# Maths Problem Set- Measure Theory

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## Exercise 1.3

- Consider $A \in \mathcal{G}_1 \implies A$ open on $\mathbb{R} \implies A^c$ is either closed on $\mathbb{R}$ or semi-open on $\mathbb{R}$. So $A^c \notin \mathcal{G}_1$ as it is not a purely open interval. Hence $\mathcal{G}_1$ is not a $\sigma$- algebra nor an algebra.

- Consider $A_n \in \mathcal{G}_2, n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} A_n \notin \mathcal{G}_2$ since $\mathcal{G}_2$ contains only sets which are finite unions of intervals of the form $(a, b], (-\infty, b], (a, \infty)$. Thus $\mathcal{G}_2$ is not a $\sigma$- algebra. Now we check whether $\mathcal{G}_2$ is an algebra. It is clear that $\phi \in \mathcal{G}_2$. Now consider any interval of the form $(a, b]$. Then it's complement is of the form $(-\infty, a] \cup (b, \infty)$ which $\in \mathcal{G}_2$. Similarly for any interval of the form $(-\infty, b]$, its complement is of the form $(b, \infty)$ which $\in \mathcal{G}_2$. Thus, for all $A \in \mathcal{G}_2$, $A^c \in \mathcal{G}_2$. Now consider $A_n \in \mathcal{G}_2$ for $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{N} A_n$ is also a finite union of disjoint intervals of the form $(-\infty, b], (a, b]$ and $(a, \infty)$. Hence $\mathcal{G}_2$ is an algebra (but not a $\sigma$-algebra).

- Now consider $A_n \in \mathcal{G}_3, n \in \mathbb{N}$. The first two properties of an algebra hold in this case as they have already been proved above. Now consider $\bigcup_{n=1}^{\infty} A_n$ where $A_n \in \mathcal{G}_3$ for $n \in \mathbb{N}$. The countable union $\bigcup_{n=1}^{\infty} A_n \in \mathcal{G}_3$ as it contains countable unions of intervals of the form $(a, b], (-\infty, b]$ and $(a, \infty)$. Thus $\mathcal{G}_3$ is a $\sigma$- algebra.

## Exercise 1.7

Let $\mathcal{A}$ be any $\sigma$-algebra. By defnition of $\sigma$-algebra, $\phi \in \mathcal{A}$. Similarly, $X = \phi^c \in \mathcal{A}$ Thus $\{\phi, X\} \subset \mathcal{A}$. Now consider any $A \in \mathcal{A}$. Since $\mathcal{A}$ is a $\sigma$-algebra on $X$, $A \subset X \Rightarrow A \in \mathcal{P}(X)$. Thus $\mathcal{A} \subset \mathcal{P}(X)$. Thus $\{\phi, X\} \subset A \subset \mathcal{P}(X)$

## Exercise 1.10

Since $\{S_\alpha\}$ is a family of $\sigma$- algebras, then $\phi \in S_\alpha \forall \alpha \Rightarrow \phi \in \bigcap_\alpha S_\alpha$. Now consider any $A \in \bigcap_\alpha S_\alpha$. This means that $A \in S_\alpha$ for each $\alpha$ Since each $S_\alpha$ is a $\sigma$ - algebra $\Rightarrow A^c \in S_\alpha \forall \alpha \Rightarrow A^c \in \bigcap_\alpha S_\alpha$. Now consider $\{A_n\} \in \bigcap_\alpha (S_\alpha) \forall n \in \mathbb{N}$. Then $A_n \in S_\alpha \forall \alpha, n \in \mathbb{N}$ since each $S_\alpha$ is a $\sigma$-algebra, it means that $\bigcup_{n=1}^{\infty} A_n \in S_\alpha \forall \alpha$. This in turn means that $\bigcup_{n=1}^{\infty} A_n \in \bigcap_\alpha S_\alpha$. Hence $\bigcap_\alpha S_\alpha$ is a $\sigma$- algebra.

## Exercise 1.17

In order to prove the results we first prove a simpler result. We prove that if $\mu : \mathcal{S} \to [0, \infty]$ is a measure, then $\mu(\bigcup_{i=1}^{n=N} A_i) = \sum_{i=1}^{i=N} \mu(A_i)$ if $A_i \bigcap A_j = \phi, i \neq j$. To prove this, let all $A_i$ for $i > N = \phi$. Since $\mu(\phi) = 0$, we then get $\mu(\bigcup_{i=1}^{n=N} A_i) = \mu(\bigcup_{i=1}^{n=\infty} A_i) = \sum_{i=1}^{i=\infty} \mu(A_i) = \sum_{i=1}^{i=N} \mu(A_i)$.

