Maths Problem Set # 5

Keshav Choudhary [1]

Problem 8.1 See Jupyter Notebook for the plot and solution.

Problem 8.2 See Jupyter Notebook for the plot and solution.

Problem 8.3 Let x be the quantity of production of GI Bard Soldiers and let y be the quantity of production of Joey dolls. The revenues are given by $12x + 10y$, the raw material cost is given by $5x+3y$ and the overhead costs are given by $F+3x+4y$ where F is the overhead cost when there is no production.

The finishing labour requirement is $15x + 10y$ minutes and molding labour requirement is $2x + 2y$ minutes.

The optimization problem is thus:

$$
\begin{aligned}
\max_{x,y}\ & 4x + 3y \\
\text{subject to: } & 3x + 2y \leq 360 \\
& x + y \leq 150 \\
& y \leq 200
\end{aligned}
$$

Problem 8.4 The optimization problem is as follows-

$$
\begin{aligned}
&\min_{x_{i,j}} \big(5x_{AD} + 2x_{AB} + 2x_{BD} + 7x_{BE} + 9x_{BF} + 5x_{BC} \\
&\quad 2x_{CF} + 4x_{DE} + 3x_{EF}\big) \\
&\text{subject to:} \\
&\quad x_{AD} + x_{AB} = 10 \\
&\quad x_{BC} + x_{BD} + x_{BE} + x_{BF} - x_{AB} = 1 \\
&\quad x_{CF} - x_{BC} = -2 \\
&\quad x_{DE} - x_{AD} - x_{BD} = -3 \\
&\quad x_{EF} - x_{BE} - x_{BD} = 4 \\
&\quad x_{CF} + x_{BF} + x_{EF} = 10 \\
&\quad 0 \leq x_{i,j} \leq 6 \text{ where } i, j \text{ are nodes}
\end{aligned}
$$

Problem 8.5

(i) The initial dictionary after adding in 3 slack variables x_3, x_4, x_5 is:

$$
\begin{aligned}
\zeta_1 &= 3x_1 + x_2 \\
\hline
x_3 &= 15 - x_1 - 3x_2 \\
x_4 &= 18 - 2x_1 - 3x_2 \\
x_5 &= 4 - x_1 + x_2
\end{aligned}
$$

[1] I have worked with Navneeraj Sharma and Shekhar Kumar on this problem set

We choose x_1 as the entering variable and x_5 as the leaving variable. The new dictionary becomes:

$$\zeta_2 = 12 + 4x_2 - 3x_5$$

$$\begin{aligned} x_1 &= 4 + x_2 - x_5 \\ x_3 &= 11 - 4x_2 + x_5 \\ x_4 &= 10 - 5x_2 + 2x_5 \end{aligned}$$

We now choose x_2 as the entering variable and x_4 as the leaving variable. The new dictionary becomes:

$$\zeta_3 = 20 - (4/5)x_4 - (7/5)x_5$$

$$\begin{aligned} x_1 &= 6 - (1/5)x_4 - (3/5)x_5 \\ x_2 &= 2 - (1/5)x_4 + (2/5)x_5 \\ x_3 &= 3 + (4/5)x_4 - (13/5)x_5 \end{aligned}$$

Since both x_4, x_5 now appear with negative signs in the objective function, this is the optimum. The values are: $x_1 = 6, x_2 = 2$ and the value of the objective function is 20. This matches the answer in the Jupyter Notebook.

(ii) The initial dictionary after adding in 3 slack variables x_3, x_4, x_5 is:

$$\zeta_1 = 4x_1 + 6x_2$$

$$\begin{aligned} x_3 &= 11 + x_1 - x_2 \\ x_4 &= 27 - x_1 - x_2 \\ x_5 &= 90 - 2x_1 - 5x_2 \end{aligned}$$

We choose x_1 as the entering variable and x_4 as the leaving variable. The new dictionary becomes:

$$\zeta_2 = 108 + 2x_2 - 4x_4$$

$$\begin{aligned} x_1 &= 27 - x_2 - x_4 \\ x_3 &= 38 - 2x_2 - x_4 \\ x_5 &= 36 - 3x_2 + 2x_4 \end{aligned}$$

We now choose x_2 as the entering variable and x_5 as the leaving variable. The new dcitionary becomes:

$$\zeta_3 = 132 - (8/3)x_4 - (2/3)x_5$$

$$\begin{aligned} x_1 &= 15 - (5/3)x_4 + (1/3)x_5 \\ x_2 &= 12 + (2/3)x_4 - (1/3)x_5 \\ x_3 &= 14 - 3x_4 + (2/3)x_5 \end{aligned}$$

All the variables now appear in the objective function with a negative sign. Hence the present choice is optimal. This occurs at $x = 15, y = 12$ and the value of the objective function is 132.

