■ **Section 11.9** textbook exercises, *page 797*: # 3, 4, 5, 6, 8, 13, 15.

In this section, we will learn how to represent certain types of functions as power series by manipulating geometric series or by differentiating or integrating such a series.

We will start with the geometric series

$R=1, \quad I=(-1,1)$

$$\sum_{n=0}^{\infty} x^n = \frac{x^0}{1-x} = \frac{1}{1-x}$$

$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{x^n} = x \Rightarrow$ common ratio $= x$, converges for $|x|<1$

Thus, we get the first example of a function that is represented by a power series

$x=1 \Rightarrow \sum_{n=0}^{\infty} x^n = \infty$, $x=-1 \Rightarrow \sum_{n=0}^{\infty} (-1)^n$ diverges.

By manipulating this first example, many other functions can also be represented as power series.

Example 1: Find a power series representation for the function and determine the interval of convergence

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad |r|<1$$

(a) $\dfrac{1}{1-x^2}$

$r = x^2$

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}$$

Power series of the function $\frac{1}{1-x^2}$

When does it converge?

$|x^2|<1 \Rightarrow x^2<1 \Rightarrow x^2-1<0 \Rightarrow (x-1)(x+1)<0$

$\Rightarrow -1<x<1$

$\Rightarrow$ Interval of convergence $= (-1,1)$, $R=1$

(b) $\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{1}{2}\underbrace{\frac{1}{1-\frac{x}{2}}}_{\frac{1}{1-r} \text{ for } r=\frac{x}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$

$$\frac{1}{2-x} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2^n}x^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}x^n$$

$$\text{I.o.c.} = (-2, 2)$$

$$|r| < 1$$

$$\Rightarrow \left|\frac{x}{2}\right| < 1 \Rightarrow -1 < \frac{x}{2} < 1$$

$$\Rightarrow -2 < x < 2$$

(c) $\frac{x}{1+2x} = x\frac{1}{1+2x} = x\underbrace{\frac{1}{1-(-2x)}}_{\frac{1}{1-r} \text{ with } r=-2x}$

$$= x\sum_{n=0}^{\infty}(-2x)^n$$

$$= x\sum_{n=0}^{\infty}(-2)^n x^n = \sum_{n=0}^{\infty}(-2)^n x^{n+1}$$

$$\Rightarrow \begin{array}{l} m=n+1 \\ n=m-1 \end{array} \Rightarrow \sum_{m=1}^{\infty}(-2)^{m-1}x^m = \sum_{n=1}^{\infty}(-2)^{n-1}x^n$$

Differentiation and Integration of Power Series.

If the power series $\sum c_n(x-a)^n$ has radius of convergence $R>0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Example 2:

(a) $\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1}$

with same radius of convergence as that of $\sum_{n=0}^{\infty} x^n$

(b) $\int \left(\sum_{n=0}^{\infty} x^n\right) dx = C + \sum_{n=0}^{\infty} \int x^n\,dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d}{dx}(x^n) = \sum_{n=0}^{\infty}\frac{n x^{n-1}}{n!}$$

$$= \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!}$$

$\boxed{n! = n(n-1)!}$

$$\int\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)dx = \sum_{n=0}^{\infty}\int\frac{x^n}{n!}\,dx$$

$$= C + \sum_{n=0}^{\infty}\frac{1}{n!}\int x^n\,dx = C + \sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!}$$

By differentiation or integration, we can find power series representation for more functions.

Example 3: Find a power series representation for the function and determine the radius of convergence.

(a) $\dfrac{1}{(1-x)^2}$. **Hint:** Note that $\dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \sum_{n=0}^{\infty} n x^{n-1}$$

$$\frac{d}{dx}\left((1-x)^{-1}\right)$$

$$= (-1)(1-x)^{-1-1} \cdot \frac{d}{dx}(1-x)$$

$$= (-1)(1-x)^{-2}(-1)$$

$$= (1-x)^{-2} = \frac{1}{(1-x)^2}$$

(b) $\ln(1+x)$. **Hint:** Think about integration.

$$\ln(1+x) = \int \frac{1}{1+x}\, dx$$

$$\frac{1}{1+x} = \underbrace{\frac{1}{1-(-x)}}_{\frac{1}{1-r} \text{ for } r=-x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\int \frac{1}{1+x}\, dx = \int \sum_{n=0}^{\infty} (-1)^n x^n\, dx = C + \sum_{n=0}^{\infty} \int (-1)^n x^n\, dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \int x^n\, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\ln(1+x) = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

Put $x=0$ on both sides: $\ln 1 = C + 0 \Rightarrow C = \ln 1 = 0$

$$\Rightarrow \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

(c) $\tan^{-1}(x)$. **Hint:** Think about integration.

$$\tan^{-1}(x) = C + \int \frac{1}{1+x^2}\,dx$$

$$\frac{1}{1+x^2} = \underbrace{\frac{1}{1-(-x^2)}}_{\frac{1}{1-r} \text{ for } r=-x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\tan^{-1} x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Put $x=0$ on both sides :—

$$\tan^{-1}(0) = C + 0 \Rightarrow C = \tan^{-1}(0) = 0$$

$$\Rightarrow \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$