Indiana University - Purdue University, Indianapolis

Math16600 Section 23715

Practice Test 2

Fall 2023, October 24

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Name:____ [2 pts]

Instructions:

- No cell phones, calculators, watches, technology, hats - stow all in your bags.
- **Write your name** on this cover page. It carries 2 points.
- This test is closed book and closed notes.
- All work must be clearly shown for partial credit.
- If you wish for something not to be graded, please strike it out neatly.
- Box, circle, or otherwise clearly indicate your final answer.
- When you finish, return your test to the proctor, and leave the classroom.
- There are a total of **16 problems** including bonus problem.
  - Problems 1-10 are each worth 6 points.
  - Problems 11-15 are each worth 8 points.
  - The bonus problem is worth 8 points.
- You can score a **maximum of 110 points out of 100**.
- There are a total of **9 pages** including the cover page.

Problem 1: Evaluate the integral

(By Parts)

$$\int x \tan^{-1}(3x)\, dx$$

[6 pts]

Let $u = \tan^{-1}(3x)$, $dv = x\, dx$

$\Rightarrow du = \dfrac{3}{1+(3x)^2}\, dx$, $v = \dfrac{x^2}{2}$

$= \dfrac{x^2}{2}\tan^{-1}(3x) - \dfrac{x}{6} + \dfrac{1}{18}\tan^{-1}(3x) + C$

$$\int x \tan^{-1}(3x)\, dx = \left(\tan^{-1} 3x\right)\frac{x^2}{2} - \int \frac{x^2}{2}\,\frac{3}{1+9x^2}\, dx$$

$$\int \frac{x^2}{2}\,\frac{3}{1+9x^2}\, dx = \frac{3}{2}\int \frac{x^2}{1+9x^2}\, dx$$

$9x^2 = 1 + 9x^2 - 1$

$x^2 = \frac{1}{9}\left[1+9x^2 - 1\right]$

$$= \frac{3}{2}\left(\frac{1}{9}\right)\int \frac{1+9x^2-1}{1+9x^2}\, dx$$

$a^2 = \frac{1}{9} \Rightarrow a = \frac{1}{3} \Rightarrow \frac{1}{a} = 3$

$$= \frac{1}{6}\int\left(\cancel{\frac{1+9x^2}{1+9x^2}} - \frac{1}{1+9x^2}\right)dx = \frac{1}{6}\int\left(1 - \frac{1}{1+9x^2}\right)dx$$

$$= \frac{1}{6}\left[\int 1\, dx - \int \frac{1}{1+9x^2}\, dx\right] = \frac{1}{6}\left[x - \frac{1}{9}\int \frac{1}{x^2+\frac{1}{9}}\, dx\right] = \frac{1}{6}\left[x - \frac{1}{9}\cdot 3\tan^{-1}(3x)\right]$$

$$= \frac{x}{6} - \frac{1}{18}\tan^{-1}(3x)$$

Problem 2:Evaluate the integral

$$\int \sqrt{\cos\theta}\, \sin^3\theta\, d\theta$$

[6 pts]

$u = \cos\theta$

$\Rightarrow du = -\sin\theta\, d\theta$

$\Rightarrow -du = \sin\theta\, d\theta$

$$= \int \sqrt{\cos\theta}\, \sin^2\theta\, (\sin\theta\, d\theta)$$

$$= \int \sqrt{u}\, (1-\cos^2\theta)\,(-du)$$

$$= \int \sqrt{u}\,(1-u^2)\,(-du)$$

$\underbrace{(1-u^2)(-1) = u^2-1}$

$$= \int \sqrt{u}\,(u^2-1)\, du$$

$$= \int \left(u^{1/2}\cdot u^2 - u^{1/2}\right) du$$

$$= \int u^{2+\frac{1}{2}}\, du - \int u^{1/2}\, du = \int u^{5/2}\, du - \int u^{1/2}\, du$$

$$= \frac{u^{5/2+1}}{5/2+1} - \frac{u^{1/2+1}}{1/2+1} = \frac{u^{7/2}}{7/2} - \frac{u^{3/2}}{3/2} + C$$

