■ **Section 11.9** textbook exercises, *page 797*: # 3, 4, 5, 6, 8, 13, 15.

In this section, we will learn how to represent certain types of functions as power series by manipulating geometric series or by differentiating or integrating such a series.

We will start with the geometric series

$\sum_{n=0}^{\infty} c_n (x-a)^n$ = output

input

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \infty = \frac{1}{1-x}$$

$r = \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{x^n} = x$

Domain = Interval of Convergence

as long as $|x| < 1$ or $-1 < x < 1$

Thus, we get the first example of a function that is represented by a power series

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } x \in (-1, 1)$$

lies in

By manipulating this first example, many other functions can also be represented as power series.

Example 1: Find a power series representation for the function and determine the interval of convergence

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots \infty \qquad -1 < r < 1$$

(a) $\frac{1}{1-x^2} = 1 + x^2 + (x^2)^2 + (x^2)^3 + \cdots$

$r = x^2$

$= 1 + x^2 + x^4 + x^6 + x^8 + \cdots$

$= \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n}$

true as long as $|x^2| < 1 \Rightarrow |x|^2 < 1 \Rightarrow \underbrace{-1}_{\text{ignore}} < |x| < 1$

because $0 \leq |x| < 1$

$|x| < 1$

$\Rightarrow -1 < x < 1$

$|x| < a \Rightarrow -a < x < a$

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}, \quad IOC = (-1, 1)$$

$\boxed{\frac{1}{1-r} \to \begin{array}{l}\text{Converges if } |r|<1 \\ \text{diverges if } |r|\geq 1\end{array}}$ $\to$ For Geom. series $|r|=1$ gives divergence.

(b) $\dfrac{1}{2-x}$

$$= \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{1}{2}\left[\frac{1}{1-\frac{x}{2}}\right] = \frac{1}{2}\left[1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\cdots\infty\right]$$

r ✓

$$\left|\frac{x}{2}\right|<1 \Rightarrow |x|<2 \Rightarrow -2<x<2$$

$$\frac{1}{2-x} = \frac{1}{2}\left[1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\cdots\infty\right], \quad -2<x<2$$

$$\frac{1}{2-x} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}, \quad -2<x<2$$

$$\Rightarrow \frac{1}{2-x} = \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}, \quad \text{IOC} = (-2, 2)$$

(c) $\dfrac{x}{1+2x}$

$$= x\cdot\frac{1}{1+2x} = x\left(\frac{1}{1-(-2x)}\right) = x\left[\sum_{n=0}^{\infty}(-2x)^n\right]$$

as long as $|-2x|<1$
$\Rightarrow |2||x|<1$
$\Rightarrow |x|<\frac{1}{2}$
$\Rightarrow -\frac{1}{2}<x<\frac{1}{2}$

$r=-2x$

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

$$= x\sum_{n=0}^{\infty}(-2x)^n, \quad -\frac{1}{2}<x<\frac{1}{2}$$

$$x\cdot x^n = x^{1+n}$$

$$= \sum_{n=0}^{\infty} x\,(-2x)^n = \sum_{n=0}^{\infty} x\,(-2)^n\,x^n = \sum_{n=0}^{\infty}(-2)^n x^{n+1}$$

$$\Rightarrow \frac{x}{1+2x} = \sum_{n=0}^{\infty}(-2)^n x^{n+1}, \quad x\in\left(-\frac{1}{2}, \frac{1}{2}\right)$$

n $\quad$ $n+1$

also correct

$$\sum_{n=0}^{\infty} x^{n+1} = x+x^2+\cdots$$

$$\sum_{n=1}^{\infty} x^n = x+x^2+\cdots$$

$$= \left[(-2)^0 x^1 + (-2)^1 x^2 + (-2)^2 x^3 + \cdots\right]$$

$$\frac{x}{1+2x} = \sum_{n=1}^{\infty}(-2)^{n-1}x^n, \quad \text{IOC} = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$$\sum_{n=0}^{\infty}(-2)^n x^{n+1} = \sum_{n=1}^{\infty}(-2)^{n-1}x^n$$

Differentiation and Integration of Power Series.

