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# MATH 171 BASIC LINEAR ALGEBRA

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## 1. Lines in two-dimensional space

The equation

$$2x - y = 3 \tag{1}$$

describes a line in two-dimensional space. The coefficients of $x$ and $y$ in the equation are just the components of the normal vector, here the normal vector is $\langle 2, -1\rangle$. Another line can either (1) be the same line so that their intersection is the line itself, (2) intersect the original line in one point or (3) be a distinct but parallel line where there is no intersection. There are no other possibilities.

The equation

$$-10x + 5y = -15 \tag{2}$$

describes the same line as Equation 1 since it is a multiple of Equation 1. Any point $(x, y)$ that satisfies one equation satisfies the other which illustrates case (1).

The equation

$$4x - 2y = 5 \tag{3}$$

describes a distinct parallel line to the line described by Equation 1. This is because the coefficients of $x$ and $y$ in Equation 3 are the same multiple of the corresponding coefficients in the Equation 1, so they have parallel normal vectors, but the equations are not multiples of each other. The lines do not intersect so there is no point $(x, y)$ that satisfies both equations. This illustrates case (3).

Finally, the equation

$$3x + 2y = 1 \tag{4}$$

describes a line that intersects the line described by Equation 1 at the single point $(1, -1)$, since they have normal vectors $\langle 2, -1\rangle$ and $\langle 3, 2\rangle$ and one is not a multiple of the other. It is the only point that satisfies both equations and this illustrates case (2).

When presented with two equations of lines we can determine which of the three cases the lines describe by *eliminating a variable.* Equations 1 and 2 are the pair

$$\begin{array}{rrcr} 2x & -y & = & 3 \\ -10x & +5y & = & -15. \end{array} \tag{5}$$

To eliminate the variable $y$ we multiply Equation 1 by 5. This will not change the intersection since the multiplied equation still describes the same line. Then we add the two equations together. Since any point that satisfies both equations will satisfy

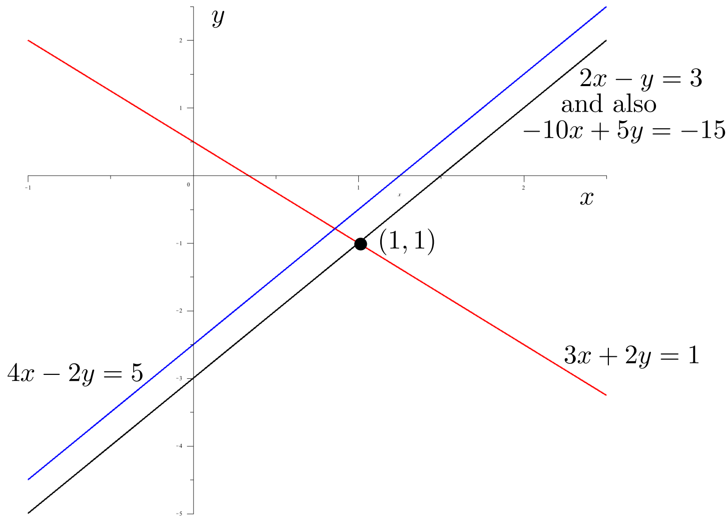


FIGURE 1. The equations $2x - y = 3$ and $-10x + 5y = -15$ give the same line, while $4x - 2y = 5$ gives a parallel line to the original line that does not intersect it, and $3x + 2y = 1$ gives a line that intersects the original line in a single point $(1, 1)$.

their sum we have not eliminated any possible intersection points. This procedure produces the new equations

$$\begin{array}{rrcr} 10x & -5y & = & 15 \\ -10x & +5y & = & -15 \\ ---- & --- & ---- & ---- \\ 0 & +0 & = & 0. \end{array}$$

The conclusion is that each equation is a multiple of the other and they describe the same line.

Equations 1 and 3 are the pair

$$\begin{array}{rrcr} 2x & -y & = & 3 \\ 4x & -2y & = & 5. \end{array} \tag{6}$$

To eliminate the variable $y$ we multiply the first equation by $-2$. By the same reasoning as before this will not change the intersection. Then we add the two equations together. Any point that satisfies both equations will satisfy their sum. The procedure produces the new equations

$$\begin{array}{rrcr} -4x & +2y & = & -6 \\ 4x & -2y & = & 5 \\ ---- & --- & ---- & ---- \\ 0 & +0 & = & -1. \end{array}$$

This is a contradiction since $0x + 0y \neq -1$ and the conclusion is that no point satisfies both equations so the lines do not intersect.

Finally, we consider Equations 1 and 4 which are

$$\begin{array}{rrcr} 2x & -y & = & 3 \\ 3x & +2y & = & 1. \end{array} \tag{7}$$

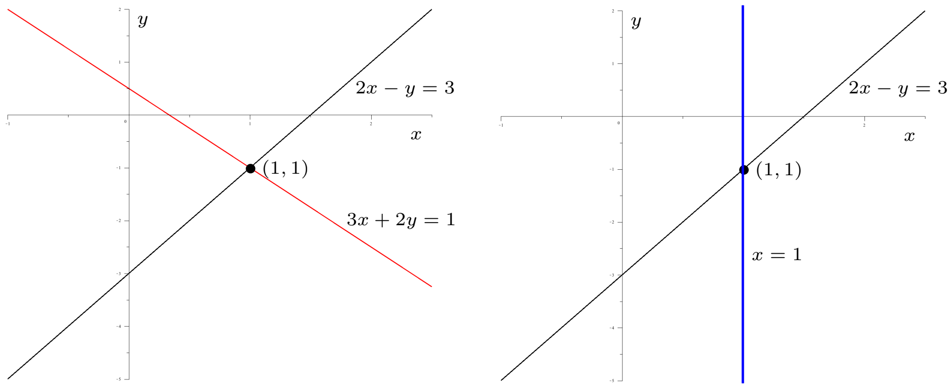


FIGURE 2. **Elimination of the variable $y$:** The lines given by $2x - y = 3$ and $3x+2y = 1$ (on the left) intersect in exactly the same point(s) as do the lines given by $2x - y = 3$ and $x = 1$ (on the right). However, the latter pair is easier to understand because the second equation does not depend on $y$ (so that it gives a vertical line).

To eliminate the variable $y$ we multiply the first equation by 2. As before this will not change the intersection. Then we add the two equations together. Again any point that satisfies both equations will satisfy their sum. The procedure produces the new equations

$$\begin{array}{rrcr} 4x & -2y & = & 6 \\ 3x & +2y & = & 1 \\ ---- & --- & ---- & ---- \\ 7x & +0 & = & 7. \end{array} \tag{8}$$

Consequently, $x = 1$ and substituting into either Equation 1 or 4 produces $y = -1$. The point $(1, -1)$ is the intersection of the two lines described by the two equations and is the only point that satisfies both.

There is an important point here. We have replaced the equation $3x + 2y = 1$ with the equation $7x = 7$ which is a line parallel to the $y$-axis and which gives a much simpler picture. A point satisfies the two equations $4x - 2y = 6$ and $7x = 7$ if and only if it satisfies the original pair of Equations 8. The reason is that we can recover the equation $3x + 2y = 1$ from the pair $4x - 2y = 6$ and $7x = 7$.

To summarize, suppose there are two equations in two variables

$$\begin{array}{rrcl} a_{11}x & +a_{12}y & = & b_1 \\ a_{21}x & +a_{22}y & = & b_2. \end{array}$$

The equations define two lines in two-dimensional space and we say each of the equations is a *linear equation.* We also say $a_{11}$ and $a_{12}$ are the *coefficients* in the first equation and $b_1$ is the *constant term* in the first equation. We assume that $a_{11}$ and $a_{12}$ are not both 0. Likewise we say $a_{21}$ and $a_{22}$ are the coefficients and $b_2$ is the constant term in the second equation and $a_{21}$ and $a_{22}$ are not both 0. The *solution set* for the system of equations is the intersection of the two lines which is the set of all $(x, y)$ that satisfy both equations. From the previous discussion we know there are three possibilities.

(1) The two equations describe the same line. This happens when there is a number $k$ so that $ka_{11} = a_{21}$, $ka_{12} = a_{22}$ and $kb_1 = b_2$.

(2) The two equations describe lines that intersect in one point. In this case we eliminate a variable to solve for the point of intersection. To eliminate $y$ we multiply the first equation by $a_{22}$, the second by $-a_{12}$ and add the resulting equations to see that $(a_{11}a_{22} - a_{12}a_{21})x = a_{22}b_1 - a_{12}b_2$. Since we are not in case (1) or (3) the quantity $a_{11}a_{22} - a_{12}a_{21}$ is not 0 and we solve for $x$. Then substitute the value of $x$ into either of the equations to find $y$. The result is

$$x = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}, \quad y = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}. \tag{9}$$

(3) The two equations describe distinct parallel lines. This happens when there is a number $k$ so that $ka_{11} = a_{21}$, $ka_{12} = a_{22}$ and $kb_1 \neq b_2$.

**Problems**

(1) Describe and sketch all possible types of intersections of two lines in $\mathbb{R}^2$. Give equations of a pair of lines that illustrate each type of intersection.

(2) On one page graph all of the the lines below. Then for each pair in the list eliminate a variable to determine if the pair define the same line, are parallel or intersect at one point. If a pair intersects at one point, find the point.
   (a) $3x - 2y = 4$
   (b) $2x + 5y = 7$
   (c) $-6x + 4y = -2$
   (d) $9x - 6y = 12$
   (e) $8x - 3y = 5$

(3) Describe and sketch all possible types of intersections of **three** lines in $\mathbb{R}^2$. Give equations of a triple of lines that illustrate each type of intersection.

(4) Eliminate variables to determine if each triple of lines in the list above, intersect or do not intersect. If a triple intersects at one point find the point.

## 2. Planes in three-dimensional space

The equation

$$2x - y + z = 3 \tag{10}$$

describes a plane in three-dimensional space. The normal vector is given by the coefficients and in this case is $\langle 2, -1, 1 \rangle$. Another plane can either (1) be the same plane so that their intersection is the plane itself, (2) intersect the original plane in a line or (3) be a distinct but parallel plane where there is no intersection. There are no other possibilities.

The equation

$$6x - 3y + 3z = 9 \tag{11}$$

describes the same plane as the plane described by Equation 10 because it is a multiple of Equation 10. Any point $(x, y, z)$ that satisfies one equation satisfies the other so this illustrates case (1).

The equation

$$-4x + 2y - 2z = 4 \tag{12}$$

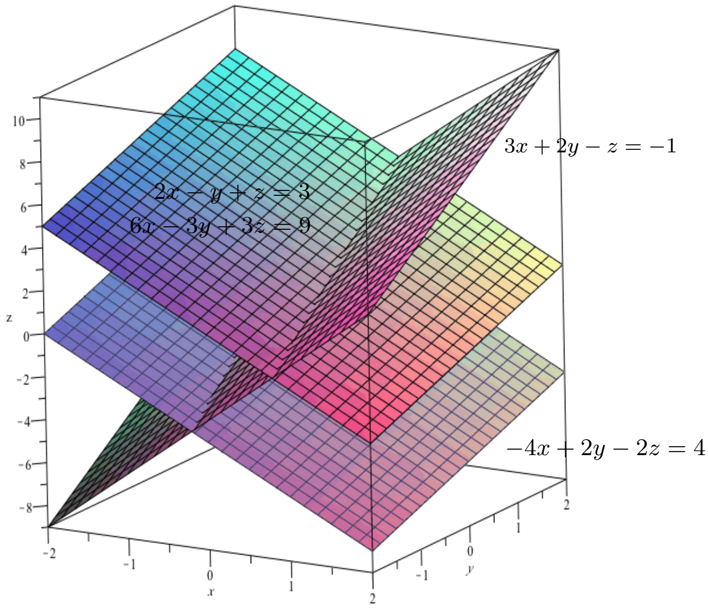


FIGURE 3. The equations $2x - y + z = 3$ and $6x + -3y + 3z = 9$ give the same plane, while $-4x + 2y - 2z = 4$ gives a plane that is parallel to the original one (below it in the figure) that does not intersect it, and $3x + 2y - z = -1$ gives another plane that intersects the original plane in a line.

describes a distinct parallel plane to the plane described by Equation 10 because the coefficients of $x$, $y$ and $z$ in Equation 12 are the same multiple of the corresponding coefficients in Equation 10, this means they have parallel normal vectors, but the equations are not multiples of each other. The planes do not intersect so there is no point $(x, y, z)$ that satisfies both equations. This illustrates case (3).

Finally, the equation

$$3x + 2y - z = -1 \tag{13}$$

describes a plane that intersects the plane described by Equation 10 in the line defined by the parametric equations

$$x = t, \qquad y = 2 - 5t, \qquad z = 5 - 7t.$$

The planes are not the same or parallel because one normal vector is not a multiple of the other. These are the points that satisfy both equations and it is any example of case (2).

