# Maths Problem Set- Spectral Theory

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**Q 4.2**[1] The eigenvalue of this linear differential operator $D[p](x) = p'(x)$ is 0.
Also, the eigenspace is $\sum_\lambda(D) = \{a + bx + cx^2 \in V | b = c = 0\}$
Algebraic and geometric multiplicities of $D$ are the dimension of $\sum_\lambda(D)$, which is 1
and the algebraic multiplicity of $\lambda_i = 0$ is 1.

**Ex 4.4** *Proof of (i)* Note that by the definition of eigenvalues, eigenvalues in a
2 x 2 matrix satisfy the following;

$$p(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Using quadratic formula,

$$(a + d)^2 - 4(ad - bc) = (a - d)^2 + 4bc$$

Now $A^H$ is Hermitian i.e $A^H = A$. This implies that $a$ and $d$ are real numbers, and the
multiplication of $b = \bar{c}$ and $c = \bar{b}$ results in positive number. Thus, $(a-d)^2 + 4bc > 0$,
implying that the solutions of characteristic equations are all real.
*Proof of (ii)*
If we find the quadratic formula of this 2nd order polynomial equation, then

$$D = (a - d)^2 + 4bc = -(a_1 - d_1)^2 - 4(abs(b)^2) < 0$$

## Q. 4.6

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*Proof.* Let $A$ be an upper triangular matrix. Consider $det(\lambda I - A) = 0$. This results in the following equation;

$$\prod_{i=1}^{n}(\lambda - a_i i)$$

, where $a_{ii}$ is the $i$th diagonal entry. This expression equals to zero, iff $\lambda = a_{ii}$ for some i. Thus, the diagonal entries of the matrix are the eigenvalues.

**Q. 4.8**

*Proof of (i).* Set the following matrix;

$$\begin{bmatrix} sin(t_1) & cos(t_1) & sin(2t_1) & cos(2t_1) \\ sin(t_2) & cos(t_2) & sin(2t_2) & cos(2t_2) \\ sin(t_3) & cos(t_3) & sin(2t_3) & cos(2t_3) \\ sin(t_4) & cos(t_4) & sin(2t_4) & cos(2t_4) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

with $t_1 = 0, t_2 = \pi/2, t_3 = \pi, t_4 = 3/2\pi$. This results in $c_1 = c_2 = c_3 = c_4 = 0$.

*Proof of (ii).* Let $D[p](x) := p'(x)$. By calculation, the matrix that represents this differential operator is the following;

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

*Answer of (iii).* Let $V_1 = span(\{sin(x), cos(x)\})$ and $V_1 = span(\{sin(2x), cos(2x)\})$.

**Q 4.13**

*Answer.*

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 0.7454 & 0.7454 \\ -0.4714 & 0.9428 \end{bmatrix}$$

**Q 4.15**

*Proof.* Due to the assumption that $A$ is semi-simple, $A$ is diagonalizable, i.e. $\exists P \ s.t. P^{-1}AP = D$, where $D$ is diagonal matrix. Then, $A^k = PD^kP^{-1}$. Then,

$$
\begin{aligned}
f(A) &= a_0 I + a_1 A + ... + a_n A^n \\
&= a_0 PIP^{-1} + a_1 PDP^{-1} + ... + a_n PD^n P^{-1} \\
&= P(a_0 + I + a_1 D + ... + a_n D^n)P^{-1}
\end{aligned}
$$

Thus, the eigenvalues of $f(A)$ are $(f(\lambda_i))_{i=1}^n$

## Q 4.16

*Proof of (i).* The Markov Chain which this matrix $A^T$ represents is irreducible and aperiodic. Thus, there exists a distribution $\pi$ such that $A\pi = \pi$. If we solve this, then $\pi = (2/3, 1/3)$, which is exactly the same with the first and the second columns of $\lim_{n\to\infty} A^n$

*Answer of (ii).* Yes, $\| \lim A^n \|_\infty = 4/3$, and $\| \lim A^n \|_F = \sqrt{10}/3$