$= \frac{2}{7} u^{7/2} - \frac{2}{3} u^{3/2} + C$

$= \frac{2}{7} (\cos\theta)^{7/2} - \frac{2}{3} (\cos\theta)^{3/2} + C$

Problem 3: Evaluate the integral

$$\int \frac{\sqrt{1+x^2}}{x^4}\,dx$$

[6 pts]

$x = \tan\theta$

$\Rightarrow dx = \sec^2\theta\, d\theta$

$= \int \frac{\sqrt{1+\tan^2\theta}}{\tan^4\theta} \sec^2\theta\, d\theta$

$\tan\theta = \frac{x}{1} = \frac{P}{B}$

$H = \sqrt{x^2+1}$

$\sin\theta = \frac{P}{H} = \frac{x}{\sqrt{x^2+1}}$

$= \int \frac{\sqrt{\sec^2\theta}}{\tan^4\theta} \sec^2\theta\, d\theta = \int \frac{\sec\theta}{\tan^4\theta}\sec^2\theta\, d\theta$

$= \int \frac{\sec^3\theta}{\tan^4\theta}\, d\theta = \int \frac{\frac{1}{\cos^3\theta}}{\frac{\sin^4\theta}{\cos^4\theta}}\, d\theta = \int \frac{1}{\cancel{\cos^3\theta}} \frac{\cos^{\cancel{4}}\theta}{\sin^4\theta}\, d\theta$

$= \int \frac{\cos\theta}{\sin^4\theta}\, d\theta = \int \frac{1}{u^4}\, du = \int u^{-4}\, du = \frac{u^{-4+1}}{-4+1} = \frac{u^{-3}}{-3} = \frac{-1}{3u^3}$

$u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$

$= \frac{-1}{3\sin^3\theta} + C = \frac{-1}{3\left(\frac{x}{\sqrt{x^2+1}}\right)^3} + C$

$\frac{-1}{3} \frac{\left(\sqrt{x^2+1}\right)^3}{x^3} + C$

$= \frac{-(x^2+1)^{3/2}}{3x^3} + C$

Problem 4: Evaluate the integral

$$\int \frac{x+2}{(2x+1)(x-1)^2}\,dx$$

[6 pts]

$\frac{x+2}{(2x+1)(x-1)^2} = \frac{a}{2x+1} + \frac{b}{x-1} + \frac{c}{(x-1)^2}$

$\Rightarrow x+2 = a(x-1)^2 + b(x-1)(2x+1) + c(2x+1)$

$x=1 \Rightarrow 3 = c(2(1)+1) \Rightarrow 3 = 3c \Rightarrow c = 1$

$x=0 \Rightarrow 2 = a - b + c \Rightarrow 2 = a - b + 1 \Rightarrow a - b = 1$ (I)

$x=-1 \Rightarrow 1 = 4a + 2b - c \Rightarrow 1 = 4a + 2b - 1 \Rightarrow 4a + 2b = 2$
$\Rightarrow 2a + b = 1$ (II)

Adding (I) and (II) we have :—

$a - \cancel{b} + 2a + \cancel{b} = 1 + 1 \Rightarrow 3a = 2 \Rightarrow a = 2/3 \Rightarrow \frac{2}{3} - b = 1 \Rightarrow b = \frac{2}{3} - 1$
(using (I)) $\Rightarrow b = \frac{-1}{3}$

$\Rightarrow \int \frac{x+2}{(2x+1)(x-1)^2}\,dx = \frac{2}{3}\int \frac{1}{2x+1}\,dx - \frac{1}{3}\int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx$

