Maths Problem Set- Measure Theory

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Exercise 1.3

- Consider $A \in \mathcal{G}_1 \implies A$ open on $\mathbb{R} \implies A^c$ is either closed on $\mathbb{R}$ or semi-open on $\mathbb{R}$. So $A^c \notin \mathcal{G}_1$ as it is not a purely open interval. Hence $\mathcal{G}_1$ is not a σ- algebra nor an algebra.

- Consider $A_n \in \mathcal{G}_2, n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} A_n \notin \mathcal{G}_2$ since $\mathcal{G}_2$ contains only sets which are finite unions of intervals of the form $(a, b], (-\infty, b], (a, \infty)$. Thus $\mathcal{G}_2$ is not a σ- algebra. Now we check whether $\mathcal{G}_2$ is an algebra. It is clear that $\phi \in \mathcal{G}_2$. Now consider any interval of the form $(a, b]$. Then it's complement is of the form $(-\infty, a] \cup (b, \infty)$ which $\in \mathcal{G}_2$. Similarly for any interval of the form $(-\infty, b]$, its complement is of the form (b, ∞) which $\in \mathcal{G}_2$. Thus, for all $A \in \mathcal{G}_2$, $A^c \in \mathcal{G}_2$. Now consider $A_n \in \mathcal{G}_2$ for $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{N} A_n$ is also a finite union of disjoint intervals of the form $(-\infty, b], (a, b]$ and (a, ∞). Hence $\mathcal{G}_2$ is an algebra (but not a σ-algebra).

- Now consider $A_n \in \mathcal{G}_3, n \in \mathbb{N}$. The first two properties of an algebra hold in this case as they have already been proved above. Now consider $\bigcup_{n=1}^{\infty} A_n$ where $A_n \in \mathcal{G}_3$ for $n \in \mathbb{N}$. The countable union $\bigcup_{n=1}^{\infty} A_n \in \mathcal{G}_3$ as it is contains countable unions of intervals of the form $(a, b], (-\infty, b]$ and (a, ∞). Thus $\mathcal{G}_3$ is a σ- algebra.

Exercise 1.7

Let $\mathcal{A}$ be any σ-algebra. By defnition of σ-algebra, $\phi \in \mathcal{A}$. Similarly, $X = \phi^c \in \mathcal{A}$ Thus $\{\phi, X\} \subset \mathcal{A}$. Now consider any $A \in \mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra on X, $A \subset X \Rightarrow A \in \mathcal{P}(X)$. Thus $\mathcal{A} \subset \mathcal{P}(X)$