Now to prove monotonicity, consider two sets $A, B \in \mathcal{S}, A \subset B$. Now define $C = A^c \cap B$. Since $\mathcal{S}$ is a $\sigma$- algebra $A^c \cap B \in \mathcal{S}$. Furthermore, $A \cap C = \phi$. Since $\mu$ is a measure, $\mu(A \cup C) = \mu(A) + \mu(B) \Rightarrow \mu(B) = \mu(A) + \mu(C)$. Since the range of $\mu$ is non-negative, $\mu(C) \geq 0$. Thus $\mu(B) \geq \mu(A)$.

Now we prove countable sub-additivity. Consider 2 sets $A_1, A_2$. We can write, $A_1 \cup A_2 = ({A_1}^c \cap A_2) \cup ({A_2}^c \cap A_1) \cup (A_1 \cap A_2)$ i.e as a union of disjoint sets. Using the result proved above, we get $\mu(A_1 \cup A_2) = \mu({A_1}^c \cap A_2) + \mu({A_2}^c \cap A_1) + \mu(A_1 \cap A_2) \leq \mu({A_1}^c \cap A_2) + \mu(A_1 \cap A_2) + \mu({A_2}^c \cap A_1) + \mu(A_1 \cap A_2) = \mu(A_1) + \mu(A_2)$. Thus we have $\mu(A_1 \cup A_2) \leq \mu(A_1) + \mu(A_2)$ The same argument can be carried out inductively for all $n \in \mathbb{N}$. For example in the case of three sets, $A_1, A_2$ and $A_3$, we can assume $A_1 \cup A_2 = A$ and proceed as before. Therefore $\mu(\bigcup_{i=1}^{i=\infty} A_i) \leq \sum_{i=1}^{i=\infty} \mu(A_i)$.

## Exercise 1.18

$\lambda(\phi) = \mu(\phi \cap B) = \mu(\phi) = 0$. Let $\{A_i\}_{i=1}^{i=\infty}$ be a collection of disjoint sets. We have, $\lambda(\bigcup_{i=1}^{i=\infty} A_i) = \mu(B \cap \bigcup_{i=1}^{i=\infty} A_i) = \mu(\bigcup_{i=1}^{i=\infty} (B \cap A_i))$ where we have used De-Morgan's laws in the last step. Since all $A_i$'s are disjoint, so are $(B \cap A_i)$'s. Now since $\mu$ is a measure, we have, $\mu(\bigcup_{i=1}^{i=\infty} (B \cap A_i)) = \sum_{i=1}^{i=\infty} \mu(B \cap A_i) = \sum_{i=1}^{i=\infty} \lambda(A_i)$. Thus $\lambda(\bigcup_{i=1}^{i=\infty} A_i) = \sum_{i=1}^{i=\infty} \lambda(A_i)$. Hence $\lambda$ is a measure.

## Exercise 1.20

Let $A_1 \supset A_2 \supset ... \supset A_n$. This is equivalent to saying $(A_1 - A_1 = \phi) \subset (A_1 - A_2) \subset (A_1 - A_3)... \subset (A_1 - A_n)$ From the previous result, we have $\lim_{n\to\infty} \mu(A_1 - A_n) = \mu(\bigcup_{n=1}^{n=\infty} (A_1 - A_n)) = \mu(A_1 - \bigcap_{n=1}^{n=\infty} A_n)$ where we have used De Morgan's Law in the last step. We have already proved previously, the property of finite additivity of a measure. Therefore we have $\mu(A_1) - \lim_{n\to\infty} \mu(A_n) = \mu(A_1) - \mu(\bigcap_{n=1}^{n=\infty} A_n)$. Since $\mu(A_1) < \infty$, we can cancel it out from both sides to get the result.

## Exercise 2.10

To prove this result, we note that countable subadditivity of an outer-measure $\Rightarrow$ finite subadditivity. This can be seen by taking $A_i = \phi$ for $i > N$. Since $\mu^*(\phi) = 0$, we have $\mu^*(\bigcup_{i=1}^{i=N} A_i) \leq \sum_{i=1}^{i=N} \mu^*(A_i)$ which follows from the definition of the outer-measure.

Now, we can write $B = (B \cap E) \cup (B \cap E^c)$. Therefore, using finite sub-additivity, we have $\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$. Since the inequality in the other direction is already given, we can replace the inequality with an equality.