$\Rightarrow I = \frac{2}{3}\cdot\frac{1}{2}\ln|2x+1| - \frac{1}{3}\ln|x-1| + \frac{(x-1)^{-2+1}}{-2+1} + C = \frac{1}{3}\ln|2x+1| - \frac{1}{3}\ln|x-1| - \frac{1}{x-1} + C$

Problem 5: Evaluate the integral

$$\int \frac{\sec^4\theta}{\tan^2\theta}\, d\theta$$

[6 pts]

$$\sec^4\theta\, d\theta = \underbrace{\sec^2\theta}_{(1+\tan^2\theta)=1+u^2}\ \underbrace{\boxed{\sec^2\theta\, d\theta}}_{du}$$

$u = \tan\theta$

$\Rightarrow du = \sec^2\theta\, d\theta$

$$\int \frac{\sec^4\theta}{\tan^2\theta}\, d\theta = \int \frac{\sec^2\theta}{\tan^2\theta}\,\sec^2\theta\, d\theta = \int \frac{1+u^2}{u^2}\, du$$

$$= \int\left(\frac{1}{u^2} + \frac{\cancel{u^2}}{\cancel{u^2}}\right)du = \int \frac{1}{u^2}\, du + \int 1\, du = \frac{u^{-2+1}}{-2+1} + u + C$$

$$= \frac{u^{-1}}{-1} + u + C = \frac{-1}{u} + u + C = \frac{-1}{\tan\theta} + \tan\theta + C$$

Problem 6: Use Simpson's rule with $n = 4$ to approximate the integral

$$\int_0^4 e^{4x-x^2}\, dx$$

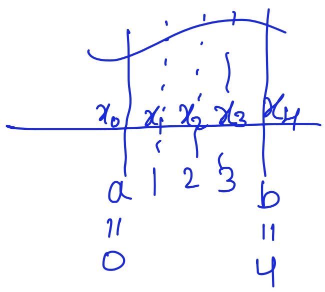
$e \approx 2.7$, $e^2 \approx 7.4$, $e^3 \approx 20$, $e^4 \approx 54.6$ [6 pts]

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$a = 0$, $b = 4$, $\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$

$x_0 = a = 0$, $x_1 = x_0 + \Delta x = 1$, $x_2 = x_1 + \Delta x = 1+1 = 2$

$x_3 = x_2 + \Delta x = 2+1 = 3$, $x_4 = x_3 + \Delta x = 3+1 = 4$



$$I \approx \frac{1}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right]$$

$f(x) = e^{4x-x^2} \Rightarrow f(0) = e^{4(0)-(0)^2} = e^0 = 1$

$\Rightarrow f(4) = e^{4(4)-(4)^2} = e^0 = 1$, $f(1) = e^{4-1^2} = e^3 = 20$,

$f(2) = e^{4(2)-(2)^2} = e^{8-4} = e^4 = 54.6$, $f(3) = e^{4(3)-(3)^2} = e^{12-9} = e^3 = 20$

$I \simeq \frac{1}{3}[1+80+109.2+80+1] = \frac{1}{3}[271.2] = 90.04$

Problem 7: Evaluate the integral

$$\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx$$

[6 pts]

$(x^{-1})' = (-1)\,x^{-1-1} = -x^{-2}$

$$\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{e^{-1/x}}{x^2}\,dx$$

$u = \frac{-1}{x} \Rightarrow \frac{du}{dx} = \frac{1}{x^2}$

$\Rightarrow du = \frac{dx}{x^2}$

$$\int \frac{e^{-1/x}}{x^2}\,dx = \int e^{-1/x}\,\frac{dx}{x^2} = \int e^u\,du$$

$$= e^u + C = e^{-1/x} + C$$

$$\int_1^t \frac{e^{-1/x}}{x^2}\,dx = e^{-1/x}\Big|_1^t = e^{-1/t} - e^{-1/1} = e^{-1/t} - e^{-1}$$

$$\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx = \lim_{t\to\infty}\left[e^{-1/t} - e^{-1}\right] = e^{\frac{-1}{\infty}} - e^{-1} = e^0 - e^{-1} = 1 - e^{-1}$$