As we did when presented with a pair of equations for lines we can determine which of the three cases a pair of planes describe by *eliminating variables*. For Equations 10 and 11 we eliminate the variable $z$ by multiplying Equation 10 by $-3$ and adding the result to Equation 11

$$
\begin{array}{rrrcr}
-6x & +3y & -3z & = & -9 \\
6x & -3y & +3z & = & 9 \\
---- & --- & ---- & ---- & --- \\
0 & +0 & +0 & = & 0.
\end{array}
\tag{14}
$$

The conclusion is that any point that satisfies the first equation satisfies the second and so the two equations describe the same plane.

For Equations 10 and 12 we eliminate the variable $z$ by multiplying Equation 10 by 2 and adding the result to Equation 12

$$
\begin{array}{rrrcr}
4x & -2y & +2z & = & 6 \\
-4x & +2y & -2z & = & 4 \\
---- & --- & ---- & ---- & --- \\
0 & +0 & +0 & = & 10.
\end{array}
\tag{15}
$$

This is a contradiction since $0x + 0y + 0z \neq 10$ and the conclusion is that no point satisfies both equations and the planes do not intersect.

Finally, for Equations 10 and 13 we eliminate the variable $z$ by adding the two

$$
\begin{array}{rrrcr}
2x & -y & +z & = & 3 \\
3x & +2y & -z & = & -1 \\
---- & --- & ---- & ---- & --- \\
5x & +y & +0 & = & 2.
\end{array}
\tag{16}
$$

Set $x$ equal to the parameter $t$ and the last equation becomes $y = 2 - 5t$. Now substitute the parametric equations for $x$ and $y$ into the either of the two original equations and solve for $z$. This produces the parametric equations

$$x = t, \qquad y = 2 - 5t, \qquad z = 5 - 7t$$

for the line of intersection.

Now we ask: what is the triple intersection of three planes in three-dimensional space? We will see that the triple intersection can be (1) a plane, (2) a line, (3) a point or (4) empty. We will also see that the method we have used of *eliminating variables* will allow us to find the intersection or solution set of a system of three equations in three variables.

Case (1), when the triple intersection of three planes in three-dimensional space is a plane happens only when all three planes are the same, meaning each of the three equations is a multiple of both of the others. This is as in the case (1) for the intersection of two planes above and we put it aside.

Case (2), when the triple intersection is a line is illustrated by Equations 10, 13 and a new equation

$$
\begin{array}{rrrcr}
2x & -y & +z & = & 3 \\
3x & +2y & -z & = & -1 \\
x & -4y & +3z & = & 7.
\end{array}
\tag{17}
$$

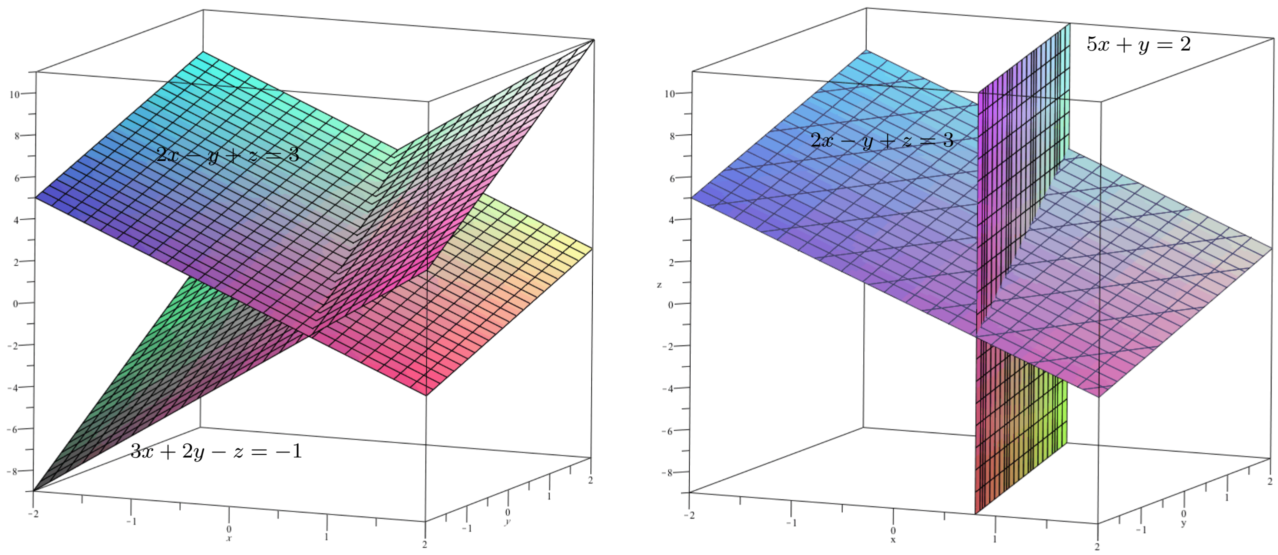


FIGURE 4. **Elimination of the variable $z$:** The planes given by $2x-y+z=3$ and $3x+2y-z=-1$ (on the left) intersect in exactly the same line as do the planes given by $3x+2y-z=-1$ and $5x+y=2$ (on the right). However, the latter pair is easier to understand because the second equation does not depend on $z$ (so that it vertical, i.e. parallel to the $z$-axis).

To eliminate the variable $z$ we start by adding the first two equations then adding 3 times the second equation to the third. This is done by

$$\begin{array}{rrrcr} 2x & -y & +z & = & 3 \\ 3x & +2y & -z & = & -1 \\ \hline 5x & +y & +0 & = & 2 \end{array}$$

and

$$\begin{array}{rrrcr} 9x & +6y & -3z & = & -3 \\ x & -4y & +3z & = & 7 \\ \hline 10x & +2y & +0 & = & 4. \end{array}$$

The two new equations in two variables $5x+y=2$ and $10x+2y=4$ describe the same line which can be parameterized by $x=t$ and $y=2-5t$. As before we substitute the parametric equations for $x$ and $y$ into any of the three original equations and solve for $z$. This produces the parametric equations

$$x = t, \qquad y = 2 - 5t, \qquad z = 5 - 7t$$

for the line of triple intersection. It is the solution set for the system of three equations.

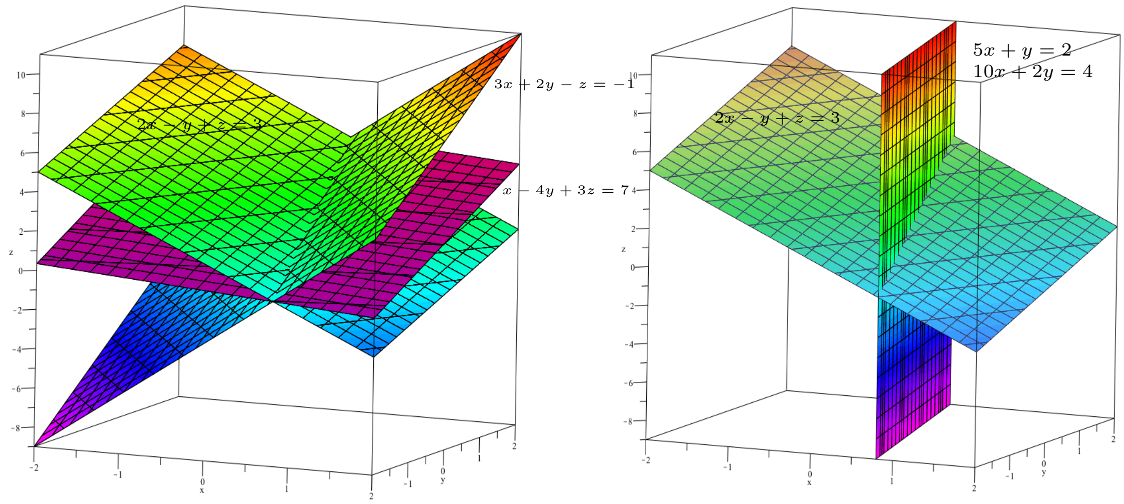


FIGURE 5. The three planes given by $2x - y + z = 3, 3x + 2y - z = -1$, and $x - 4y + 3z = 7$ (on the left) will intersect in the same line as the three planes given by $2x - y + z = 3, 5x + y = 2$, and $10x + 2y = 4$ (on the right). The second system is simpler for two reasons: (1) the latter two planes are vertical (parallel to $z$-axis) so they are simpler and (2) the latter two planes are the same, showing that the three planes will intersect in a line.

Case (3), when the triple intersection is a single point is illustrated by Equations 10, 13 and another new equation.

$$\begin{array}{rrrcr} 2x & -y & +z & = & 3 \\ 3x & +2y & -z & = & -1 \\ x & -3y & +2z & = & 2. \end{array} \tag{18}$$

To eliminate the variable $z$ we start as before by adding the first two equations to obtain the equation $5x + y = 2$. Then we add 2 times the second equation to the third obtaining

$$\begin{array}{rrrcr} 6x & +4y & +2z & = & -2 \\ x & -3y & -2z & = & 2 \\ ---- & --- & ---- & ---- & --- \\ 7x & +y & +0 & = & 0. \end{array}$$

Now we have a system of two equations in two variables

$$\begin{array}{rrcr} 5x & +y & = & 2 \\ 7x & +y & = & 0. \end{array}$$

These have the solution $x = -1$ and $y = 7$. Substituting these values of $x$ and $y$ into any of the three original equations yields $z = 12$. The triple intersection is the point $(-1, 7, 12)$ and it is the solution set for the system of equations.

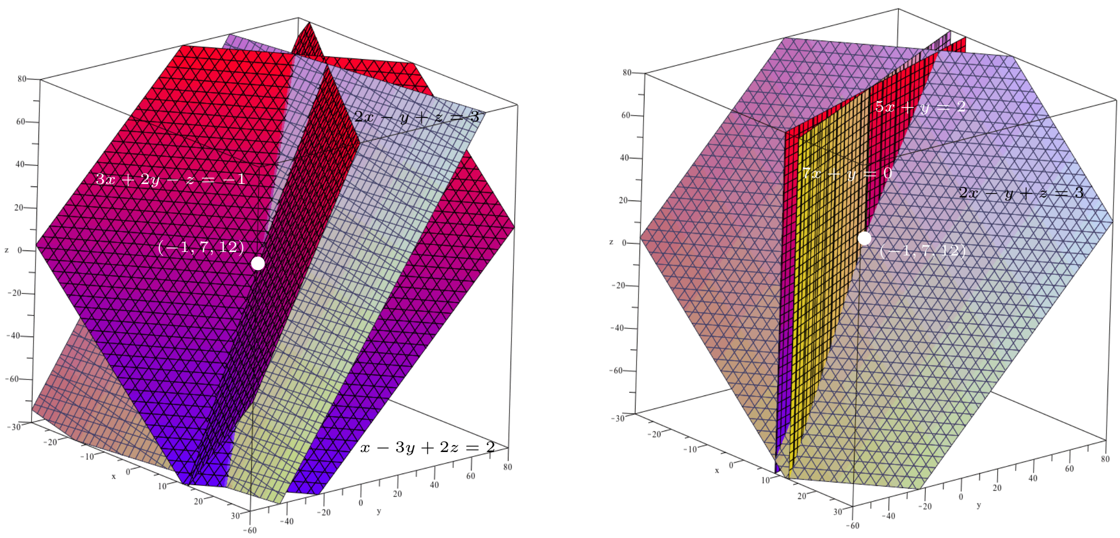


FIGURE 6. The three planes given by $2x-y+z=3, 3x+2y-z=-1$, and $x-3y+2z=2$ intersect in a point $(-1,7,12)$ (on the left). To see this, we eliminate the variable $z$ from two of the equations, replacing the three equations with $2x-y+z=3$ and the two vertical planes $5x+y=2$ and $7x+y=0$ (on the right). The latter system is simpler since the vertical planes intersect in a vertical line given by $x=-1, y=7$.

Case (4), when the triple intersection is empty is illustrated by Equations 10, 13 and a yet another equation

$$\begin{array}{rrrcr} 2x & -y & +z & = & 3 \\ 3x & +2y & -z & = & -1 \\ x & -4y & +3z & = & 2. \end{array} \tag{19}$$

To eliminate the variable $z$ we start as we have before by adding the first two equations to obtain the equation $5x+y=2$. Then we add 3 times the second equation to the third obtaining

$$\begin{array}{rrrcr} 9x & +6y & -3z & = & -3 \\ x & -4y & +3z & = & 2 \\ ---- & --- & ---- & ---- & --- \\ 10x & +2y & +0 & = & -1. \end{array}$$

We have produced a system of two equations in two variables

$$\begin{array}{rrcr} 5x & +y & = & 2 \\ 10x & +2y & = & -1. \end{array}$$

These describe two distinct parallel lines that do not intersect. It means the solution set is empty and the triple intersection of the planes is empty.

