x}x, \sigma_y\sqrt{1-r^2}\right)$$

Exercise 2(B)

Part A (Short answer questions)

1. Define a two-dimensional RV. Give an example for the outcome of a random experiment, that is a two-dimensional RV.
2. Define the joint pmf of a two-dimensional discrete RV.
3. Define the joint pdf of a two-dimensional continuous RV.
4. Write down the joint pdf of a bivariate normal distribution.
5. Define the cdf of a two-dimensional RV and write down the formulas for, finding the cdf of (X, Y), when (X, Y) is (i) a discrete RV and (ii) a continuous RV.
6. State the properties of the cdf of a two-dimensional RV (X, Y).
7. Define the marginal probability distributions of X and Y, when (X, Y) is a discrete RV.
8. Define the marginal probability density functions of X and Y, when (X, Y) is a continuous RV.
9. Define independence of 2 RVs X and Y, both in the discrete case and in the continuous case.
10. Define the conditional probability distributions of X and Y, given Y and X respectively, when (X, Y) is a discrete RV.
11. Define the conditional probability density functions of X and Y given Y and X respectively, when (X, Y) is a continuous RV.
12. Find the probability distribution of $(X + Y)$ from the bivariate distribution of (X, Y) given below.

X	Y: 1	Y: 2
1	0.1	0.2
2	0.3	0.4

13. Find the marginal distributions of X and Y from the bivariate distribution of (X, Y) given in Q.12.
14. Find the conditional distribution of X, when $Y = 1$, from the bivariate distribution of (X, Y) given in Q.12.
15. Find the value of k, if $f(x, y) = k(1-x)(1-y)$, for $0 < x, y < 1$, is to be a joint density function.

16. If $f(x, y) = k(1 - x - y)$, $0 < x, y < \frac{1}{2}$, is a joint density function, find k.
17. If the joint pdf of (X, Y) is $f(x, y) = \frac{1}{4}$, $0 \le x, y \le 2$, find $P(X + Y \le 1)$.
18. If the joint pdf of (X, Y) is $f(x, y) = 6e^{-2x-3y}$, $x \ge 0, y \ge 0$, find the marginal density of X and conditional density of Y given X.
19. The j pdf of (X, Y) is given by $f(x, y) = e^{-(x+y)}$, $0 \le x, y < \infty$. Are X and Y independent? Why?
20. Define a random vector with an example.
21. Define the joint density and distribution functions of an n-dimensional RV. How are they related?

Part B

22. If X denotes the number of aces and Y the number of queens obtained when 2 cards are drawn at random (without replacement) from a deck of cards, obtain the joint probability distribution of (X, Y).
23. The joint probability function of two discrete RVs X and Y is given by $f(x, y) = c(2x + y)$, where x and y can assume all integers such that $0 \le x \le 2$ and $0 \le y \le 3$, and $f(x, y) = 0$ otherwise. (a) find the value of c and (b) find $P(X \ge 1, Y \le 2)$. (MKU — Apr. 96)

Note *$f(x, y)$ should not be mistaken as pdf, It is used instead of $p(x_i, y_j)$*

24. The joint probability distribution of a two-dimensional discrete RV (X, Y) is given below:

Y	X					
	0	1	2	3	4	5
0	0	0.01	0.03	0.05	0.07	0.09
1	0.01	0.02	0.04	0.05	0.06	0.08
2	0.01	0.03	0.05	0.05	0.05	0.06
3	0.01	0.02	0.04	0.06	0.06	0.05

(i) Find $P(X > Y)$ and $P\{\max(X, Y) = 3\}$ and
(ii) Find the probability distribution of the RV $Z = \min(X, Y)$.

25. The input to a binary communication system, denoted by a RV X, takes one of two values 0 or 1 with probabilities 3/4 and 1/4 respectively. Because of errors caused by noise in the system, the output Y differs from the input occasionally. The behaviour of the communication system is modeled by the conditional probabilities given below:

$$P(Y = 1/X = 1) = 3/4 \text{ and } P(Y = 0/X = 0) = 7/8$$

Find (i) $P(Y = 1)$, (ii) $P(Y = 0)$ and (iii) $P(X = 1/Y = 1)$.

26. The following table represents the joint probability distribution of the discrete RV (X, Y). Find all the marginal and conditional distributions.

Y	X		
	1	2	3
1	1/12	1/6	0
2	0	1/9	1/5
3	1/18	1/4	2/15

27. The joint distribution of X_1 and X_2 is given by $f(x_1, x_2) = \frac{x_1 + x_2}{21}$, $x_1 = 1, 2$ and 3; $x_2 = 1$ and 2. Find the marginal distributions of X_1 and X_2. (MU — Nov. 96).
28. If the joint pdf of a two-dimensional RV (X, Y) is given by

$$f(x, y) = x^2 + \frac{xy}{3};\ 0 < x < 1, 0 < y < 2$$
$$= 0, \text{ elsewhere}$$

find (i) $P\left(X > \frac{1}{2}\right)$, (ii) $P(Y < X)$ and (iii) $P\left(Y < \frac{1}{2} / X < \frac{1}{2}\right)$.

29. If the joint pdf of a two-dimensional RV (X, Y) is given by

$$f(x, y) = k(6 - x - y);\ 0 < x < 2, 2 < y < 4$$
$$= 0, \text{ elsewhere}$$

find (i) the value of k, (ii) $P(X < 1, Y < 3)$, (iii) $P(X + Y < 3)$ and (iv) $P(X < 1/Y < 3)$.

30. The joint density function of the RVs X and Y is given by

$$f(x, y) = 8xy;\ 0 < x < 1, 0 < y < x$$
$$= 0, \text{ elsewhere,}$$

find $P\left(Y < \frac{1}{8} / X < \frac{1}{2}\right)$. (MU — Nov. 96)

31. Given that the joint pdf of (X, Y) is

$$f(x, y) = e^{-y};\ x > 0, y > x$$
$$= 0, \text{ elsewhere}$$

find (i) $P(X > 1/Y < 5)$ and (ii) the marginal distributions of X and Y.

32. If the joint pdf of a two-dimensional RV (X, Y) is given by

$$f(x, y) = 2;\ 0 < x < 1, 0 < y < x$$
$$= 0, \text{ otherwise}$$

find the marginal density functions of X and Y. (BDU — Nov. 96)

33. If the joint pdf of (X, Y) is given by $f(x, y) = k$, $0 \le x < y \le 2$, find k and also the marginal and conditional density functions.

34. The joint density function of a RV (X, Y) is $f(x, y) = 8xy$, $0 < x < 1$, $0 < y < x$. find the conditional density function $f(y/x)$. (MU — Apr. 96)
35. The joint density function of a RV (X, Y) is given by $f(x, y) = axy$, $1 \le x \le 3$, $2 \le y \le 4$, and $= 0$, elsewhere.
Find (i) the value of a, (ii) the marginal densities of X and Y and (iii) the conditional densities of X and Y, given Y and X respectively.

36. Let X_1 and X_2 be two RVs with joint pdf given by $f(x_1, x_2) = e^{-(x_1 + x_2)}$; $x_1, x_2 \ge 0$, and $= 0$, otherwise. Find the marginal densities of X_1 and X_2. Are they independent? Also find $P[X_1 \le 1, X_2 \le 1]$ and $P(X_1 + X_2 \le 1)$.
(BDU — Apr. 96)
37. The joint pdf of the RVs X and Y is given by $p(x, y) = xe^{-x(y+1)}$ where $0 \le x, y < \infty$. (i) Find $p(x)$ and $p(y)$ and (ii) Are the RVs independent?
(BU — Nov. 96).
38. If the joint pdf of the RV (X, Y) is given by $f(x, y) = k(x^3y + xy^3)$, $0 \le x \le 2$, $0 \le y \le 2$, find (i) the value of k, (ii) the marginal densities of X and Y and (iii) the conditional densities of X and Y.
39. If the joint pdf of (X, Y) is given by

$$f(x, y) = \frac{9(1 + x + y)}{2(1 + x)^4 (1 + y)^4} \quad x > 0 \; y > 0$$

find the marginal densities of X and Y. Are they independent?
40. Trains X and Y arrive at a station at random between 8 A.M. and 8.20 A.M. Train A stops for 4 min and train B stops for 5 min. Assuming that the trains arrive independently of each other, find the probability that (i) X will arrive before Y, (ii) the trains will meet and (iii) X arrived before Y, assuming that they met.
41. If the two-dimensional RV (X, Y) follows a bivariate normal distribution $N(0, 0; \sigma_x, \sigma_y; r)$, find the marginal density function of Y and the conditional density function of X, given Y.
42. The two-dimensional RV (X, Y) has the joint density

$$f(x, y) = 8xy, \; 0 < x < y < 1$$
$$= 0, \text{ otherwise}$$

(i) Find $P(X < 1/2 \cap Y < 1/4)$,
(ii) Find the marginal and conditional distributions, and
(iii) Are X and Y independent? Give reasons for your answer.
(BDU — Apr. 97)

ANSWERS

Exercise 2(A)

6. Assign the values 0, 1, 2 to X, when the outcome consists of 2 tails, 1 tail and 1 head and 2 heads respectively. Then the required probability distribution of X is

x:	0	1	2
p_x:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

9. **Example:** $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, $-\infty < x < \infty$, is the equation of the normal curve, viz., the probability curve of the normal distribution.

11.

X:	0	1	2	3	4	5
p_x:	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

12. Required probability $= \dfrac{P\left\{\left(\frac{1}{2} < X < \frac{7}{2}\right) \cap (X > 1)\right\}}{P(X > 1)}$

$$= \frac{P(X = 2 \text{ or } 3)}{P(X = 2, 3 \text{ or } 4)} = \frac{0.5}{0.6} = \frac{5}{6}$$

16. $f(x) \ge 0$; $\int_{-1}^{1} |x| \, dx = 2 \int_{0}^{1} x \, dx = 1$

$\therefore$ $f(x)$ can be the pdf of continuous RV.

17. $\int_0^3 kx^2 \, dx = 1$; $9k = 1$

$\therefore \quad k = \frac{1}{9}$

18. $P\{|X| > 1\} = 1 - P\{|X| < 1\} = 1 - \int_{-1}^{1} \frac{1}{4} \, dx = \frac{1}{2}$

19. $\int_0^\infty kx \, e^{-x} \, dx = 1$;

$k[x(-e^{-x}) - 1 \times (e^{-x})]^\infty_0 = 1$

$\therefore \quad k = 1$

20. Required probability $= \dfrac{P(X > 1.5)}{P(X > 1)} = \dfrac{\left(\frac{x^2}{4}\right)^2_{1.5}}{\left(\frac{x^2}{4}\right)^2_1} = \dfrac{7}{12}$

21. $\Sigma P_x = 1 \therefore k = 0.1$; $E(X) = -0.8 - 0.1 + 0 + 0.3 = -0.6$

22. The probability distribution of X is

X:	1	2	3	4	4	5	6
p_x:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$F(x) = P(X \le x) = 0, \text{ if } x < 1; = \frac{1}{6}, \text{ if } 1 \le x < 2;$$

$$= \frac{2}{6} \text{ if } 2 \le x < 3; = \frac{3}{6}, \text{ if } 3 \le x < 4; = \frac{4}{6}, \text{ if } 4 \le x < 5;$$

$$= \frac{5}{6}, \text{ if } 5 \le x < 6 \text{ and } = 1, \text{ if } 6 \le x$$

23. When $x < 0$, $F(x) < 0$; when $0 \le x < 1$, $F(x) = x^2$; when $1 \le x$, $F(x) = 1$.

24. $f(x) = \dfrac{dF}{dx} = \lambda e^{-\lambda x}$, when $x > 0$ and $= 0$ when $x < 0$.