Problem 8: Set up an integral for length of the curve $y = x^2 e^x$, $-1 \le x \le 1$. [6 pts]

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx, \qquad a=-1,\ b=1$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 e^x\right) = (x^2)'\,e^x + x^2\,(e^x)' = 2x\,e^x + x^2 e^x$$

$$L = \int_{-1}^{1} \sqrt{1+\left[2x e^x + x^2 e^x\right]^2}\,dx$$

$$= \int_{-1}^{1} \sqrt{1+\left[x e^x (2+x)\right]^2}\,dx$$

$$= \int_{-1}^{1} \sqrt{1+x^2 e^{2x} (2+x)^2}\,dx$$

Problem 9: Find the area of the surface obtained by rotating the curve $y = x^3$, $0 \le x \le 1/\sqrt[4]{3}$ about the x-axis. [6 pts]

$$A = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad , \quad y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$$

$$= \int_0^{\frac{1}{\sqrt[4]{3}}} 2\pi x^3 \sqrt{1+(3x^2)^2}\, dx = \int_0^{\frac{1}{\sqrt[4]{3}}} 2\pi x^3 \sqrt{1+9x^4}\, dx$$

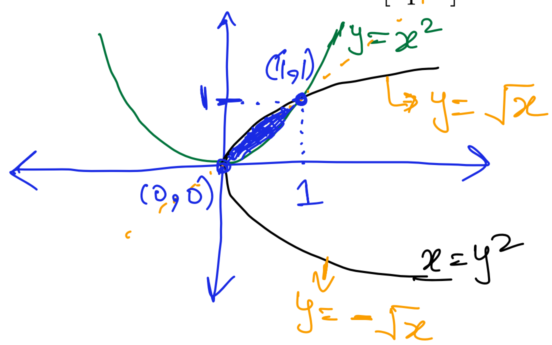
$$u = 1+9x^4 \Rightarrow du = 36x^3\, dx \Rightarrow \frac{1}{36} du = x^3 dx$$

$$x=0 \Rightarrow u = 1+9(0)^4 = 1 \quad , \quad x = \frac{1}{\sqrt[4]{3}} \Rightarrow u = 1+9\left(\frac{1}{3}\right) = 4$$

$$A = \int_1^4 2\pi \sqrt{u}\, \frac{du}{36} = \frac{2\pi}{36}\int_1^4 u^{1/2}\, du = \frac{2\pi}{36} \left.\frac{u^{3/2}}{3/2}\right|_1^4 = \frac{\pi}{27}[8-1]$$

$$= \frac{7\pi}{27}$$

Problem 10: Find the coordinates of the centroid for the region bounded by the curves $y = x^2$ and $x = y^2$. [6pts]



Pt. of intersection $\quad y = x^2$ and $x = y^2$

$$y = (y^2)^2 \Rightarrow y = y^4 \Rightarrow y^4 - y = 0$$

$$\Rightarrow y(y^3-1) = 0 \Rightarrow y = 0 \text{ or } y^3 = 1$$

$$y^3 = 1 \Rightarrow y = 1 \Rightarrow x = y^2 = 1^2 = 1$$

$$y = 0 \Rightarrow x = y^2 = 0^2 = 0$$

$(0,0)$ and $(1,1)$

$$\bar{x} = \frac{1}{A}\int_a^b x\left(f(x) - g(x)\right) dx \quad , \quad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right] dx$$

$g(x) = x^2$ [$y = x^2$ is the lower curve]