## Exercise 2.14

Let $\mathcal{A} = \{A : A$ is a countable union of intervals of the form $(a, b], (-\infty, b]$ and $(a, \infty)\}$. We first show that $\sigma(\mathcal{A}) \subset \sigma(\mathcal{O})$. To see this, let $A \in \sigma(\mathcal{A})$. We can write $(a, b] = \bigcap_{n=1}^{n=\infty}(a, b - 1/n), (-\infty, b] = \bigcap_{n=1}^{n=\infty}(-\infty, b - 1/n)$. Thus $A$ can be written as a countable union of intervals of the form $\bigcap_{n=1}^{n=\infty}(a, b - 1/n), \bigcap_{n=1}^{n=\infty}(-\infty, b-1/n), (a, \infty)$. By the property of a $\sigma$- algebras, each of these terms, being countable intersections of open intervals, belong to $\sigma(\mathcal{O})$. Thus the countable union of these terms also belongs to $\sigma(\mathcal{O})$. Thus $\sigma(\mathcal{A}) \subset \sigma(\mathcal{O})$.
Now we show that $\sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$. To see this, let $A \in \sigma(\mathcal{O})$. Thus $A$ is an open interval. Let $A = (a, b)$. We can write $A = (a, b) = \bigcup_{n=1}^{n=\infty}(a, b-1/n]$. Similarly, any interval of the form $(-\infty, b)$ can be written as $\bigcup_{n=1}^{n=\infty}(-\infty, b-1/n]$ and any interval of the form $(a, \infty)$ can be written as $\bigcup_{n=1}^{n=\infty}[a-1/n, \infty)$. Note that each of the terms is a countable union of sets that $\in \mathcal{A}$ which $\Rightarrow$ that they $\in \sigma(\mathcal{A})$. Thus any countable union of open sets also $\in \sigma(\mathcal{A})$. Thus $\sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$.
We thus have $\sigma(\mathcal{A}) = \sigma(\mathcal{O})$. By Caratheodory's Theorem, $\mathcal{B}(\mathbb{R}) \subset \mathcal{M}$.

## Exercise 3.1

Let $X \subset \mathbb{R}$ be a countable set. Let $x_1, x_2, x_3...$ be the elements of $X$. For every $\epsilon > 0$, define, $A_n = (x_n - \frac{\epsilon}{2^{n+2}}, x_n + \frac{\epsilon}{2^{n+2}}) \forall n \in \mathbb{N}$. Let $\mu$ denote the Lebesgue Measure. Therefore $\mu(\bigcup_{n=1}^{n=\infty} A_n) = \sum_{n=1}^{n=\infty} \frac{\epsilon}{2^{n+1}}$. Summing the terms of the Geometric Progression on the RHS, we get $\mu(\bigcup_{n=1}^{n=\infty})A_n = \epsilon/2$. Since $\epsilon$ is arbitrary, we get $\mu(\bigcup_{n=1}^{n=\infty})A_n = 0$. Now each $x_n \in X$ also implies $x_n \in A_n$ as $A_n$ has been defined in a manner that includes $x_n$. Thus $X \subset \bigcup_{n=1}^{n=\infty} A_n$. By the monotonicity property, $\mu(X) \leq \mu(\bigcup_{n=1}^{n=\infty} A_n) = 0$. Thus $\mu(X) = 0$ since the range of $\mu$ is non-negative.

## Exercise 3.4

We show that the following conditions are equivalent:

1. $\{x \in X : f(x) < a\} \in \mathcal{M}$
2. $\{x \in X : f(x) \geq a\} \in \mathcal{M}$
3. $\{x \in X : f(x) > a\} \in \mathcal{M}$
4. $\{x \in X : f(x) \leq a\} \in \mathcal{M}$

(1) $\implies$ (2): Suppose $\{x \in X : f(x) < a\} \in \mathcal{M}$. Observe that $f^{-1}([a, \infty)) = (f^{-1}(-\infty, a))^c$. $\mathcal{M}$ is closed under complements, therefore $f^{-1}([a, \infty)) \in \mathcal{M}$.

(2) $\implies$ (3): Suppose $\{x \in X : f(x) \geq a\} \in \mathcal{M}$. Observe that $f^{-1}((a, \infty)) = \cap_{n=1}^{\infty} f^{-1}([a - \frac{1}{n}, \infty))$. By assumption, each of the sets in this intersection is in $\mathcal{M}$. $\mathcal{M}$ is closed under countable intersections. Therefore, $f^{-1}(a, \infty) \in \mathcal{M}$.