25. $P(X > 1/X < 3)$

$$= \frac{P(1 < X < 3)}{P(0 < X < 3)} = \frac{F(3) - F(1)}{F(3) - F(0)} = \frac{\frac{9}{16} - \frac{1}{16}}{\frac{9}{16} - 0} = \frac{8}{9}$$

26. For the binomial distribution $B(n, p)$, mean $= np$ and variance $= npq$.

27. Mean = variance = λ for the Poisson distribution with parameter λ.

29. $P(|X| < 1) = P(-1 < X < 1)$

$$= \int_0^1 e^{-x}\, dx = \frac{e-1}{e}$$

Part B

38. $P(X = r) = 4\, C_r \left(\frac{1}{2}\right)^4$, $r = 0, 1, 2, 3, 4$

39. $P(X = 0) = \frac{1}{64}$, $P(X = 1) = \frac{9}{64}$, $P(X = 2) = \frac{27}{64}$, $P(X = 3) = \frac{27}{64}$

40. $P(X = 0) = q$, $P(X = 1) = p$; when $x < 0$, $F(x) = 0$; when $0 \le x < 1$, $F(x) = q$; when $1 \le x$, $F(x) = q + p = 1$

41. $P(X = r) = p(X = 14 - r) = \dfrac{(r-1)}{36}$, $r = 2, 3, 4, 5, 6, 7$

42. $P(X = 0) = \frac{5}{16}$, $P(X = 1) = \frac{6}{16}$, $P(X = 2) = \frac{5}{16}$; $E(X) = 1$; $V(X) = \frac{5}{8}$

43. $a = \frac{1}{18}$; $P(X < 3) = \frac{1}{9}$; $V(X) = 4.4719$; $F(x) = 0$ in $x < 0$, $F(x) = \frac{1}{81}$ in $0 \le x < 1$, $F(x) = \frac{4}{81}$ in $1 \le x < 2$, $F(x) = \frac{9}{81}$ in $2 \le x < 3$ etc. $F(x) = \frac{64}{81}$ in $7 \le x < 8$ and $F(x) = 1$ in $8 \le x$.

44. $P(Y = 0) = 0.1$; $P(Y = 3) = 0.3$; $P(Y = 8) = 0.5$; $P(Y = 15) = 0.1$; $E(Y) = 6.4$; $V(Y) = 16.24$.

45. $C = \frac{2}{7}$; $P = \frac{16}{37}$; $F(x) = 0$, when $x < 0$; $= \frac{12}{49}$, when $0 \le x < 1$; $= \frac{28}{49}$, when $1 \le x < 2$ and $= 1$, when $2 \le x$; $x = 0$; $x = 1$.

46. $k = \frac{1}{100}$; $P = \frac{8}{99}$; $E(X) = 3.54$; $V(X) = 0.4684$; $F(x) = 0$, when $x < 1$; $= \frac{1}{100}$, when $1 \le x < 2$; $= \frac{9}{100}$, when $2 \le x < 3$; $= \frac{36}{100}$, when $3 \le x < 4$ and $= 1$, when $4 \le x$.

47. $0 < a < 1$

48. $k = a$; $E(X) = \frac{1}{a}$; $V(X) = \frac{1}{a}\left(\frac{1}{a} - 1\right)$

49. $k = 1$; $E(X) = e^t$; $V(X) = e^t(e^t - 1)$.

50. $k = \frac{3}{4}$; $E(X) = 1$; $V(X) = \frac{1}{5}$; $F(x) = 0$, when $x < 0$; $= \frac{1}{4}(3x^2 - x^3)$, when $0 \le x < 2$; $= 1$, when $2 \le x$.

51. (i) $k = 6$;
(ii) $F(x) = 0$, when $x < 0$; $= 3x^2 - 2x^3$, when $0 \le x < 1$; $= 1$, when $1 \le x$;
(iii) the root of the equation $6a^3 - 9a^2 + 2 = 0$ that lies between 0 and 1;
(iv) $\frac{1}{2}$.

52. $k = \frac{1}{8}$; $F(x) = 0$, when $x < 0$; $= \frac{x^2}{16}$, when $0 \le x < 2$; $= \frac{1}{4}(x - 1)$, when $2 \le x < 4$; $= -\frac{1}{16}(20 - 12x + x^2)$, when $4 \le x < 6$; $= 1$, when $6 \le x$.

53. (i) $\frac{9}{16}$ (ii) $\frac{27}{64}$

54. $\frac{1}{3}$

55. $k = 5$; $P = \frac{1}{33}$

56. (a) $\frac{1}{4}$ (b) $\frac{4}{9}$ (c) 5

57. $P = 1 - \frac{1}{e}; f(x) = \frac{k}{2} e^{-k|x|}$

58. $f(x) = 1$ in $0 \le x \le 1$ and $= 0$, elsewhere; $\frac{1}{6}; \frac{1}{2}$.

59. $\frac{1}{5}$

60. $\frac{2}{3}$

61. (i) λ; (ii) $\frac{\lambda^2 t}{1+\lambda t}$

62. (i) 0.8849 (ii) 0.8819 (iii) 0.8593

Exercise 2(B)

4. $f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}}$

$$\exp\left[-\frac{1}{2(1-r^2)}\left\{\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2r(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right\}\right],$$

$-\infty < x, y < \infty$

12.

$(X+Y)$:	2	3	4
p:	0.1	0.5	0.4

13.

X:	1	2	and	Y:	1	2
p_x:	0.3	0.7		p_y:	0.4	0.6

14.

X:	1	2
$p_{X/Y} = 1$:	0.25	0.75

15. $k \int_0^1\int_0^1 (1-x)(1-y)\,dx\,dy = 1;$

$$k\left\{\frac{(1-x)^2}{-2}\right\}_0^1 \left\{\frac{(1-y)^2}{-2}\right\}_0^1 = 1$$

i.e., $\frac{k}{4} = 1$

$\therefore$ $k = 4$

16. $\int_0^{1/2}\int_0^{1/2} k(1-x-y)\,dx\,dy = 1$

$$k\int_0^{1/2}\left(x - \frac{x^2}{2} - yx\right)_0^{\frac{1}{2}} dy = 1;$$

$$k\left(\frac{3}{8y} - \frac{y^2}{4}\right)_0^{\frac{1}{2}} = 1$$

$\therefore$ $k = 8$

17. $P = \int_0^1\int_0^{1-y} \frac{1}{4}\,dx\,dy = \frac{1}{4}\int_0^1 (1-y)\,dy = \frac{1}{8}$

18. $f_X(x) = 6\int_0^{\infty} e^{-2x} e^{-3y}\,dy = 2e^{-2x}; x \ge 0$

$f_{Y/X}(y) = \frac{f(x, y)}{f_X(x)} = 3e^{-3y}; y \ge 0$

19. $f_X(x) = e^{-x}$ and $f_Y(y) = e^{-y}$

$f(x, y) = f_X(x), f_Y(y)$

$\therefore$ X and Y are independent

22.

$Y \backslash X$	0	1	2
0	0.71	0.13	0.01
1	0.13	0.01	0
2	0.01	0	0

23. $c = \frac{1}{42}; P = \frac{4}{7}$

24. (i) 0.75, 0.21 (ii)

Z	0	1	2	3
P	0.28	0.30	0.25	0.17

25. (i) $\frac{9}{32}$ (ii) $\frac{23}{32}$ (iii) $\frac{2}{3}$

26.

$\{i, p_{i*}\}$

$x = i$	p_{i*}
1	5/36
2	19/36
3	1/3

$\{j, p_{*j}\}$

$y = j$	p_{*j}
1	1/4
2	14/45
3	79/180

CPD of $X/Y = 1$

$x = i$	p_{i1}/p_{*1}
1	1/3
2	2/3
3	0

CPD of $X/Y = 2$

$x = i$	p_{i2}/p_{*2}
1	0
2	5/14
3	9/14

CPD of $X/Y = 3$

$x = i$	p_{i3}/p_{*3}
1	10/79
2	45/79
3	24/79

CPD of $Y/X = 1$

$y = j$	p_{1j}/p_{1*}
1	3/5
2	0
3	2/5

CPD of $Y/X = 2$

$x = j$	p_{2j}/p_{2*}
1	6/19
2	4/19
3	9/19

CPD of $Y/X = 3$

$y = j$	p_{3j}/p_{3*}
1	0
2	3/5
3	2/5

27.

M.D. of X_1

$X_1 = i$	$pi*$
1	5/21
2	7/21
3	9/21

M.D. of X_2

$X_2 = j$	$p*j$
1	9/21
2	12/21

28. (i) $\frac{5}{6}$ (ii) $\frac{7}{24}$ (iii) $\frac{5}{32}$

29. (i) 1/8 (ii) 3/8 (iii) 5/24 (iv) 3/5

30. $\frac{31}{256}$

31. (i) $\frac{e^4 - 5}{e^5 - 6}$ (ii) $f_X(x) = e^{-x}, x > 0; f_Y(y) = y\, e^{-y}, y > 0.$

32. $f_X(x) = 2x$ in $0 < x < 1$; $f_Y(y) = 2(1 - y)$ in $0 < y < 1$.

33. $k = 1/2$; $f_X(x) = 1/2\,(2 - x), 0 \le x \le 2$; $f_Y(y) = (1/2)\, y, 0 \le y \le 2$; $f(x/y) = 1/y; 0 < x < y$; $f(y/x) = \frac{1}{2 - x}, x < y < 2.$

34. $f(y/x) = 2y/x^2, 0 < y < x$

35. (i) $a = 1/24$
(ii) $f_X(x) = x/4, 1 \le x \le 3$; $f_Y(y) = y/6, 2 \le y \le 4$
(iii) $f(x/y) = x/4, 1 < x < 3$; $f(y/x) = y/6, 2 < y < 4$

36. $f_{X_1}(x_1) = e^{-x_1}$; $f_{X_2}(x_2) = e^{-x_2}$; X_1 and X_2 are independent; $(1 - e^{-1})^2$; $(1 - 2e^{-1})$

37. $f_X(x) = e^{-x}, 0 < x < \infty$; $f_Y(y) = (y + 1)^{-2}, 0 < y < \infty$; X and Y are not independent.

38. $k = 1/16$; $f_X(x) = 1/8\,(x^3 + 2x), 0 \le x \le 2$; $f_Y(y) = 1/8\,(y^3 + 2y), 0 \le y \le 2$

$f(x/y) = x/2\, \frac{(x^2 + y^2)}{(y^2 + 2)}\ 0 < x < 2$; $f(y/x) = y/2\, \frac{(x^2 + y^2)}{(x^2 + 2)}, 0 < y < 2$

39. $f_X(x) = (3/4)\, \frac{2x + 3}{(1 + x)^4}\ x > 0$ and $f_Y(y) = (3/4)\, \frac{2x + 3}{(1 + y)^4}\ y > 0$;

Not independent.

40. (i) 1/2 (ii) 0.3988 (iii) 0.4514

41. $f_Y(y) = \frac{1}{\sigma_y \sqrt{2\pi}} \exp\left(-\frac{y^2}{2\sigma_y^2}\right) - \infty < y < \infty$

$f(x/y) = N\left(\frac{r\sigma_x}{\sigma_y} \cdot y, \sigma_x \sqrt{1 - r^2}\right)$

42. (i) 1/256
(ii) $f_X(x) = 4x\,(1 - x^2)$ in $(0, 1)$; $f_Y(y) = 4\,y^3$ in $(0, 1)$; $f(y/x) = \left|\frac{2y}{1 - x^2}\right|$ $f(x/y) = \frac{2x}{y^2}$
(iii) X and Y are not independent.

Chapter 4

Statistical Averages

A discrete random variable (RV) is no doubt completely described by its probability mass function or probability distribution. Similarly, a continuous RV is completely described by its probability density function. For many purposes, this description is often considered to consist of too many details. It is sometimes simpler and more convenient to describe a RV or to characterise its distribution by a few parameters or summary measures that are representative of the distribution. These parameters or characteristic numbers are the various expected values or statistical averages of the RV.

Definitions: If X is a discrete RV, then ***the expected value*** or the mean value of $g(X)$ is defined as

$$E\{g(X)\} = \sum_i g(x_i)p_i,$$

where $p_i = P(X = x_i)$ is the probability mass function of X.

If X is a continuous RV with pdf $f(x)$, then

$$E\{g(X)\} = \int_{R_X} g(x)f(x)\,dx$$

Two expected values which are most commonly used for characterising a RV X are ***its mean*** μ_X and ***variance*** σ^2_X, which are defined as follows:

$$\mu_X = E(X)$$

$$= \sum_i x_i\, p_i\text{, if } X \text{ is discrete}$$

$$= \int_{R_X} xf(x)\,dx\text{, if } X \text{ is continuous}$$

$$\text{Var}(X) = \sigma^2_X = E\{(X - \mu_X)^2\}$$

$$= \sum_i (x_i - \mu_X)^2\, p_i\text{, if } X \text{ is } \textit{discrete}$$

$$\int_{R_x} (x-\mu_X)^2 f(x)\,dx, \text{ if } X \text{ is continuous}$$

The square root of variance is called ***the standard deviation***. The mean of a RV is its average value and the variance is a measure of the spread or dispersion of the values of the RV.