$f(x) = \sqrt{x}$ [$x = y^2$ is the upper curve]

$$y^2 = x \Rightarrow y = \pm\sqrt{x}$$

$$A = \int_a^b \left(f(x) - g(x)\right) dx \qquad a = 0,\ b = 1$$

$$A = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \int_0^1 \sqrt{x}\, dx - \int_0^1 x^2\, dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_0^1 - \frac{x^3}{3}\Bigg|_0^1$$

$$= \frac{x^{3/2}}{3/2}\Bigg|_0^1 - \left(\frac{1^3}{3} - \frac{0^3}{3}\right) = \frac{2}{3}\left(1^{3/2} - 0^{3/2}\right) - \frac{1}{3}$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \qquad \Rightarrow A = \frac{1}{3} \Rightarrow \frac{1}{A} = 3$$

$$\bar{x} = 3\int_0^1 x\left(\sqrt{x} - x^2\right) dx = 3\int_0^1 x^{1+\frac{1}{2}}\, dx - 3\int_0^1 x^3\, dx$$

$$= 3\int_0^1 x^{3/2}\, dx - 3\int_0^1 x^3\, dx = 3\,\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\Bigg|_0^1 - 3\,\frac{x^4}{4}\Bigg|_0^1$$

$$= 3\left(\frac{2}{5}\right)\left(1^{5/2} - 0^{5/2}\right) - \frac{3}{4}\left(1^4 - 0^4\right) = \frac{6}{5} - \frac{3}{4} = \frac{24-15}{20} = \frac{9}{20}$$

$$\bar{y} = 3\int_0^1 \frac{1}{2}\left[\left(\sqrt{x}\right)^2 - \left(x^2\right)^2\right] dx = \frac{3}{2}\int_0^1 \left(x - x^4\right) dx$$

$$= \frac{3}{2}\int_0^1 x\, dx - \frac{3}{2}\int_0^1 x^4\, dx = \frac{3}{2}\,\frac{x^2}{2}\Bigg|_0^1 - \frac{3}{2}\,\frac{x^5}{5}\Bigg|_0^1$$

$$= \frac{3}{2}\left(\frac{1}{2}\right) - \frac{3}{2}\left(\frac{1}{5}\right) = \frac{3}{4} - \frac{3}{10} = \frac{15-6}{20} = \frac{9}{20}$$

$\Rightarrow$ The centroid of the given region is at $\left(\frac{9}{20}, \frac{9}{20}\right)$

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Problem 11: Evaluate the integral

$u = x^2 \Rightarrow du = 2x\,dx$

$dv = \sin x\,dx \Rightarrow v = -\cos x$

$$\int x^2 \sin x\,dx$$

[8 pts]

$$\int x^2 \sin x\,dx = x^2(-\cos x) - \int(-\cos x)\,2x\,dx$$

$$= -x^2\cos x + 2\int x\cos x\,dx$$

$$\int \underbrace{x}_{u}\underbrace{\cos x\,dx}_{dv} = x\sin x - \int \sin x\,dx = x\sin x - (-\cos x)$$

$$= [x\sin x + \cos x]$$

$u = x \Rightarrow du = dx$

$dv = \cos x\,dx$

$\Rightarrow v = \sin x$

$$\Rightarrow \int x^2\sin x\,dx = -x^2\cos x + 2[x\sin x + \cos x] + C$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

Problem 12: Evaluate the integral

$x = 3\sin\theta$

$$\int \frac{x^3}{\sqrt{9-x^2}}\,dx$$

$\Rightarrow dx = 3\cos\theta\,d\theta$

[8 pts]

$$I = \int \frac{27\sin^3\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,d\theta = \int \frac{27\sin^3\theta}{\sqrt{9\cos^2\theta}}\,3\cos\theta\,d\theta$$

$$= \int \frac{27\sin^3\theta}{\cancel{3\cos\theta}}\,\cancel{3\cos\theta}\,d\theta = 27\int \sin^3\theta\,d\theta$$

$$\sin^3\theta = \sin^2\theta\,\sin\theta = (1-\cos^2\theta)\sin\theta$$

$$\int \sin^3\theta\,d\theta = \int (1-\cos^2\theta)\underbrace{\sin\theta\,d\theta}_{-du},$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$-du = \sin\theta\,d\theta$