To summarize and generalize the previous discussion define a *linear equation in $n$ variables* to be of the form

$$a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} + a_nx_n = b \tag{20}$$

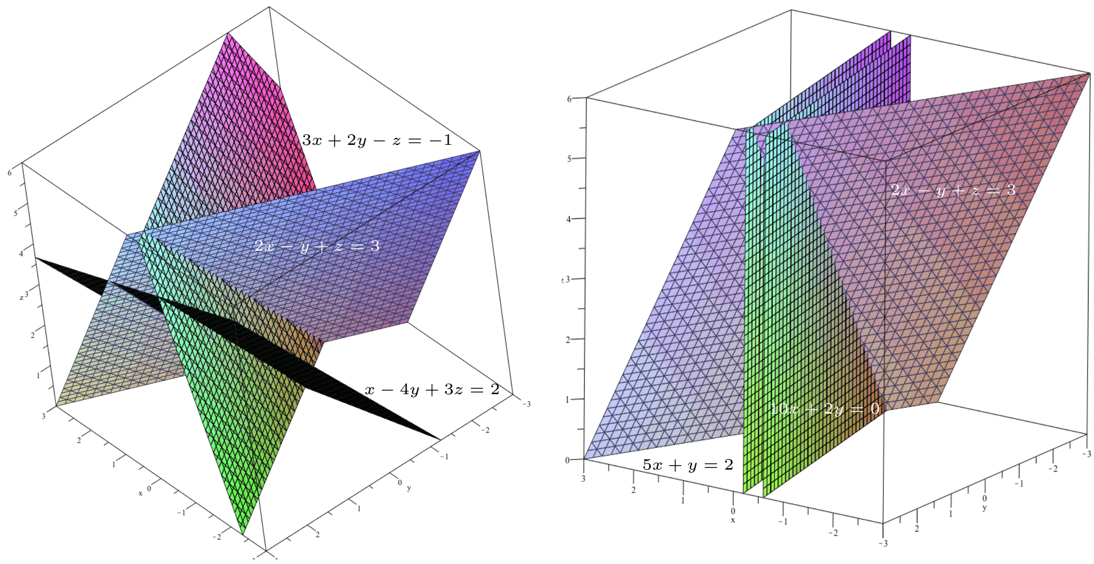


FIGURE 7. The three planes given by $2x-y+z=3, 3x+2y-z=-1$, and $x-4y+3z=2$ do not intersect (on the left). To see this, we eliminate the variable $z$ from two of the equations, replacing the three equations with $2x-y+z=3$ and the two vertical planes $5x+y=2$ and $10x+2y=0$ (on the right). The latter system is simpler since two of the planes are vertical planes and also they are parallel so that they do not intersect.

where $a_1, a_2, \ldots, a_{n-1}, a_n$ are real numbers and are called the *coefficients* of the equation, $x_1, x_2, \ldots, x_{n-1}, x_n$ are the *variables* (like $x, y, z$) of the equation, and $b$ is a real number called the *constant term* of the equation. The equation is called *linear* because the exponent of each variable is 1.

Next define a *system of $m$ linear equations in $n$ variables* to be of the form

$$\begin{array}{rrcrrcl}
a_{11}x_1 & +a_{12}x_2 & +\cdots+ & a_{1(n-1)}x_{n-1} & +a_{1n}x_n & = & b_1 \\
a_{21}x_1 & +a_{22}x_2 & +\cdots+ & a_{2(n-1)}x_{n-1} & +a_{2n}x_n & = & b_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{(m-1)1}x_1 & +a_{(m-1)2}x_2 & +\cdots+ & a_{(m-1)(n-1)}x_{n-1} & +a_{(m-1)n}x_n & = & b_{(m-1)} \\
a_{m1}x_1 & +a_{m2}x_2 & +\cdots+ & a_{m(n-1)}x_{n-1} & +a_{mn}x_n & = & b_m.
\end{array}$$

There are $m$ linear equations in $n$ variables, each has $n$ coefficients and 1 constant term. We say it is an $m$ by $n$ or $m \times n$ system of linear equations. We have examined $2 \times 2$, $2 \times 3$ and $3 \times 3$ systems of linear equations. The *solution set* for the system of linear equations is the set of all $n$-tuples of real numbers, $(x_1, x_2, \ldots, x_{n-1}, x_n)$, that satisfy all $m$ equations. Given any $m \times n$ system of linear equations, with care and patience the solution set can be found by *eliminating variables.*

You can imagine that each equation describes an $(n-1)$-dimensional "plane" in $n$-dimensional space. A line is a 1-dimensional plane in $\mathbb{R}^2$. We define a point in $\mathbb{R}^2$ to be a 0-dimensional plane. In $\mathbb{R}^3$, a point is a 0-dimensional plane, a line is a 1-dimensional plane and the usual type of plane is a 2-dimensional plane . The equation

$$2x_1 - 3x_2 - 5x_3 + x_4 = 2 \tag{21}$$

defines a 3-dimensional plane in four dimensional space, $\mathbb{R}^4$. The space $\mathbb{R}^4$ contains 0, 1, 2 and 3 dimensional planes. Observe that in our examples of systems of linear equations the solution set is always either empty or a plane. With the correct definition of "planes" this is always true. The solution set for an $m \times n$ system of linear equations is either empty or a plane in $\mathbb{R}^n$. We discuss this in the exercises.

**Problems**

(1) Describe and sketch all possible types of intersections of two planes in $\mathbb{R}^3$.
(2) Eliminate variables to determine if each pair of planes in the list below, define the same plane, intersect in a line or do not intersect. If the pair intersects in a line find parametric equations for the line of intersection.
  (a) $3x - 2y + z = 2$
  (b) $2x + y - z = 2$
  (c) $-x + 3y + 2z = 4$
  (d) $4x + 2y - 2z = 4$
  (e) $-6x + 4y - 2z = -2$
(3) Eliminate variables to determine if each triple of planes in problem 2, define the same plane, intersect in a line, intersect in one point or do not intersect. If the triple intersects in a line find parametric equations for the line of intersection. If the triple intersect at one point find the point.
(4) Use the parametric equations for the line of intersection of the planes $(a)$ and $(b)$ in problem 2 to find the intersection of this line with the planes defined by $(c)$, $(d)$ and $(e)$. Compare this with the answers in problem 3 you obtained by eliminating variables.

## 3. Matrices and elementary row operations

A *matrix* is a rectangular array of numbers. A matrix is usually labeled by a capital letter.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -4 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 3 \\ 2 & 1 \\ 0 & -2 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 & -2 \\ 4 & -1 & 1 \\ -2 & 1 & -3 \end{bmatrix}$$

$A$, $B$, $C$ and $D$ are all matrices. Matrix $A$ has two rows and two columns so we say it is a 2 by 2 or $2 \times 2$ matrix. Matrix $B$ has two rows and three columns and we say it is a $2 \times 3$ matrix. Then matrix $C$ is a $3 \times 2$ matrix and $D$ is a $3 \times 3$ matrix. An $m \times n$ or $m$ by $n$ matrix $E$ has $m$ rows, $n$ columns and $m \cdot n$ entries. In the general, the entries of the $m \times n$ matrix $E$ are labeled by $E_{ij}$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Using this labeling we see that $A_{11} = 2$, $A_{22} = -1$, $B_{12} = 0$, $B_{23} = -4$, $C_{12} = 3$, $C_{32} = -2$, $D_{22} = -1$ and $D_{32} = 1$.

Row $i$ of the matrix $E$ is the $1 \times n$ matrix $[E_{i1}\ E_{i2} \cdots E_{in}]$. A $1 \times n$ matrix is often called a *row vector*. Column $j$ of $E$ is the $m \times 1$ matrix

$$\begin{bmatrix} E_{1j} \\ E_{2j} \\ \vdots \\ E_{mj} \end{bmatrix}.$$

A $m \times 1$ matrix is often called a *column vector*.

A matrix is *square* if the number of rows is the same as the number of columns. A square $m \times m$ matrix $E$ has a *diagonal* which is the ordered collection of $m$ numbers $E_{11}, E_{22}, \ldots, E_{mm}$. The trace of a square matrix is the sum of the elements on the diagonal, $E_{11} + E_{22} + \cdots + E_{mm}$.

Every matrix has a *transpose*. An $m \times n$ matrix $A$ has as its transpose the $n \times m$ matrix $A^T$ which is defined by the relationship that the $ij$ entry of the transpose matrix is the $ji$ entry of the original matrix, which mean $(A^T)_{ij} = A_{ji}$. Two examples are

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 & 5 \\ 3 & -2 & 1 \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 5 & 1 \end{bmatrix}.$$

We can associate to a system of linear equations such as (Equations 7 from above)

$$\begin{array}{rrcr} 2x & -y & = & 3 \\ 3x & +2y & = & 1 \end{array}$$

several matrices. The *augmented matrix* associated to the system is the $2 \times 3$ matrix

$$\left[\begin{array}{rr|r} 2 & -1 & 3 \\ 3 & 2 & 1 \end{array}\right].$$

There are two parts to the augmented matrix. One part is the *coefficient matrix*

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

and the last column of the augmented matrix is the *constant column vector*

$$\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

We use the vertical bar in the augmented matrix as a reminder that we are thinking of the augmented matrix as the coefficient matrix together with the constant column vector.

Using these ideas we see that the system of linear equations (Equations 18 from above)

$$\begin{array}{rrrcr} 2x & -y & +z & = & 3 \\ 3x & +2y & -z & = & -1 \\ x & -3y & +2z & = & 2. \end{array}$$

has the associated matrices

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 3 \\ 3 & 2 & -1 & -1 \\ 1 & -3 & 2 & 2 \end{array}\right] \qquad A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

When we were presented with a system of linear equations we eliminated variables to find the solution set. Another approach is to start with a system of linear equations, such as

$$\begin{array}{rrcr} 2x & -y & = & 3 \\ 3x & +2y & = & 1, \end{array}$$

then write down the associated augmented matrix

$$\left[\begin{array}{rr|r} 2 & -1 & 3 \\ 3 & 2 & 1 \end{array}\right].$$

Now, instead of manipulating equations we manipulate the rows of the matrix. First, as we did earlier, we multiply the first row of the augmented matrix by 2 to produce the matrix

$$\left[\begin{array}{rr|r} 4 & -2 & 6 \\ 3 & 2 & 1 \end{array}\right]$$

and then add the first row to the second to obtain the matrix

$$\left[\begin{array}{rr|r} 4 & -2 & 6 \\ 7 & 0 & 7 \end{array}\right].$$

At this point we see that $7x = 7$ or $x = 1$ and we can substitute into the equation $4x - 2y = 6$ to find $y$. Or, we can continue to simplify the matrix. To continue we divide the second row by 7 and interchange the rows to obtain the matrix

$$\left[\begin{array}{rr|r} 1 & 0 & 1 \\ 4 & -2 & 6 \end{array}\right].$$

Then we add $-4$ times the first row to the second to obtain the matrix

$$\left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & -2 & 2 \end{array}\right].$$

Next multiply the second row by $-1/2$ to produce the matrix

$$\left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right].$$

From this matrix we immediately read off the solution set for the system of equations. It consists of the single point $(1, -1)$.

We have performed three types of operations on the rows of the matrices. These are the *elementary row operations*. They are the only three operations allowed and they do not change the solution set of the associated system of linear equations. The three elementary row operations are:

(1) interchanging two rows of the matrix,
(2) multiplying a row by a nonzero number,
(3) adding a multiple of one row to another.

The crucial point about elementary row operations is that they are reversible. It means that if $B$ is a matrix obtained from a matrix $A$ by an elementary row operation then $A$ can be obtained from $B$ by an elementary row operation. The consequence is that the system of equations associated to $A$ and the system of equations associated to $B$ have the the same solution set.

Let us find the solution set for a system of three linear equations in three unknowns using the elementary row operations. The system is

$$\begin{array}{rrrcr} 2x & -y & +z & = & 3 \\ 3x & +2y & -z & = & -1 \\ x & -3y & +2z & = & 2. \end{array}$$

and we have already seen that the solution set consists of the single point $(-1, 7, 12)$. First we write down the associated augmented matrix

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 3 \\ 3 & 2 & -1 & -1 \\ 1 & -3 & 2 & 2 \end{array}\right].$$

Then we interchange the first and third row to obtain

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 2 \\ 3 & 2 & -1 & -1 \\ 2 & -1 & 1 & 3 \end{array}\right].$$

Next we add $-3$ times the first row to the second and $-2$ times the first row to the third which results in the matrix

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 2 \\ 0 & 11 & -7 & -7 \\ 0 & 5 & -3 & -1 \end{array}\right].$$

Now add $-2$ times the third row to the second giving

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & -5 \\ 0 & 5 & -3 & -1 \end{array}\right].$$

Then add $-5$ times the second row to the third to obtain

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 2 & 24 \end{array}\right].$$

Finally we multiply the third row by $1/2$ which results in the matrix

$$\left[\begin{array}{rrr|r} 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 12 \end{array}\right].$$

From this matrix we read off the solution set. The third row tell us that $z = 12$. Substituting the value of $z$ into the second row tells us that $y = 7$. Then substituting the values for $y$ and $z$ into the first row tell us that $x = -1$. The solution set consists of the single point $(-1, 7, 12)$.