Note

$Var(X) = E(X^2) - \{E(X)\}^2$ *(BU — Apr. 96)*

$$\begin{aligned} Var(X) &= E\{(X-\mu_x)^2\} \\ &= E\{X^2 - 2\mu_x X + \mu_x^2\} \\ &= E(X^2) - 2\mu_x E(X) + \mu_x^2 \text{ (since } \mu_X \text{ is a constant)} \\ &= E(X^2) - \mu_x^2 \text{ [since } \mu_x = E(X)] \\ &= E(X^2) - E(X)\}^2 \end{aligned}$$

This modified formula for var(X) holds good for both discrete and continuous RVs.

Note

If X is a discrete RV and a is a constant, then (i) $E(aX) = a\,E(X)$, (ii) $Var(aX) = a^2 Var(X)$. *(BU — Apr. 96)*

(i) $E(aX) = \sum_j ax_j pi$

$$= a\sum_j x_j pi$$

$$= aE(X)$$

(ii) $Var(aX) = E(a^2X^2) - \{E(aX)\}^2$ (by Note 1)

$$\begin{aligned} &= a^2 E(X^2) - \{aE(X)\}^2 \\ &= a^2[E(X^2) - \{E(X)^2] \\ &= a^2 Var(X) \end{aligned}$$

This result holds good for a continuous RV also.

Moments: If X is a discrete or continuous RV, $E(X^n)$ is called nth order raw moment of X about the origin and denoted by μ'_n.

$E\{(X-\mu_X)^n\}$ is called the *n*th order central moment of X and denoted by μ_n.

$E\{|X|^n\}$ and $E\{|X-\mu_X|^n\}$ are called absolute moments of X.

$E\{X-a)^n\}$ and $E\{|X-a|^n\}$ are called generalised moments of X.

Expected Values of a Two-Dimensional RV

If (X, Y) is a two-dimensional discrete RV with joint probability mass function p_{ij}, then $E\{g(X,Y)\} = \sum_j\sum_i g(x_i, y_i)p_{ij}$.

If (X, Y) is a two-dimensional continuous RV with joint pdf $f(x, y)$, then

$$E\{g(X,Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,dx\,dy$$

Properties of Expected Values

We give below the proofs of the properties for continuous RVs. Students can prove the properties for discrete RVs.

(i) $E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$, where $f_X(x)$ is the marginal density of X.

Proof

$$\begin{aligned} E\{g(X)\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)f(x,y)\,dx\,dy \\ &= \int_{-\infty}^{\infty} g(x)\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]dx \\ &= \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \end{aligned}$$

(ii) $E\{h(Y)\} = \int_{-\infty}^{\infty} h(y) f_Y(y)\,dy$

where $f_Y(y)$ is the marginal density of Y.
(Proof is left as an exercise to the student.)

(iii) $E(X+Y) = E(X) + E(Y)$

Proof

$$\begin{aligned} E(X+Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)f(x,y)\,dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\,dx\,dy \\ &= E(X) + E(Y) \end{aligned}$$

(iv) In general, $E(XY) \neq E(X) \times E(Y)$, but if X and Y are independent RVs, $E(XY) = E(X) \times E(Y)$.

Proof

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\,dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)\,f_Y(y)\,dx\,dy \end{aligned}$$

(since X and Y are independent)

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx \times \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

$$= E(X) \times E(Y)$$

In general, if X and Y are independent,

$$E\{g(X) \times h(Y)\} = E\{g(X)\} \times E\{h(Y)\}$$

Conditional Expected Values

If (X, Y) is a two-dimensional discrete RV with joint probability mass function p_{ij}, then the conditional expectations of $g(X, Y)$ are defined as follows:

$$E\{g(X, Y)/Y = Y_j\} = \sum_i g(x_i, y_j) \times P(X = x_i / Y = y_j)$$

$$= \sum_i g(x_i, y_j) \frac{p_{ij}}{p_{*j}}$$

and $E\{g(X, Y)/X = x_i\} = \sum_j g(x_i, y_j) p_{ij} / p_{i*}$

If (X, Y) is a two-dimensional continuous RV with joint pdf $f(x, y)$, then

$$E\{g(X, Y)/Y\} = \int_{-\infty}^{\infty} g(x, y) \times f(x/y)\,dx \text{ and}$$

$$E\{g(X, Y)/X\} = \int_{-\infty}^{\infty} g(x, y) \times f(y/x)\,dy$$

In particular, the conditional means are defined as

$$\mu_{Y/X} = E(Y/X) = \int_{-\infty}^{\infty} y f(y/x)\,dy \text{ and}$$

$$\mu_{X/Y} = E(X/Y) = \int_{-\infty}^{\infty} x f(x/y)\,dx$$

The conditional variances are defined as

$$\sigma_{Y/X}^2 = E\{(Y - \mu_{Y/X})^2\} = \int_{-\infty}^{\infty} (y - \mu_{Y/X})^2 f(y/x)\,dy \text{ and}$$

$$\sigma_{X/Y}^2 = E\{(X - \mu_{X/Y})^2\} = \int_{-\infty}^{\infty} (x - \mu_{X/Y})^2\} f(x/y)\,dx$$

Properties

(1) If X and Y are independent RVs, then $E(Y/X) = E(Y)$ and $E(X/Y) = E(X)$.

Proof

$$E(Y/X) = \int_{-\infty}^{\infty} y f(y/x)\,dy$$

$$= \int_{-\infty}^{\infty} y \frac{f(x, y)}{f_X(x)}\,dy$$

$$= \int_{-\infty}^{\infty} y \frac{f_X(x) \times f_Y(y)}{f_X(x)}\,dy$$

(since X and Y are independent)

$$= \int_{-\infty}^{\infty} y f_Y(y)\,dy = E(Y)$$

A similar proof can be given for the other result.

(2) $E[E\{g(X, Y)/X\}] = E\{g(X, Y)\}$

Proof

$$E\{g(X, Y)/X\} = \int_{-\infty}^{\infty} g(x, y) f(y/x)\,dy$$

Since $E\{g(X, Y)/X\}$ is a function of the RV X,

$$E[E\{g(X, Y)/X\}] = \int_{-\infty}^{\infty} E\{g(X, Y)/X\} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f(y/x) f_X(x)\,dx\,dy \text{ [from (1)]}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f(x, y)\,dx\,dy$$

$$= E\{g(X, Y)\}$$

In particular,

$$E\{E(Y/X)\} = E(Y) \text{ and similarly}$$

$$E\{E(X/Y)\} = E(X).$$

(3) $E\{g_1(X) \times g_2(Y)\} = E[g_1(X) \times E\{g_2(Y)/X\}]$

Proof

$$E\{g_1(X) \times g_2(Y)\} = E[E\{g_1(X) \times g_2(Y)/X] \text{ (by Property (2))}$$

$$= E[g_1(X) \times E\{g_2(Y)/X\}] \text{ (since } X \text{ is given)}$$

In particular,

$$E(XY) = E[X \times E(Y/X)] \text{ and}$$

$$E(X^2Y^2) = E[X^2 \times E(Y^2/X)]$$

Worked Example 4(A)

Example 1

A lot is known to contain 2 defectives and 8 non-defective items. If these items are inspected at random, one after another, what is the expected number of items that must be chosen in order to remove, both the defective ones?

Let the random variable X denote the number of items that must be drawn in order to remove both defective items.

Clearly X takes the values 2, 3, 4, ..., 10.

$$P(X = r) = P(r \text{ items are to be drawn to remove both defectives})$$

$$= P\{\text{the first } (r-1) \text{ items drawn should contain 1 defective and } r\text{th item drawn should be defective}\}$$

$$= \frac{2C_1 \times 8C_{r-2}}{10C_{r-1}} \times \frac{1}{10-(r-1)} = \frac{2 \times 8C_{r-2}}{10C_{r-1}(11-r)}$$

$$(r = 2, 3, \ldots, 10)$$

The probability distribution of X will then be as follows:

$X = r$	2	3	4	5	6	7	8	9	10
p_r	1/45	2/45	3/45	4/45	5/45	6/45	7/45	8/45	9/45

$$E(X) = \sum_{r=2}^{10} rp_r = \frac{22}{3}$$

Example 2

A box contains 2^n tickets of which nC_r tickets bear the number r ($r = 0, 1, 2, \ldots, n$). Two tickets are drawn from the box. Find the expectation of the sum of their numbers.

Total number of tickets in the box.

$$\sum_{r=0}^{n} nC_r = nC_0 + nC_1 + \cdots + nC_n$$

$$= (1+1)^n = 2^n, \text{ as given.}$$

Let the RVs X and Y represent the numbers on the first and second tickets respectively.

Then $\quad E(X + Y) = E(X) + E(Y)$

X can take the values 0, 1, 2, $\cdots$, n with probabilities $\frac{nC_0}{2^n}, \frac{nC_1}{2^n}, \cdots \frac{nC_n}{2^n}$ respectively.

$$\therefore \quad E(X) = 1 \times \frac{nC_1}{2^n} + 2 \times \frac{nC_2}{2^n} + \cdots + n \times \frac{nC_n}{2^n}$$

$$= \frac{n}{2^n}\{(n-1)C_0 + (n-1)C_1 + \cdots + {}^{(n-1)}C_{n-1}\}$$

$$= \frac{n}{2^n}(1+1)^{n-1} = \frac{n}{2}$$

Similarly, $\quad E(Y) = \frac{n}{2}$

$\therefore \quad E(X + Y) = n$

Example 3

Find the mean and variance of the Pascal's (negative binomial distribution) distribution, given by $P(X = k) = \binom{n+k-1}{k} p^n q^k,\ k = 0, 1, 2, \cdots$

$$E(X) = \sum_{k=0}^{\infty} k \binom{n+k-1}{k} p^n q^k$$

$$= p^n[1 \times nC_1 q^1 + 2(n+1)C_2 q^2 + 3(n+2)C_3 q^3 + \cdots]$$

$$= np^n q[1 + (n+1)C_1 q + (n+2)C_2 q^2 + \cdots]$$

$$= np^n q(1-q)^{-(n+1)} = \frac{nq}{p}$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 \binom{n+k-1}{k} p^n q^k$$

$$= p^n[1^2 nC_1 q^1 + 2^2(n+1)C_2 q^2 + 3^2(n+2)C_3 q^3 + \cdots]$$

$$= p^n\left[nq + \frac{1 \times 2 + 2}{2!}(n+1)nq^2 + \frac{(2 \times 3 + 3)}{3!}(n+2)(n+1)nq^3 + \cdots\right]$$

$$= np^n q\left[\left\{1 + \frac{(n+1)}{1!}q + \frac{(n+1)(n+2)}{2!}q^2 + \cdots\right\} + (n+1)q \times \left\{1 + \frac{(n+2)}{1!}q + \frac{(n+2)(n+3)}{2!}q^2 + \cdots\right\}\right]$$

$$= np^n q[(1-q)^{-(n+1)} + (n+1)q(1-q)^{-(n+2)}]$$

$$= np^n q\left[\frac{1}{p^{n+1}} + \frac{(n+1)q}{p^{n+2}}\right]$$

$$= \frac{nq}{p} + \frac{n(n+1)q^2}{p^2}$$

$$\therefore \quad \mathrm{Var}(X) = E(X^2) - \{E(X)\}^2$$

$$= \frac{nq}{p} + \frac{n(n+1)q^2}{p^2} - \frac{n^2q^2}{p^2}$$

$$= \frac{nq}{p}\left(1+\frac{q}{p}\right) = \frac{nq}{p^2}$$

Example 4

If the continuous RV X has Rayleight density $f(x) = \frac{X}{\alpha^2} e^{-x^2/2\alpha^2} \times U(x)$, find $E(X^n)$ and deduce the values of $E(X)$ and var(X).

