M16600 Lecture Notes

Section 11.10: Taylor and Maclaurin Series

■ **Section 11.10** textbook exercises, *page 811*: #6, 8, 9, 19, 21, 23, 25, 35, 37, 54. For #54, use the series representation for $\sin x$ in Table 1, page 808.

Taylor Series is a power series with a formula for the coefficient c_n. How do we find the formula for the coefficients? We will start out with the general form for power series

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-2)^3 + c_4(x-a)^4 + \cdots ,$$

then compute $f(a)$, $f'(a)$, $f''(a)$, $f'''(a)$, *etc.* and see if we can find a pattern for c_n:

$$\rightarrow f(a) = \sum_{n=0}^{\infty} c_n(a-a)^n = \underbrace{c_0(a-a)^0}_{c_0 \times 1} + \underbrace{c_1(a-a)^1}_{0} + \underbrace{c_2(a-a)^2}_{0} + \underbrace{\cdots\cdots}_{0}$$

$$\parallel$$

$$c_0(x-a)^0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

$$\Rightarrow f(a) = c_0$$

$$f'(x) = \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x-a)^n = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$= c_1(x-a)^{1-1} + 2c_2(x-a)^{2-1} + 3c_3(x-a)^{3-1} + \cdots$$

$$= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

$$f'(a) = c_1$$

Taylor Series of $f(x)$ at a.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A special case of Taylor series is when the center $a = 0$. This special is given a name called ***Maclaurin series***.

Maclaurin series (Taylor series centered at 0). $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f''(x) = 2c_2 + (3\cdot 2)c_3(x-a) + 4\cdot 3\, c_4 (x-a)^2 + \cdots$$

$$\downarrow$$

$$f''(a) = 2c_2$$

$$f'''(x) = (3\cdot 2\cdot 1)c_3 + (4\cdot 3\cdot 2)c_4(x-a) + (5\cdot 4\cdot 3)\cdot c_5(x-a)^2 + \cdots$$

$$\downarrow$$

$$f'''(a) = 3!\, c_3$$

$$\vdots$$

$$f^{(n)}(x) = n!\, c_n + (n+1)\cdots 2\, c_{n+1}\cancel{(x-a)}^{\,0} + (n+2)\cdots 3\, c_{n+2}\cancel{(x-a)^2}^{\,0} + \cancel{\cdots}^{\,0}$$

$$f^{(n)}(a) = n!\, c_n$$

$$\Rightarrow \boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

Example 1: Use the definition of Taylor series to find the first four nonzero terms of the series for $f(x) = \ln x$ centered at $a = 1$.

n^{th} term of the Taylor series for f about $a=1 \Rightarrow \frac{f^{(n)}(1)}{n!}(x-1)^n$

$n=0 \Rightarrow \frac{f^{(0)}(1)}{0!}(x-1)^0 = f(1) = \ln 1 = 0$

$n=1 \Rightarrow \frac{f'(1)}{1!}(x-1)^1 = \frac{1}{1!}(x-1) = (x-1)$

$n=2 \Rightarrow \frac{f^{(2)}(1)}{2!}(x-1)^2 = \frac{(-1)}{2!}(x-1)^2 = -\frac{(x-1)^2}{2}$

$n=3 \Rightarrow \frac{f^{(3)}(1)}{3!}(x-1)^3 = \frac{2}{3!}(x-1)^3 = \frac{(x-1)^3}{3}$

$f'(x) = \frac{1}{x}$

$f''(x) = \frac{-1}{x^2}$

$f^{(3)}(x) = \frac{2}{x^3}$

$f^{(4)}(x) = \frac{2(-3)}{x^4}$

$n=4 \Rightarrow \frac{f^{(4)}(1)}{4!}(x-1)^4 = \frac{-6}{24}(x-1)^4 = -\frac{(x-1)^4}{4}$

$$\ln x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$$

Example 2: Find the Taylor series for $f(x) = \dfrac{1}{1+x}$ centered at $a = 2$.

