Bayesian Time Series Forecasting With Change Point and Anomaly Detection

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Time Series Model

Observation equaton

$$y_t = \mu_t + \gamma_t + z_t^a o_t + (1 - z_t^a)\varepsilon_t$$

Transition Equations

Trend:

$$\mu_t = \mu_{t-1} + \delta_{t-1} + z_t^c r_t + (1 - z_t^c) u_t$$
$$\delta_t = \delta_{t-1} + v_t$$

Seasonality:

$$\gamma_t = -\sum_{s=1}^{S-1} \gamma_{t-s} + w_t$$

Anomaly point

$\{z_t^a\}_{t=1}^n \sim \mathrm{Ber}(p_a)$, i.i.d.

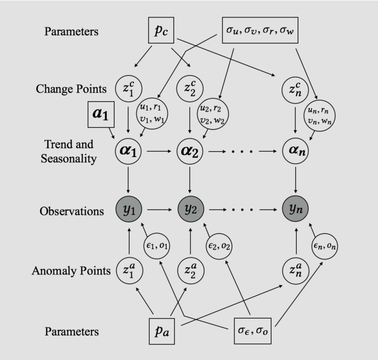
Change point

$\{z_t^c\}_{t=1}^n \sim \mathrm{Ber}(p_c)$, i.i.d.

Noise

$o_t, \varepsilon_t, r_t, u_t, v_t, z_t$ - independent zeros mean normal noises

Graphical Model



Latent Variables

$$\{\alpha_t\}_{t=1}^n = (\mu_t, \delta_t, \gamma_t, \ldots, \gamma_{t-S+2})_{t=1}^n$$

$$\{z_t\}_{t=1}^n = \{(z_t^a, z_t^c)\}_{t=1}^n$$

Parameters

$$a_1 = (\mu_o, \delta_0, \gamma_0, \ldots, \gamma_{2-S})$$
$$p = (p_a, p_c)$$
$$\sigma = (\sigma_\varepsilon, \sigma_o, \sigma_u, \sigma_r, \sigma_v, \sigma_w)$$

Algorithm

Part I: Initialization

1. Initialize $\sigma_\varepsilon, \sigma_o, \sigma_u, \sigma_r, \sigma_v, \sigma_w$ with the empirical standard deviation.
2. Initialize a_1: $a_1[0] = \frac{1}{S}\sum_{t=1}^{S} y_t$, $a_1[1:] = 0$.
3. Initialize $p_a = p_c = \frac{1}{n}$, $\{z_t^a\}_{t=1}^n \sim \text{Ber}(p_a)$, $\{z_t^c\}_{t=1}^n \sim \text{Ber}(p_c)$

Algorithm

Part II: Inference

while $L_{a_1,p,\sigma}(y,\alpha,z)$ does not converges:

1. $\alpha \sim p_{a_1,p,\sigma}(\alpha|y,z)$ by Kalman filter, Kalman smoothing and "fake-path" trick
2. $z_t^a \sim p_{a_1,p,\sigma}(z_t^a|y,\alpha) = \text{Ber}(p_t^a)$,
   $z_t^c \sim p_{a_1,p,\sigma}(z_t^c|y,\alpha) = \text{Ber}(p_t^c)$
   $\{p_t^a\}_{t=1}^n = \mathbb{P}(z_t^a = 1|y,\alpha)$, $\{p_t^c\}_{t=1}^n = \mathbb{P}(z_t^c = 1|y,\alpha)$
3. Segment control on z_c: requirement on the length of segment among two consecutive change points
4. Using α and z, update σ by the empirical standard deviation
5. Update a_1. $a_1[:2] = \alpha_1[:2]$, $a_1[2:] = \alpha_{S+1}[2:]$
6. Calculate $L_{a_1,p,\sigma}(y,\alpha,z)$

Algorithm

The Joint Likelihood function

$$L_{a_1,p,\sigma}(y,\alpha,z) =$$

$$= \prod_{t:z_t^a=0} \mathcal{N}(y_t|\mu_t+\gamma_t,\sigma_\varepsilon) \cdot \prod_{t:z_t^a=1} \mathcal{N}(y_t|\mu_t+\gamma_t,\sigma_o) \cdot$$

$$\cdot \prod_{t:z_t^c=0} \mathcal{N}(\mu_t|\mu_{t-1}+\delta_{t-1},\sigma_u) \cdot \prod_{t:z_t^c=1} \mathcal{N}(\mu_t|\mu_{t-1}+\delta_{t-1},\sigma_r) \cdot$$

$$\cdot \prod_{t=1}^{n} \mathcal{N}(\delta_t|\delta_{t-1},\sigma_v) \cdot \prod_{t=1}^{n} \mathcal{N}(\gamma_t| - \sum_{s=1}^{S-1} \gamma_{t-s},\sigma_w) \cdot$$

$$\cdot \prod_{t=1}^{n} (p_a)^{z_t^a} (1-p_a)^{1-z_t^a} (p_c)^{z_t^c} (1-p_c)^{1-z_t^c}$$