By definition,

$$E(X^n) = \int_0^\infty x^n \times \frac{x}{\alpha^2} e^{-x^2/2\alpha^2}\, dx$$

$$= \int_0^\infty (2\alpha^2 t)^{n/2} e^{-t}\, dt \qquad \left(\text{putting } \frac{x^2}{2\alpha^2} = t\right)$$

$$= 2^{n/2} \alpha^n \int_0^\infty t^{n/2} e^{-t}\, dt$$

$$= 2^{n/2}\alpha^n \Gamma(k+1) \text{ if } n = 2k$$

$$= 2^{n/2}\alpha^n \lfloor k = 2^{n/2} \alpha^n \lfloor n/2, \text{ if } n \text{ is even}$$

$$E(X^n) = 2^{n/2} \alpha^n \Gamma\left(\frac{2k+3}{3}\right) \text{ if } n = 2k+1$$

$$= 2^{n/2} \alpha^n \frac{2k+1}{2} \times \frac{2k-1}{2} \cdots \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)$$

$$= 2^{n/2} \alpha^n \frac{1 \times 3 \times 5 \times \cdots \times n}{2^{(n+1)/2}} \sqrt{\pi}$$

$$= 1 \times 3 \times 5 \times \ldots \times n\alpha^n \sqrt{\pi/2} \quad \text{if } n \text{ is odd}$$

$$\therefore \quad E(X) = \alpha\sqrt{\pi/2}\,;\ E(X^2) = 2\alpha^2;\ \mathrm{var}(X) = \left(2 - \frac{\pi}{2}\right)\alpha^2$$

Example 5

A line of length a units is divided into two parts. If the first part is of length X, find $E(X)$, var(X) and $E\{X(a-X)\}$.

Since the positions of the point of division are equally likely, X is uniformly distributed in $(0, a)$.

$$\therefore \quad f(x) = \frac{1}{a}$$

$$E(X) = \int_0^a xf(x)\, dx = \frac{1}{a}\int_0^a x\, dx = \frac{a}{2}$$

$$E(X^2) = \int_0^a x^2 f(x)\, dx = \frac{a^2}{3}$$

$$\therefore \quad \mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{a^2}{3} - \frac{a^2}{4} = \frac{a^2}{12}$$

$$E\{X(a-X)\} = a\,E(X) - E(X^2) = \frac{a^2}{3} - \frac{a^2}{4} = \frac{a^2}{12}$$

Example 6

If X is a continuous RV, prove that

$$E(X) = \int_0^\infty [1 - F(x)]\, dx - \int_{-\infty}^0 F(x)\, dx.$$

$$E(X) = \int_{-\infty}^\infty xf(x)\, dx = \int_{-\infty}^0 x\, dF(x) - \int_0^\infty x\, d\{1 - F(x)\}$$

[since $F'(x) = f(x)$]

$$= [xF(x)]_{-\infty}^0 - \int_{-\infty}^0 F(x)\, dx - [x\{1 - F(x)\}]_0^\infty + \int_0^\infty \{1 - F(x)\}\, dx$$

$$= \int_0^\infty \{1 - F(x)\}\, dx - \int_{-\infty}^0 F(x)\, dx$$

[since $F(-\infty) = 0$ and $F(\infty) = 1$]

Example 7

If the random variable X follows $N(0, 2)$ and $Y = 3X^2$, find the mean and variance of Y.

Since X follows $N(0, 2)$, $E(X) = 0$ and var(X) $= 4$

$\therefore \quad E(X^2) = \text{var}(X) + \{E(X)\}^2 = 4$

Now $\quad E(Y) = E(3X^2) = 3 \times 4 = 12$

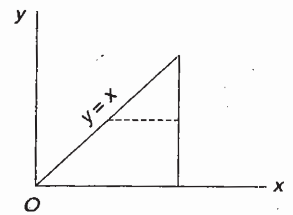
$$E(Y^2) = E(9X^4) = 9 \times 3 \times 2^4$$

[since for the normal distribution $N(0, \sigma)$, $E(X^{2r}) = \dfrac{(2r)!\,\sigma^{2r}}{2^r\, r!}$]

$$\text{Var}(Y) = E(Y^2) - \{E(Y)\}^2$$
$$= 27 \times 2^4 - 12^2 = 288$$

Example 8

If the joint pdf of (X, Y) is given by $f(x, y) = 24y(1-x)$, $0 \le y \le x \le 1$, find $E(XY)$.



$$E(XY) = \int_0^1 \int_y^1 xyf(x, y)\, dx\, dy$$
$$= 24 \int_0^1 \int_y^1 xy^2(1-x)\, dx\, dy$$
$$= 24 \int_0^1 y^2 \left(\frac{1}{6} - \frac{y^2}{2} + \frac{y^3}{3}\right) dy$$
$$= \frac{4}{15}$$

Example 9

If X and Y are two independent RVs with $f_X(x) = e^{-x}U(x)$ and $f_Y(y) = e^{-y}\, U(y)$ and $Z = (X - Y)\, U(X - Y)$, prove that $E(Z) = 1/2$.

$$U(X - Y) = \begin{cases} 1 & \text{if } X > Y \\ 0 & \text{if } X < Y \end{cases}$$

$$\therefore \quad Z = \begin{cases} X - Y & \text{if } X > Y \\ 0 & \text{if } X < Y \end{cases}$$

$$E(Z) = \int_0^\infty \int_0^\infty z\, e^{-(x+y)}\, dx\, dy$$

[since X and Y are independent, $f(x, y) = f_X(x) \times f_Y(y)$]

$$= \int_0^\infty \int_y^\infty (x - y)\, e^{-(x+y)}\, dx\, dy$$
$$= \int_0^\infty e^{-y}[(x-y)(-e^{-x}) - e^{-x}]_y^\infty\, dy$$
$$= \int_0^\infty e^{-2y}\, dy = \frac{1}{2}$$

Example 10

The joint pdf of (X, Y) is given by $f(x, y) = 24xy$; $x > 0$, $y > 0$, $x + y \le 1$, and $f(x, y) = 0$, elsewhere, find the conditional mean and variance of Y, given X.

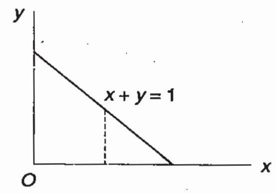


Fig. 4.1

$$f_X(x) = \int_0^{1-x} 24xy\, dy$$
$$= 12x\,(1 - x)^2,\ 0 < x < 1$$

Now $\quad f(y/x) = \dfrac{f(x, y)}{f_X(x)} = \dfrac{2y}{(1-x)^2},\ 0 < y < 1 - x$

$$E(Y/X = x) = \int_0^{1-x} yf(y/x)\, dy$$
$$= \int_0^{1-x} \frac{2y^2}{(1-x)^2}\, dy = \frac{2}{3}(1 - x)$$
$$E(Y^2/x) = \int_0^{1-x} y^2 \times f(y/x)\, dy = \frac{1}{2}(1 - x)^2$$
$$\text{Var}(Y^2/x) = E(Y^2/x) - \{E(Y/x)\}^2$$

$$= \frac{1}{2}(1-x)^2 - \frac{4}{9}(1-x)^2$$

$$= \frac{1}{18}(1-x)^2$$

Example 11

If (X, Y) is uniformly distributed over the semicircle bounded by $y = \sqrt{1-x^2}$ and $y = 0$, find $E(X/Y)$ and $E(Y/X)$. Also verify the $E\{E(X/Y)\} = E(X)$ and $E\{E(Y/X)\} = E(Y)$.

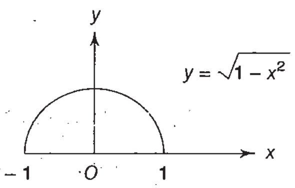


Fig. 4.2

$$f(x, y) = k$$

$$\int\int f(x, y)\,dy\,dx = 1$$

i.e., $$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} k\,dy\,dx = 1$$

i.e., $$2k\int_0^1 \sqrt{1-x^2}\,dx = 1$$

$\therefore$ $$k = \frac{2}{\pi}$$

$$f_X(x) = \int_0^{\sqrt{1-x^2}} \frac{2}{\pi}\,dy = \frac{2}{\pi}\sqrt{1-x^2}, \ -1 \le x \le 1$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\pi}\,dx = \frac{4}{\pi}\sqrt{1-y^2}, \ 0 \le y \le 1$$

$$f(x/y) = \frac{f(x, y)}{f_Y(y)} = \frac{1}{2\sqrt{1-y^2}}, \ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$$

$$f(y/x) = \frac{1}{\sqrt{1-x^2}}, \ 0 \le y \le \sqrt{1-x^2}$$

$$E(X) = \int_{-1}^{1} x f_X(x)\,dx = \frac{2}{\pi}\int_{-1}^{1} x\sqrt{1-x^2}\,dx = 0$$

(since the integrand is odd)

$$E(X/Y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x f(x/y)\,dx$$

$$= \frac{1}{2\sqrt{1-y^2}}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx = 0$$

(since the integrand is odd)

$\therefore$ $$E\{E(X/Y)\} = E\{0\} = 0 = E(X)$$

$$E(Y) = \int_0^1 y f_Y(y)\,dy = \frac{4}{\pi}\int_0^1 y\sqrt{1-y^2}\,dy = \frac{4}{3\pi}$$

$$E(Y/X) = \int_0^{\sqrt{1-x^2}} y f(y/x)\,dy = \frac{1}{\sqrt{1-x^2}}\left(\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}} = \frac{1}{2}\sqrt{1-x^2}$$

$\therefore$ $$E\{E(Y/X)\} = E\left\{\frac{1}{2}\sqrt{1-X^2}\right\}$$

$$= \int_{-1}^{1} \frac{1}{2}\sqrt{1-x^2}\, f_X(x)\,dx$$

$$= \frac{2}{\pi}\int_0^1 (1-x^2)\,dx = \frac{4}{3\pi}$$

$\therefore$ $$E\{E(Y/X)\} = E(Y)$$

Example 12

If (X, Y) follows a bivariate normal distribution $N(0, 0; \sigma_X, \sigma_Y; r)$, find $E(Y/X)$, $E(Y^2/X)$, $E(XY)$ and $E(X^2Y^2)$.

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_x^2} - \frac{2rxy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}\right)\right\}$$

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left(\frac{y}{\sigma_y} - \frac{rx}{\sigma_x}\right)^2 - \frac{x^2}{2\sigma_x^2}\right\}$$

$$f_X(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp(-x^2/2\,\sigma_X^2)$$

[refer to the worked Example 12 in Chapter 2 on two-dimensional RVs]

$\therefore$ $$f(y/x) = \frac{f(x, y)}{f_X(x)} = \frac{1}{\sigma_y\sqrt{1-r^2}\,\sqrt{2\pi}}$$

$$\exp\left\{-\frac{1}{2(1-r^2)\sigma_y^2}\left(y-\frac{rx\sigma_y}{\sigma_x}\right)^2\right\}$$

which is a $N\left(\frac{rx\sigma_y}{\sigma_x}, \sigma_y\sqrt{1-r^2}\right)$

$\therefore \quad E(Y/X = x) = \int_{-\infty}^{\infty} yf(y/x)\,dy = \frac{rx\sigma_y}{\sigma_y}$

and $\quad \text{Var}(Y/X = x) = \sigma_y^2\,(1 - r^2)$

$$\text{Var}(Y/X) = E(Y^2/X) - \{E(Y/X)\}^2$$

$\therefore \quad E(Y^2/X) = \sigma_y^2\,(1 - r^2) + r^2x^2\sigma_y^2/\sigma_x^2$

By Property 3 of conditional expected values,

$$E(XY) = E\{XE(Y/X)\}$$

$$= E\left\{r\frac{\sigma_y}{\sigma_x}X^2\right\} = r\frac{\sigma_y}{\sigma_x} \times \sigma_x^2 = r\sigma_x\sigma_y$$

Again, by the same property

$$E(X^2Y^2) = E\{X^2 \times E(Y^2/X)\}$$

$$= E\left[\sigma_y^2(1-r^2)X^2 + \frac{r^2\sigma_y^2}{\sigma_x^2}X^4\right]$$

$$= \sigma_y^2\,(1 - r^2)E(X^2) + \frac{r^2\sigma_y^2}{\sigma_x^2}\,E(X^4)$$

$$= \sigma_x^2\,\sigma_y^2\,(1 - r^2) + \frac{r^2\sigma_y^2}{\sigma_x^2} \times 3\sigma_x^4$$

$$= (1 + 2r^2)\sigma_x^2\sigma_y^2$$

Exercise 4(A)

Part A (Short answer questions)

1. Define the expected value of $g(X)$, where X is a RV.
2. Define the mean and variance of a RV.
3. Prove that $\text{Var}(X) = E(X^2) - E^2(X)$
4. If X is a RV, prove that $E(X^2) \geq \{E(X)\}^2$.
5. If X is a discrete/continuous RV prove that $E(aX + b) = aE(X) + b$ and $\text{Var}(aX) = a^2\text{Var}(X)$.