(3) $\implies$ (4): Suppose $\{x \in X : f(x) > a\} \in \mathcal{M}$. Observe that $f^{-1}((-\infty, a]) = (f^{-1}(a, \infty))^c$. $\mathcal{M}$ is closed under complements, therefore $f^{-1}((-\infty, a]) \in \mathcal{M}$.

(4) $\implies$ (1): Suppose $\{x \in X : f(x) \leq a\} \in \mathcal{M}$. Observe that $f^{-1}((-\infty, a)) = \cap_{n=1}^{\infty} f^{-1}((-\infty, a + \frac{1}{n}))$. By assumption, each of the sets in this intersection is in $\mathcal{M}$. $\mathcal{M}$ is closed under countable intersections. Therefore, $f^{-1}((a, \infty)) \in \mathcal{M}$.

# Exercise 3.7

Suppose $f$ and $g$ are measurable functions on $(X, \mathcal{M})$. Then the following are measurable:

1. $f + g$
2. $f \cdot g$
3. $\max(f, g)$
4. $\min(f, g)$
5. $|f|$

We can prove (3), (4), and (5) directly from the definition of measurable functions and use results from Exercise 3.4 to rewrite the condition for measurability in equivalent forms.

1. Consider $F(f(x) + g(x)) = f(x) + g(x)$. Then $F$ is continuous and by part 4 of Theorem 3.6, measurable. Therefore, $f + g$ is measurable.

2. Consdier $F(f(x) + g(x)) = f(x)g(x)$. Then $F$ is continuous and by part 4 of Theorem 3.6, measurable. Therefore, $f \cdot g$ is measurable.

3. Because $f$ and $g$ are measurable functions on $(X, \mathcal{M})$, we have that for all $a \in \mathbb{R}$, $\{x \in X : f(x) < a\} \in \mathcal{M}$ and $\{x \in X : g(x) < a\} \in \mathcal{M}$. Therefore, it follows that $\{x \in X : \max(f(x), g(x)) < a\} = \{x \in X : f(x) < a\} \cap \{x \in X : g(x) < a\}$. $\mathcal{M}$ is closed under countable intersections, therefore, $\{x \in X : \max(f(x), g(x)) < a\} \in \mathcal{M}$, so that $\max(f(x), g(x))$ is measurable.

4. The proof that $\min(f, g)$ is measurable is analogous to the proof of (3). The key observation here is that $\{x \in X : \min(f(x), g(x)) > a\} = \{x \in X : f(x) > a\} \cap \{x \in X : g(x) > a\}$. $\mathcal{M}$ is closed under countable intersections, therefore, $\{x \in X : \min(f(x), g(x)) > a\} \in \mathcal{M}$, so that $\min(f(x), g(x))$ is measurable.

5. Observe that $\{x \in X : |f(x)| > a\} = \{x \in X : f(x) < -a\} \cup \{x \in X : f(x) > a\}$. Both of these sets are in $\mathcal{M}$. $\mathcal{M}$ is closed under countable unions, therefore, $\{x \in X : |f(x)| > a\} \in \mathcal{M}$, so that $|f(x)|$ is measurable.

## Exercise 3.14

Let $f$ be bounded, and fix $\epsilon > 0$. Then, there exists an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in X$. Therefore, $x \in E_i^M$ for some $i$ and all $x \in X$. Observe that there is an $N \in \mathbb{R}$ and $N \geq M$ such that $\frac{1}{2^N} < \epsilon$. Therefore, for all $x \in X$ and $n \geq N$, $||s_n(x) - f(x)|| < \epsilon$ .Therefore, the convergence in part (1) of Theorem 3.13 is uniform.

## Exercise 4.13

To show that $f \in \mathscr{L}^1(\mu, E)$, we must show that both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite.

Recall that $||f|| = f^+ + f^-$. Also note that $0 \leq f^+$ and $0 \leq f^-$ by definition. Because $||f|| < M$ on $E$, then $0 \leq f^+ < M$ and $0 \leq f^- < M$ on $E$.

Then, by Proposition 4.5, because $\mu(E) < \infty$, we have that,

$\int_E f^+ d\mu < M\mu(E) < \infty$ and $\int_E f^- d\mu < M\mu(E) < \infty$

Therefore, both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite. Then by definition, $f \in \mathscr{L}^1(\mu, E)$.