Or, we can also continue to simplify the matrix by adding the third row to the second and $-2$ times the third row to the first arriving at the matrix

$$\left[\begin{array}{rrr|r} 1 & -3 & 0 & -22 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 12 \end{array}\right].$$

Finally add 3 times the second row to the first and obtain the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 12 \end{array}\right].$$

Now you can immediately see that the solution set consists of the single point $(-1, 7, 12)$.

Let us examine three of the matrices we have produced using elementary row operations. They are

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & 12 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 12 \end{array}\right].$$

Notice the similarities of the shapes of the matrices. Two of the matrices are in reduced row-echelon form.

We say a matrix is in *reduced row-echelon form* if the following four conditions hold.

(1) The first nonzero entry in each nonzero row is a 1. This 1 is called a *leading* 1.
(2) The rows of all 0's are at the bottom of the matrix.
(3) The first row has the first (left-most) leading 1, the second row has the second (left-most) leading 1 and so forth.
(4) The columns containing leading 1's have all other entries 0.

If you go back and check you will see that none of the other matrices we produced meet all of these conditions. The first and third of these matrices meet all for of the conditions. The second fails to meet the fourth condition. We will see that matrices in reduced row-echelon form have special properties.

The method of using elementary row operations to put a matrix in reduced row-echelon form is called *Gaussian or Gauss-Jordan elimination.*

**Problems**

(1) For each pair of equations in problem 2, section 1 (Lines in two-dimensional space), write down the associated augmented matrix, the coefficient matrix and the constant vector.
(2) For each triple of equations in problem 2, section 2 (Planes in three-dimensional space,) write down the associated augmented matrix, the coefficient matrix and the constant vector.
(3) Determine which of the following matrices are in reduced row-echelon form.
    (a)

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

    (b)

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(4) There are 15 leading 1 patterns for $3 \times 4$ matrices in reduced row-echelon form (the matrix

$$\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} \qquad * = \text{unknown}$$

is one pattern). Write down the other 14.

## 4. Gaussian elimination

Now we will formalize the discussion of Gaussian elimination from the previous section. Given any matrix we can use Gaussian elimination to put it in reduced row-echelon form. Here is an example. Start with the matrix

$$\begin{bmatrix} 0 & -1 & 2 & 4 & -3 \\ 2 & -4 & 6 & -2 & 4 \\ 3 & -4 & 7 & -7 & 6 \\ 1 & -1 & 2 & -3 & -1 \\ -2 & 7 & -8 & -2 & -3 \end{bmatrix}.$$

First we interchange rows so that the (1,1) entry is nonzero.

$$\begin{bmatrix} 2 & -4 & 6 & -2 & 4 \\ 0 & -1 & 2 & 4 & -3 \\ 3 & -4 & 7 & -7 & 6 \\ 1 & -1 & 2 & -3 & -1 \\ -2 & 7 & -8 & -2 & -3 \end{bmatrix}$$

The second step is to multiply the first row by a number to make the (1,1) entry 1. It will be the first leading 1.

$$\begin{bmatrix} 1 & -2 & 3 & -1 & 2 \\ 0 & -1 & 2 & 4 & -3 \\ 3 & -4 & 7 & -7 & 6 \\ 1 & -1 & 2 & -3 & -1 \\ -2 & 7 & -8 & -2 & -3 \end{bmatrix}$$

We use this leading 1 to eliminate all other nonzero entries in its column. Add a multiple of the first row to each of the other rows so that the first entry of all but the first row is 0. This results in the matrix

$$\begin{bmatrix} 1 & -2 & 3 & -1 & 2 \\ 0 & -1 & 2 & 4 & -3 \\ 0 & 2 & -2 & -4 & 0 \\ 0 & 1 & -1 & -2 & -3 \\ 0 & 3 & -2 & -4 & 1 \end{bmatrix}.$$

Multiply the second row by $-1$ to get the next leading 1 and use it to eliminate all nonzero terms in the second column. This produces the matrix

$$\begin{bmatrix} 1 & 0 & -1 & -9 & 8 \\ 0 & 1 & -2 & -4 & 3 \\ 0 & 0 & 2 & 4 & -6 \\ 0 & 0 & 1 & 2 & -6 \\ 0 & 0 & 4 & 8 & -8 \end{bmatrix}.$$

Next we divide row three by 2 to produce the third leading 1 and use it to eliminate all other nonzero terms in the third column. This produces the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -7 & 5 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

Finally we divide row four by -3 to produce the fourth leading 1 and use it to eliminate all other nonzero terms in the fifth column. This produces the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix is now in reduced row-echelon form.

Any matrix can be put into reduced row-echelon form and the reduced row-echelon is unique. An algorithm to put a matrix into reduced row-echelon form follows. Follow the previous example as you read through the algorithm.

**Algorithm.** Let $A$ be an $m \times n$ matrix.

(1) Find the first column with a nonzero entry and call it $j_1$. Interchange rows so that the $(1, j_1)$ entry is nonzero.
(2) Multiply the first row by a number so that the $(1, j_1)$ entry is 1.
(3) Add a multiple of the first row to each other row so that every entry in the $j_1$ column except the 1 in the $(1, j_1)$ entry is 0.
(4) Excluding the first row find the first column with a nonzero entry and call it $j_2$. Interchange rows so that the $(2, j_2)$ entry is nonzero.
(5) Multiply the second row by a number so that the $(2, j_2)$ entry is 1.
(6) Add a multiple of the second row to each other row so that every entry in the $j_2$ column except the 1 in the $(2, j_2)$ entry is 0.
(7) Continue until the matrix is in reduced row-echelon form.

### Problems

(1) Use Gaussian elimination to put the following matrices in reduced row-echelon form.

(a)
$$\left[\begin{array}{rrrr|r} 3 & 3 & -4 & -2 & 1 \\ 2 & 2 & -3 & 1 & 3 \\ 1 & 1 & -2 & 4 & 5 \end{array}\right]$$

(b)
$$\left[\begin{array}{rrrr|r} 2 & 3 & 3 & -1 & 3 \\ 1 & 1 & -2 & 3 & 4 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right]$$

(c)
$$\left[\begin{array}{rr|r} 2 & -4 & 6 \\ -2 & 3 & -3 \\ 3 & 7 & 5 \end{array}\right]$$

(d)
$$\left[\begin{array}{rrr|r} 2 & -4 & 6 & 8 \\ 3 & 7 & 5 & 3 \\ -1 & -1 & 17 & 19 \end{array}\right]$$

(e)
$$\left[\begin{array}{rrrrr|r} -3 & -6 & 9 & 6 & -12 & -3 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ 1 & 2 & -3 & -2 & 4 & 1 \end{array}\right]$$

## 5. Reduced row-echelon matrices and solution sets

Given a system of linear equations we have seen how to write down the augmented matrix and then use Gaussian elimination to put the matrix in reduced row-echelon form. Now we will examine carefully how a matrix in reduced row-echelon form determines the solution set and conversely how the solution set determines the reduced row-echelon matrix.

We will rework the examples of the first two sections using the augmented matrices and their reduced row-echelon forms. The first examples are of two lines in $\mathbb{R}^2$. From Equations 5 we get the augmented matrix with its reduced row-echelon form.

$$\left[\begin{array}{rr|r} 2 & -1 & 3 \\ -10 & 5 & -15 \end{array}\right] \qquad \left[\begin{array}{lr|r} 1 & -1/2 & 3/2 \\ 0 & 0 & 0 \end{array}\right].$$

There is only one non-zero row and the leading 1 is in the first column. This means the variable $y$ corresponding to the second column is free. We set $y$ equal to a parameter $t$ and solve for $x$. This produces the parametric equations for a line

$$x = (3/2) + (1/2)t, \qquad y = t$$

which is the solution set.

From Equations 6 we get the augmented matrix with its reduced row-echelon form.

$$\left[\begin{array}{rr|r} 2 & -1 & 3 \\ 4 & -2 & 5 \end{array}\right] \qquad \left[\begin{array}{lr|r} 1 & -1/2 & 3/2 \\ 0 & 0 & 1 \end{array}\right].$$

The second row is non-zero and the leading 1 is in the column corresponding to the constant term. This means we have the equation $0x + 0y = 1$ which is impossible so the solution set is empty.

We have already seen that Equations 7 have the augmented matrix and its reduced row-echelon form

$$\left[\begin{array}{rr|r} 2 & -1 & 3 \\ 3 & 2 & 1 \end{array}\right] \qquad \left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right].$$

The second row is non-zero with a leading 1 in the second column. This tells us that the variable $y$ corresponding to the second column is equal to $-1$. Then the first row with a leading 1 in the column corresponding to the variable $x$ tells us that $x$ is equal to 1. The solution set consists of the single point $(1, -1)$ as we have already seen.

The next examples are of two planes is $\mathbb{R}^3$. We look at the system of equations given by Equations 10 and 11 to obtain the augmented matrix with its reduced row-echelon form

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 3 \\ 6 & -3 & 3 & 9 \end{array}\right] \qquad \left[\begin{array}{rrr|r} 1 & -1/2 & 1/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

There is only one non-zero row where the leading 1 is in the first column. This tells us that the variables $y$ corresponding to the second column and the variable $z$ corresponding to the third column are free. We set $z$ equal to a parameter $t$ and $y$ equal to a parameter $s$ and solve for $x$. This produces the parametric equations

$$x = (3/2) + (1/2)s - (1/2)t, \qquad y = s, \qquad z = t$$

which describes the plane $2x - y + z = 3$. The plane is the solution set.

The system of equations given by Equations 10 and 12 produce the augmented matrix with its reduced row-echelon form

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 3 \\ -4 & 2 & -2 & 4 \end{array}\right] \qquad \left[\begin{array}{rrr|r} 1 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The second row is non-zero but the leading 1 is in the last column corresponding to the constant term. This means we have the equation $0x + 0y + 0z = 1$ which is impossible so the solution set is empty.

The system of equations given by Equations 10 and 13 produce the augmented matrix with its reduced row-echelon form

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 3 \\ 3 & 2 & -1 & -1 \end{array}\right] \qquad \left[\begin{array}{rrr|r} 1 & 0 & 1/7 & 5/7 \\ 0 & 1 & -5/7 & -11/7 \end{array}\right].$$

The second row is non-zero and the leading 1 is in the second column. This means the variable $z$ corresponding to the third column is free. We set $z$ equal to a parameter $t$ and use the second row to solve for $y = -(11/7) + (5/7)t$. Then we use the first row and the parametric equations for $z$ and $y$ to solve for $x$. This produces the parametric equations for the solution set

$$x = (5/7) - (1/7)t, \qquad y = -(11/7) + (5/7)t, \qquad z = t$$

which is the line described in section 2.

Then there are examples of three planes in $\mathbb{R}^3$. The system given by Equations 17 produces the augmented matrix with its reduced row-echelon form

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 3 \\ 3 & 2 & -1 & -1 \\ 1 & -4 & 4 & 7 \end{array}\right] \qquad \left[\begin{array}{rrr|r} 1 & 0 & 1/7 & 5/7 \\ 0 & 1 & -5/7 & -11/7 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The second row is the last non-zero row and it is the same as the second row in the previous example. We proceed as we did there. Then the first row is also the same as the first row in the previous example so we again proceed as we did there. We see that the solution set for this system of linear equations as the solution set for the system of linear equations in the previous example.

We have already worked through the system given by Equations 18 but we will review it. The system of linear equations produces the augmented matrix with its reduced row-echelon form

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 3 \\ 3 & 2 & -1 & -1 \\ 1 & -3 & 2 & 2 \end{array}\right] \qquad \left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 12 \end{array}\right].$$

The last non-zero row is the third one and its leading 1 in the third column. This tells us that the variable $z$ corresponding to the third column is equal to 12. Then the second row with a leading 1 in the column corresponding to the variable $y$ tells us that $y$ is equal to 7. Finally, the first row with a leading 1 in the first column tells us that $x$ is $-1$. The solution set consists of the single point $(-1, 7, 12)$ as we have already seen.

The system given by Equations 19 produces the the augmented matrix with its reduced row-echelon form

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 3 \\ 3 & 2 & -1 & -1 \\ 1 & -4 & 3 & 2 \end{array}\right] \qquad \left[\begin{array}{rrr|r} 1 & 0 & 1/7 & 0 \\ 0 & 1 & -5/7 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The last non-zero row is the third one non-zero but the leading 1 is in the last column corresponding to the constant term so the solution set is empty.