6. If μ_X and σ_X are the mean and SD of the RV X, find μ_Y and σ_Y, where $Y = \frac{1}{\sigma_X}(X - \mu_X)$.
7. Define the raw and central moments of a RV and state the relation between them.
8. The probability distribution of a RV X is given by

X:	0	1	2	3
p_X:	0.1	0.3	0.4	0.2

   find $E(Y)$, where $Y = X^2 + X$.
9. Find the mean of the RV X, if its pmf is given by $P(x = j) = (1 - a)\,a^j, j = 0, 1, 2, \cdots, \infty$.
10. Find the mean of the RV X if its pdf is $f(x) = 6x(1 - x), 0 \leq x \leq 1$.
11. Find the mean and variance of the uniform distribution in (a, b).
12. Find the mean and variance of a RV X, that is uniformly distributed in $(2, 8)$.
13. If X is uniformly distributed in $(1, 2)$ and $Y = X^3$, find the mean of Y.
14. Obtain the mean of the binomial distribution $B(n; p)$.
15. Obtain the mean of the Poisson distribution $P(\lambda)$.
16. Find the binomial distribution whose mean is 6 and SD is $\sqrt{2}$.
17. If X is a binomial RV with mean 2.4 and variance 1.44, find $P(X = 7)$.
18. if X is binomially distributed with $n = 5$ such that $P(X = 1) = 2P(X = 2)$, find $E(X)$ and $\text{Var}(X)$.
19. If X is binomially distributed with $n = 6$ such that $P(X = 2) = 9\,P(X = 4)$, find $E(X)$ and $\text{Var}(X)$.
20. X is a Poisson RV such that $P(X = 1) = P(X = 2)$, find $E(X)$ and $E(X^2)$.
21. Find the mean of the geometric distribution given by $P(X = r) = pq^r$ $(r = 0, 1, 2, \cdots)$. where $p + q = 1$.
22. On the average, how many times must a dice be thrown until a '6' is obtained?
23. Find the mean and variance of the exponential distribution given by $f(x) = \lambda e^{-\lambda x}, x > 0$.
24. If the RV X follows $N(0, 2)$, find $E(X^2)$.
25. Define the expected value of $g(X, Y)$, where (X, Y) is a two-dim nsional continuous RV with joint pdf $f(x, y)$.
26. If (X, Y) is a two-dimensional continuous RV, express $E[g(X$ ar $E[h(Y)\}$ in terms of the marginal densities of X and Y.
27. If X and Y are independent RVs prove that $E(XY) = E(X) \times E(Y)$.
28. If X and Y are independent RVs with means 2 and 3 and variances 1 and 2 respectively, find the mean and variance of $Z = 2X - 5Y$.
29. If (X, Y) is a two-dimensional continuous RV, define $E\{g(X, Y)/X\}$ and $E\{g(X, Y)/Y\}$.
30. If (X, Y) is a two-dimensional continuous RV, define conditional mean and conditional variance of X, given Y.

31. If X and Y are independent RVs, prove that $E(Y/X) = E(Y)$.
32. If X and Y are independent RVs, prove that $E\{E(Y/X)\} = E(Y)$.
33. If X and Y are independent RVs, prove that $E(XY) = E\{X.E(Y/X)\}$.
34. If the joint pdf of (X, Y) is given by $f(x, y) = 2 - x - y$, in $0 \le x < y \le 1$, find $E(X)$(and $E(Y)$.
35. If the joint pdf of (X, Y) is given by $f(x, y) = 2$, in $0 \le x < y \le 1$, find $E(X)$.

Part B

36. What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability p of success?
37. What is the expectation of (i) the sum of the points on n dice? and (ii) the product of the points on n dice?
38. Three tickets are chosen at random without replacement from 100 tickets, numbered 1, 2, 3, ..., 100. Find the expectation of the sum of the numbers.
39. From an urn containing 3 red and 2 black balls, a man is to draw 2 balls at random without replacement, being promised Rs. 20/- for each red ball he draws and Rs. 10/- for each black ball. Find his expectation.
40. If X follows a uniform distribution in (a, b), find $E(X)$ and Var(X).
41. Find the mean and variance of the geometric distribution given by $P(X = r) = pq^r, r = 0, 1, 2, \ldots; p + q = 1$.
42. Find the mean and variance of the binomial distribution $B(n; p)$.
43. Find the mean and variance of the Poisson distribution $P(\lambda)$.
44. If the continuous RV X follows a normal distribution $N(0, \sigma)$, prove that

$$\text{(i) } E(X^n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 \times 3 \times 5 \cdots (n-1)\,\sigma^n & \text{if } n \text{ is even} \end{cases}$$

$$\text{(ii) } E(|X|^n) = \begin{cases} \sqrt{\dfrac{2}{\pi}} \times 2^{\frac{n-1}{2}} \lfloor (n-1)/2\,\sigma^n & \text{if } n \text{ is odd} \\ 1 \times 3 \times 5 \cdots (n-1)\,\sigma^n & \text{if } n \text{ is even} \end{cases}$$

45. If the continuous RV X has a Maxwell density, given by

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{x^2}{\alpha^3} e^{-x^2/2\alpha^2} U(x), \text{ prove that}$$

$$E(X^n) = \begin{cases} 1, 3 \cdots (2k+1)\,\alpha^{2k} & \text{if } n = 2k \\ \sqrt{\dfrac{\pi}{2}} \times 2^k k \times \alpha^{2k-1} & \text{if } n = 2k-1 \end{cases}$$

Hence find the mean and variance of the distribution.

46. If X has a Rayleigh density with parameter α and $Y = a + b X^2$, prove that $\sigma_y^2 = 4b^2\alpha^4$.
47. (i) If $Y = aX + b$, show that $\sigma_Y = a\sigma_X$ and
(ii) If $Y = (X - \mu_X)/\sigma_X$, find μ_Y and σ_Y.

48. If X is a RV for which $E(X) = 10$ and Var$(X) = 25$, for what positive values of a and b does $Y = aX - b$ have expectation 0 and variance 1?
49. If X is uniformly distributed in (1, 2) and $Y = X^3$, find the mean and variance of Y.
50. If the continuous RV X has the density function $f(x) = 2xe^{-x^2}, x \ge 0$, and if $Y = X^2$, find the mean and variance Y.
51. If X and Y are independent random variables with density functions $f_X(x) = \dfrac{8}{x^3}, x > 2$, and $f_Y(y) = 2y, 0 < y < 1$, respectively, and $Z = XY$, find $E(Z)$.
52. If each of the independent RVs X and Y follows $N(0, \sigma)$ and $Z = |X - Y|$, prove that $E(Z) = 2\sigma/\sqrt{\pi}$ and $E(Z^2) = 2\sigma^2$.
53. If the joint pdf of (X, Y) is given by $f(x, y) = 2, 0 \le x < y \le 1$, find the conditional mean and conditional variance of X, given that $Y = y$.
54. If the joint pdf of (X, Y) is given by $f(x, y) = 21x^2y^3, 0 \le x < y \le 1$, find the conditional mean and variance of X, given that $Y = y, 0 < y < 1$.
55. If the joint pdf of (X, Y) is given by $f(x, y) = 3xy(x + y), 0 \le x, y \le 1$, verify that $E\{E(Y/X)\} = E(Y) = \dfrac{17}{24}$.

LINEAR CORRELATION

In many situations, the outcome of a random experiment will have two measurable characteristics, viz., will result in two random variables X and Y. Often we will be interested in finding whether the two different R.V.'s are related to each other. If they are related, we will try to determine the nature of relationship and degree of relationship (correlation). Assuming that there is some correlation between X and Y, we will then try to find a formula expressing the relationship and use this formula to predict the most likely value of one R.V. corresponding to any given value of the other R.V.

To examine whether the two R.V.'s are inter-related, we collect n pairs of values of X and Y corresponding to n repetitions of the random experiment. Let them be $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. Then we plot the points with co-ordinates $(x_1, y_1), \cdots, (x_n, y_n)$ on a graph paper. The simple figure consisting of the plotted points is called a *scatter diagram*. From the scatter diagram, we can form a fairly good, though vague, idea of the relationship between X and Y. If the points are dense or closely packed, we may conclude that X and Y are correlated. On the other hand if the points are widely scattered throughout the graph paper, we may conclude that X and Y are either not correlated or poorly correlated.

Further if the points in the scatter diagram appear to lie near a straight line, we assume that the R.V.'s have *linear correlation*. If they cluster round a well defined curve other than a straight line, the R.V.'s are assumed to be *non-linear*. In this section we will assume linear correlation between the concerned R.V.'s and discuss how to measure the degree of linear correlation.

CORRELATION COEFFICIENT

As the variance $E\{X - E(X)\}^2$ measures the variations of the R.V. X from its mean value $E(X)$, the quantity $E\{[X - E(X)]\,[Y - E(Y)]\}$ measures the simultaneous variation of two R.V.'s X and Y from their respective means and hence it is called *the covariance of* X, Y and denoted as Cov (X, Y).

Cov $(X, Y) = E\{[X - E(X)]\,[Y - E(Y)]\}$ is also called the *product moment* of X and Y and is also denoted as $p(X, Y)$.

Though $p(X, Y)$ is a useful measure of the degree of correlation between X and Y, it is to be expressed in mixed units of X and Y. To avoid this difficulty and to express the degree of correlation in absolute units, we divide $p(X, Y)$ by $\sigma_x \cdot \sigma_y$, so that $\frac{p(x, y)}{\sigma_x \sigma_y}$ is a mere number, free from the units of X and Y.

$\frac{p(x, y)}{\sigma_x \sigma_y}$ is a measure of intensity of linear relationship between X and Y and is called *Karl Pearson's Product Moment Correlation Coefficient* or simply *correlation coefficient* between X and Y. It is denoted by $r(X, Y)$ or r_{XY} or simply r.

Thus
$$r_{XY} = \frac{E\{[X - E(X)][Y - E(Y)]\}}{\sqrt{E\{X - E(X)\}^2 E\{Y - E(Y)\}^2}} \quad (1)$$

since σ_x, the standard deviation of X is the positive square root of the variance of X.

Now
$$\begin{aligned} & E[\{X - E(X)\}\,\{Y - E(Y)\}] \\ &= E[XY - E(Y)\cdot X - E(X)\cdot Y + E(X)\cdot E(Y)] \\ &= E(XY) - E(Y)\cdot E(X) - E(X)\cdot E(Y) + E(X)\cdot E(Y) \\ & \qquad [\because E(X) \text{ and } E(Y) \text{ are non-random constants}] \\ &= E\{XY\} - E(X)\cdot E(Y) \qquad (2) \end{aligned}$$

Also we know that $E\{X - E(X)\}^2 = E(X^2) - \{E(X)\}^2 \quad (3)$

and $E\{Y - E(Y)\}^2 = E\{Y^2\} - \{E(Y)\}^2 \quad (4)$

Using (2), (3) and (4) in (1), we get
$$r_{XY} = \frac{E(XY) - E(X)\cdot E(Y)}{\sqrt{\{E(X^2) - E^2(X)\}\{E(Y^2) - E^2(Y)\}}} \quad (5)$$

where $E^2(X)$ means $\{E(X)\}^2$.

We will mainly deal with linear correlation of discrete R.V.'s X and Y. X will take the values $x_1, x_2, \cdots, x_n$ with frequency 1 each and Y will simultaneously take the values $y_1, y_2, \cdots, y_n$ with frequency 1 each. Hence $E(X) = \frac{1}{n}\sum x_i$; $E(X^2) = \frac{1}{n}\sum x_i^2$, $E(XY) = \frac{1}{n}\sum x_i y_i$ etc. Using these values in (5), the working formula for the compution of r_{XY} is got as

$$r_{XY} = \frac{\frac{1}{n}\Sigma x_i y_i - \frac{1}{n}\Sigma x_i \cdot \frac{1}{n}\Sigma y_i}{\sqrt{\left\{\frac{1}{n}\Sigma x_i^2 - \left(\frac{1}{n}\Sigma x_i\right)^2\right\}\left\{\frac{1}{n}\Sigma y_i^2 - \left(\frac{1}{n}\Sigma y_i\right)^2\right\}}} \quad (6)$$

or
$$r_{XY} = \frac{n\,\Sigma xy - \Sigma x \cdot \Sigma y}{\sqrt{\{n\Sigma x^2 - (\Sigma x)^2\}\{n\Sigma y^2 - (\Sigma y)^2\}}} \quad (7)$$

Properties of Correlation Coefficient

1. $-1 \le r_{XY} \le 1$ or $|\text{Cov}(X, Y)| \le \sigma_X \cdot \sigma_Y$.