Algorithm

State Space Equations

$y_t = Z_t\alpha_t + A_t o_t + (1 - A_t)\varepsilon_t$
$\alpha_{t+1} =$
$T_t\alpha_t + R_t C_t \eta_t + R_t(1 - C_t)\xi_t,$
where
$o_t \sim \mathcal{N}(0, H_t^o)$
$\varepsilon_t \sim \mathcal{N}(0, H_t^\varepsilon)$
$\eta_t \sim \mathcal{N}(0, H_t^\eta)$
$\xi_t \sim \mathcal{N}(0, H_t^\xi),$
$A_t \sim \text{Bernoulli}(p_a)$
$C_t \sim \text{Bernoulli}(p_c)$

Transition equations

$\alpha_t = (\mu_t, \delta_t, \gamma_t, \ldots, \gamma_{t-S+2})^T$
$\eta_t = (r_t, v_t, w_t),$
$\xi_t = (u_t, v_t, w_t)$
$Z_t = (1, 0, 1, 0, \ldots, 0)$
$T_t = \text{diag}(T_\mu, T_\gamma)$
$R = \text{diag}(R_\mu, R_\gamma)$
$H_t^o = \sigma_o^2,$
$H^\varepsilon = \sigma_\varepsilon^2$
$Q^\eta = \text{diag}(\sigma_r^2, \sigma_v^2, \sigma_w^2)$
$Q^\xi = \text{diag}(\sigma_u^2, \sigma_v^2, \sigma_w^2)$

Algorithm

Kalman Filter Recursion

$v_t = y_t - Z_t a_t$
$F_t = Z_t P_t Z_t^T + p_t^a H_t^o + (1 - p_t^a) H_t^\varepsilon$
$a_{t|t} = a_t + P_t Z_t^T F_t^{-1} v_t$
$P_{t|t} = P_t Z_t F_t^{-1} Z_t P_t$
$K_t = T_t P_t Z_t^T F_t^{-1}$

Update equations:

$a_{t+1} = T_t a_t + K_t v_t$
$P_{t+1} = T_t P_{t|t} T_t^T + R_t (p_t^c Q_t^\eta + (1 - p_t^c Q_t^\xi)) R_t^T$

Kalman Smoothing Recursion

Inputs from Kalman Filter:

a_t, P_t, v_t, F_t, K_t
$r_n = 0, N_n = 0$

Update equations:

$L_t = T_t - K_t Z_t$
$r_{t-1} = Z_t F_t^{-1} v_t + L_t r_t$
$\hat{\alpha}_t = a_t + P_t r_{t-1}$
$N_{t-1} = Z_t^T F_t^{-1} Z_t + L_t^T N_t L_t$
$V_t = P_t - P_t N_{t-1} P_t$

Algorithm

"Fake-path" Trick

The purpose is to obtain posterior distribution $\mathbb{P}_{\alpha_1, p, \sigma(\alpha|y,z)}$. All hidden variables z, p, σ are given:

1. Pick some vector $\tilde{a}_1$ and generate a sequence of time series $\tilde{y}$ from it. We also observe $\tilde{\alpha}$.
2. Obtain $\{\mathbb{E}(\tilde{\alpha}_t|\tilde{y})\}_{t=1}^n$ from $\tilde{y}$ by Kalman filter and Kalman smoothing.
3. Use $\{\tilde{\alpha}_t - \mathbb{E}(\tilde{\alpha}_t|\tilde{y}) + \mathbb{E}(\alpha_t|y)\}_{t=1}^n$ as sampling distribution from the conditional distribution.

Algorithm

Segment control on change points

Denote $t_1 < t_2 < \ldots$ to be all the indexes such that $z_{t_i}^c = 1$

while there exists i such that $|t_{i+1} - t_i| < l$ ***do*** **:**

1. Check if $|\mu_{t_i - 1} - \mu_{t_{t+1}+1}| \leq 2$. If so, exclude both of them from change points by setting $z_{t_i}^c = z_{t_{i+1}}^c = 0$. Otherwise, randomly exclude one of them by setting the corresponding coordinate in z^c to be 0.
2. Update all the indexes of change points in z^c.

end

Algorithm

Part III: Forecasting

1. With a_n and σ, generate future time series y_{future} with length m. Repeat generative procedure to obtain future paths $y^{(1)}_{future}, y^{(2)}_{future}, ..., y^{(N)}_{future}$.
2. Combine all the predictive paths and give the distribution for the future time series forecasting. Calculate the point-wise quantile intervals.

Algorithm

Generative Procedure

1. Generate the indexes of anomalies or change points occur
$$\{z_t^a\}_{t=1}^n \sim \mathrm{Ber}(p_a), \quad \{z_t^c\}_{t=1}^n \sim \mathrm{Ber}(p_c).$$
2. Gnerate $\varepsilon, o, u, r, v, w$ as independent normal r.v.'s with zero mean and standard deviations $\sigma_\varepsilon, \sigma_o, \sigma_u, \sigma_r, \sigma_v, \sigma_w$.
3. Generate ${\alpha_t}_{t=1}^m$ by transition functions.
4. Generate time series $\{y_t\}_{t=1}^m$ by the observation function.

Experiments