We already have an algorithm which put a matrix in reduced row-echelon form. Now we formulate an algorithm that will read off the solution set from a matrix in reduced row-echelon form.

**Algorithm.** Let $A$ be an $m \times (n+1)$ augmented matrix corresponding to a system of $m$ linear equations in $n$ variables and $E$ the $m \times (n+1)$ matrix in reduced row-echelon form derived from $A$. To find the solution set for the system of linear equations proceed as follows.

(1) Find last non-zero row of $E$ and call it the $i_1$st row.
(2) If the leading 1 of the $i_1$st row of $E$ is in the $(n+1)$st column corresponding to the constant terms of the equations then the solution set is empty. Otherwise the solution set is not empty.
(3) If the leading 1 is the $n$th column, set $x_n = E_{i_1(n+1)}$.
(4) If the leading 1 is the $j_1 < n$ column of $E$ set

$$x_{(j_1+1)} = t_{(j_1+1)}, x_{(j_1+2)} = t_{(j_1+2)}, \ldots, x_n = t_n,$$

for parameters $t_{(j_1+1)}, t_{(j_1+2)}, \ldots, t_n$.

(5) Use these parametric equations and row $i_1$ of $E$ to solve for $x_{j_1}$.
(6) Consider the $(i_1 - 1)$st row of $E$.
(7) If the leading 1 is in the $j_1-1$ column of $E$ use the equations for $x_{j_1}, x_{j_1+1}, \ldots, x_n$ and the $(i_1 - 1)$st row of $E$ to solve for $x_{j_1-1}$.
(8) If the leading 1 is in the $j_2 < (j_1 - 1)$ column of $E$ set

$$x_{(j_2+1)} = t_{(j_2+1)}, x_{(j_2+2)} = t_{(j_2+2)}, \ldots, x_{(j_1-1)} = t_{(j_1-1)},$$

for parameters $t_{(j_2+1)}, t_{(j_2+2)}, \ldots, t_{(j_1-1)}$.

(9) Use the equations for $x_{(j_2+1)}, \ldots, x_n$ and row $(i_1 - 1)$ of $E$ to solve for $x_{j_2}$.
(10) Continue in this way working up through the rows of $E$ until all variables $x_1, x_2, \ldots, x_n$ have been solved for.
(11) These values and/or parametric equations for the $x_i$ compose the solution set for the original system of linear equations.

The reduced row-echelon form of an augmented matrix specifies in a simple way the solution set for the original equations. Suppose we know the size of the reduced row-echelon matrix and we know a nonempty solution set, does this determine the reduced row-echelon matrix? Let us examine the case where there are three equations in three unknowns so that the associated reduced row-echelon matrix is $3 \times 4$ matrix. Suppose the solution set is a single point, for example $(2, -3, 6)$. Then it is easily seen that the reduced row-echelon matrix is

$$\left[\begin{array}{ccc|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 6 \end{array}\right].$$

Next suppose the solution set is a line given by the parametric equations

$$x = 2 - 4t, \qquad y = 5 + 3t, \qquad z = t.$$

The reduced row-echelon matrix derived from the augmented matrix is

$$\left[\begin{array}{ccr|c} 1 & 0 & 4 & 2 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

A slightly more complicated situation is when the solution set is a line given by the parametric equations

$$x = 2 + 3t, \qquad y = -3 + 4t, \qquad z = 6 - 2t.$$

Reparameterize the line by setting $z = 6 - 2t = s$ so that $t = 3 - (1/2)s$. Substituting for $t$ results in the parametric equations

$$x = 11 - (3/2)s, \qquad y = 9 - 2s, \qquad z = s$$

Then we can write down the reduced row-echelon matrix which is

$$\left[\begin{array}{ccr|r} 1 & 0 & 3/2 & 11 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The last case we will examine is when the solution set is a plane, for example

$$2x - 3y + 7z = 6.$$

This is the same plane as described by the equation

$$x - (3/2)y + (7/2)z = 3.$$

The plane is described by the parametric equations

$$x = 3 + (3/2)s - (7/2)t, \qquad y = s, \qquad z = t$$

which means the reduced row-echelon matrix is

$$\left[\begin{array}{ccc|c} 1 & -3/2 & 7/2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

**Problems**

(1) Use elementary row operations to put each matrix from problem 1, section 3 (Matrices and elementary row operations), in reduced row-echelon form. Use this to write down the solution set and compare the answer to the ones you obtained in problem 2, section 1 (Lines in two-dimensional space).

(2) Use elementary row operations to put each matrix from problem 2, section 3 (Matrices and elementary row operations), in reduced row-echelon form. Use this to write down the solution set and compare the answer to the ones you obtained in problem 2, section 2 (Planes in two-dimensional space).

(3) For each reduced row-echelon matrix obtained in problem 1, section 4 (Gaussian elimination), write down the solution set for the system of linear equations associated to the original matrix.

(4) Each of the following matrices is in reduced row-echelon form and represents a system of linear equations. Write down the solution set for the system of equations associated to each matrix.

(a)

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right] \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{ccc|c} 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(b)

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 3 & 0 & 7 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(5) Suppose we start with a system of three linear equations in three variables, write down the associated augmented matrix, put it in reduced row-echelon form and find the solution set. For each solution set below, find the reduced row-echelon matrix it came from.

(a) The solution set is the point $(-4, 3, 1)$.

(b) The solution set is the line $x = 1 - 4t$, $y = 3 + 3t$, $z = 2 - t$.

(c) The solution set is the line $x = 3 + 4t$, $y = 1 + 4t$, $z = 5$.

(d) The solution set is the plane $3x - 4y + 6z = 9$.

## 6. Matrix arithmetic

Matrix addition and scalar multiplication are defined just as vector addition and scalar multiplication are defined.

Two matrices of the same size ($m \times n$) can be added entry by entry to form a new matrix of the same size. For example

$$\begin{bmatrix} 2 & -1 & 1 & -3 \\ 3 & 0 & -1 & -2 \\ 5 & -4 & 4 & 7 \end{bmatrix} + \begin{bmatrix} -3 & -1 & 2 & 5 \\ 4 & 1 & 0 & -4 \\ -2 & -4 & 5 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 3 & 2 \\ 7 & 1 & -1 & -6 \\ 3 & -8 & 9 & 6 \end{bmatrix}.$$

Any matrix can be multiplied by any real number. For example

$$3 \begin{bmatrix} 2 & -1 & 1 & -3 \\ 3 & 0 & -1 & -2 \\ 5 & -4 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 6 & -3 & 3 & -9 \\ 9 & 0 & -3 & -6 \\ 15 & -12 & 12 & 21 \end{bmatrix}.$$

Matrix multiplication is much more interesting and has many important uses. Two matrices can be multiplied if the number of entries in the rows of the first matrix (number of columns) is equal to the number of entries in each column of the second matrix (number of rows). The resulting matrix will have the same number of rows as the first and the same number of columns as the second. This means an $m \times n$ matrix $A$ can be multiplied on the right by an $n \times p$ matrix $B$ and the result will be an $m \times p$ matrix $AB$. The $ij$ entry of $AB$ is given by the formula

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{i(n-1)}B_{(n-1)j} + A_{in}B_{nj}.$$

Computing the $ij$ entry of $AB$ is like taking the dot product of row $i$ of $A$ and column $j$ of $B$. An example is

$$\begin{bmatrix} 2 & -1 & 1 & -3 \\ 3 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 3 \\ -3 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -3 & 2 \\ 2 & -7 & 5 \end{bmatrix}.$$

This is true because the 11 entry of the product matrix is

$$\langle 2, -1, 1, -3 \rangle \cdot \langle 1, -2, -3, 2 \rangle = 2 \cdot 1 + (-1) \cdot (-2) + 1 \cdot (-3) + (-3) \cdot 2 = -5,$$

the 22 entry of the product is

$$\langle 3, 0, -1, -2 \rangle \cdot \langle -1, 0, 2, 1 \rangle = 3 \cdot (-1) + 0 \cdot 0 + (-1) \cdot 2 + (-2) \cdot 1 = -7$$

and so forth. When multiplying matrices it helps to use your fingers, your left forefinger goes across the appropriate row of the matrix on the left and your right forefinger goes down the appropriate column of the matrix on the right. It takes a little practice. It is important to notice that in the example we just did the order of multiplication cannot be reversed. The sizes of the matrices do not fit together. As an example where the order multiplication can be reversed let

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 \\ 4 & -2 \end{bmatrix}.$$

then compute

$$AB = \begin{bmatrix} 2 & 0 \\ 13 & -7 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} -3 & 2 \\ 10 & -2 \end{bmatrix}.$$

The matrices $AB$ and $BA$ are not equal. This is an example of an arithmetic operation that *does not commute.*

Just as for the arithmetic operations for real numbers or vectors there are a number of properties of the matrix arithmetic operations.

**Properties of matrix arithmetic.** Suppose that $A$, $B$ and $C$ are matrices with sizes so that the listed operation makes sense and that $c$ and $d$ a real numbers then the following rules for matrix arithmetic hold.

(1) $A + B = B + A$, matrix addition is commutative
(2) $(A + B) + C = A + (B + C)$, matrix addition is associative
(3) $c(A + B) = cA + cB$, scalar multiplication distributes over matrix addition
(4) $(c + d)A = cA + dA$, scalar multiplication distributes over scalar addition
(5) $(cd)A = c(dA)$, scalar multiplication is associative
(6) $(AB)C = A(BC)$, matrix multiplication is associative
(7) $A(B + C) = AB + AC$, left matrix multiplication distributes over addition
(8) $(A + B)C = AC + BC$, right matrix multiplication distributes over addition
(9) $c(AB) = (cA)B$, scalar and matrix multiplication are associative

An expression of the form $A^2 - AB + 3B$ when the matrices $A$ and $B$ are of appropriate sizes is an *arithmetic matrix expression.* If $A$ and $B$ are the 2 by 2 matrices above then

$$A^2 - AB + 3B = \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 13 & -7 \end{bmatrix} + \begin{bmatrix} -3 & 3 \\ 12 & -6 \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ -6 & 9 \end{bmatrix}.$$

Recall the earlier Example 18 of a system of three linear equations in three unknowns

$$\begin{array}{rrrcr} 2x & -y & +z & = & 3 \\ 3x & +2y & -z & = & -1 \\ x & -3y & +2z & = & 2. \end{array}$$

It has the associated coefficient matrix and constant column vector

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

If we define a *variable column vector*

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

then the system of linear equations is transformed into the single *matrix equation* $A\vec{x} = \vec{b}$. In general, a system of $m$ linear equations in $n$ unknowns is equivalent to the matrix equation $A\vec{x} = \vec{b}$ where $A$ is the associated $m \times n$ coefficient matrix, $\vec{x}$ is the $n \times 1$ variable column vector with $x_i$ in row $i$ and $\vec{b}$ is the associated $m \times 1$ constant column vector.

## Problems

(1) Given the matrices

$$E = \begin{bmatrix} -1 & 3 & 5 \\ -4 & 2 & 2 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 3 & -5 & 1 \\ 6 & -2 & -1 \end{bmatrix}$$

compute
 (a) $E + F$
 (b) $3E$
 (c) $4E - 2F$.

(2) Given the matrices

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 5 \\ -4 & 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -5 \\ 6 & -2 \\ 0 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

compute all possible products of pairs of the matrices.

(3) Given the matrices matrices $A$ and $D$ in the previous problems compute
 (a) $A^2 - 5A$
 (b) $D^3 + 2D$.

## 7. The multiplicative identity and inverse matrices

The two matrices

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are identity matrices. The first is the $2 \times 2$ *identity matrix* and the second is the $3 \times 3$ *identity matrix*. Every identity matrix is square with 1's down the diagonal and 0's everywhere else. By $I_m$ we mean the $m \times m$ identity matrix. If the size of the matrix is not in doubt it is common to simply use $I$ to denote the matrix. They are called identity matrices because

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & -3 & 2 \\ 2 & -7 & 5 \end{bmatrix} = \begin{bmatrix} -5 & -3 & 2 \\ 2 & -7 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -3 & 2 \\ 2 & -7 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -3 \\ 3 & -2 & 4 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -3 \\ 3 & -2 & 4 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -3 \\ 3 & -2 & 4 \\ 1 & 0 & 5 \end{bmatrix}$$

The same relationships hold for all sizes of matrices.

Consider the pair of square matrices

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

and compute

$$AC = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = CA = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We say that $C$ *is the inverse matrix of* $A$ and we use $A^{-1}$ to stand for the inverse matrix of $A$.