Problem 8.6 After adding 3 slack variables w_1, w_2, w_3 , the initial dictionary is:

$$\zeta_1 = 4x + 3y$$

$$\begin{aligned} w_1 &= 360 - 3x - 2y \\ w_2 &= 150 - x - y \\ w_3 &= 200 - y \end{aligned}$$

We choose as x as the entering variable and w_1 as the leaving variable. The new dictionary becomes:

$$\zeta_2 = 480 + (1/3)y - (4/3)w_1$$

$$\begin{aligned} x &= 120 - (2/3)y - (1/3)w_1 \\ w_2 &= 30 - (1/3)y + (1/3)w_1 \\ w_3 &= 200 - y \end{aligned}$$

Next, we choose y as the entering variable and w_2 as the leaving variable.

$$\zeta_3 = 510 - 10w_2 - (5/3)w_1$$

$$\begin{aligned} x &= 60 + 20w_2 + (1/3)w_1 \\ y &= 90 - 30w_2 - w_1 \\ w_3 &= 110 + 30w_2 + w_1 \end{aligned}$$

As all the terms in the objective function appear with a negative sign, we are at the optimum. The value of the objective function i.e profit is \$510 and $x = 60, y = 90$

Problem 8.7

1. The origin is not part of the feasible set. This can be seen from the Jupyter Notebook where the feasible set is plotted. We therefore set up an auxiliary problem first by subtracting x_0 from all the constraints. The dictionary for the auxiliary problem is:

$$\begin{aligned}
&\zeta_1 = -x_0 \\
\hline
&x_3 = -8 + 4x_1 + 2x_2 + x_0 \\
&x_4 = 6 + 2x_1 - 3x_2 + x_0 \\
&x_5 = 3 - x_1 + x_0
\end{aligned}$$

We pivot x_0 and x_1. The new dictionary becomes:

$$\begin{aligned}
&\zeta_2 = -x_0 \\
\hline
&x_1 = 2 - (1/2)x_2 + (1/4)x_3 - (1/4)x_0 \\
&x_4 = 10 - 4x_2 + (1/2)x_3 + (1/2)x_0 \\
&x_5 = 1 + (5/2)x_2 - (1/4)x_3 + (5/4)x_0
\end{aligned}$$

We can see that this dictionary is optimal as all points are feasible and the objective function is 0 Thus $x_1 = 2, x_2 = 0$, is a feasible point for the original problem. We can remove x_0 from the main problem and replace x_1 in terms of the non-basic variables. The new dictionary becomes:

$$\begin{aligned}
&\zeta_3 = 2 - (1/2)x_2 + 2x_2 + (1/4)x_3 \\
\hline
&x_1 = 2 - (1/2)x_2 + (1/4)x_3 \\
&x_4 = 10 - 4x_2 + (1/2)x_3 \\
&x_5 = 1 + (1/2)x_2 - (1/4)x_3
\end{aligned}$$

We pivot x_3, x_5. The new dictionary becomes:

$$\begin{aligned}
&\zeta_4 = 3 + 2x_2 - x_5 \\
\hline
&x_1 = 3 - x_5 \\
&x_3 = 4 + 2x_2 - 4x_5 \\
&x_4 = 12 - 3x_2 - 2x_5
\end{aligned}$$

We again pivot x_2, x_4 and obtain the dictionary:

$$
\begin{array}{l}
\zeta_5 = 11 - (2/3)x_4 - (7/3)x_5 \\
\hline
x_1 = 3 - x_5 \\
x_2 = 4 - (1/3)x_4 - (2/3)x_5 \\
x_3 = 12 - (2/3)x_4 - (16/3)x_5
\end{array}
$$

This dictionary is optimal as all the terms appear with a negative sign. The optimal values are $x_1 = 3, x_2 = 4$. This can also be confirmed from the diagram in the Jupyter Notebook.

