Indiana University, Indianapolis

Spring 2025 Math-I 165

Practice Test 2a

Instructor: Keshav Dahiya

Name:____

Instructions:

- No cell phones, calculators, watches, technology, hats - stow all in your bags.
- **Write your name** on this cover page.
- This test is **closed book and closed notes**.
- All work must be clearly shown for partial credit.
- If you wish for something not to be graded, please strike it out neatly.
- Box, circle, or otherwise clearly indicate your final answer.
- When you finish, return your test to the proctor, and leave the classroom.
- There are a total of **12 problems including 2 bonus problems**.
  - Problems 1-10 are each worth 10 points.
  - The bonus problems are each worth 5 points.
- You can score a **maximum of 110 points out of 100**.
- There are a total of **7 pages** including the cover page.

Problem 1. Evaluate the limit: $\lim_{x\to-\infty} \frac{\sqrt{2x^3+1}}{x\sqrt{x}-1}$. [10 pts]

$= \text{d.n.e}$

because domain of given function does not contain $-$ve number.

$$\lim_{x\to\infty} \frac{\sqrt{2x^3+1}}{x\sqrt{x}-1} = ??$$

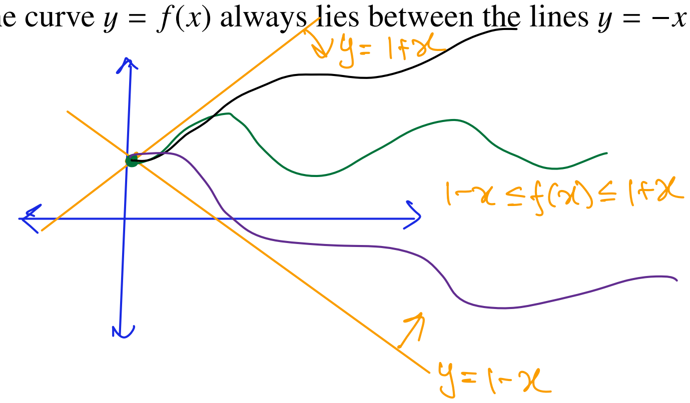
Divide by the highest Power in denominator

$$\Rightarrow \lim_{x\to+\infty} \frac{\frac{1}{x^{3/2}}\sqrt{2x^3+1}}{\frac{1}{x^{3/2}}(x\sqrt{x}-1)} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^3}}\sqrt{2x^3+1}}{\frac{x\sqrt{x}}{x^{3/2}} - \frac{1}{x^{3/2}}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{2x^3}{x^3}+\frac{1}{x^3}}}{1-\frac{1}{x^{3/2}}} = \frac{\sqrt{\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{1}{x^3}}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty}\frac{1}{x^{3/2}}} = \frac{\sqrt{2+0}}{1-0}$$

$$= \sqrt{2}$$

Problem 2. Suppose a function f is defined on $[0,\infty)$. If $f(0) = 1$ and $|f'(x)| \le 1$, then show that the curve $y = f(x)$ always lies between the lines $y = -x+1$ and $y = x+1$.



Mean Value Theorem

Take a Closed interval $[0,x]$

$$\frac{f(x)-f(0)}{x-0} = f'(c)$$

for some c between 0 and x.

For every x, $|f'(c)| \le 1$

$$\Rightarrow -1 \le f'(c) \le 1$$

$$\Rightarrow -1 \le \frac{f(x)-f(0)}{x-0} \le 1 \Rightarrow -1 \le \frac{f(x)-1}{x-0} \le 1$$

$$\Rightarrow -x \le f(x)-1 \le x$$

($x>0$ so can multiply with x and inequalities remain the same)

$$\Rightarrow 1-x \le f(x) \le 1+x$$

Hence proved

Problem 3. Consider the function $f(x) = \dfrac{x+1}{x-1}$. Find the intervals where f increasing and the points of local maximum and minimum. [10 pts]

First derivative test

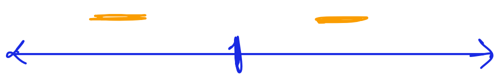
$$f'(x) = \frac{(x-1)(x+1)' - (x+1)(x-1)'}{(x-1)^2} = \frac{x-1-(x+1)}{(x-1)^2}$$

$$= \frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$-2 \rightarrow$ always -ve

$(x-1)^2$ always +ve

Critical numbers : $(x-1)^2 = 0 \Rightarrow x = 1$



$1 \rightarrow$ Neither a lo max nor a lo min

f is always decreasing $\Rightarrow$ No interval where it is increasing.