This allows us to immediately solve a set of linear equations where the coefficient matrix is $A$. For example suppose the system of equations is

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

then we can compute the solution by multiplying both sides of the equation by $C$

$$\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -26 \\ 11 \end{bmatrix}.$$

Suppose we start with a $2 \times 2$ matrix

$$B = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$$

and try to solve for its inverse by writing

$$\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-3c & b-3d \\ -2a+6c & -2b+6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We see that the 11 entry produces the equation $a - 3c = 1$ and the 21 entry produces the equation $-2a + 6c = 0$. These two equations contradict each other so there can be no inverse matrix for $B$.

In general, if $A$ is a square matrix and $C$ is a square matrix satisfying

$$AC = CA = I$$

we say that $C$ *is the inverse matrix of* $A$ and we use $A^{-1}$ for inverse matrix of $A$. If $A$ has an inverse matrix we say it is *invertible* and we say a matrix $B$ is *singular* if it does not have an inverse matrix.

In the $3 \times 3$ case consider the matrices

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ -3 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 3 & -2 \\ 2 & 0 & 4 \\ 3 & 1 & 2 \end{bmatrix}.$$

Given the matrix

$$A^{-1} = \begin{bmatrix} 5 & 1 & 2 \\ 4 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$$

we can compute to see that it is the inverse matrix for the matrix $A$. Before long we will learn how to show that the matrix $B$ is singular.

If we are given a $2 \times 2$ matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

and we want to determine if it is invertible or singular we can calculate

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} aA_{11}+cA_{12} & bA_{11}+dA_{12} \\ aA_{21}+cA_{22} & bA_{21}+dA_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We see that $aA_{11}+cA_{12} = 1$ and $aA_{21}+cA_{22} = 0$ so that if $\langle A_{11}, A_{12}\rangle$ and $\langle A_{21}, A_{22}\rangle$ are multiples of each other there is a contradiction and there can be no inverse matrix. The condition that $\langle A_{11}, A_{12}\rangle$ and $\langle A_{21}, A_{22}\rangle$ are multiples of each other is the same as the condition that $A_{11}/A_{21} = A_{12}/A_{22}$ (if $A_{21}, A_{22} \neq 0$) which in general is the condition that $A_{11}A_{22} - A_{12}A_{21} = 0$. This means that if $A_{11}A_{22} - A_{12}A_{21} = 0$ the matrix $A$ is singular. If $A_{11}A_{22} - A_{12}A_{21} \neq 0$ we write down the matrix

$$A^{-1} = \left(\frac{1}{A_{11}A_{22} - A_{12}A_{21}}\right)\begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} \tag{22}$$

and compute to see that it is indeed the inverse matrix for $A$. We have shown that a $2\times 2$ matrix $A$ is invertible if and only if $A_{11}A_{22} - A_{12}A_{21} \neq 0$. We have already seen this expression in Equation 9 and later we will identify $A_{11}A_{22} - A_{12}A_{21}$ as the determinant of the matrix $A$.

Now let us change our attention to $3\times 3$ matrices. Consider our two examples above

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ -3 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 3 & -2 \\ 2 & 0 & 4 \\ 3 & 1 & 2 \end{bmatrix}.$$

We will find an algorithm which will tell us whether or not any square matrix is invertible and if the matrix is invertible the algorithm will produce the inverse matrix. For matrix $A$ write down the matrix

$$[A|I] = \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & -1 & -2 & 0 & 1 & 0 \\ -3 & 3 & 1 & 0 & 0 & 1 \end{array}\right].$$

Use elementary row operations to put $A$ on the left side of the matrix into reduced row-echelon form. Each time you do an elementary row operation apply it to the entire row of $[A|I]$. Using the first row to get 0s in the first entry of the second and third rows produces the matrix

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right].$$

Adding the second row to the first gives the matrix

$$\left[\begin{array}{rrr|rrr} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right].$$

Finally using the last row to put the left side of the matrix into reduced row-echelon form produces the matrix

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 5 & 1 & 2 \\ 0 & 1 & 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right].$$

This matrix is $[I|A^{-1}]$. The $3 \times 3$ matrix on the left is the identity matrix and the $3 \times 3$ matrix on the right is the inverse matrix of $A$.

Now we repeat the same procedure for the matrix $B$ starting with

$$[B|I] = \left[\begin{array}{ccc|ccc} 5 & 3 & -2 & 1 & 0 & 0 \\ 2 & 0 & 4 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right].$$

First we interchange the first and second rows then multiply the new first row by 1/2 giving the matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1/2 & 0 \\ 5 & 3 & -2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right].$$

Now use the first row to eliminate the first entry of the second and third rows which results in the matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1/2 & 0 \\ 0 & 3 & -12 & 1 & -5/2 & 0 \\ 0 & 1 & -4 & 0 & -3/2 & 1 \end{array}\right].$$

Finally, interchange the second and third rows using the new second to eliminate the second element of the third row. This produces the matrix

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1/2 & 0 \\ 0 & 1 & -4 & 0 & -3/2 & 1 \\ 0 & 0 & 0 & 1 & 2 & -3 \end{array}\right].$$

The $3 \times 3$ matrix on the left is the reduced row-echelon form of $B$. It is not the identity matrix which means that $B$ is a singular matrix.

We have an algorithm to determine whether or not an $m \times m$ matrix $A$ is invertible. The algorithm will produce the inverse matrix when the matrix $A$ is invertible.

**Algorithm.** Let A be an $m \times m$ matrix.

(1) Write down the $m \times 2m$ matrix $[A|I]$.
(2) Apply elementary row operations (Gaussian elimination) to $[A|I]$ until the left-hand matrix $A$ is in reduced row-echelon form.
(3) If the reduced row-echelon form of $A$ is the identity matrix $I$ then the right-hand $m \times m$ matrix is the inverse matrix $A^{-1}$ for $A$.
(4) If the reduced row-echelon matrix is not the identity matrix then the matrix $A$ is singular.

Let us see why this algorithm works. We consider the case for $3 \times 3$ matrices but the same reasoning holds for all square matrices. Consider the three standard vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If $A$ is invertible, each of the systems of linear equation $A\vec{x} = \vec{e}_1$, $A\vec{x} = \vec{e}_2$ and $A\vec{x} = \vec{e}_3$ has a unique solution. If $A$ is not invertible, at least one of them does not have a solution. We try to solve all three systems using elementary row operations and putting $A$ in reduced row-echelon form. If we do this simultaneously we begin with the "augmented matrix" $[A|I]$. If the reduced row-echelon form of $A$ is the identity then the solution to the system $A\vec{x} = \vec{e}_1$ is the first column of the resulting matrix on the right. The solution to $A\vec{x} = \vec{e}_2$ is the second column and the solution to $A\vec{x} = \vec{e}_3$. This means the resulting matrix on the right is $A^{-1}$. If the reduced row-echelon form of $A$ is not the identity matrix then at least one of the systems of equations does not have a solution and $A$ is not invertible.

Recall again the earlier Example 18 of a system of three linear equations in three unknowns which has the associated coefficient matrix and constant column vector

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

It is equivalent to the single matrix equation $A\vec{x} = \vec{b}$. The matrix $A$ has the inverse matrix

$$A^{-1} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 7/2 & -3/2 & -5/2 \\ 11/2 & -5/2 & -7/2 \end{bmatrix}.$$

We can find the solution set for the system of linear equations using $A^{-1}$ by computing

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{x} = I\vec{x} = (A^{-1}A)\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -1/2 & 1/2 & 1/2 \\ 7/2 & -3/2 & -5/2 \\ 11/2 & -5/2 & -7/2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 12 \end{bmatrix}.$$

We have now found the solution set for this system of linear equations in three ways. The first way was by eliminating variables. The second method was to write down the associated augmented matrix and put it into reduced row-echelon form. The third way was to compute the inverse of the associated coefficient matrix and then multiply the constant column vector by it.

**Problems**

(1) Determine whether each matrix is invertible or not and find the inverse matrix for each invertible matrix. Check your answer by multiplying the matrices.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix} \quad \begin{bmatrix} 3 & -4 \\ 6 & 8 \end{bmatrix} \quad \begin{bmatrix} 1/2 & -1/3 \\ 1/4 & 1/2 \end{bmatrix}$$

(2) Determine whether each matrix is invertible or not and find the inverse matrix for each invertible matrix. Check your answer by multiplying the matrices.

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} -2 & 1 & 3 \\ 3 & -1 & 5 \\ 12 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 8 \\ 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 1/4 & 0 \\ -1/3 & 1/4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(3) Use an inverse matrix from above to solve the system of linear equations

$$\begin{array}{rrcr} -x & +2y & = & 3 \\ x & +5y & = & -1 \end{array}$$

(4) Use an inverse matrix from above to solve the system of linear equations

$$\begin{array}{rrrcr} x & +y & -2z & = & -2 \\ x & +2y & +3z & = & 5 \\ 3x & +5y & +6z & = & 3 \end{array}$$

(5) Use an inverse matrix from above to solve the system of linear equations

$$\begin{array}{rrrcr} (1/3)x & +(1/4)y & & = & 8 \\ (-1/3)x & +(1/4)y & & = & -3 \\ & & 3z & = & 4 \end{array}$$

## 8. Determinants

In this section we will define the determinant of a square matrix, relate the determinant to vector products and discuss some of their geometric properties.

If $A$ is an $n \times n$ matrix we will let $\det A$ or $|A|$ stand for the determinant of the matrix $A$. First we say that the determinant of a $1 \times 1$ matrix $[a]$ is the real number $a$. Then for a $2 \times 2$ matrix

$$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \quad \text{we say} \quad \det \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = 3 \cdot 2 - 1 \cdot 5 = 1.$$

In general, $\det A = |A| = A_{11}A_{22} - A_{21}A_{12}$ which is a real number.

The columns of $A$ are the 2-dimensional vectors $\langle 3, 1 \rangle$ and $\langle 5, 2 \rangle$. If we think of them as 3-dimensional vectors and form the cross product we see that

$$\langle 3, 1, 0 \rangle \times \langle 5, 2, 0 \rangle = \langle 0, 0, \det A \rangle.$$

This holds in general for all $2 \times 2$ matrices. We know that the length of the vector $\langle 3, 1, 0 \rangle \times \langle 5, 2, 0 \rangle$ is the area of the parallelogram determined by the vectors $\langle 3, 1, 0 \rangle$ and $\langle 5, 2, 0 \rangle$ which is the area of the parallelogram in the $xy$-plane determined by the 2-dimensional vectors $\langle 3, 1 \rangle$ and $\langle 5, 2 \rangle$. This means $|\det A|$ is the area of the parallelogram determined by its columns.

Now let's see what the sign ($\pm$) of the determinant tells us. We know that two vectors have an angle $\theta$ between them and that $0 \le \theta \le \pi$. To go from the vector $\langle 3, 1 \rangle$ to the vector $\langle 5, 2 \rangle$ we either turn through the angle $+\theta$ (counterclockwise) or $-\theta$ (clockwise). In this example we turn through $+\theta$. If we turn through the angle $+\theta$ the sign of the determinant is positive and if we turn through the angle $-\theta$ the sign of the determinant is negative. In other words, the sign of the determinant depends on the *orientation* of the vectors. Let us check to see if this is true by reversing which of the column vectors comes first. If this is correct then the sign of the determinant should change. Compute

$$\det \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} = 5 \cdot 1 - 2 \cdot 3 = -1$$

which agrees with what we just said.

In Section 7, Equation 22 we computed the inverse of a $2 \times 2$ matrix and saw that a matrix had an inverse if and only if the term $A_{11}A_{22} - A_{21}A_{12}$ is not 0. This term is just the determinant of $A$. Let us summarize what we know about $2 \times 2$ matrices.

**Determinants of $2 \times 2$ matrices**

Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be a $2 \times 2$ matrix. Then

(1) $\det A = |A| = A_{11}A_{22} - A_{21}A_{12}$,

(2) $|\det A|$ is the area of the parallelogram determined by the column vectors $\langle A_{11}, A_{21}\rangle$ and $\langle A_{12}, A_{22}\rangle$,
(3) the sign of $\det A$ depends on the orientation of the column vectors, and
(4) $A$ is invertible if and only if $\det A \neq 0$.

**Determinants of $3 \times 3$ matrices** Let

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

be a $3 \times 3$ matrix. Then

$$\det A = |A| = A_{11} \cdot \det \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} - A_{21} \cdot \det \begin{bmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{bmatrix} + A_{31} \cdot \det \begin{bmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{bmatrix}.$$

There are several things to notice. The first term is

$$A_{11} \cdot \det \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix},$$

where there is the $A_{11}$ entry of the matrix $A$ times the determinant of the matrix obtained by deleting the first row and first column of $A$. In the second term of the sum there is the $A_{21}$ entry of the matrix $A$ times the determinant of the matrix obtained by deleting the first column and second row of $A$. The equivalent statement holds for the third term. We express this by saying that we are *expanding* along the first column. Another point to observe is that the signs of the terms alternate, the term that begins with $A_{11}$ is positive, the term that begins with $A_{21}$ is negative and the term that begins with $A_{31}$ is positive.

