Maths Problem Set- Measure Theory

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Exercise 1.3

- Consider $A \in \mathcal{G}_1 \implies A$ open on $\mathbb{R} \implies A^c$ is either closed on $\mathbb{R}$ or semi-open on $\mathbb{R}$. So $A^c \notin \mathcal{G}_1$ as it is not a purely open interval. Hence $\mathcal{G}_1$ is not a σ- algebra nor an algebra.
- Consider $A_n \in \mathcal{G}_2, n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} A_n \notin \mathcal{G}_2$ since $\mathcal{G}_2$ contains only sets which are finite unions of intervals of the form $(a, b], (-\infty, b], (a, \infty)$. Thus $\mathcal{G}_2$ is not a σ- algebra. Now we check whether $\mathcal{G}_2$ is an algebra. It is clear that $\phi \in \mathcal{G}_2$. Now consider any interval of the form $(a, b]$. Then it's complement is of the form $(-\infty, a] \cup (b, \infty)$ which $\in \mathcal{G}_2$. Similarly for any interval of the form $(-\infty, b]$, its complement is of the form (b, ∞) which $\in \mathcal{G}_2$. Thus, for all $A \in \mathcal{G}_2$, $A^c \in \mathcal{G}_2$. Now consider $A_n \in \mathcal{G}_2$ for $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{N} A_n$ is also a finite union of disjoint intervals of the form $(-\infty, b], (a, b]$ and (a, ∞). Hence $\mathcal{G}_2$ is an algebra (but not a σ-algebra).
- Now consider $A_n \in \mathcal{G}_3, n \in \mathbb{N}$. The first two properties of an algebra hold in this case as they have already been proved above. Now consider $\bigcup_{n=1}^{\infty} A_n$ where $A_n \in \mathcal{G}_3$ for $n \in \mathbb{N}$. The countable union $\bigcup_{n=1}^{\infty} A_n \in \mathcal{G}_3$ as it contains countable unions of intervals of the form $(a, b], (-\infty, b]$ and (a, ∞). Thus $\mathcal{G}_3$ is a σ- algebra.

Exercise 1.7

Let $\mathcal{A}$ be any σ-algebra. By defnition of σ-algebra, $\phi \in \mathcal{A}$. Similarly, $X = \phi^c \in \mathcal{A}$ Thus $\{\phi, X\} \subset \mathcal{A}$. Now consider any $A \in \mathcal{A}$. Since $\mathcal{A}$ is a σ-algebra on X, $A \subset X \Rightarrow A \in \mathcal{P}(X)$. Thus $\mathcal{A} \subset \mathcal{P}(X)$. Thus $\{\phi, X\} \subset A \subset \mathcal{P}(X)$

Exercise 1.10

Since $\{S_\alpha\}$ is a family of σ- algebras, then $\phi \in S_\alpha \forall \alpha \Rightarrow \phi \in \bigcap_\alpha S_\alpha$. Now consider any $A \in \bigcap_\alpha S_\alpha$. This means that $A \in S_\alpha$ for each α Since each S_α is a σ - algebra $\Rightarrow A^c \in S_\alpha \forall \alpha \Rightarrow A^c \in \bigcap_\alpha S_\alpha$. Now consider $\{A_n\} \in \bigcap_\alpha (S_\alpha) \forall n \in \mathbb{N}$. Then $A_n \in S_\alpha \forall \alpha, n \in \mathbb{N}$ since each S_α is a σ-algebra, it means that $\bigcup_{n=1}^{\infty} A_n \in S_\alpha \forall \alpha$. This in turn means that $\bigcup_{n=1}^{\infty} A_n \in \bigcap_\alpha S_\alpha$. Hence $\bigcap_\alpha S_\alpha$ is a σ- algebra.

Exercise 1.17

In order to prove the results we first prove a simpler result. We prove that if $\mu : \mathcal{S} \to [0, \infty]$ is a measure, then $\mu(\bigcup_{i=1}^{n=N} A_i) = \sum_{i=1}^{i=N} \mu(A_i)$ if $A_i \bigcap A_j = \phi, i \neq j$. To prove this, let all A_i for $i > N = \phi$. Since $\mu(\phi) = 0$, we then get $\mu(\bigcup_{i=1}^{n=N} A_i) = \mu(\bigcup_{i=1}^{n=\infty} A_i) = \sum_{i=1}^{i=\infty} \mu(A_i) = \sum_{i=1}^{i=N} \mu(A_i)$.

Now to prove monotonicity, consider two sets $A, B \in \mathcal{S}, A \subset B$. Now define $C = A^c \cap B$. Since $\mathcal{S}$ is a σ- algebra $A^c \cap B \in \mathcal{S}$. Furthermore, $A \cap C = \phi$. Since μ is a measure, $\mu(A \cup C) = \mu(A) + \mu(B) \Rightarrow \mu(B) = \mu(A) + \mu(C)$. Since the range of μ is non-negative, $\mu(C) \geq 0$. Thus $\mu(B) \geq \mu(A)$.