$\Rightarrow$ No points of local max/min.

Problem 4. Let $f(x) = \dfrac{x+1}{x-1}$ be as in problem 2. Find all the asymptotes (vertical and horizontal) to the curve $y = f(x)$. [10 pts]

Vertical Asymptotes

$x - 1 = 0 \Rightarrow x = 1$

Horizontal Asymptotes

$$\lim_{x\to\infty} \frac{x+1}{x-1} = \lim_{x\to\infty} \frac{\frac{1}{x}(x+1)}{\frac{1}{x}(x-1)} = \lim_{x\to\infty} \frac{\frac{x}{x}+\frac{1}{x}}{\frac{x}{x}-\frac{1}{x}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{1+0}{1-0} = 1$$

$$\lim_{x\to-\infty} \frac{x+1}{x-1} = \lim_{x\to-\infty} \frac{\frac{1}{x}(x+1)}{\frac{1}{x}(x-1)} = \lim_{x\to-\infty} \frac{\frac{x}{x}+\frac{1}{x}}{\frac{x}{x}-\frac{1}{x}} = \lim_{x\to-\infty} \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{1+0}{1-0} = 1$$

$\Rightarrow y = 1$

Problem 5. Let $f(x) = \dfrac{x+1}{x-1}$ be as in problem 2. Find the intervals of concavity and the points of inflection of f. [10 pts]

$$f'(x) = \frac{-2}{(x-1)^2} = -2(x-1)^{-2}$$

$$f''(x) = (-2)(-2)(x-1)^{-3} = \frac{4}{(x-1)^3}$$

$-$ | 1 | $+$ sign of $f''(x)$

Concave down in $(-\infty, 1)$

Concave up in $(1, \infty)$

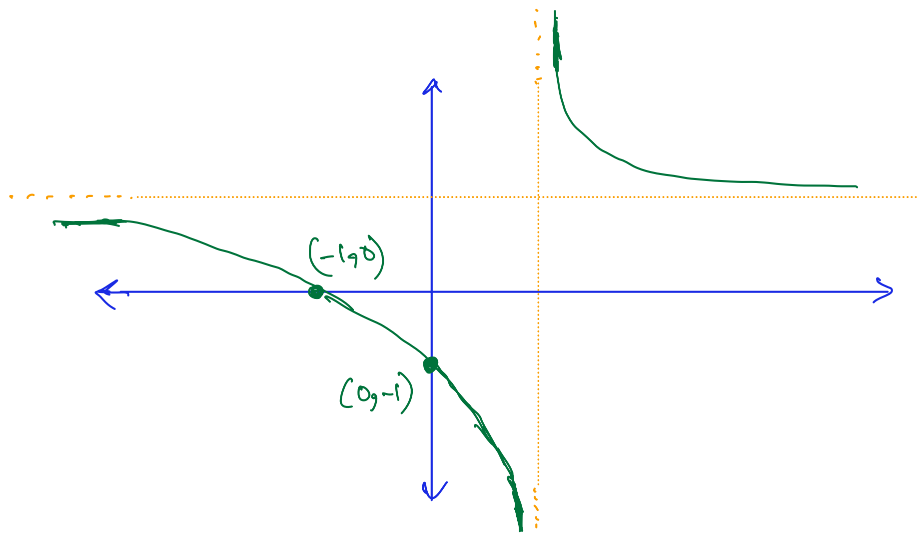
$\Rightarrow$ There is no point of inflection since $x=1$ is not in the domain of f.