$$= \int (1-u^2)(-du) = \int (u^2-1)\,du$$

$$= \int u^2\,du - \int 1\,du = \frac{u^3}{3} - u + C = \frac{\cos^3\theta}{3} - \cos\theta + C$$

$$= \frac{1}{81}(9-x^2)^{3/2} - \frac{\sqrt{9-x^2}}{3} +$$

$\sin\theta = \frac{x}{3} \Rightarrow P = x,\ H = 3 \Rightarrow B = \sqrt{9-x^2} \Rightarrow \cos\theta = \frac{\sqrt{9-x^2}}{3}$ $+ C$

$\Rightarrow I = 27\int \sin^3\theta\, d\theta = \frac{1}{3}(9-x^2)^{3/2} - 9\sqrt{9-x^2} + C$

Problem 13: Evaluate the integral

$$\int \frac{1}{(x^2-1)^2}\, dx$$

[8 pts]

$$\frac{1}{(x^2-1)^2} = \frac{1}{[(x-1)(x+1)]^2} = \frac{1}{(x-1)^2(x+1)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1} + \frac{d}{(x+1)^2}$$

$$\Rightarrow 1 = a(x-1)(x+1)^2 + b(x+1)^2 + c(x-1)^2(x+1) + d(x-1)^2$$

$x = 1 \Rightarrow 1 = b(1+1)^2 \Rightarrow 1 = 4b \Rightarrow b = \frac{1}{4}$

$x = -1 \Rightarrow 1 = d(-1-1)^2 \Rightarrow 1 = 4d \Rightarrow d = \frac{1}{4}$

$x = 0 \Rightarrow 1 = -a + b + c + d \Rightarrow 1 = -a + \frac{1}{4} + c + \frac{1}{4} \Rightarrow 1 - \frac{1}{2} = -a + c$

$\Uparrow$

$-a + c = \frac{1}{2}$ (I)

$x = 2 \Rightarrow 1 = 9a + 9b + 3c + d \Rightarrow 1 = 9a + \frac{9}{4} + 3c + \frac{1}{4} \Rightarrow 1 = 9a + 3c + \frac{5}{2}$

$\Rightarrow 9a + 3c = 1 - \frac{5}{2} = -\frac{3}{2}$

$\Rightarrow \cancel{3}(3a + c) = -\frac{\cancel{3}}{2}$ (II)

$$\left.\begin{aligned} -a + c &= \frac{1}{2} \\ 3a + c &= -\frac{1}{2} \end{aligned}\right\} \text{subtract} \Rightarrow -a - 3a = \frac{1}{2} - \left(-\frac{1}{2}\right)$$

$\Rightarrow -4a = 1 \Rightarrow a = -\frac{1}{4}$

$\frac{1}{4} + c = \frac{1}{2} \Rightarrow c = \frac{1}{2} - \frac{1}{4} \Rightarrow c = \frac{1}{4}$

Problem 14: Use your answer to the above integral to find length of the curve $y = \frac{1}{2}x^2$, $0 \le x \le 1$. [8 pts]

$$\frac{1}{(x^2-1)^2} = \frac{-1}{4}\frac{1}{x-1} + \frac{1}{4}\frac{1}{(x-1)^2} + \frac{1}{4}\frac{1}{x+1} + \frac{1}{4}\frac{1}{(x+1)^2}$$

$$\Rightarrow \int \frac{1}{(x^2-1)^2}\,dx = \frac{-1}{4}\int \frac{1}{x-1}\,dx + \frac{1}{4}\int \frac{1}{(x-1)^2}\,dx + \frac{1}{4}\int \frac{1}{x+1}\,dx + \frac{1}{4}\int \frac{1}{(x+1)^2}\,dx$$

$$= \frac{-1}{4}\ln|x-1| + \frac{1}{4}\frac{(x-1)^{-1}}{-1} + \frac{1}{4}\ln|x+1| + \frac{1}{4}\frac{(x+1)^{-1}}{-1} + C$$