Now we prove countable sub-additivity. Consider 2 sets A_1, A_2. We can write, $A_1 \cup A_2 = ({A_1}^c \cap A_2) \cup ({A_2}^c \cap A_1) \cup (A_1 \cap A_2)$ i.e as a union of disjoint sets. Using the result proved above, we get $\mu(A_1 \cup A_2) = \mu({A_1}^c \cap A_2) + \mu({A_2}^c \cap A_1) + \mu(A_1 \cap A_2) \leq \mu({A_1}^c \cap A_2) + \mu(A_1 \cap A_2) + \mu({A_2}^c \cap A_1) + \mu(A_1 \cap A_2) = \mu(A_1) + \mu(A_2)$. Thus we have $\mu(A_1 \cup A_2) \leq \mu(A_1) + \mu(A_2)$ The same argument can be carried out inductively for all $n \in \mathbb{N}$. For example in the case of three sets, A_1, A_2 and A_3, we can assume $A_1 \cup A_2 = A$ and proceed as before. Therefore $\mu(\bigcup_{i=1}^{i=\infty} A_i) \leq \sum_{i=1}^{i=\infty} \mu(A_i)$.

Exercise 1.18

$\lambda(\phi) = \mu(\phi \cap B) = \mu(\phi) = 0$. Let $\{A_i\}_{i=1}^{i=\infty}$ be a collection of disjoint sets. We have, $\lambda(\bigcup_{i=1}^{i=\infty} A_i) = \mu(B \cap \bigcup_{i=1}^{i=\infty} A_i) = \mu(\bigcup_{i=1}^{i=\infty}(B \cap A_i))$ where we have used De-Morgan's laws in the last step. Since all A_i's are disjoint, so are $(B \cap A_i)$'s. Now since μ is a measure, we have, $\mu(\bigcup_{i=1}^{i=\infty}(B \cap A_i)) = \sum_{i=1}^{i=\infty} \mu(B \cap A_i) = \sum_{i=1}^{i=\infty} \lambda(A_i)$. Thus $\lambda(\bigcup_{i=1}^{i=\infty} A_i) = \sum_{i=1}^{i=\infty} \lambda(A_i)$. Hence λ is a measure.

Exercise 1.20

Let $A_1 \supset A_2 \supset ... \supset A_n$. This is equivalent to saying $(A_1 - A_1 = \phi) \subset (A_1 - A_2) \subset (A_1 - A_3) ... \subset (A_1 - A_n)$ From the previous result, we have $\lim_{n\to\infty} \mu(A_1 - A_n) = \mu(\bigcup_{n=1}^{n=\infty}(A_1 - A_n)) = \mu(A_1 - \bigcap_{n=1}^{n=\infty} A_n)$ where we have used De Morgan's Law in the last step. We have already proved previously, the property of finite additivity of a measure. Therefore we have $\mu(A_1) - \lim_{n\to\infty} \mu(A_n) = \mu(A_1) - \mu(\bigcap_{n=1}^{n=\infty} A_n)$. Since $\mu(A_1) < \infty$, we can cancel it out from both sides to get the result.

Exercise 2.10

To prove this result, we note that countable subadditivity of an outer-measure $\Rightarrow$ finite subadditivity. This can be seen by taking $A_i = \phi$ for $i > N$. Since $\mu^*(\phi) = 0$, we have $\mu^*(\bigcup_{i=1}^{i=N} A_i) \leq \sum_{i=1}^{i=N} \mu^*(A_i)$ which follows from the definition of the outer-measure.

Now, we can write $B = (B \cap E) \cup (B \cap E^c)$. Therefore, using finite sub-additivity, we have $\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$. Since the inequality in the other direction is already given, we can replace the inequality with an equality.

Exercise 2.14

Let $\mathcal{A} = \{A : A$ is a countable union of intervals of the form $(a, b], (-\infty, b]$ and $(a, \infty)\}$. We first show that $\sigma(\mathcal{A}) \subset \sigma(\mathcal{O})$. To see this, let $A \in \sigma(\mathcal{A})$. We can write $(a, b] = \bigcap_{n=1}^{n=\infty}(a, b - 1/n), (-\infty, b] = \bigcap_{n=1}^{n=\infty}(-\infty, b - 1/n)$. Thus A can be written as a countable union of intervals of the form $\bigcap_{n=1}^{n=\infty}(a, b - 1/n), \bigcap_{n=1}^{n=\infty}(-\infty, b-1/n), (a, \infty)$. By the property of a σ- algebras, each of these terms, being countable intersections of open intervals, belong to $\sigma(\mathcal{O})$. Thus the countable union of these terms also belongs to $\sigma(\mathcal{O})$. Thus $\sigma(\mathcal{A}) \subset \sigma(\mathcal{O})$.
Now we show that $\sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$. To see this, let $A \in \sigma(\mathcal{O})$. Thus A is an open interval. Let $A = (a, b)$. We can write $A = (a, b) = \bigcup_{n=1}^{n=\infty}(a, b-1/n]$. Similarly, any interval of the form $(-\infty, b)$ can be written as $\bigcup_{n=1}^{n=\infty}(-\infty, b-1/n]$ and any interval of the form (a, ∞) can be written as $\bigcup_{n=1}^{n=\infty}[a-1/n, \infty)$. Note that each of the terms is a countable union of sets that $\in \mathcal{A}$ which $\Rightarrow$ that they $\in \sigma(\mathcal{A})$. Thus any countable union of open sets also $\in \sigma(\mathcal{A})$. Thus $\sigma(\mathcal{O}) \subset \sigma(\mathcal{A})$.
We thus have $\sigma(\mathcal{A}) = \sigma(\mathcal{O})$. By Caratheodory's Theorem, $\mathcal{B}(\mathbb{R}) \subset \mathcal{M}$.