Let us consider
$$E[a\{X - E(X)\} + \{Y - E(Y)\}]^2 = a^2\sigma_x^2 + 2a\,C_{XY} + \sigma_Y^2 \quad (1)$$

The R.H.S. expression is a quadratic expression in a, that is a real quantity. It is positive, as it is the expected value of a perfect square.

Hence, by the property of quadratic expressions, the discriminant of the R.H.S. ≤ 0

i.e., $4C_{XY}^2 - 4\sigma_X^2\sigma_Y^2 \le 0$

i.e., $C_{XY}^2 \le \sigma_X^2 \cdot \sigma_Y^2 \quad (2)$

i.e., $\frac{C_{XY}^2}{\sigma_X^2 \cdot \sigma_Y^2} \le 1$

i.e., $r_{XY}^2 \le 1$

$\therefore$ $|r_{XY}| \le 1$ or $-1 \le r_{XY} \le 1$

From step (2), it is clear that $|C_{XY}| \le \sigma_X \cdot \sigma_Y$

Note: When $0 < r_{XY} \le 1$, the correlation between X and Y is said to be *positive* or direct.

When $-1 \le r_{XY} \le 0$, the correlation is said to be *negative* or *inverse*.

When $-1 \le r_{XY} \le -0.5$ or $0.5 \le r_{XY} \le 1$, the correlation is assumed to be high, otherwise the correlation is assumed to be poor.

12. Find the equations of the regression lines from the following data. Also estimate the value of Y when $X = 71$ and the value of X when $Y = 70$.

X:	65	66	67	67	68	69	70	72
Y:	67	68	65	68	72	72	69	71

13. Find the equation of the regression line of Y on X using the method of least squares from the following data. Find the value of Y corresponding to $X = 18$.

X:	5	10	15	20	25
Y:	16	19	23	26	30

14. Obtain the line of regression of X on Y using the method of least squares from the following data. Find the value of X when $Y = 45$.

X:	4.7	8.2	12.4	15.8	20.7	24.9	31.9	35.0	39.1	38.8
Y:	4.0	8.0	12.5	16.0	20.0	25.0	31.0	36.0	40.0	40.0

15. Find the most likely price in Mumbai corresponding to the price of Rs. 70 at Chennai and that in Chennai corresponding to the price of Rs. 75 at Mumbai from the following:

	Chennai	Mumbai
Mean	65	67
S.D.	2.5	3.5

Coefficient of correlation between the prices in the two cities is 0.8.

16. In a partially destroyed laboratory record of an analysis of correlation data, the following results only are legible.
Variance of $X = 9$. Regression equations are $8x - 10y + 66 = 0$ and $40x - 18y = 214$. What were (i) the mean values of X and Y?
(ii) the correlation coefficient between X and Y and (iii) the standard deviation of Y?

17. The equations of two regression lines got in a correlation analysis are $3x + 12y = 19$ and $3y + 9x = 46$. Obtain (i) the correlation coefficient between X and Y, (ii) the mean values of X and Y and (iii) the ratio of the coefficient of variation of X to that of Y.

18. The equations of lines of regression are given by $x + 2y - 5 = 0$ and $2x + 3y - 8 = 0$ and variance of X is 12. Compute the values of $\bar{x}$, $\bar{y}$, σ_Y^2 and r_{XY}.

19. The regression lines of Y on X and of X on Y are respectively $y = a + bx$ and $x = c + dy$. Find the values of $\bar{x}$, $\bar{y}$ and r_{XY}. Can you find S_X snd S_Y from them ?

20. If the lines of regression of Y on X and X on Y are respectively $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, prove that $a_1b_2 \le a_2b_1$. Find also the coefficient of correlation between X and Y and the ratio of the coefficient of variability of Y to that of X.

Characteristic Function

Although higher order moments of a RV X may be obtained directly by using the definition of $E(X^n)$, it will be easier in many problems to compute them through the characteristic function or equivalently through the moment generating function of the RV X. While the characteristic function always exists, the moment generating function need not.

Moment Generating Function (MGF) of a RV X (discrete or continuous) is defind as $E(e^{tX})$, where t is a real variable and denoted as $M(t)$.

If X is discrete, then $M(t) = \sum_r e^{tx_r} p_r$,

where X takes the values $x_1, x_2, x_3, \ldots$, with probabilities $p_1, p_2, p_3, \ldots$
If X is a continuous RV with density function $f(x)$, then

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

Properties of MGF

(Proofs of the properties are omitted, as the proofs of the corresponding properties of characteristic function will be given later.)

1. $M(t) = \sum_{n=0}^{\infty} t^n E(X^n)/\underline{|n}$

   i.e., $E(X^n) = \mu'_n$ is the co-efficient of $\dfrac{t^n}{\underline{|n}}$ in the expansion of $M(t)$ in series of powers of t.

2. $\mu'_n = E(X^n) = \left[\dfrac{d^n}{dt^n} M(t)\right]_{t=0}$

3. If the MGF of X is $M_X(t)$ and if $Y = aX + b$, then $M_Y(t) = e^{bt} M_X(at)$.
4. If X and Y are independent RVs and $Z = X + Y$, then $M_Z(t) = M_X(t) M_Y(t)$.

Characteristic function of a RV X (discrete or continuous) is defined as $E(e^{i\omega X})$ and denoted as $\phi(\omega)$.

If X is a discrete RV that can take the values $x_1, x_2, \ldots$, such that $P(X = x_r) = p_r$, then

$$\phi(\omega) = \sum_r e^{i\omega x_r} p_r$$

If X is a continuous RV with density function $f(x)$, then

$$\phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx$$

Properties of Characteristic Function

1. $\mu'_n = E(X^n) =$ the coefficient of $\dfrac{i^n \omega^n}{n!}$ in the expansion of $\phi(\omega)$ in series of ascending powers of $i\omega$.

Proof

$$\phi(\omega) = E(e^{i\omega X})$$

$$= E\left(1 + \frac{i\omega X}{\lfloor 1} + \frac{i^2\omega^2 X^2}{\lfloor 2} + \cdots + \frac{i^n \omega^n X^n}{\lfloor n} \cdots\right)$$

$$= 1 + E(X) \times \frac{i\omega}{\lfloor 1} + E(X^2) \times \frac{i^2\omega^2}{\lfloor 2} + \cdots$$

$$= 1 + \mu'_1 \frac{i\omega}{\lfloor 1} + \mu'_2 \frac{i^2\omega^2}{\lfloor 2} + \cdots$$

i.e.,
$$\phi(\omega) = \sum_{n=0}^{\infty} \mu'_n \frac{i^n \omega^n}{\lfloor n} \quad (1)$$

Hence the result.

2. $\mu'_n = \frac{1}{i^n}\left[\frac{d^n}{d\omega^n}\phi(\omega)\right]_{\omega=0}$

Proof

Differentiating both sides of (1) with respect to ω, n times and then putting $\omega = 0$,

$$[\phi^{(n)}(\omega)]_{\omega=0} = \mu'_n i^n$$

$$\therefore \quad \mu'_n = \frac{1}{i^n}[\phi^{(n)}(\omega)]_{\omega=0}$$

3. If the characteristic function of a RV X is $f_X(\omega)$ and if $Y = aX + b$, then

$$\phi_Y(\omega) = e^{ib\omega}\phi_X(a\omega)$$ (BDU — Apr. 96)

Proof

$$\phi_Y(\omega) = E\{e^{i\omega Y}\}$$
$$= E\{e^{i\omega(aX+b)}\}$$
$$= e^{ib\omega} E\{e^{i(a\omega)X}\}$$
$$= e^{ib\omega}\phi_X(a\omega)$$

4. If X and Y are independent RVs, then

$$\phi_{X+Y}(\omega) = \phi_X(\omega) \times \phi_Y(\omega)$$ (BDU — Apr. 96)

Proof

$$\phi_{X+Y}(\omega) = E\{e^{i\omega(X+Y)}\}$$
$$= E\{e^{i\omega X} \times e^{i\omega Y}\}$$
$$= E(e^{i\omega X}) \times E(e^{i\omega Y})$$

[since X and Y are independent]

$$= \phi_X(\omega) \times \phi_Y(\omega)$$

5. If the characteristic function of a continuous RV X with density function $f(x)$ is $\phi(\omega)$, then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(\omega)\, e^{-ix\omega}\, d\omega$.

Proof

$$\phi(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{i\omega x}\, dx$$

RS can be identified as the Fourier transform of $f(x)$.
Therefore, by Fourier inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(\omega)\, e^{-i\omega x}\, d\omega$$

6. If the density function of X is known, the density function of $Y = g(X)$ can be found from the CF of Y, provided $Y = g(X)$ is one-to-one.

Proof

Let the density function of X be $f_X(x)$.

Then $\phi_{g(X)}(\omega) = \int_{-\infty}^{\infty} e^{ig(x)\omega} f_X(x)\, dx$

Put $g(X) = Y$ and hence $g(x) = y$

Then $\phi_Y(\omega) = \int_{-\infty}^{\infty} e^{iy\omega} h(y)\, dy$, say

Therefore, $h(y)$ is the density function of Y.

Cumulant Generating Function (CGF)

If $M(t)$ is the MGF of a RV X, then $\log_e M(t)$ is called the cumulant generating function of X and denoted by $K(t)$.

The coefficient of $\frac{t^r}{r!}$ in the expansion of $K(t)$ in ascending powers of t is called the *rth order cumulant of* X and denoted by λ_r.

i.e.,
$$K(t) = \sum_{r=1}^{\infty} \frac{\lambda_r t^r}{\lfloor r}$$

Also
$$\lambda_r = \left\{\frac{d^r}{dt^r}K(t)\right\}_{t=0}$$

If $\phi(\omega)$ is the characteristic function of a RV X, then $\log_e \phi(\omega)$ is called the *second characteristic function of* X and denoted by $\psi(\omega)$.

The coefficient of $\frac{i^r \omega^r}{\lfloor r}$ in the expansion of $\psi(\omega)$ in ascending powers of ω is the rth order cumulant of X and denoted by λ_r.

Thus $$\psi(\omega) = \sum_{r=1}^{\infty} \lambda_r i^r \omega^r / r!$$

Also $$\lambda_r = \frac{1}{i^r} \left\{ \frac{d^r}{d\omega^r} \psi(\omega) \right\}_{\omega=0}$$

Joint Characteristic Function

If (X, Y) is a two-dimensional RV, then $E\left(e^{i\omega_1 X + i\omega_2 Y}\right)$ is called the joint characteristic function of (X, Y) and denoted by $\phi_{XY}(\omega_1, \omega_2)$.
It is easily verified that

(i) $\phi_{XY}(0, 0) = 1$

(ii) $E\{X^m Y^n\} = \frac{1}{i^{m+n}} \left[\frac{\partial^{m+n}}{\partial \omega_1^m \partial \omega_2^n} \phi_{XY}(\omega_1, \omega_2) \right]_{\omega_1 = 0, \omega_2 = 0}$

(iii) $\phi_X(\omega) = \phi_{XY}(\omega, 0)$ and $\phi_Y(\omega) = \phi_{XY}(0, \omega)$

(iv) If X and Y are independent
$\phi_{XY}(\omega_1, \omega_2) = \phi_X(\omega_1) \times \phi_Y(\omega_2)$ and conversely.