*Answer of (iii).* By the Theorem 4.3.12, $f(\lambda_1) = 3 + 5 * \lambda_1 + \lambda_1^3 = 9$, and $f(\lambda_2) = 3 + 5 * \lambda_2 + \lambda_2^3 = 5.0640$

## Q 4.18

*Proof.* Note that

$$
det(A^T - \lambda I) = det((A - \lambda I)^T) = det(A - \lambda I) = 0
$$

Thus, $A^T x = \lambda x$, and with transposition, $x^T A = \lambda x^T$

## Q 4.20

*Proof.* Note that $A^H = A$. Using the notations in the Definition 4.4.1,

$$
B^H = (U^H A U)^H = U^H A^H U = U^H A U = B
$$

## Q. 4.24

*Proof.* Due to the assumption that the matrix $A$ is hermitian, all the eigenvalues of

$A$ are real, because;

$$v^H A v = \lambda v^H v$$
$$\lambda v^H v = v^H A v = (v^H A v)^H = \bar{\lambda} v^H v$$

Thus, $\lambda = \bar{\lambda}$, meaning that the eigenvalues are real. Also, due to this fact, the matrix $A$ is positive semi-definite. Thus, by Proposition 4.5.6, the eigenvectors of $A$ corresponding to distinct eigenvalues are orthogonal. Then, any vector $x$ can be expressed in the following;

$$x = \sum_{j=1}^{n} c_j v_j = Vc$$

, where $v_j$s are orthonormal eigenvectors.

This results in the following;

$$\rho(x) = \frac{x^H A x}{x^H x}$$
$$= \frac{c^H V^H A V c}{c^H V^H V c}$$
$$= \frac{c^H \mathbf{A} c}{c^H c}$$

where $A = V \mathbf{A} V^H$, and $\mathbf{A}$ is diagonal matrix. Thus,

$$\rho(x) = \frac{\lambda_1 |c_1|^2 + \cdots \lambda_n |c_n|^2}{|c_1|^2 + \cdots |c_n|^2 +}$$

Thus, $\rho(x)$ are real with hermitian matrix, $A$. We can do the same proof for Skewed matrix. **Q. 4.25**

*Proof of (i).* Note that the following holds due to the assumption that $[x_1, ..., x_n]$ are orthonormal vectors;

$$x_i^H x_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Thus,

$$(x_1 x_1^H + \cdots + x_n x_n^H) x_j = x_j = I x_j$$

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for all $j$. Thus, the statement holds.

*Proof of (ii).*    Note that by the Thoerem 4.4.14, $A$ is orthonormally diagonaliz-able, i,e.

$$A = UTU^H$$

with $T$ being diagonal, and $U$ being orhonormal. This results in;

$$A = \sum_{i=1}^{n} t_{ii} u_i u_i^H$$

## Q. 4.27

Note that the positive-definite matrix $A$ satisfies the following;

$$\forall x \neq 0, x^H A x > 0$$

If we feed the standard basis vector to $x$, then

$$e_i^H A e_i = a_{ii} > 0$$

Thus, all the diagonal entries are real and positive.   Q.E.D

## Q. 4.28

*Proof.* Note that by the same logic of Ex. 4.27, one can show that the diagonal entries of any semi-positive definite matrix are non-negative. For the first inequality, we need to show that $AB$ is a semi-positive definite matrix.