Problem 6. Find the x and y intercepts, and the domain of the function $f(x) = \dfrac{x+1}{x-1}$. Use this along with the information obtained from problems 2-5 to sketch the curve $y = f(x)$. [10 pts]

Domain = All real numbers except 1 = $(-\infty, 1) \cup (1, \infty)$

x-intercept: $y=0 \Rightarrow \dfrac{x+1}{x-1} = 0 \Rightarrow x+1=0 \Rightarrow x=-1 \Rightarrow (-1, 0)$

y-intercept: $(0, f(0)) = \left(0, \dfrac{0+1}{0-1}\right) = (0, -1)$



Problem 7. Find all the points of local maximum and minimum of the function $f(x) = \sqrt[3]{x^2 - 1}$. [10 pts]

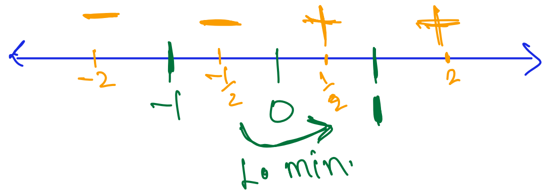
$$f'(x) = \frac{d}{dx}\left((x^2-1)^{1/3}\right) = \frac{1}{3}(x^2-1)^{\frac{1}{3}-1}\left(\frac{d}{dx}(x^2-1)\right)$$

$$= \frac{1}{3}(x^2-1)^{-2/3}(2x) \qquad \text{(chain rule)}$$

$$= \frac{2x}{3(x^2-1)^{2/3}}$$

Critical numbers $\Rightarrow f'(x)=0 \Rightarrow x=0$

$(x^2-1)^{2/3}=0 \Rightarrow x^2-1=0 \Rightarrow x=\pm 1$



Sign of $f'(x)$

$f'(2) = \frac{+ve}{3(4-1)^{2/3}} > 0$

$f'\left(\frac{1}{2}\right) = \frac{1}{3\left(\frac{1}{4}-1\right)^{2/3}} = \frac{1}{3\left(\frac{-1}{2}\right)^{2/3}} = \frac{1}{3\left(\frac{1}{4}\right)^{1/3}} > 0$ → +ve

$f'\left(-\frac{1}{2}\right) = \frac{-1}{3\left(\frac{1}{4}-1\right)^{2/3}} < 0$, $f'(-2) = \frac{-4}{+ve} < 0$ → +ve

$\Rightarrow x=0$ is pt of l. min. and there is no pt. of l. max.

Problem 8. Find the intervals of concavity of the function $f(x) = 2x - \tan x$, $\pi/2 < x < \pi/2$. [10 pts]

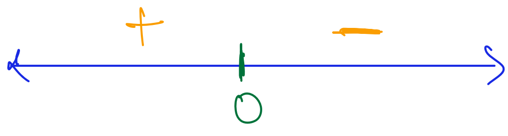
$-\frac{\pi}{2} < x < \frac{\pi}{2}$

$$f'(x) = 2 - \sec^2 x$$

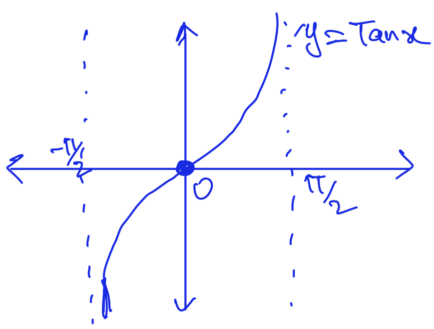
$$f''(x) = 0 - [2\sec x][\sec x]'$$

$$= -2\sec x(\sec x \tan x) = -2\underbrace{\sec^2 x}_{\text{always +ve}}\tan x$$

does not contribute to sign change



sign of $f''(x)$



f is concave up in $\left(-\frac{\pi}{2}, 0\right)$

and concave down in $\left(0, \frac{\pi}{2}\right)$

Problem 9. Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$. [10 pts]

Let $(x, \sqrt{x})$ be a point on $y = \sqrt{x}$

$$d(x) = \sqrt{(x-3)^2 + (\sqrt{x}-0)^2} \quad \text{(distance formula)}$$

$d(x)$ is minimum when $(d(x))^2$ is minimum.