2. The origin is not part of the feasible set as the third constraint appears with a negative sign. The auxiliary problem is:

$$
\begin{array}{l}
\zeta_1 = -x_0 \\
\hline
x_3 = 15 - 5x_1 - 3x_2 + x_0 \\
x_4 = 15 - 3x_1 - 5x_2 + x_0 \\
x_5 = -12 - 4x_1 + 3x_2 + x_0
\end{array}
$$

We pivot x_0, x_5 and get the new dictionary as:

$$
\begin{array}{l}
\zeta_2 = 12 + 4x_1 - 3x_2 + x_5 \\
\hline
x_0 = 12 + 4x_1 - 3x_2 + x_5 \\
x_3 = 27 - x_1 - 6x_2 + x_5 \\
x_4 = 27 + x_1 - 8x_2 + x_5
\end{array}
$$

Again, we pivot x_2, x_4 and obtain:

$$
\begin{array}{l}
\zeta_3 = -(15/8) - (29/8)x_1 - (3/8)x_4 - (5/8)x_5 \\
\hline
x_3 = (27/4) - (7/4)x_1 + (3/4)x_4 + (1/4)x_5 \\
x_2 = (27/8) + (1/8)x_1 - (1/8)x_4 + (1/8)x_5 \\
x_0 = (15/8) + (29/8)x_1 + (3/8)x_4 + (5/8)x_5
\end{array}
$$

This dictionary is optimal since all the coefficients in the objective function are negative. However, at this optimum, $x_0 \neq 0$. Hence the problem is infeasible.

3. After adding in the slack variables, the initial dictionary is :

$$\begin{array}{l} \zeta_1 = -3x_1 + x_2 \\ \hline x_3 = 4 - x_2 \\ x_4 = 6 + 2x_1 - 3x_2 \end{array}$$

We pivot, x_2, x_4 and obtain the following dictionary:

$$\begin{array}{l} \zeta_2 = 2 - (7/3)x_1 - (1/3)x_4 \\ \hline x_2 = 2 + (2/3)x_1 - (1/3)x_4 \\ x_3 = 2 - (2/3)x_1 + (1/3)x_3 \end{array}$$

This dictionary is optimal as all the coefficients in the objective function have negative signs. The optimal solution is $x_1 = 0, x_2 = 2$ and the value of the objective function is 2.

Problem 8.8

$$\begin{aligned} \max_{x,y,z} -3x + y + 3z \\ \text{subject to: } y \le 4 \\ -2x + 3y + 4z \le 10 \end{aligned}$$

Problem 8.9 As per proposition 8.3.1, if the coefficient of the variable in objective function is positive and the coefficients in the constraints are non-negative then the optimization problem will be unbounded.Consider the problem below

$$\begin{aligned} \max_{x,y,z} ax + 5y + 3z \\ \text{subject to: } mx - y - z \le 1 \\ 3y + 4z \le 1 \end{aligned}$$

If a, m are positive then the problem would be unbounded.

Problem 8.10 As seen in the problem 8.7(ii), if the constraints are such that their solution set is null then the optimization problem will be infeasible.

$$\begin{aligned}\max_{x,y,z} 2x + 9y + 3z \\ \text{subject to: } x + y + z \leq 1 \\ 3x + 2y + 5z \leq 4 \\ x - y - z \leq -10000\end{aligned}$$

The last constraint ensures that the intersection between the three constraints is null.Hence, the optimization is infeasible.

Problem 8.11 Section 8.3.2 provides the condition for the origin **0** vector to be infeasible. The intuition is that $Ax \preceq b$ has to be true for $x = 0$ and if vector **b** has any one component has a negative value then the origin becomes infeasible. Problem 8.7(i) is an example of such a case in 2-dimensions.

The dual problem for 8.18 can be an example for a 3-D case. I try to give another solution below:-

$$\begin{aligned}\max_{x,y,z} 2x + 9y + 3z \\ \text{subject to: } x + y + z \leq 1 \\ 3x + 2y + 5z \leq -1 \\ x - 2y - 3z \leq -4\end{aligned}$$

The last two constraints ensure that some components of **b** vector are negative. Hence the origin will be infeasible. As I have already solved 8.18 dual problem. See it as an example for auxiliary problem method.