Note that $\forall x \neq 0, x^T A x, x^T B x \geq 0$. Then,

$$(x^T A x)(x^T B x) = (x^T A)(x x^T)(B x) \geq 0$$

Note that $x x^T$ is positive scalar when $x \neq 0$. Thus, $x^T A B x \geq 0$, implying that $AB$ is positive semi-definite. This leads to the result that

$$tr(AB) \geq 0$$

Lastly, by using Cauchy-Scwartz inequality, we have

$$tr(AB) \leq tr(A)tr(B)$$

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**Ex. 4.31**

*Proof of (i).* Note that we want to show $\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}^H\mathbf{A})}$
We can first simply prove when $\mathbf{P}$ is hermitian,

$$\lambda_{\max} = \max_{\|\mathbf{x}\|_2=1} \mathbf{x}^H\mathbf{P}\mathbf{x}$$

That's because when $\mathbf{P}$ is Hermitian, there exists one and only one unitary matrix $\mathbf{U}$ that can diagonalize $\mathbf{P}$ as $\mathbf{U^HPU} = \mathbf{D}$ (so $\mathbf{P} = \mathbf{UDU^H}$), where $\mathbf{D}$ is a diagonal matrix with eigenvalues of $\mathbf{P}$ on the diagonal, and the columns of $\mathbf{U}$ are the corresponding eigenvectors. Let $\mathbf{y} = \mathbf{U^Hx}$ and substitute $\mathbf{x} = \mathbf{Uy}$ to the optimization problem, we obtain

$$\max_{\|\mathbf{x}\|_2=1} \mathbf{x}^H\mathbf{P}\mathbf{x} = \max_{\|\mathbf{y}\|_2=1} \mathbf{y}^H\mathbf{D}\mathbf{y} = \max_{\|\mathbf{y}\|_2=1} \sum_{i=1}^{n} \lambda_i|y_i|^2 \leq \lambda_{\max} \max_{\|\mathbf{y}\|_2=1} \sum_{i=1}^{n} |y_i|^2 = \lambda_{\max}$$

Thus, just by choosing $\mathbf{x}$ as the corresponding eigenvector to the eigenvalue $\lambda_{\mathbf{max}}$, $\max_{\|\mathbf{x}\|_2=1} \mathbf{x}^H\mathbf{P}\mathbf{x} = \lambda_{\max}$. This proves $\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}^H\mathbf{A})}$

*Proof of (ii).* Note that if the matrix $A$ is invertible, then;

$$A^{-1} = (U\Sigma V^H)^{-1} = (V^H)^{-1}\Sigma^{-1}U^{-1} = \hat{U}\Sigma^{-1}\hat{V}^H$$

Note that $\hat{U}$ and $\hat{V}$ are all trivially orthornormal. Also, inverted diagonal matrix $\Sigma^{-1}$ takes the inverse values of its diagonal entries on its diagonal line. Thus, $\|A^{-1}\|_2 = \sigma_n^{-1}$

*Proof of (iii).*

Note that according to the property of singular values(positive and real), the following holds;
$$\Sigma = \Sigma^T = \Sigma^H$$

Thus, by *(i)*, $\|A\|_2 = \|A^H\|_2 = \|A^T\|_2$
Also,
$$A^H A = V\Sigma^H U^H U\Sigma V^H = V\Sigma^H \Sigma V^H = V\Sigma^2 V^H$$

Thus, $\|A^H A\|_2 = \|A\|_2^2$

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*Proof of (iv).*

Note that

$$UAV = UU_1 \Sigma V_1^H V = \hat{U} \Sigma \hat{V}^H$$

Note that $\hat{U}$ and $\hat{V}$ are orthonormal. For example,

$$\hat{U}^H \hat{U} = (UU_1)^H UU_1 = U_1^H U^H UU_1 = U_1^H U_1 = I$$

The same argument can be used to prove that $\hat{V}$ is orthonormal. Thus, $\|UAV\|_2 = \|A\|_2$. **Q. 4.32**

We first prove $(ii)$, and then use the result to prove $(i)$.