The statement that the determinant of a $2 \times 2$ matrix gives the area of the parallelogram determined by the columns of the matrix generalizes. Let

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & -1 & -3 \\ 2 & 4 & 5 \end{bmatrix}.$$

Computing the determinant by expanding down the first column gives $\det A = 3(-1 \cdot 5 - 4 \cdot -3) - (-2)(1 \cdot 5 - 4 \cdot 2) + 2(1 \cdot (-3) - (-1) \cdot 2) = 13$. Next consider the three column vectors of $A$. They are $\langle 3, -2, 2\rangle$, $\langle 1, -1, 4\rangle$ and $\langle 2, -3, 5\rangle$. They determine a parallelepiped in $\mathbb{R}^3$. Earlier we proved that the absolute value of the triple scalar product of three vectors is the volume of the parallelepiped they determine. We compute to see

$$\langle 3, -1, 2\rangle \cdot (\langle 1, -1, 4\rangle \times \langle 2, -3, 5\rangle) = \det A.$$

This means that the absolute value of the determinant is the volume of the parallelepiped determined by the columns of $A$.

As for the $2 \times 2$ matrices we can consider the significance of the sign ($\pm$) of the determinant. When the determinant is nonzero the parallelepiped has nonzero volume. Taking the columns in order we can ask if they obey a right-handed rule or a left-handed rule. The first two column vectors determine a plane and the third column vector sticks out of the plane to the right-handed or left-handed side. If it is the right-handed side the determinant is positive and if it is the left-handed side the determinant is negative. As in the $2 \times 2$ case the sign of the determinant is determined by the *orientation* of the column vectors. Let us interchange the first and second columns of the matrix above. If what we have said is true the determinant of the new matrix should be $-13$. Compute

$$\det \begin{bmatrix} 1 & 3 & 2 \\ -1 & -2 & -3 \\ 4 & 2 & 5 \end{bmatrix} = ((-2)\cdot 5 - 2\cdot 4) - (3)((-1)\cdot 5 - (-3)\cdot 4) + (-2)((-1)\cdot 2 - (-3)\cdot 1) = -13.$$

In the $2 \times 2$ case we showed that a matrix is invertible if and only if the determinant is not zero. The same is true for $3 \times 3$ matrices but we will not prove it. We summarize as before.

**Determinants of $3 \times 3$ matrices**

Let

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

be a $3 \times 3$ matrix. Then

(1)

$$\det A = |A| = A_{11} \cdot \det \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} - A_{21} \cdot \det \begin{bmatrix} A_{12} & A_{33} \\ A_{13} & A_{31} \end{bmatrix} + A_{31} \cdot \det \begin{bmatrix} A_{12} & A_{23} \\ A_{13} & A_{22} \end{bmatrix},$$

(2) $|\det A|$ is the volume of the parallelepiped determined by the column vectors $\langle A_{11}, A_{21}, A_{31}\rangle$, $\langle A_{12}, A_{22}, A_{32}\rangle$ and $\langle A_{13}, A_{23}, A_{33}\rangle$,

(3) the sign of $\det A$ depends on the orientation of the column vectors, and

(4) $A$ is invertible if and only if $\det A \neq 0$.

Let's reformulate the definition of determinant. Suppose $A$ is an $n \times n$ matrix. Define $A(i|j)$ to be the $(n-1) \times (n-1)$ matrix obtained by deleting row $i$ and column $j$ from $A$. If $A$ is a $3 \times 3$ matrix there are nine different matrices $A(i|j)$ and if $A$ is a $4 \times 4$ there are sixteen. Now define determinants step by step.

(1) If $A = [A_{11}]$ is a $1 \times 1$ matrix define $\det A = A_{11}$.

(2) If

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is a $2 \times 2$ matrix define $\det A = (-1)^{1+1} A_{11} \det A(1|1) + (-1)^{2+1} A_{21} \det A(2|1)$.

(3) If

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

is a $3 \times 3$ matrix define

$$\det A = (-1)^{1+1} A_{11} \det A(1|1) + (-1)^{2+1} A_{21} \det A(2|1) + (-1)^{3+1} A_{31} \det A(3|1).$$

(4) If $A$ is an $n \times n$ matrix define

$$\det A = (-1)^{1+1} A_{11} \det A(1|1) + (-1)^{2+1} A_{21} \det A(2|1) + \cdots + (-1)^{n+1} A_{1n} \det A(n|1).$$

This allows us to define and compute the determinant for any square matrix.

In our definition of determinants we always expanded along the first column. This is not necessary. We can compute the determinant by expanding along any column or any row. For example let us compute the determinant of our matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -2 & -1 & -3 \\ 2 & 4 & 5 \end{bmatrix}$$

by expanding down the second column

$$\begin{aligned} \det A &= (-1)^{1+2}(1)\det A(1|2) + (-1)^{2+2}(-1)\det A(2|2) + (-1)^{3+2}(4)\det A(3|2) \\ &= -1\cdot(-4) - 1\cdot(11) - 4\cdot(-5) = 13 \end{aligned}$$

which agrees with our previous computation. It is true that the determinant can be computed by expanding along any row or column (taking care with signs) but we will not prove it here.

**Problems**

(1) Compute the determinants of the matrices

$$\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}.$$

(2) Compute the determinants of the matrices in two ways, first by expanding along a row and then by expanding down a column.

$$\begin{bmatrix} 3 & -2 & -1 \\ 1 & -4 & 2 \\ 3 & 0 & -5 \end{bmatrix} \qquad \begin{bmatrix} -2 & 3 & 1 \\ 5 & -2 & 5 \\ 1 & 4 & 7 \end{bmatrix}.$$

(3) Compute the determinants of the matrices

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}.$$

(4) Compute the determinants of the matrices

$$\begin{bmatrix} 3 & -2 & 1 \\ 0 & -4 & 9 \\ 0 & 0 & -2 \end{bmatrix} \qquad \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}.$$

(5) Compute the determinants of the matrices

$$\begin{bmatrix} 3 & 0 & 0 \\ 5 & -4 & 0 \\ 7 & -9 & -2 \end{bmatrix} \qquad \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

(6) Compute the determinant of the matrix in two ways, first by expanding along a row and then by expanding down a column.

$$\begin{bmatrix} 2 & -3 & 1 & 5 \\ -1 & 4 & 0 & 2 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

(7) Given a $2 \times 2$ matrix $A$ with a nonzero determinant, what is the determinant of its inverse matrix $A^{-1}$?
(8) Given $A$ and $B$ two $2 \times 2$ matrices show that $\det(AB) = \det A \det B$.
(9) Suppose $A$ is a $3 \times 3$ matrix and $\det A = 5$. Using the properties of $3 \times 3$ determinants find the determinant of;
  (a) a matrix obtained by interchanging two rows of $A$,
  (b) a matrix obtained by multiplying a row by 3,
  (c) a matrix obtained by multiplying a row by $-2$,
  (d) the matrix $3A$,
  (e) the matrix $(-2)A$,
  (f) the matrix obtained by adding the second row of $A$ to the first row of $A$,
  (g) the transpose of $A$, $A^T$.
(10) Given $A$ and $B$ two $3 \times 3$ matrices show that $\det(AB) = \det A \det B$.
(11) Given a $3\times3$ matrix $A$ with a nonzero determinant, show that the determinant of its inverse matrix $A^{-1}$ is $1/\det A$.
(12) Given a $3 \times 3$ matrix $A$ with a nonzero determinant, define the $3 \times 3$ matrix $C$ by

$$C_{ij} = (-1)^{i+j} \det A(i|j)$$

and show that $AC^T = (\det A)I_3$.

## 9. Functions

Now it is time to use the geometric and algebraic tools that we have developed. We have systems of linear equations, lines and planes, geometric intersections of lines and planes, solution sets to systems of linear equations, matrices, row operations, reduced row-echelon form, invertible and noninvertible matrices, determinants and volumes. The obvious question is, what do all these things have in common? The common denominator is to view a matrix as defining a linear function between two copies of Euclidian space. A simple but beautiful geometric picture allows us to understand the meaning and consequences of all the problems we have examined.

An $m \times n$ matrix $A$ defines a function from $\mathbb{R}^n$ into $\mathbb{R}^m$ as follows. Think of a point $(x_1, \ldots, x_n)$ as the terminal point of the $n$-dimensional column vector

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and define the function} \quad \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \cdots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

using matrix multiplication.

For example if

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & 4 & -2 \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} 2 & 1 & -1 \\ -3 & 4 & -2 \end{bmatrix} \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ 27 \end{bmatrix}.$$

The matrix $A$ defines a function from $\mathbb{R}^3$ into $\mathbb{R}^2$ and the value of the function at $(-7, 2, 1)$ in $\mathbb{R}^3$ is $(-13, 27)$ in $\mathbb{R}^2$.

If $A$ is an $n \times n$ square matrix then it maps $\mathbb{R}^n$ into $\mathbb{R}^n$. An example for a $3 \times 3$ matrix is

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \\ -10 \end{bmatrix}.$$

The function maps $\mathbb{R}^3$ into $\mathbb{R}^3$ and the value of the function at $(1, 2, -3)$ in $\mathbb{R}^3$ is $(11, 10, -10)$ which is also in $\mathbb{R}^3$.

A function defined by a matrix is *linear*. This means two things. If $A$ is an $m \times n$ matrix it maps $\mathbb{R}^n$ into $\mathbb{R}^m$ and if $\vec{x}$, $\vec{y}$ are $n$-dimensional column vectors and $a$ is a real number then

(1) $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$ and
(2) $A(a\vec{x}) = a(A\vec{x})$.

We can see this from our example

(1)

$$\begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \left( \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix},$$

(2)

$$\begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \left( 5 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right) = 5 \left( \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right).$$

A very important idea is the image of the function defined by the matrix $A$. If $A$ is an $m \times n$ matrix it maps $\mathbb{R}^n$ into $\mathbb{R}^m$ and the *image of* $A$ is $\{A\vec{x} \in \mathbb{R}^m : \vec{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$. The image of $A$ is the set of all points in $\mathbb{R}^m$ that can be hit by the function defined by $A$. The crucial point is that a system of linear equations

$$A\vec{x} = \vec{b}$$

has a nonempty solution set exactly when $\vec{b}$ is in the image of $A$. In the case when $A$ is a $2 \times 3$ matrix the image of $A$ is a subset of $\mathbb{R}^2$ which must be one of three things. It can be

(1) the set $\{(0, 0)\}$, which happens when the matrix $A$ is all 0's,
(2) a line in $\mathbb{R}^2$ passing through the origin or
(3) all of $\mathbb{R}^2$.

There are no other possibilities. When $A$ is a $3 \times 3$ matrix the image of $A$ is a subset of $\mathbb{R}^3$ which must be one of four things. It can be

(1) the set $\{(0, 0, 0)\}$, which happens when the matrix $A$ is all 0's,
(2) a line in $\mathbb{R}^3$ passing through the origin,
(3) a plane in $\mathbb{R}^3$ containing the origin or
(4) all of $\mathbb{R}^3$.

This generalizes to $m \times n$ matrices.

An important observation is that an $n \times n$ matrix is invertible if and only if its image is all of $\mathbb{R}^n$. This means the function defined by $A$ has an inverse. It also agrees with the fact that the absolute value of the determinant of a $2 \times 2$ matrix is the area of the parallelogram determined by its rows and that the absolute value of the determinant of a $3 \times 3$ matrix is the volume of the parallelepiped determined by its rows.

Let

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & -1 \\ -3 & 4 & 2 \end{bmatrix}.$$

The matrix $A$ defines a linear function from $\mathbb{R}^3$ to $\mathbb{R}^3$ and the matrix $B$ defines a function from $\mathbb{R}^3$ to $\mathbb{R}^2$. We can compose the two functions by first applying the function defined by $A$ and then applying the function defined by $B$. This results in a new function from $\mathbb{R}^3$ to $\mathbb{R}^2$. The matrix that defines this new function is the matrix

$$BA = \begin{bmatrix} 15 & 0 & -9 \\ -3 & 1 & 4 \end{bmatrix}.$$

So matrix multiplication corresponds to composition of functions.

**Problems**

(1) Carry out the computations on both sides of the equation to see that they are equal.

$$\begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \left( \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$$

(2) Carry out the computations on both sides of the equation to see that they are equal.

$$\begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \left( 5 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right) = 5 \left( \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right)$$

(3) Given the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 0 \\ -5 & 2 \end{bmatrix}$$

determine which of the following vectors are in the image of $A$

$$\begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} -5 \\ -8 \\ 12 \end{bmatrix}.$$

(4) Given the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ -3 & 3 & 1 \end{bmatrix}$$

determine which of the following vectors are in the image of $A$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -5 \\ -4 \\ -3 \end{bmatrix}.$$

(5) In the previous problem use the inverse of $A$ to find the vectors in $\mathbb{R}^3$ that are mapped by $A$ to the three vectors listed.