Problem 8.12 The initial dictionary after adding in the slack variables is:

$$\begin{aligned}&\zeta_1 = 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \hline &x_5 = -0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\ &x_6 = -0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\ &x_7 = 1 - x_1\end{aligned}$$

Using Bland's rule, we pivot x_1, x_5. The new dictionary is:

$$\begin{aligned}&\zeta_2 = -27x_2 + x_3 - 44x_4 - 20x_5 \\ \hline &x_1 = 3x_2 + x_3 - 2x_4 - 2x_5 \\ &x_6 = 4x_2 + 2x_3 - 8x_4 + x_5 \\ &x_7 = 1 - 3x_2 - x_3 + 2x_4 + 2x_5\end{aligned}$$

We now pivot x_3, x_7 and obtain:

$$\begin{aligned}
&\zeta_3 = 1 - 30x_2 - 42x_4 - 18x_5 - x_7 \\
\hline
&x_1 = 1 - x_7 \\
&x_6 = 2 - 2x_2 - 4x_4 + 5x_5 - 2x_7 \\
&x_3 = 1 - 3x_2 + 2x_4 + 2x_5 - x_7
\end{aligned}$$

This dictionary is optimal as all the coefficients appear with negative sign in the objective function. The optimal points are $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0$ and the value of the objective function is 1.

Problem 8.15 Using the definitions of primal and dual problems where $\boldsymbol{x}, \boldsymbol{y}$ are feasible points of the primal and the dual respectively, we have

$$\begin{aligned}
\boldsymbol{A^T y} &\succeq \boldsymbol{c} \\
\Rightarrow \boldsymbol{x^T A^T y} &\geq \boldsymbol{x^T c} \\
\Rightarrow (\boldsymbol{Ax})^{\boldsymbol{T}} \boldsymbol{y} &\geq (\boldsymbol{x^T c}) \\
\Rightarrow \boldsymbol{b^T y} &\geq (\boldsymbol{Ax})^{\boldsymbol{T}} \boldsymbol{y} \geq \boldsymbol{x^T c} \\
\Rightarrow \boldsymbol{b^T y} &\geq \boldsymbol{x^T c} \\
&= \boldsymbol{c^T x} \\
\Rightarrow \boldsymbol{b^T y} &\geq \boldsymbol{c^T x}
\end{aligned}$$

Problem 8.17

Consider the primal problem

$$\begin{aligned}
&\max \mathbf{c^T x} \\
\text{subject to } &A\mathbf{x} \preceq \mathbf{b} \\
&\mathbf{x} \succeq 0
\end{aligned}$$

The dual of the problem is

$$\begin{aligned}
&\min \mathbf{b^T y} \\
\text{subject to } &A^T\mathbf{y} \succeq \mathbf{c} \\
&\mathbf{y} \succeq 0
\end{aligned}$$

The dual problem can be re-written as

$$\begin{aligned} &\max(-\mathbf{b^T y}) \\ \text{subject to } &-A^T\mathbf{y} \preceq \textbf{-c} \\ &\mathbf{y} \succeq 0 \end{aligned}$$

Let us rename

$$-b^T = r^T, -A^T = K, -c = p$$

The dual problem thus becomes-

$$\begin{aligned} &\max(\mathbf{r^T y}) \\ \text{subject to } &K\mathbf{y} \preceq \mathbf{p} \\ &\mathbf{y} \succeq 0 \end{aligned}$$

This is in the same form as a linear optimization problem. The dual of this problem is:

$$\begin{aligned} &\min \mathbf{p^T x} \\ \text{subject to } &K^T\mathbf{x} \succeq \mathbf{r} \\ &\mathbf{x} \succeq 0 \end{aligned}$$

On replacing the terms defined above we get:

$$\begin{aligned} &\max \mathbf{c^T x} \\ \text{subject to } &A\mathbf{x} \preceq \mathbf{b} \\ &\mathbf{x} \succeq 0 \end{aligned}$$

Which is nothing but the primal problem. Hence the dual of the dual is the primal problem.