Worked Example 4(D)

Example 1

If X represents the outcome, when a fair die is tossed, find the MGF of X and hence find $E(X)$ and $\mathrm{Var}(X)$.
The probability distribution of X is given by

$$p_i = p(X = i) = \frac{1}{6}, \; i = 1, 2, \ldots, 6$$

$$M(t) = \sum_i e^{tx_i} p_i = \sum_{i=1}^{6} e^{ti} p_i$$

$$= \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$$

$$E(X) = [M'(t)]_{t=0} = \frac{7}{2}$$

$$E(X^2) = [M''(t)]_{t=0}$$

$$= \frac{1}{6}[e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]_{t=0} = \frac{91}{6}$$

$$\therefore \quad \mathrm{Var}(x) = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

Example 2

If a RV X has the MGF $M(t) = \frac{3}{3-t}$, obtain the standard deviation of X.

$$M(t) = \frac{3}{3\left(1 - \frac{t}{3}\right)} = 1 + t/3 + t^2/9 + \cdots + \infty \qquad (1)$$

$$E(X) = \text{coefficient of } \frac{t}{\lfloor 1} \text{ in (1)} = \frac{1}{3}$$

$$E(X^2) = \text{coefficient of } \frac{t^2}{\lfloor 2} \text{ in (1)} = \frac{2}{9}$$

$$\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{1}{9}$$

$$\therefore \quad \sigma_X = \frac{1}{3}$$

Example 3

Find the MGF of the binomial distribution and hence find its mean and variance.
Binomial distribution is given by

$$p_r = p(X = r) = nC_r p^r q^{n-r}, \; r = 0, 1, 2, \ldots, n$$

$$\therefore \quad M(t) = \sum_{r=0}^{n} e^{tr} p_r$$

$$= \sum_{r=0}^{n} e^{tr} nC_r p^r q^{n-r}$$

$$= \sum_{r=0}^{n} nC_r (p e^t)^r q^{n-r}$$

$$= (p e^t + q)^n$$

$$M'(t) = n(p e^t + q)^{n-1} \times p e^t$$

$$M''(t) = np[(p e^t + q)^{n-1} \times e^t + (n-1)(p e^t + q)^{n-2} p e^{2t}]$$
$$E(X) = M'(0) = np \qquad \text{(since } p + q = 1\text{)}$$
$$E(X^2) = M''(0) = np\,[1 + (n-1)p]$$
$$\text{Var}(X) = E(X^2) - \{E(X)\}^2 = np - np^2 = npq$$

Example 4

Find the characteristic function of the Poisson distribution and hence find the values of the first four central moments.

Poisson distribution is given by

$$p_r = P(X = r) = e^{-\lambda}\lambda^r/\lfloor r \quad r = 0, 1, 2, \cdots, \infty$$

$$\therefore \quad \phi(\omega) = \sum_{r=0}^{\infty} e^{i\omega r} e^{-\lambda}\lambda^r/\lfloor r$$
$$= \sum_{r=0}^{\infty} e^{-\lambda}(\lambda e^{i\omega})^r/\lfloor r$$
$$= e^{-\lambda} e^{\lambda e^{i\omega}} = e^{-\lambda(1-e^{i\omega})}$$
$$\phi^{(1)}(\omega) = e^{-\lambda} e^{\lambda e^{i\omega}} i\lambda e^{i\omega}$$
$$\phi^{(2)}(\omega) = i\lambda e^{-\lambda}\{e^{\lambda e^{i\omega}} e^{i\omega} i + e^{\lambda e^{i\omega}} i\lambda e^{i2\omega}\}$$
$$= i^2\lambda e^{-\lambda}\{e^{i\omega} + \lambda e^{i2\omega}\} e^{\lambda e^{i\omega}}$$
$$\phi^{(3)}(\omega) = i^2 \lambda e^{-\lambda}$$
$$[\{i e^{i\omega} + 2i\lambda e^{i2\omega}\} e^{\lambda e^{i\omega}} + \{e^{i\omega} + \lambda e^{i2\omega}\} e^{\lambda e^{i\omega}} i\lambda e^{i\omega}]$$
$$= i^3\lambda e^{-\lambda} e^{\lambda e^{i\omega}}\{e^{i\omega} + 3\lambda e^{i2\omega} + \lambda^2 e^{i3\omega}\}$$
$$\phi^{(4)}(\omega) = i^3 \lambda e^{-\lambda}[e^{\lambda e^{i\omega}} i\lambda e^{i\omega}\{e^{i\omega} + 3\lambda e^{i2\omega} + \lambda^2 e^{i3\omega}\}$$
$$+ e^{\lambda e^{i\omega}}\{ie^{i\omega} + 6i\lambda e^{i2\omega} + 3\, i\lambda^2 e^{i3\omega}\}$$
$$= i^4 \lambda e^{-\lambda} e^{\lambda e^{i\omega}}\{e^{i\omega} + 7\lambda e^{i2\omega} + 6\lambda^2 e^{i3\omega} + \lambda^3 e^{i4\omega}\}$$

$$\therefore \quad E(X) = \frac{1}{i}\phi^{(1)}(0) = \lambda$$
$$E(X^2) = \frac{1}{i^2}\phi^{(2)}(0) = \lambda(1 + \lambda)$$
$$E(X^3) = \frac{1}{i^3}\phi^{(3)}(0) = \lambda(1 + 3\lambda + \lambda^2)$$
$$E(X^4) = \frac{1}{i^4}\phi^{(4)}(0) = \lambda(1 + 7\lambda + 6\lambda^2 + \lambda^3)$$

The central moments are given by

$$\mu_k = E\{X - \mu_X\}^k\}$$

$$\therefore \quad \mu_1 = E\{(X - \lambda)\} = 0$$
$$\mu_2 = E\{(X - \lambda)^2\} = \lambda$$
$$\mu_3 = E\{(X - \lambda)^3\} = \lambda$$
$$\mu_4 = E\{(X - \lambda)^4\} = 3\lambda^2 + \lambda$$

Example 5

Find the characteristic function of the geometric distribution given by $P(X = r) = q^r p$, $r = 0, 1, 2, \ldots, \infty$, $p + q = 1$.

Hence find the mean and variance.

$$\phi(\omega) = \sum_{r=0}^{\infty} e^{i\omega r} pq^r$$
$$= p\sum_{r=0}^{\infty}(qe^{i\omega})^r = p(1 - qe^{i\omega})^{-1}$$
$$\phi^{(1)}(\omega) = p(1 - qe^{i\omega})^{-2}\, i\, q\, e^{i\omega}$$
$$\phi^{(2)}(\omega) = i^2 pq\,[2(1 - q\, e^{i\omega})^{-3}\, qe^{i2\omega} + (1 - qe^{i\omega})^{-2} e^{i\omega}]$$
$$\therefore \quad E(X) = \frac{1}{i}\phi^1(0) = \frac{q}{p} \text{ and } E(X^2) = \frac{1}{i^2}\phi^{(2)}(0) = \frac{q}{p^2}(1 + q)$$
$$\therefore \quad \mu_X = \frac{q}{p} \text{ and } \sigma_X^2 = \frac{q}{p^2}$$

Example 6

Obtain the characteristic function of the normal distribution. Deduce the first four central moments. (MKU — Apr. 96)

Let X follow $N(\mu, \sigma)$

Then $Z = \dfrac{X - \mu}{\sigma}$ follows $N(0, 1)$, i.e., the standard normal distribution whose density function is $\phi(z) = \dfrac{1}{\sqrt{2\pi}} e^{-z^2/2}$.

Now
$$\phi_Z(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2} e^{i\omega z}\,dz$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - i\omega)^2 + \frac{i^2\omega^2}{2}}\,dz$$
$$= e^{-\omega^2/2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2}\cdot\sqrt{2}\,dt, \text{ on putting } \frac{z - i\omega}{\sqrt{2}} = t$$

$$= e^{-\omega^2/2}\left[\text{since} \int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\Gamma\left(\tfrac{1}{2}\right)} = \sqrt{\pi}\right]$$

$\therefore$ $$\phi_X(\omega) = f_{\sigma Z + \mu}(\omega)$$
$$= e^{i\mu\omega}\,\phi_Z(\sigma\omega) \text{ (by Property 3)}$$
$$= e^{i\mu\omega}\, e^{-\sigma^2\omega^2/2}$$

Now $$\phi_X(\omega) = e^{i\omega(\mu + i\sigma^2\omega/2)}$$
$$= 1 + \frac{1}{\lfloor 1}\, i\omega(\mu + i\sigma^2\omega/2) + \frac{1}{\lfloor 2}\, i^2\omega^2(\mu + i\sigma^2\omega/2)^2 + \ldots$$
$$= 1 + \frac{i\omega\mu}{\lfloor 1} + \frac{i^2\omega^2}{\lfloor 2}(\sigma^2 + \mu^2) + \frac{i^3\omega^3}{\lfloor 3}(3\mu\sigma^2 + \mu^3)$$
$$+ \frac{i^4\omega^4}{\lfloor 4}(3\sigma^4 + 6\mu^2\sigma^2 + \mu^4) + \ldots \qquad (1)$$

$\therefore$ $$\mu'_1 = E(X) = \text{coefficient of } \frac{i\omega}{\lfloor 1} \text{ in (1)} = \mu$$

Similarly, $E(X^2) = \sigma^2 + \mu^2$;

$E(X^3) = 3\mu\sigma^2 + \mu^3$ and,

$E(X^4) = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$.

Using the relation $\mu_k = E\{(X - \mu)^k\}$, we get

$$\mu_1 = 0,\ \mu_2 = \sigma^2,\ \mu_3 = 0 \text{ and } \mu_4 = 3\sigma^4$$

Example 7

Find the characteristic function of the Erlang distribution given by $f(x) = \frac{\lambda^n}{\lfloor n-1}\, x^{n-1} e^{-\lambda x}\, U(x)$ and hence find its mean and variance.

$$\phi(\omega) = \frac{\lambda^n}{\lfloor n-1} \int_0^{\infty} x^{n-1} e^{-(\lambda - i\omega)x}\,dx$$
$$= \frac{\lambda^n}{\lfloor n-1\,(\lambda - i\omega)^n} \int_0^{\infty} t^{n-1} e^{-t}\,dt, \text{ on putting } (\lambda - i\omega)x = t$$
$$= \left(\frac{\lambda}{\lambda - i\omega}\right)^n \frac{1}{\lfloor n-1}\,\Gamma(n) = \frac{1}{\left(1 - \frac{i\omega}{\lambda}\right)^n}$$

$$\left[\text{since } \Gamma(n) = \lfloor n-1 \text{ when } n \text{ is a positive integer}\right]$$

Now $$\phi(\omega) = \left(1 - \frac{i\omega}{\lambda}\right)^{-n}$$
$$= 1 + \frac{n}{\lfloor 1}\frac{i\omega}{\lambda} + \frac{n(n+1)}{\lfloor 2}\frac{i^2\omega^2}{\lambda^2} + \cdots + \infty$$

$\therefore$ $$E(X) = \frac{n}{\lambda} \text{ and } E(X^2) = \frac{n(n+1)}{\lambda^2}$$

$\therefore$ $$\text{Var}(X) = \frac{n}{\lambda^2}$$

Note *If $n = 1$, Erlang's distribution becomes the exponential distribution. Hence the mean and variance of an exponential distribution with parameter λ are $1/\lambda$ and $1/\lambda^2$ respectively.*

Example 8

Find the characteristic function of the Laplace distribution with pdf $f(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, $-\infty < x < \infty$. Hence find its mean and variance

$$\phi(\omega) = \frac{\alpha}{2}\int_{-\infty}^{\infty} e^{-\alpha|x|} e^{i\omega x}\,dx$$
$$= \frac{\alpha}{2}\left[\int_{-\infty}^{0} e^{(\alpha + i\omega)x}\,dx + \int_0^{\infty} e^{-(\alpha - i\omega)x}\,dx\right]$$
$$= \frac{\alpha}{2}\left\{\frac{1}{\alpha + i\omega} + \frac{1}{\alpha - i\omega}\right\} = \frac{\alpha^2}{\alpha^2 + \omega^2}$$

Now $$\phi(\omega) = \left(1 + \frac{\omega^2}{\alpha^2}\right)^{-1} = 1 - \frac{\omega^2}{\alpha^2} + \frac{\omega^4}{\alpha^4} + \cdots + \infty$$

$$E(X) = 0 \text{ and } E(X^2) = \text{Var}(X) = \frac{2}{\alpha^2}$$

Example 9

If X_1 and X_2 are two independent RVs that follow Poisson distribution with parameters λ_1 and λ_2, prove that $(X_1 + X_2)$ also follows a Poisson distribution with parameter $(\lambda_1 + \lambda_2)$. (MKU — Apr. 97

(This property is called the reproductivity property of the Poisson distribution)

$$\phi_{X_1}(t) = e^{\lambda_1 (e^{i\omega} - 1)}$$

$$\phi_{X_2}(t) = e^{\lambda_2 (e^{i\omega} - 1)}$$

since X_1 and X_2 are independent RVs,

$\phi_{X_1 + X_2}(t) = e^{(\lambda_1 + \lambda_2)(e^{i\omega} - 1)}$, which is the characteristic function of a Poisson distribution with parameter $(\lambda_1 + \lambda_2)$. Hence the result.

Example 10

Show that the distribution for which the characteristic function is $e^{-|\omega|}$ has the density function $f(x) = \frac{1}{\pi} \times \frac{1}{1+x^2}$ $-\infty < x < \infty$.