$f(x) = \frac{1}{1+x}$ about $a=2$ $\qquad \frac{f^{(n)}(2)}{n!}(x-2)^n$

$n=0 \Rightarrow \frac{f(2)}{0!} = f(2) = \frac{1}{1+2} = \frac{1}{3}$

$n=1 \Rightarrow f'(2) = \frac{-1}{(1+2)^2} = \frac{-1}{3^2}$

$n=2 \Rightarrow f''(2) = \frac{2}{(1+2)^3} = \frac{2}{3^3}$

$n=3 \Rightarrow f'''(2) = \frac{-(3\cdot2\cdot1)}{(1+2)^4} = -\frac{(3\cdot2\cdot1)}{3^4}$

$f(x) = \frac{1}{1+x}$

$f'(x) = \frac{-1}{(1+x)^2}$

$f''(x) = \frac{2}{(1+x)^3}$

$f'''(x) = \frac{2(-3)}{(1+x)^4}$

Example 3: Use the definition of Maclaurin series to find the Maclaurin series of $f(x) = e^x$.

$$\Rightarrow f^{(n)}(2) = \frac{(-1)^n n!}{3^{n+1}}$$

n^{th} term of Taylor series $= \dfrac{f^{(n)}(2)}{n!}(x-2)^n$

$$= \frac{1}{\cancel{n!}} \frac{(-1)^n \cancel{n!}}{3^{n+1}}(x-2)^n$$

$$= \frac{(-1)^n}{3^{n+1}}(x-2)^n$$

$$\Rightarrow \frac{1}{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(x-2)^n$$

Example 4: Use the result in example 3 to find the Maclaurin series for

(a) $f(x) = e^{-x^2}$

Example 3 $\quad f(x) = e^x$ about $x = 0 \Rightarrow \dfrac{f^{(n)}(0)}{n!} x^n$

want to find $f^{(n)}(0)$

n	$f(x)$	$f(0)$
0	e^x	$e^0 = 1$
1	e^x	$e^0 = 1$
2	e^x	$e^0 = 1$
$\vdots$		
n	e^x	$e^0 = 1$

$\Rightarrow f^{(n)}(0) = 1$ for every n

$\Downarrow$

n^{th} term $= \dfrac{1}{n!} x^n$

$$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

(b) $f(x) = xe^x$

Example 4 (a) $f(x) = e^{-x^2}$, (b) $f(x) = xe^{x}$

$$e^{x} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Replace x with $-x^2$ on both the sides

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n (x^2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

(b) $f(x) = xe^{x}$

$$e^{x} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

multiply both the sides by x :—

$$xe^{x} = x \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1}$$

Example 5: (a) Evaluate $\int e^{-x^2}\,dx$ as an infinite series. (Note, we cannot compute this indefinite integral using any of the integral techniques we've learned in chapter 7)

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

$$\int e^{-x^2} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} dx = C + \sum_{n=0}^{\infty} \int \frac{(-1)^n}{n!} x^{2n} dx$$

$$\int e^{-x^2} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{(2n+1)}$$

$$\int x^{2n} dx = \frac{x^{2n+1}}{2n+1}$$

(b) Evaluate $\int_0^1 e^{-x^2}\,dx$ using the first four terms of the power series you found in part (a).

$$\int e^{-x^2} dx \approx \frac{(-1)^0}{0!} \frac{x^{2(0)+1}}{(2(0)+1)} + \frac{(-1)^1}{1!} \frac{x^{2(1)+1}}{2(1)+1} + \frac{(-1)^2}{2!} \frac{x^{2(2)+1}}{2(2)+1} + \underbrace{\frac{(-1)^3}{3!} \frac{x^{2(3)+1}}{2(3)+1}}_{6\times 7=42}$$

$$\approx x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}$$

$$\int_0^1 e^{-x^2} dx = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42}$$

$$\approx 1 - 0.33 + 0.1 - 0.025$$

$$\approx 0.77 - 0.025 = 0.745$$