## Exercise 4.14

We prove the contrapositive of this statement. Suppose there exists a measurable set $\hat{E} \subset E$ such that $f$ is infinite on $\hat{E}$. Here, we assume that $f$ reaches positive infinity (without loss of generality, the proof for negative infinity or mixed between positive and negative infinity is analogous). It follows that,

$$\infty = \int_{\hat{E}} f d\mu \leq \int_E f d\mu \leq \int_E ||f|| d\mu \tag{1}$$

The first inequality is proved in 4.16, below. However, this implies that $f \notin \mathscr{L}^1(\mu, E)$.

## Exercise 4.15

Let $f, g \in \mathscr{L}^1(\mu, E)$. Define the set of simple functions $B(f) = \{s : 0 \leq s \leq f, s \text{ simple, measurable}\}$. Let $f \leq g$. If follows that $f^+ \leq g^+$ and $f^- \geq g^-$. Then following a similar proof to Proposition 4.7, we have that $B(f^+) \subset B(g^+)$ and $B(g^-) \subset B(f^-)$. These two relationships imply that $\int_E f^+ d\mu \leq \int_E g^+ d\mu$ and $\int_E f^- d\mu \geq \int_E g^- d\mu$. Then by the definition of the Lebesgue integral, we observe that,

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu \leq \int_E g^+ d\mu - \int_E g^- d\mu = \int_E g d\mu \tag{2}$$

Therefore, we have that,

$$\int_E f d\mu \leq \int_E g d\mu \tag{3}$$

## Exercise 4.16

Following Definition 4.1, fix a simple function $s(x) = \sum_{i=1}^{N} c_i \chi_{E_i}$, where $E_i \in \mathcal{M}$. Let $A \subset E \in \mathcal{M}$. Then, by the monotonicity of measures, we have that $\mu(A \cap E_i) \leq \mu(E \cap E_i)$ for all $i$. Therefore, combining this result with Definition 4.1, we have that,

$$\int_A s d\mu = \sum_{i=1}^{N} c_i \mu(A \cap E_i) \leq \sum_{i=1}^{N} c_i \mu(E \cap E_i) = \int_E s d\mu \tag{4}$$

Now, by Definition 4.2, we have that,

$$\int_A f d\mu = \sup\{\int_A s d\mu : 0 \leq s \leq f, s \text{ simple, measurable}\}$$

and

$$\int_E f d\mu = \sup\{\int_E s d\mu : 0 \leq s \leq f, s \text{ simple, measurable}\}$$

Now because our choice of $s$ was arbitrary, we have by Equation (4) that,

$$\int_A f d\mu \leq \int_E f d\mu \tag{5}$$

Because $f \in \mathscr{L}^1(\mu, E)$, by definition we have that $\int_E ||f|| d\mu < \infty$. Therefore, $\int_E f d\mu < \infty$. Finally, it follows that $\int_A f d\mu < \infty$, which in turn implies $\int_A f^+ d\mu < \infty$ and $\int_A f^- d\mu < \infty$, so that $f \in \mathscr{L}^1(\mu, A)$.

## Exercise 4.21

Let $A, B \in \mathcal{M}$, $B \subset A$, $\mu(A - B) = 0$, and $f \in \mathscr{L}^1$. Then, by Proposition 4.6. we have that,

$$\int_{A-B} f d\mu = 0. \tag{6}$$

Recall that $f^+$ and $f^-$ are non-negative $\mathcal{M}$-measurable functions because $f \in \mathscr{L}^1$. By Theorem 4.19, we have that $\mu_1(A) = \int_A f^+ d\mu$ and $\mu_2(A) = \int_A f^- d\mu$ are measures on $\mathcal{M}$. Therefore, by the definition of the Lesbesgue integral,

$$\int_A f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu = \mu_1(A) - \mu_2(A) \tag{7}$$

Now, consider the disjoint union $A = (A - B) \cup B$. Because both $\mu_1(A)$ and $\mu_2(A)$ are measures, we have that $\mu_i(A) = \mu_i(A - B) + \mu_i(B)$ for $i = 1, 2$, because measures are additively separable on disjoint sets. Therefore, we have that $\mu_i(A) = \mu_i(B)$ for $i = 1, 2$ because $\mu(A - B) = 0$. Therefore,

$$\int_A f d\mu = \mu_1(B) - \mu_2(B) = \int_B f d\mu \tag{8}$$

This result clearly implies that

$$\int_A f d\mu \leq \int_B f d\mu \tag{9}$$