$\Rightarrow$ ignore the overall square root

want to minimize $g(x) = (x-3)^2 + (\sqrt{x}-0)^2$

$$= x^2 - 6x + 9 + x = x^2 - 5x + 9$$

$g'(x) = 2x - 5 = 0 \Rightarrow x = \frac{5}{2} \Rightarrow \left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$ is the point on $y = \sqrt{x}$ closest to $(3, 0)$

$-$ $+$ sign of $g'(x)$

$\frac{5}{2} \rightarrow$ gives absolute minimum

Problem 10. Use the closed interval method to find the absolute maximum and minimum values of the function $f(x) = \sin x + \cos^2 x$ on the interval $[0, \pi]$. [10 pts]

$$f'(x) = \cos x + 2\cos x(\cos x)' = \cos x - 2\sin x \cos x$$

$$= \cos x(1 - 2\sin x)$$

$f'(x) = 0 \Rightarrow \cos x = 0$ or $1 - 2\sin x = 0 \Rightarrow \sin x = \frac{1}{2}$

critical numbers in $[0, \pi]$: $\cos x = 0 \Rightarrow x = \frac{\pi}{2}$

$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$, $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} + \cos^2\frac{\pi}{6} = \frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$$

$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos^2\frac{\pi}{2} = 1 + 0^2 = 1$, $f\left(\frac{5\pi}{6}\right) = \sin\frac{5\pi}{6} + \cos^2\frac{5\pi}{6}$

$$= \frac{1}{2} + \left(\frac{-\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$$

End Points : $f(0) = \sin 0 + \cos^2 0 = 0 + 1^2 = 1$

$$f(\pi) = \sin\pi + \cos^2\pi = 0 + (-1)^2 = 1$$

$\Rightarrow$ Absolute min value $= 1$, Absolute max value $= \frac{5}{4}$

Bonus Problem 1. Find two numbers whose difference is 100 and whose product is a minimum. [5 pts]

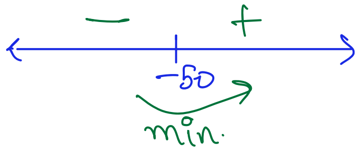
Let x, y be the two numbers.

We have $x - y = 100$ and want to minimize xy.

$\Rightarrow x = y + 100$

$P(y) = xy = (y+100)y = y^2 + 100y$

$P'(y) = 2y + 100 \Rightarrow P'(y) = 0 \Rightarrow 2y + 100 = 0 \Rightarrow y = -50$



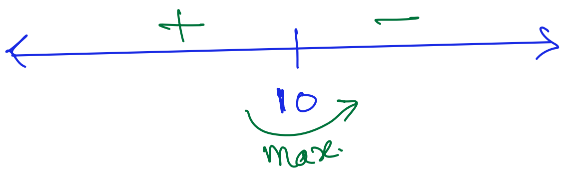
One critical pt. $\Rightarrow$ we have absolute min.

$\Rightarrow$ The numbers are $y = -50$, $x = -50 + 100 = 50$

Bonus Problem 2. The cost function of a firm is $C(x) = 1000 + 40x - x^2$. If the demand function is given by $p(x) = 100 - 4x$, find the production level that maximizes the profit. [5 pts]

$$\text{Profit} = x\,p(x) - C(x)$$
$$= x(100 - 4x) - (1000 + 40x - x^2)$$
$$= 100x - 4x^2 - 1000 - 40x + x^2$$
$$\Rightarrow P(x) = 60x - 3x^2 - 1000$$

$P'(x) = 60 - 6x \Rightarrow P'(x) = 0 \Rightarrow 60 - 6x = 0 \Rightarrow x = 10$



$\Rightarrow$ To maximize profit, the firm must produce 10 items.