

Derivatives of logarithmic functions

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \frac{du}{dx}.$$

$$\frac{d}{dx}(\log_b u) = \frac{1}{u \ln b} \frac{du}{dx}.$$

Example 1. Differentiate $y = \log_2 x^2$ with respect to x .

$$\begin{aligned} y' &= [\log_2 x^2]' = \frac{1}{x^2 \ln 2} \cdot [x^2]' \\ &= \frac{1}{x^2 \ln 2} \cdot 2x = \frac{2}{x \ln 2} \end{aligned}$$

Handwritten notes: $b=2$ points to the base 2; $u=x^2$ points to the argument x^2 . The final result $\frac{2}{x \ln 2}$ is highlighted in yellow.

Example 2. Differentiate $T = \log_{10}(v^2 + v)$ with respect to v .

$$\begin{aligned} \frac{dT}{dv} &= \frac{d}{dv} (\log_{10}(v^2 + v)) \quad \text{chain rule.} \\ &= \frac{1}{(v^2 + v) \ln 10} \cdot \frac{d}{dv} (v^2 + v) \\ &= \frac{1}{(v^2 + v) \ln 10} \cdot (2v + 1) \\ &= \frac{2v + 1}{(v^2 + v) \ln 10} \end{aligned}$$

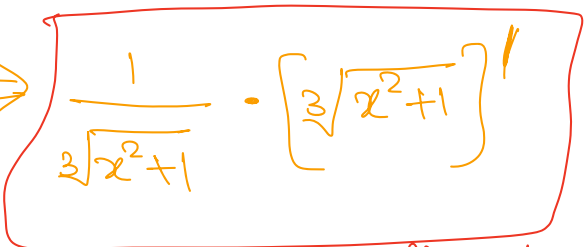
Handwritten notes: The final result $\frac{2v + 1}{(v^2 + v) \ln 10}$ is highlighted in yellow.

Example 3. Differentiate $y = \ln \sec x$ with respect to x .

$$\begin{aligned}
 y' &= [\ln(\sec x)]' = \frac{1}{\sec x} \cdot [\sec x]' \\
 &= \frac{\cancel{\sec x} \tan x}{\cancel{\sec x}} \\
 &= \tan x
 \end{aligned}$$

Example 4. Find dy/dx if $y = \ln \sqrt[3]{x^2 + 1}$.

use properties of \ln



$$\frac{1}{\sqrt[3]{x^2+1}} \cdot [\sqrt[3]{x^2+1}]'$$

more complicated.

$$y = \ln \sqrt[3]{x^2+1}$$

$$= \ln \underbrace{(x^2+1)}_m^{\frac{1}{3} \leftarrow k} = \frac{1}{3} \ln(x^2+1)$$

$$(\ln m^k = k \ln m)$$

$$y' = \frac{1}{3} [\ln(x^2+1)]' = \frac{1}{3} \frac{1}{x^2+1} \cdot (x^2+1)'$$

$$= \frac{1}{3} \frac{2x}{x^2+1} = \frac{2x}{3(x^2+1)}$$

Example 5. Find the derivative of $y = \ln(\sin^2 x/x)$.

 do not diff. directly.

$$y = \ln\left(\frac{\sin^2 x}{x}\right)$$

$$= \ln(\sin^2 x) - \ln x$$

$$= \ln(\sin x)^2 - \ln x$$

$$= 2 \ln(\sin x) - \ln x$$

$$y' = 2 [\ln(\sin x)]' - [\ln x]'$$

$$= 2 \frac{1}{\sin x} \cdot [\sin x]' - \frac{1}{x}$$

$$= 2 \frac{\cos x}{\sin x} - \frac{1}{x} = 2 \cot x - \frac{1}{x}$$

Logarithmic Differentiation: Differentiate $y = [f(x)]^{g(x)}$.

1. *Step 1:* Take \ln on both sides so that $\ln y = g(x) \ln f(x)$.
2. *Step 2:* Simplify the RHS if possible.
3. *Step 3:* Differentiate both sides with respect to x .

Note that the LHS always differentiates to $\frac{1}{y} \frac{dy}{dx}$.

4. *Step 4:* Multiply both sides with y to obtain $\frac{dy}{dx}$.

Example 6. Differentiate $y = x^x$.

Step 1 : $\ln y = \ln x^x = x \ln x$

Step 2 : No more simplification possible.

Step 3 : $\frac{1}{y} \frac{dy}{dx} = [x \ln x]'$ Product rule

$$= [x]' \ln x + x [\ln x]'$$

$$= 1 \cdot \ln x + \cancel{x} \cdot \frac{1}{\cancel{x}} = \ln x + 1$$

Step 4 $\frac{dy}{dx} = y (\ln x + 1) = x^x (\ln x + 1)$

Example 7. Differentiate $y = (\sin x)^{\cos x}$.

Step 1 $\ln y = \ln (\sin x)^{\cos x} = \cos x \ln(\sin x)$

Step 2 No simplification possible.

Step 3 $\frac{1}{y} \frac{dy}{dx} = [\underbrace{\cos x}_u \cdot \underbrace{\ln(\sin x)}_v]'$ (Product rule)

$$= (\cos x)' \ln(\sin x) + \cos x \cdot [\ln(\sin x)]'$$

$$= -\sin x \cdot \ln(\sin x) + \cos x \cdot \left[\frac{1}{\sin x} \cdot \underbrace{[\sin x]'}_{\cos x} \right]$$

$$= -\sin x \cdot \ln(\sin x) + \cos x \cdot \cot x \quad \rightarrow \frac{\cos x}{\sin x}$$

Step 4

$$\frac{dy}{dx} = y \left[-\sin x \cdot \ln(\sin x) + \cos x \cdot \cot x \right]$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\cos x} \left[-\sin x \cdot \ln(\sin x) + \cos x \cdot \cot x \right]$$

HW.

$$(x)^{\sin x}, \quad (\tan x)^x$$