**M16600 Lecture Notes**

*Section 6.7: Hyperbolic Functions*

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■ **Section 6.7** exercises, *page 489*: #1, 3, 7, 8, 9, 30, 31, 32, 33, 36, 37, 38, 59, 60, 61, 62, 63, 64.

## SUMMARY

- Definitions of Hyperbolic Functions and their graphs
- Some indentities
- Derivatives of Hyperbolic Functions. Hence, we get some more integral formulas.

Certain even and odd combinations of the exponential functions $e^x$ and $e^{-x}$ arise so frequently in mathematics and its applications that they deserve to be given special names. These are the ***Hyperbolic Functions***. In many ways, the hyperbolic functions are analogous to the trigonometric functions.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

## Graphs of Hyperbolic Functions

$\sinh(0) = \dfrac{e^0 - e^{-0}}{2} = \dfrac{1-1}{2} = 0$, $\cosh(0) = 1$

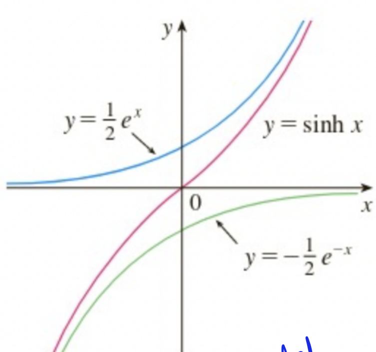


**FIGURE 1**
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

odd

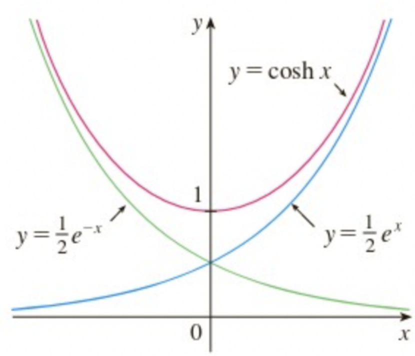


**FIGURE 2**
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

even

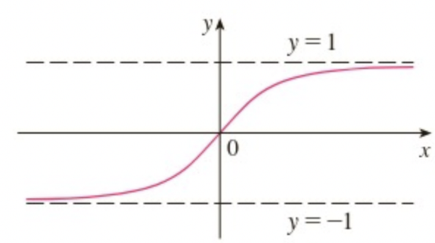


**FIGURE 3**
$y = \tanh x$

odd

$\lim_{x \to \infty} \tanh x = 1$, $\lim_{x \to -\infty} \tanh x = -1$

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities.

**Hyperbolic Identities**

sinh is odd function

Cosh is even function

$$\sinh(-x) = -\sinh(x) \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$\cos^2 x + \sin^2 x = 1$ $\qquad$ $1 + \tan^2 x = \sec^2 x$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Here are the derivative formulas of Hyperbolic Functions. Note that from these formulas, we also obtain integral formulas.

**Derivatives of Hyperbolic Functions**

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

**Inverse Hyperbolic Functions:** See textbook, page 486.

*Example 1*: Compute the derivative of $y = \tanh^5(x^5)$

$y = [\operatorname{Tanh}(x^5)]^5 = z^5 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(z^5) = \frac{d}{dz}(z^5)\frac{dz}{dx}$

Let $z = \operatorname{Tanh}(x^5)$

$= 5z^4 \frac{dz}{dx} = 5\operatorname{Tanh}^4(x^5)\frac{dz}{dx}$

Let $u = x^5 \Rightarrow z = \operatorname{Tanh}(u)$

$\Rightarrow \frac{dz}{dx} = \frac{d}{dx}(\operatorname{Tanh}(u)) = \frac{d}{du}(\operatorname{Tanh}(u))\frac{du}{dx} = \operatorname{sech}^2(u)(5x^4)$

$= 5x^4 \operatorname{sech}^2(x^5)$

$\Rightarrow \frac{dy}{dx} = (5\operatorname{Tanh}^4(x^5))(5x^4)\operatorname{sech}^2(x^5)$

$$= 25x^4 \tanh^4(x^5)\,\text{sech}^2(x^5)$$

*Example 2*: Evaluate the integral

(a) $\displaystyle\int \frac{\sinh(\ln x)}{x}\,dx$

$$= \int \sinh(\ln x)\,\underbrace{\frac{1}{x}\,dx}$$

let $u = \ln x$

$$\Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow \underbrace{du} = \frac{1}{x}\,dx$$

$$= \int \sinh(u)\,du$$

$$= \cosh(u) + C = \cosh(\ln x) + C$$

(b) $\displaystyle\int \frac{\sinh x}{1+\cosh x}\,dx$

$$= \int \frac{1}{1+\cosh x}\,\underbrace{\sinh x\,dx}$$

let $u = 1 + \cosh x$

$$\Rightarrow \frac{du}{dx} = 0 + \sinh x$$

$$\Rightarrow \underbrace{du} = \sinh x\,dx$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln|1+\cosh(x)| + C$$

(c) **What about** $\displaystyle\int \frac{\sinh x}{1+\cosh^2 x}\,dx$?

$$\int \frac{\sinh x}{1+\cosh^2 x}\,dx$$

Let $u = \cosh x$

$\Rightarrow \frac{du}{dx} = \sinh x$

$\Rightarrow du = \sinh(x)\,dx$

$$= \int \frac{1}{1+\cosh^2 x}\,\sinh(x)\,dx$$

$$= \int \frac{1}{1+u^2}\,du$$

$$= \tan^{-1}(u) + C = \tan^{-1}(\cosh x) + C$$