If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ $\xleftarrow{+0} \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$ $n=0$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Example 2:

(a) $\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(x^n) = \sum_{n=0}^{\infty} n x^{n-1}$ → because for $n=0$

$a=0$

$C_n=1$

$0 \cdot x^{0-1} = \underline{\underline{0}}$

$= \sum_{n=1}^{\infty} n x^{n-1}$

(b) $\int \left(\sum_{n=0}^{\infty} x^n\right) dx = \sum_{n=0}^{\infty} \int x^n\,dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

$= C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

$= C + \frac{x^{0+1}}{0+1} + \frac{x^{1+1}}{1+1} + \frac{x^{2+1}}{2+1} + \frac{x^{3+1}}{3+1} + \cdots\cdots\infty$

$= C + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\cdots\infty$

$= C + \sum_{n=1}^{\infty} \frac{x^n}{n}$

By differentiation or integration, we can find power series representation for more functions.

Example 3: Find a power series representation for the function and determine the radius of convergence.

(a) $\dfrac{1}{(1-x)^2}$. **Hint:** Note that $\dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)$ $\rightarrow \dfrac{d}{dx}\left((1-x)^{-1}\right)$

$$= (-1)(1-x)^{-1-1}\frac{d}{dx}(1-x)$$
$$= (-1)(1-x)^{-2}(-1)$$
$$= (1-x)^{-2}$$
$$= \frac{1}{(1-x)^2}$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$R = 1 \quad \{|x|<1\}$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=0}^{\infty}\frac{d}{dx}(x^n)$$

R does not change.

$$= \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n-1}$$

Standard

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n, \quad \boxed{IOC = (-1,1)}$$

$a=0,\ R=1$

also correct

$\rightarrow 1x^{1-1} + 2x^{2-1} + 3x^{3-1} + \cdots$

$1 + 2x + 3x^2 + 4x^3 + \cdots$

$x^0 \quad x^1 \quad x^2 \quad x^3$

$n = 0,\ 1,\ 2,\ 3, \ldots$

$C_n = 1,\ 2,\ 3,\ 4, \ldots$

(b) $\ln(1+x)$. **Hint:** Think about integration.

$$\ln(1+x) = \int \frac{1}{1+x}\,dx$$

$|-x|<1 \Rightarrow |x|<1$

$$\frac{1}{1+x} = \frac{1}{1-\underset{r}{(-x)}} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^n\,dx = \sum_{n=0}^{\infty}\int \underbrace{(-1)^n}_{\text{constant} \mid \text{independent of } x} x^n\,dx$$

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\int x^n\,dx = \underbrace{C}_{\text{needs to be determined.}} + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$$

$$\frac{(-1)^0 x^{0+1}}{0+1} + \frac{(-1)^1 x^{1+1}}{1+1} + \cdots$$

$x=0 \longrightarrow x,\ x^2,\ x^3$

Put $x=0$

$$\ln 1 = C + 0 \Rightarrow \boxed{0 = C}$$

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}, \quad IOC = (-1,1)$$

(c) $\tan^{-1}(x)$. **Hint:** Think about integration.

$$\tan^{-1}(x) = \int \frac{1}{1+x^2}\,dx$$

$$|-x^2| < 1 \Rightarrow |x^2| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$$

$$\frac{1}{1+x^2} = \frac{1}{1-\underbrace{(-x^2)}_{r}} = \sum_{n=0}^{\infty} (-x^2)^n, \quad IOC = (-1,1)$$

$$\tan^{-1}(x) = \int \sum_{n=0}^{\infty} (-x^2)^n\,dx = \sum_{n=0}^{\infty} \int (-x^2)^n\,dx$$

$$= \sum_{n=0}^{\infty} \int (-1)^n (x^2)^n\,dx = \sum_{n=0}^{\infty} (-1)^n \int x^{2n}\,dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \longrightarrow \frac{(-1)^0 x^{2(0)+1}}{2(0)+1} + \frac{(-1)^1 x^{2(1)+1}}{2(1)+1} + \cdots$$

C needs to be determined.

$x=0 \longrightarrow x, \quad x^3, \quad x^5,$

$x=0$

$$\tan^{-1}(0) = C + 0 \Rightarrow \boxed{0 = C}$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad IOC = (-1,1)$$

$\sin(x) \rightsquigarrow$ Power series : sum of infinite terms.

Take some $\underline{N}$ number of terms.

$$\tan^{-1}(1) \approx \sum_{n=0}^{100} (-1)^n \frac{(1)^{2n+1}}{2n+1} = \sum_{n=0}^{100} \frac{(-1)^n}{2n+1} \approx \frac{\pi}{4}$$

machine computes every term.