(6) Given the matrix
$$B = \begin{bmatrix} 5 & 3 & -2 \\ 2 & 0 & 4 \\ 3 & 1 & 2 \end{bmatrix}$$
determine which of the following vectors are in the image of $B$
$$\begin{bmatrix} 9 \\ 6 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

(7) Let $A$ be a $3 \times 3$ matrix. Show that the image under the function defined by $A$ of a line through the origin is the point $(0, 0, 0)$ or a line through the origin. Find an example for each case.

(8) Let $A$ and $B$ be $3 \times 3$ matrices. Each defines a function from $\mathbb{R}^3$ to $\mathbb{R}^3$. Show that the matrix $BA$ defines the function obtained by composing the function defined by $B$ with the function defined by $A$ (first apply $A$ then $B$). What happens when the order of applying the functions is reversed?

(9) Let $A$ be a $2 \times 2$ matrix. Show that the image of the unit square $[0, 1] \times [0, 1]$ is a parallelogram with area $|\det A|$. Then show that the image of any square in the plane is a parallelogram with area equal to $\det A$ times the area of the original square.

(10) Let $A$ be a $3 \times 3$ matrix. Show that the image of the unit cube $[0, 1]^3$ is a parallelepiped with volume $|\det A|$. Then show the image of any cube is a parallelepiped with volume equal to $|\det A|$ times the volume of the original cube. Use this to how that if $A$ and $B$ are two $3 \times 3$ matrices then $\det(AB) = \det A \cdot \det B$.

## 10. Eigenvalues and eigenvectors

Suppose $A$ is an $n \times n$ square matrix. We know it defines a function that takes $\mathbb{R}^n$ into $\mathbb{R}^n$. If the matrix acts on an $n$-dimensional vector as multiplication by a real number we say the vector is an eigenvector and the real number is an eigenvalue.

**Definition.** If $A$ is an $n \times n$ matrix and $\vec{x}$ is a nonzero $n$-dimensional column vector with $A\vec{x} = \lambda\vec{x}$ for some real number $\lambda$ we say $\lambda$ is an *eigenvalue* for $A$ and $\vec{x}$ is an *eigenvector* for $A$ corresponding to $\lambda$. Note that any nonzero multiple of $\vec{x}$ is also an eigenvector for $A$ corresponding to $\lambda$.

We begin the discussion by examining $2 \times 2$ matrices. Let
$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}.$$
The matrix $A$ is invertible and has two eigenvalues. The eigenvalues are $1/2$ and $2$ with corresponding eigenvectors
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The matrix $B$ is not invertible and has two eigenvalues. The eigenvalues are 0 and 3 with corresponding eigenvectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The matrix $C$ is invertible but has only one eigenvalue. The eigenvalue is 1 with corresponding eigenvector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The matrix $D$ is invertible but has no eigenvalues.

The questions become, how do you know if a matrix has an eigenvalue, how do you find the them and how do you find the corresponding eigenvectors?

To answer this we take a given a matrix $A$ and ask if there is a real number $\lambda$ and a vector $\vec{x}$ with

$$A\vec{x} = \lambda\vec{x} \qquad \text{or} \qquad A\vec{x} - \lambda\vec{x} = (A - \lambda I)\vec{x} = \vec{0}.$$

By our previous discussions of matrices and their inverses that means $(A - \lambda I)$ would be a noninvertible matrix. With what we know about determinants it means that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{we search for the roots of the polynomial} \quad \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}.$$

It means finding eigenvalues is the same as finding the roots of the polynomial

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc).$$

Let us do that for each of the above matrices.

$$\det(A - \lambda I) = \lambda^2 - (3/2)\lambda + 1, \qquad \det(B - \lambda I) = \lambda(\lambda - 3)$$

$$\det(C - \lambda I) = (\lambda - 1)^2, \quad \det(D - \lambda I) = \lambda^2 - \sqrt{3}\lambda + 1.$$

The polynomial $c_A(\lambda) = \det(A - \lambda I)$ is the *characteristic polynomial of the matrix* $A$. The roots of the characteristic polynomial are the eigenvalues of the matrix $A$. Using the quadratic equation we solve for the roots of the characteristic polynomials and see that the roots agree with the previous observations. Once we have found the eigenvalues for a matrix we can solve a system of linear equations to find the eigenvectors. As an example take the matrix and its characteristic polynomial

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad c(\lambda) = \lambda^2 - 3\lambda + 1.$$

The roots of the characteristic polynomial are the eigenvalues

$$\frac{3 \pm \sqrt{5}}{2}.$$

We solve the equations

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{3 \pm \sqrt{5}}{2} \begin{bmatrix} x \\ y \end{bmatrix}$$

to find eigenvectors

$$\begin{bmatrix} 2 \\ -1+\sqrt{5} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 2 \\ -1-\sqrt{5} \end{bmatrix}$$

corresponding to the eigenvalues $\frac{3+\sqrt{5}}{2}$ and $\frac{3-\sqrt{5}}{2}$. Note that $\frac{3+\sqrt{5}}{2} > 1$ and $0 < \frac{3-\sqrt{5}}{2} < 1$ and that their product is equal to the determinant of the matrix which is 1.

Another example is the matrix and its characteristic polynomial

$$\begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix}, \qquad c(\lambda) = \lambda^2 - \lambda + 4.$$

The roots of the characteristic polynomial are

$$\frac{1 \pm \sqrt{-15}}{2} = \frac{1}{2} \pm \frac{\sqrt{15}}{2} i,$$

which are (nonreal) complex numbers. So the matrix has no eigenvalues. Note here that both of the roots are complex numbers, which are complex conjugates, with modulus 2 and their product is the determinant of the matrix which is 4.

Next we turn to $3 \times 3$ matrices where the ideas of the characteristic polynomial, eigenvalues and eigenvectors are the same but the computations are more complicated.

Let

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}.$$

Computing the characteristic polynomial gives

$$c_A(\lambda) = \det \begin{bmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{bmatrix} = -(\lambda^3 - 11\lambda^2 + 39\lambda - 45).$$

Because of our discussion of complex numbers we know this polynomial must have three roots (counting multiplicities) and that there can be three real roots or one real and two complex (conjugate) roots. Testing (by plugging in) we see that 3 is a root and that the polynomial factors by

$$c_A(\lambda) = -(\lambda^3 - 11\lambda^2 + 39\lambda - 45) = -(\lambda - 3)^2(\lambda - 5).$$

The eigenvalues are 3 and 5. Then we solve the equations

$$\begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 5 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

to find the eigenvectors. There are two eigenvectors that are not multiples of each other corresponding to 3 and 1 eigenvector corresponding to 5. They are

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{for 3 and} \quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{for 5.}$$

A problem here is how to find the roots of the characteristic polynomial. A third degree polynomial must have one real root and if that is found we can divide its factor out of the characteristic polynomial and then use the quadratic formula on remaining degree two polynomial. If there is an integer root it must be a divisor of the constant term. By trying each we can see if there are any integer roots. Let us try another example with

$$B = \begin{bmatrix} 6 & -3 & -2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}.$$

Compute the characteristic polynomial

$$c_B(\lambda) = \det \begin{bmatrix} 6-\lambda & -3 & -2 \\ 4 & -1-\lambda & -2 \\ 10 & -5 & -3-\lambda \end{bmatrix} = -(\lambda^3 - 2\lambda^2 + \lambda - 2).$$

By trial and testing we see that 2 is a root of the polynomial so we divide out the factor $\lambda - 2$ to get $c_B(\lambda) = -(\lambda-2)(\lambda^2+1)$. The other two roots are the complex conjugates $\pm i$. So 2 is the only eigenvalue for $B$. We solve to find that

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

is an eigenvector corresponding to 2.

## Problems

(1) Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 5 & 1 \\ 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}.$$

(2) Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \quad \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}.$$

(3) Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 5 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 5 \\ -1 & 3 \end{bmatrix}$$

(4) Find the eigenvalues for the matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 2 & 1 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -2 \end{bmatrix}$$

(5) Find the eigenvalues for the matrices

$$\begin{bmatrix} -1 & 2 & 0 \\ 3 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & 2 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -2 & 2 & -2 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

(6) Show geometrically by looking at the image of the function defined by a $2 \times 2$ matrix that a $2 \times 2$ matrix is not invertible if and only if it has 0 for an eigenvalue.

(7) Show that for each $c \in \mathbb{R}$ the function defined by the matrix
$$A = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$
maps the graph of the hyperbola $xy = c$ to itself.

(8) Show that the function defined by the matrix
$$D = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$
maps every circle centered at the origin to itself. Then prove it rotates the circle by $30^0$.

## 11. Complex eigenvalues and eigenvectors

In the previous section we insisted that the eigenvalues for a matrix be real numbers. They are the real roots of the characteristic polynomial. This makes geometric sense when we consider the matrix as defining a linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$. However, in other settings it makes sense to allow eigenvalues to be complex numbers and eigenvectors to have complex entries. One of the methods used to solve systems of linear differential equations makes use of complex eigenvalues and eigenvectors. You will learn this method when you take a course in differential equations. We will discuss the problem of complex eigenvalues here.

**Definition.** If $A$ is an $n \times n$ matrix and $\vec{x}$ is a nonzero $n$-dimensional column vector with complex numbers as entries and $A\vec{x} = \lambda\vec{x}$ for some complex number $\lambda$ we say $\lambda$ is a *complex eigenvalue* for $A$ and $\vec{x}$ is a *complex eigenvector* for $A$ corresponding to $\lambda$. Note that any nonzero complex multiple of $\vec{x}$ is also an eigenvector for $A$ corresponding to $\lambda$.

Let's examine an example from the previous section. Let
$$D = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$
with characteristic polynomial $c_D(\lambda) = \lambda^2 - \sqrt{3}\lambda + 1$. This has complex roots
$$\frac{\sqrt{3} \pm \sqrt{3-4}}{2} = \frac{\sqrt{3}}{2} \pm \frac{1}{2}i.$$
To find the eigenvectors we solve the linear equations with complex coefficients
$$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \left(\frac{\sqrt{3}}{2} \pm \frac{1}{2}i\right) \begin{bmatrix} z \\ w \end{bmatrix}.$$
We find complex eigenvalues
$$\begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 \\ i \end{bmatrix}$$
corresponding to $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $\frac{\sqrt{3}}{2} - \frac{1}{2}i$ respectively. Note that the two vectors are not complex multiples of each other. Further observe that $\cos 30^0 = \frac{\sqrt{3}}{2}$ and $\sin 30^0 = \frac{1}{2}$ which is the reason the matrix $D$ considered as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ maps every

circle centered at the origin to itself and rotates it by $30^0$ (exercise 8 in the previous section).

Another example is

$$A = \begin{bmatrix} -3 & -5 \\ 3 & 1 \end{bmatrix}.$$

with characteristic polynomial $c_A(\lambda) = \lambda^2 + 2\lambda + 12$. This has complex roots

$$-1 \pm \sqrt{11}i.$$

Solve the linear equations with complex coefficients to find complex eigenvectors

$$\begin{bmatrix} 5 \\ -2 - \sqrt{11}i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 \\ -2 + \sqrt{11}i \end{bmatrix}$$

corresponding to $-1 + \sqrt{11}i$ and $-1 - \sqrt{11}i$ respectively. There is another slight complication that arises when computing complex eigenvectors. Note that

$$\begin{bmatrix} -2 + \sqrt{11}i \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 - \sqrt{11}i \\ 3 \end{bmatrix}$$

are also complex eigenvectors corresponding to $-1 + \sqrt{11}i$ and $-1 - \sqrt{11}i$. This is because

$$\begin{bmatrix} -2 + \sqrt{11}i \\ 3 \end{bmatrix} = \left( \frac{-2}{5} + \frac{\sqrt{11}}{5} i \right) \begin{bmatrix} 5 \\ -2 - \sqrt{11}i \end{bmatrix}$$

and

$$\begin{bmatrix} -2 - \sqrt{11}i \\ 3 \end{bmatrix} = \left( \frac{-2}{5} - \frac{\sqrt{11}}{5} i \right) \begin{bmatrix} 5 \\ -2 + \sqrt{11}i \end{bmatrix}.$$

They are complex multiples of each other and this fact may not be easy to recognize.

**Problems**

(1) Find the complex eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} -4 & -3 \\ 6 & 2 \end{bmatrix} \quad \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ 5 & 4 \end{bmatrix}.$$

(2) Find the complex eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \quad \begin{bmatrix} 5 & 7 \\ -2 & -1 \end{bmatrix}.$$