*Proof of (ii).*

$$\begin{aligned}
\|A\|_F^2 &= tr(AA^H) = tr(U\Sigma V^H V \Sigma^H U^H) \\
&= tr(U\Sigma \Sigma^H U^H) \\
&= tr(\Sigma \Sigma^H U^H U) \\
&= tr(\Sigma \Sigma^H) \\
&= \sigma_1^2 + ... + \sigma_n^2
\end{aligned}$$

*Proof of (i).*

$$\begin{aligned}
\|U_1 AV_1\|_F^2 &= tr((U_1 AV_1)(U_1 AV_1)^H) \\
&= tr(U_1 AV_1 V_1^H A^H U_1^H) \\
&= tr(U_1 AA^H U_1^H) \\
&= tr(AA^H U_1^H U_1) \\
&= tr(AA^H) \\
&= tr(\Sigma \Sigma^H) \\
&= \sigma_1^2 + ... + \sigma_n^2
\end{aligned}$$

Thus, $\|A\|_F = \|U_1 AV_1^H\|$  Q.E.D

**Q. 4.33**

*Proof.*

$$\begin{aligned}|y^H A x| &= |y^H (U\Sigma V^H)x| \\ &= |y^H (\sum_{i=1}^{r} \sigma_i u_i v_i^H|)x \\ &\leq \sigma_{max} |\sum_{i=1}^{r} y^H u_i v_i^H x|\end{aligned}$$

Note that $\|y^H u_i v_i^H\|_2 \leq \|y^H\|\|u_i\|\|v_i^H\| \leq 1 \times 1 \times 1 = 1$. Thus,

$$\sigma_{max}|\sum_{i=1}^{r} y^H u_i v_i^H x| \leq \sigma_{max}|\sum_{i=1}^{r} x_u| \leq \sigma_{max}$$

We can attain equality when $\|y^H u_i v_i^H\|_2 = 1$, and $\sum_{i=1}^{r}(x_i)^2 = 1$, which is possibly chosen due to the assumption that $x$ and $y$ are free variable of supremum, and $U$ and $V$, which are the matrices of SVD of A and orthonormal, are arbitrary. **Q. 4.36**

*Answer.* Try any non-symmetric matrix. For example,

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

gives $\lambda_1 = 1, \lambda_2 = 5$ as eigenvalues, but it gives $\sigma_1 = 0.9262, \sigma_2 = 5.3983$ as singular values.

**Q. 4.38**

*Proof of (i).*

$$\begin{aligned}AA^+A &= (U\Sigma V^H)(V\Sigma^{-1}U^H)(U\Sigma V^H) \\ &= U\Sigma\Sigma^{-1}U^H U\Sigma V^H \\ &= U\Sigma V^H \\ &= A\end{aligned}$$

*Proof of (ii).*

$$A^+AA^+ = (V\Sigma^{-1}U^H)(U\Sigma V^H)(V\Sigma^{-1}U^H)$$
$$= U\Sigma^{-1}V^H$$
$$= A^+$$

*Proof of (iii).*

$$(AA^+)^H = (A^+)^H A^H$$
$$= (V\Sigma^{-1}U^H)^H(U\Sigma V^H)^H$$
$$= UU^H$$
$$= AA^+$$

*Proof of (v).*

We use the facts above and the fact that if $X = YZ$, then $\mathcal{R}(X) \subseteq \mathcal{R}(Y)$. We need only to show that the matrices $AA^+$ and $A^+A$ are Hermitian, idempotent, and their ranges are equal to the subspaces on which they are supposed to project.

Both $AA^+$ and $A^+A$ are obviously Hermitian; see $(iii)$ and $(iv)$. In addition, $(i)$ and $(ii)$ imply that they are idempotent. It remains to show that $\mathcal{R}(AA+) = R(A)$ and $\mathcal{R}(A^+A) = \mathcal{R}(A^H)$. Clearly, $\mathcal{R}(AA^+) \subseteq \mathcal{R}(A)$; $\mathcal{R}(A) \subseteq \mathcal{R}(AA^+)$ follows from $(i)$. From $(iv)$, we have $A^+A = A^H(A^+)^H$, so $\mathcal{R}(A^+A) \subseteq \mathcal{R}(A^H)$. From $(i)$ and $(iv)$, $A^H = A^+AA^H$, so $\mathcal{R}(A^H) \subseteq \mathcal{R}(A^+A)$.