(BU – Apr. 96)

By inversion Property 5,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\omega)\, e^{-ix\omega}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|\omega|} (\cos x\omega - i \sin x\omega)\, d\omega$$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-\omega} \cos x\omega\, d\omega$$

(by properties of odd and even function)

$$= \frac{1}{\pi} \left[\frac{e^{-\omega}}{1+x^2} (-\cos x\omega + x \sin x\omega) \right]_0^{\infty}$$

$$= \frac{1}{\pi} \times \frac{1}{1+x^2}, \ -\infty < x < \infty$$

Example 11

Find the density function $f(x)$ corresponding to the characteristic function defined as

$$\phi(t) = \begin{cases} 1 - |t| & \text{for } |t| \le 1 \\ 0 & \text{for } |t| > 1 \end{cases}$$

Note *The letter t is used in the place of ω.*

By inversion Property 5,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t)\, e^{-ixt}\, dt$$

$$= \frac{1}{2\pi} \int_{-1}^{1} \{1 - |t|\}\, e^{-ixt}\, dt$$

$$= \frac{1}{\pi} \int_0^1 (1 - t) \cos xt\, dt$$

$$= \frac{1}{\pi} \left\{ (1-t) \frac{\sin xt}{x} - \frac{\cos xt}{x^2} \right\}_0^1$$

$$= \frac{1}{\pi} \frac{(1 - \cos x)}{x^2}$$

Example 12

Express the first four cumulants in terms of central moments.

By definition, $K(t) = \log \{M(t)\}$

$$\therefore \quad \sum_{r=1}^{\infty} \frac{\lambda t}{\lfloor r} = \log \left\{ \sum_{r=0}^{\infty} \frac{\mu'_r t^r}{\lfloor r} \right\}$$

$$= \log \left\{ 1 + \mu'_1 t + \frac{\mu'_2}{\lfloor 2} t^2 + \frac{\mu'_3}{\lfloor 3} t^3 + \cdots \right\}$$

$$= t \left(\mu'_1 + \frac{\mu'_2}{2} t + \frac{\mu'_3 t^2}{6} + \frac{\mu'_4 t^3}{24} + \cdots \right)$$

$$- \frac{t^2}{2} \left(\mu'_1 + \frac{\mu'_2}{2} t + \frac{\mu'_3 t^2}{6} + \frac{\mu'_4 t^3}{24} + \cdots \right)^2 + \cdots$$

Comparing like coefficients, we get,

$$\lambda_1 = \mu'_1;\ \lambda'_2 = \mu'_2 - \mu'^2_1 = \mu_2$$

$$\lambda_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 = \mu_3$$

$$\lambda_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu'^2_2 + 12\mu'^1_2\, \mu'^2_1 - 6\,\mu'^4_1$$

$$= \mu_4 - 3\mu_2^2$$

Example 13

If X and Y are two independent RVs, prove that the cumulant of $(X + Y)$ of any order is the sum of the cumulants of X and Y of the same order.

By Property 4 of characteristic functions, $\phi_{X+Y}(\omega) = \phi_X(\omega) \times \phi_Y(\omega)$, when X and Y are independent.

$\therefore \quad \log \phi_{X+Y}(\omega) = \log \phi_X(\omega) + \log \phi_Y(\omega)$

i.e., $\quad \psi_{X+Y}(\omega) = \psi_X(\omega) + \psi_Y(\omega)$

i.e., $$\sum_{r=1}^{\infty} \lambda_r (X+Y) \frac{i^r \omega^r}{\lfloor r} = \sum_{r=1}^{\infty} \lambda_r (X) \frac{i^r \omega^r}{\lfloor r} + \sum_{r=1}^{\infty} \lambda_r (Y) \frac{i^r \omega^r}{\lfloor r}$$

$\therefore \quad \lambda_r(X+Y) = \lambda_r(X) + \lambda_r(Y)$

Example 14

If the RV X follows $N(0, \sigma)$, find the density function of $Y = aX^2$, using the characteristic function technique.

$$\phi_{aX^2}(\omega) = \int_{-\infty}^{\infty} e^{iax^2\omega} \times \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \, dx \quad (1)$$

Put $\quad y = ax^2$ in (1)

$$\therefore \quad dx = \frac{dy}{2ax} = \frac{dy}{2\sqrt{ay}}$$

$$\therefore \quad \phi_Y(\omega) = 2\int_0^{\infty} e^{i\omega y} \times \frac{1}{2\sigma\sqrt{2\pi ay}} \times e^{-y/2a\sigma^2} \, dy \quad (2)$$

But $$\phi_Y(\omega) = \int_{-\infty}^{\infty} e^{i\omega y} f_Y(y) \, dy \quad (3)$$

Comparing (2) and (3), we get

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi a y}} e^{-y/2a\sigma^2} U(y)$$

Example 15

Two RVs X and Y have the joint characteristic function $\phi_{XY}(\omega_1, \omega_2) = \exp(-2\omega_1^2 - 8\omega_2^2)$. Show that X and Y are both zero mean RVs and also that they are uncorrelated. (BDU — Apr. 97)

By the property of joint CF

$$E(X) = \frac{1}{i}\left[\frac{\partial}{\partial \omega_1}\left(e^{-2\omega_1^2 - 8\omega_2^2}\right)\right]_{\omega_1 = 0, \omega_2 = 0}$$

$$= \left[e^{-8\omega_2^2} e^{-2\omega_1^2} 4i\omega_1\right]_{\omega_1 = 0, \omega_2 = 0}$$

$$= 0$$

$$E(Y) = \left[e^{-2\omega_1^2} e^{-8\omega_2^2} 16 i\omega_2\right]_{\omega_1 = 0, \omega_2 = 0}$$

$$= 0$$

$$E(XY) = \frac{1}{i^2}\left[\frac{\partial^2}{\partial\omega_1 \partial\omega_2}\left(e^{-2\omega_1^2 - 8\omega_2^2}\right)\right]_{\omega_1 = 0, \omega_2 = 0}$$

$$= \frac{\partial}{\partial\omega_1}\left\{e^{-2\omega_1^2} e^{-8\omega_2^2} 16\omega_2\right\}_{\omega_1 = 0, \omega_2 = 0}$$

$$= \left\{-64\,\omega_1\,\omega_2\, e^{-2\omega_1^2 + 8\omega_2^2}\right\}_{\omega_1 = 0, \omega_2 = 0}$$

$$= 0$$

$$C_{XY} = E(X) - E(X) \times E(Y) = 0$$

Therefore, X and Y are uncorrected RVs.

Exercise 4(D)

Part A (Short answer questions)

1. Define the MGF of a RV X. Why is it called so?
2. State the properties of the MGF of a RV
3. Derive the relation between the MGFs of X and Y when $Y = aX + b$.
4. If X and Y are independent RVs and $Z = X + Y$, prove that $M_Z(t) = M_X(t) \times M_Y(t)$.
5. Define the characteristic function of a RV. How does it differ from the MGF?
6. State the properties of the characteristic function of a RV.
7. State the uniqueness theorem of characteristic functions.
8. If $Y = aX + b$, find the relation between the characteristic functions of X and Y.
9. If X and Y are 2 independent RVs prove that $\phi_{X+Y}(\omega) = \phi_X \times \phi_Y(\omega)$.
10. If the characteristic function of a continuous RV X is $\phi(\omega)$, express its density function $f(x)$ in terms of $\phi(\omega)$.
11. Define the cumulant generating function/the second characteristic function of a RV X and what is its use?
12. If the rth moment of a continuous RV X about the origin is $r!$, find the MGF of X.
13. If the MGF of a RV X is $\frac{2}{2-t}$ find the SD of X.
14. Find the MGF/CF of a uniform distribution in (a, b).
15. Find the MGF/CF of the binomial distribution.
16. Find the MGF/CF of the Poisson distribution.
17. State and prove the reproductive property of the Poisson distribution.

18. Find the CF/MGF of the geometric distribution.
19. Find the CF/MGF of the exponential distribution.
20. If the CF of the standard normal distribution $N(0, 1)$ is $e^{-\omega^2/2}$, find the CF of the general normal distribution $N(\mu, \sigma)$.
21. Find the CF of X whose pdf is given by $f(x) = \frac{1}{2} e^{-|x|}, -\infty < x < \infty$.

Part B

22. Find the MGF of a RV which is uniformly distributed over (–1, 2) and hence find its mean and variance.
23. Find the characteristic function of the binomial distribution and hence find its mean and variance.
24. Obtain the MGF of the Poisson distribution; deduce the values of the first four central moments. (MKU — Apr. 96)
25. Find the characteristic function of the negative binomial distribution given by $P(X = r) = (n + r - 1)\, C_r\, q^r p^n$, $(r = 0, 1, 2, \ldots, \infty)$, $p + q = 1$, and hence find its mean and variance.
26. Find the MGF of the two-parameter exponential distribution whose density function is given by $f(x) = \lambda e^{-\lambda(x-a)}$, $x \geq a$. Hence find its mean and variance.
27. If the desnity function of a continuous RV X is given by $f(x) = \frac{1}{2} e^{-|x|}$, $-\infty < x < \infty$, find the MGF of X. Hence find its mean and variance.
28. Find the characteristic function of the Cauchy's distribution given by $f(x) = \frac{1}{\pi} \times \frac{1}{1+x^2}$, $-\infty < x < \infty$. Comment about the first two moments.

(**Hint**: Use contour integration.)

29. Find the characteristic function for the following probability density function:

$$f_X(x) = \frac{\lambda}{\pi(\lambda^2 + x^2)}$$ (BDU—Nov. 96)

30. If X follows $N(\mu_X, \sigma_X)$ and Y follows $N(\mu_Y, \sigma_Y)$, prove, by using characteristic functions, that $(aX + bY)$ follows a normal distribution with mean $(a\mu_X + b\mu_Y)$ and variance $(a^2 \sigma_X^2 + b^2 \sigma_Y^2)$.
31. Find the density function of the distribution for which the characteristic function is given by $\phi(t) = e^{-\sigma^2 t^2/2}$. [MU—Apr. 96]
32. If the raw moments of a continuous RV X are given by $E(X^n) = \lfloor n$, find the characteristic function of X and also the density function of X. (**Hint**: Use contour integration to find pdf.)
33. Express the first 4 raw moments about the origin in terms of the cumulants.

34. Prove that the cumulants of all orders are equal for the Poisson distribution.
35. If X and Y are two jointly normal RVs whose joint pdf is $N(\mu_X; \mu_Y; \sigma_X, \sigma_Y; r)$, find the joint characteristic function of (X, Y).
36. If the random variable X is uniformly distributed over $(-\pi/2, \pi/2)$, find the pdf of $Y = \sin X$, using characteristic function technique.

Bounds on Probabilities

If we know the probability distribution of a random variable X (i.e., the pdf in the continuous case or the pmf in the discrete case), we may compute $E(X)$ and Var(X). Conversely, if $E(X)$ and Var(X) are known, we cannot construct the probability distribution of X and hence compute quantities such as $P\{|X - E(X)| \leq k\}$. Although we cannot evaluate such probabilities from a knowledge of $E(X)$ and Var(X), several approximation techniques have been developed to yield upper and/or lower bounds to such probabilities. The most important of such techniques is Tchebycheff inequality.

Tchebycheff Inequality

If X is a RV with $E(X) = \mu$ and $\text{Var}(X) = \sigma^2$, then $P\{|X - \mu| \geq c\} \leq \frac{\sigma^2}{c^2}$ where $c > 0$.

Proof

Let X be a continuous RV with pdf $f(x)$.

Then
$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$\geq \int_{-\infty}^{\mu - c} (x-\mu)^2 f(x)\, dx + \int_{\mu + c}^{\infty} (x-\mu)^2 f(x)\, dx$$

In the first integral, $x \leq \mu - c$

$\therefore \quad (x - \mu)^2 \geq c^2$

In the second integral, $x \geq \mu + c$

$\therefore \quad (x - \mu)^2 \geq c^2$

$$\therefore \quad \sigma^2 \geq c^2 \left\{ \int_{-\infty}^{\mu - c} f(x)\, dx + \int_{\mu + c}^{\infty} f(x)\, dx \right\} \tag{1}$$

RS of (1)
$$= c^2[1 - P\{\mu - c \leq X \leq \mu + c\}]$$
$$= c^2[1 - P\{-c \leq X - \mu \leq c\}]$$
$$= c^2[1 - P\{|X - \mu| \leq c\}]$$
$$= c^2 P\{|X - \mu| \geq c\} \tag{2}$$