$$= \frac{-1}{4}\ln|x-1| - \frac{1}{4}\frac{1}{x-1} + \frac{1}{4}\ln|x+1| - \frac{1}{4}\frac{1}{x+1} + C$$

$$= \frac{1}{4}\Big[\ln|x+1| - \ln|x-1|\Big] - \frac{1}{4}\left[\frac{1}{x-1} + \frac{1}{x+1}\right] + C$$

$$= \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{4}\left[\frac{x\cancel{+1} + x\cancel{-1}}{(x-1)(x+1)}\right] + C$$

$$\Rightarrow \int \frac{1}{(x^2-1)^2}\,dx = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{2}\,\frac{x}{x^2-1} + C$$

(14) Length of $y = \frac{1}{2}x^2$, $0 \le x \le 1$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \Rightarrow y = \frac{1}{2}x^2 \Rightarrow \frac{dy}{dx} = \frac{1}{2}(2x) = x$$

$$= \int_0^1 \sqrt{1+x^2}\,dx \qquad \Rightarrow x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$$

$x = 0 = \tan\theta \Rightarrow \theta = \tan^{-1}(0) = 0$

$x = 1 = \tan\theta \Rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\underbrace{1+\tan^2\theta}_{\sec^2\theta}}\;\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec\theta\,\sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^3\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos\theta}{\cos^4\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos\theta}{(1-\sin^2\theta)^2}\,d\theta$$

$\cos^4\theta = [\cos^2\theta]^2 = [1-\sin^2\theta]^2$

$u = \sin\theta$

$\Rightarrow du = \cos\theta\,d\theta$

$$= \int_{\sin 0}^{\sin\frac{\pi}{4}} \frac{1}{(1-u^2)^2}\,du = \underbrace{\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{(u^2-1)^2}\,du}_{\text{same as previous integral}}$$

$$= \left[\frac{1}{4}\ln\left|\frac{u+1}{u-1}\right| - \frac{1}{2}\,\frac{u}{u^2-1}\right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \left[\frac{1}{4}\ln\left|\frac{\frac{1}{\sqrt{2}}+1}{\frac{1}{\sqrt{2}}-1}\right| - \frac{1}{2}\,\frac{\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^2-1}\right] - \cancel{\left[\frac{1}{4}\ln\left|\frac{0+1}{0-1}\right| - \frac{1}{2}\,\frac{0}{0^2-1}\right]} \rightarrow 0$$

$$\frac{1}{4}\ln\left|\frac{1+\sqrt{2}}{1-\sqrt{2}}\right| \cancel{-\frac{1}{2}} \frac{\frac{1}{\sqrt{2}}}{\cancel{\left(\frac{1}{2}-1\right)}}$$

$$= \frac{1}{4}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) + \frac{1}{\sqrt{2}}$$

Problem 15: Evaluate the integral

$$\int_{-1}^{0} \frac{x^2}{(1+x^3)^4}\,dx$$

[8 pts]

At $x=-1$

$$\frac{x^2}{(1+x^3)^4} \longrightarrow \frac{(-1)^2}{[1+(-1)^3]^4} = \frac{1}{0^4} = \frac{1}{0} \quad (\text{discont. at } x=-1)$$

$$\lim_{t\to -1^+}\int_t^0 \frac{x^2}{(1+x^3)^4}\,dx = \lim_{t\to -1^+} \left.\frac{-1}{9(1+x^3)^3}\right|_t^0 = \lim_{t\to -1^+}\left[\frac{-1}{9} - \frac{-1}{9(1+t^3)^3}\right]$$

$$= \frac{-1}{9} + \lim_{t\to -1^+}\frac{1}{9(1+t^3)^3} = \frac{-1}{9} + \frac{1}{9(0)}$$

$$= \frac{-1}{9} + \infty = \infty$$

$\Rightarrow$ The given integral is divergent.