Problem 8.18 We first solve the primal problem using the simplex method. The initial dictionary is

$$\zeta_1 = x_1 + x_2$$

$$\begin{aligned} x_3 &= 3 - 2x_1 - x_2 \\ x_4 &= 5 - x_1 - 3x_2 \\ x_5 &= 4 - 2x_1 - 3x_2 \end{aligned}$$

We pivot x_1, x_3 and obtain the dictionary:

$$\begin{aligned}
\zeta_2 &= 1.5 + 0.5x_2 - 0.5x_3 \\
\hline
x_1 &= 1.5 - 0.5x_2 - 0.5x_3 \\
x_4 &= 3.5 - 2.5x_2 + 0.5x_3 \\
x_5 &= 1 - 2x_2 + x_3
\end{aligned}$$

We now pivot x_2, x_5 and obtain:

$$\begin{aligned}
\zeta_3 &= 1.75 - 0.25x_3 - 0.25x_5 \\
\hline
x_1 &= 1.25 - 0.75x_3 + 0.25x_5 \\
x_4 &= 2.25 - 0.75x_3 + 1.25x_5 \\
x_2 &= 0.5 + 0.5x_3 - 0.5x_5
\end{aligned}$$

This dictionary is optimal as the coefficients of the variables in the objective function appear with a negative sign. The optimal values are $x_1 = 1.25, x_2 = 0.5$ and the objective function is 1.75.

The dual problem is:

$$\begin{aligned}
&\max_{y_1,y_2,y_3} (-3y_1 - 5y_2 - 4y_3) \\
\text{subject to } &- 2y_1 - y_2 - 2y_3 \leq -1 \\
&- y_1 - 3y_2 - 3y_3 \leq -1 \\
&y_1, y_2, y_3 \geq 0
\end{aligned}$$

The origin is not part of the feasible set for this problem as the R.H.S has a negative sign in both constraints. Therefore we set up the auxiliary problem and obtain the dictionary:

$$\begin{aligned}
\zeta_1 &= -y_0 \\
\hline
y_4 &= -1 + 2y_1 + y_2 + 2y_3 + y_0 \\
y_5 &= -1 + y_1 + 3y_2 + 3y_3 + y_0
\end{aligned}$$

We pivot y_0, y_4 and obtain the following dictionary:

$$\begin{aligned}
\zeta_2 &= -1 + y_1 + ey_2 + 3y_3 - y_5 \\
\hline
y_0 &= 1 - y_1 - 3y_2 - 3y_3 + y_5 \\
y_4 &= y_1 - 2y_2 - y_3 + y_5
\end{aligned}$$

We now pivot y_1, y_0 and obtain:

$$\begin{array}{l} \zeta_3 = -y_0 \\ \hline y_1 = 1 - 3y_2 - 3y_3 + y_5 - y_0 \\ y_4 = 1 - 5y_2 - 4y_3 + 2y_5 - y_0 \end{array}$$

This dictionary is optimal as the objective function is 0 at the minimum. Since y_1 is a non basic variable we write it in terms of other basic variables, remove y_0 and obtain the following dictionary for the main problem-

$$\begin{array}{l} \zeta_4 = -3 + 4y_2 + 5y_3 - 3y_5 \\ \hline y_1 = 1 - 3y_2 - 3y_3 + y_5 \\ y_4 = 1 - 5y_2 - 4y_3 + 2y_5 \end{array}$$

We pivot y_2, y_4 and obtain the following dictionary:

$$\begin{array}{l} \zeta_5 = -(11/5) + (9/5)y_3 - (4/5)y_4 - (7/5)y_5 \\ \hline y_1 = (2/5) - (3/5)y_3 + (3/5)y_4 - (1/5)y_5 \\ y_2 = (1/5) - (4/5)y_3 - (1/5)y_4 + (2/5)y_5 \end{array}$$

Finally, we pivot y_2, y_3 and obtain:

$$\begin{array}{l} \zeta_6 = -(7/4) - (9/4)y_2 - (5/4)y_4 - (1/2)y_5 \\ \hline y_1 = (1/4) + (3/4)y_2 + (3/4)y_4 - (1/2)y_2 \\ y_3 = (1/4) - (5/4)y_2 - (1/4)y_4 + (1/2)y_5 \end{array}$$

This is the optimal dictionary as all terms appear with negative signs. The value of the objective function is $-(-7/4) = 7/4 = 1.75$ at the optimum which is the same as the primal problem. Hence both are equivalent.