$$\int \frac{x^2}{(1+x^3)^4}\,dx = \int \frac{1}{u^4}\,\frac{1}{3}\,du$$

$u = 1+x^3$

$du = 3x^2\,dx$

$\Rightarrow \frac{1}{3}du = x^2\,dx$

$$= \frac{1}{3}\int u^{-4}\,du$$

$$= \frac{1}{3}\,\frac{u^{-4+1}}{-4+1} = \frac{-1}{9}u^{-3} = \frac{-1}{9u^3} + C = \frac{-1}{9(1+x^3)^3} + C$$

Bonus Problem: Evaluate the integral

$$\int \frac{4x^3}{2x^2-3x+1}\,dx$$

[8 pts].

$$\int \frac{4x^3}{2x^2-3x+1}\,dx$$

$$\begin{array}{r|l} & 2x+3 \\ \hline 2x^2-3x+1 & \cancel{4x^3} \\ & \cancel{4x^3} - 6x^2 + 2x \\ \hline & \cancel{6x^2} - 2x \\ & \cancel{6x^2} - 9x + 3 \\ \hline & 7x-3 \end{array}$$

Dividend = Divisor X Quotient + R

$$4x^3 = (2x^2-3x+1)(2x+3) + 7x-3$$

$$\frac{4x^3}{2x^2-3x+1} = 2x+3 + \frac{7x-3}{2x^2-3x+1}$$

$$\int \frac{4x^3}{2x^2-3x+1}\,dx = \int (2x+3)\,dx + \int \frac{7x-3}{2x^2-3x+1}\,dx$$

$$= \int 2x\,dx + \int 3\,dx + \int \frac{7x-3}{2x^2-3x+1}\,dx$$

$$= x^2 + 3x + \int \frac{7x-3}{2x^2-3x+1}\,dx$$

$$2x^2-3x+1 = 2x^2 - x - 2x + 1 = x(2x-1) - 1(2x-1)$$

$$= (x-1)(2x-1)$$

$$\frac{7x-3}{2x^2-3x+1} = \frac{7x-3}{(x-1)(2x-1)} = \frac{a}{x-1} + \frac{b}{2x-1}$$

$$\Rightarrow \quad 7x-3 = a(2x-1) + b(x-1)$$

$$x = \frac{1}{2} \quad \Rightarrow \quad 7\left(\frac{1}{2}\right) - 3 = 0 + b\left(\frac{1}{2} - 1\right)$$

$$\Rightarrow \quad \frac{7}{2} - 3 = -\frac{1}{2}b \quad \Rightarrow \quad \frac{1}{2} = -\frac{1}{2}b \quad \Rightarrow \quad b = -1$$

$$x = 1 \quad \Rightarrow \quad 7(1) - 3 = a(2(1)-1) + 0$$

$$\Rightarrow \quad 4 = a \qquad \Rightarrow \quad a = 4$$

$$\int \frac{7x-3}{2x^2-3x+1}\,dx = \int \frac{4}{x-1}\,dx + \int \frac{-1}{2x-1}\,dx$$

$$= 4\int \frac{1}{x-1}\,dx - 1\int \frac{1}{2x-1}\,dx$$

$$= 4\ln|x-1| - \frac{1}{2}\ln|2x-1| + C$$

$$\int \frac{4x^3}{2x^2-3x+1}\,dx = x^2 + 3x + 4\ln|x-1| - \frac{1}{2}\ln|2x-1| + C$$

$$\lim_{t\to -1^+} \frac{1}{9(1+t^3)^3} = \lim_{h\to 0} \frac{1}{9\left[1+(-1+h)^3\right]^3}$$

$$= \lim_{h\to 0} \frac{1}{9\left[\cancel{1+(-1)^3} + h^3 + 3(-1)^2 h + 3(-1)h^2\right]^3}$$

$$= \lim_{h\to 0} \frac{1}{9\left[h^3 - 3h + 3h^2\right]^3} \to 0$$