M16600 Lecture Notes

Section 11.1: Sequences

■ **Section 11.1** textbook exercises, *page 744*: #3, 5, 13, 15, 23, 25, 27, 29, 31, 33, 35, 39, 41, 50.

Definition of a Sequence. A ***sequence*** is a list of terms written in a definite order.

E.g., $\{2, \ 4, \ 6, \ 8, \ 10, \ 12, \ 14, \ldots, 2n, \ldots\}$ is a sequence.

1st term 2nd 3rd

$2(1), 2(2), 2(3), 2(4), \ldots\ldots, 2(n), \ldots$

Find the 27^{th}-term of the above sequence.

Notation: A sequence $\{a_1, \ a_2, \ a_3, \ a_4, \ldots, a_n, \ldots\}$ could be written as $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Note: n does not have to start from 1.

For the above sequence $\{2, \ 4, \ 6, \ 8, \ 10, \ 12, \ 14, \ldots, 2n, \ldots\}$,

$a_n = 2n$. Therefore, we could write this sequence as $\{2n\}_{n=1}^{\infty}$

Here are more examples of a sequences

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{1+1}, \frac{2}{2+1}, \frac{3}{3+1}, \frac{4}{4+1}, \ldots\ldots\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\ldots\right\}$$

$$\left\{\frac{(-1)^n}{n^2}\right\} = \left\{\frac{(-1)^1}{1^2}, \frac{(-1)^2}{2^2}, \frac{(-1)^3}{3^2}, \frac{(-1)^4}{4^2}, \ldots\right\} = \left\{-1, \frac{1}{4}, \frac{-1}{9}, \frac{1}{16}, \ldots\right\}$$

$$a_n = \frac{3^n}{(n+1)!}$$

Here, for any positive integer k, $k! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot k$.

$k!$ is read "k factorial"

$1! = 1$

$2! = 1 \cdot 2 = 2$

$3! = 1 \cdot 2 \cdot 3 = 6$

$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

Example 1: Find a formula for the general term a_n of the sequence

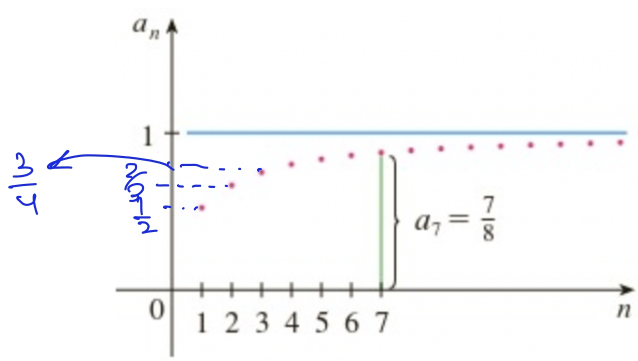
$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16} \cdots\right\}$$

$a_n = \dfrac{n}{2^n}$

$\dfrac{1}{2^1}, \dfrac{2}{2^2}, \dfrac{3}{2^3}, \dfrac{4}{2^4}, \ldots\ldots$

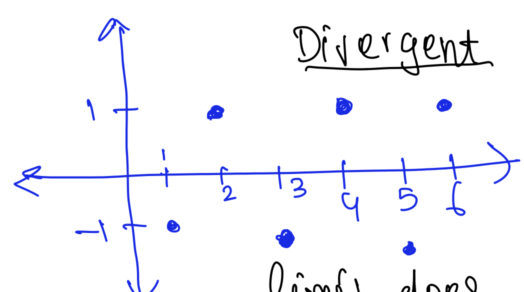
Limit of a Sequence. We write $\lim_{n\to\infty} a_n = L$ if we can make the terms a_n as close to L as we like by taking n sufficiently large.

For example, given the sequence $a_n = \dfrac{n}{n+1}$, we have $\lim_{n\to\infty} \dfrac{n}{n+1} = 1$ because the terms $a_n = \dfrac{n}{n+1}$ approaches 1 as n gets large. Below is the plot of some terms of this sequence.



$a_n = (-1)^n$

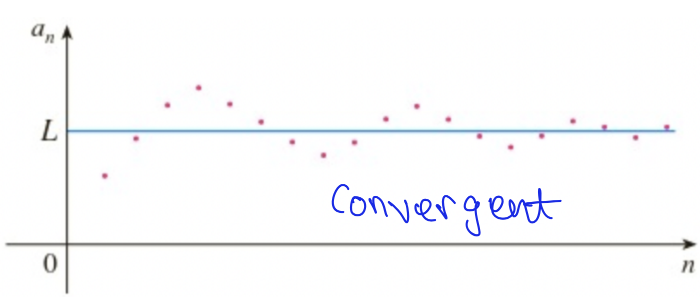
Divergent

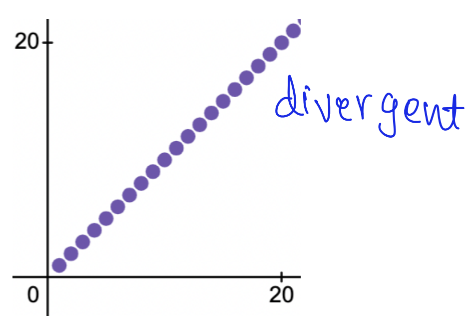


limit does not exist

Convergent or Divergent Sequence.

- If $\lim_{n\to\infty} a_n =$ (a finite number), then the sequence a_n is said to be ***convergent***.
- If $\lim_{n\to\infty} a_n = \pm\infty$ or $\lim_{n\to\infty} a_n$ does not exists , then the sequence a_n is said to be ***divergent***.





Example 2: Determine whether the sequence converges or diverges. If it converges, find the limit

$$\lim_{n\to\infty} \frac{\text{a finite number}}{n^K} = 0$$
K Positive

(a) $a_n = \dfrac{4n^2+2}{n+n^2}$. To answer this question, we want to compute $\lim_{n\to\infty} a_n$.

Method 1 (an algebra approach): Factor as many x's as we can on the numerator and on the denominator then simplify. Then compute the limit.

$$\lim_{n\to\infty} \frac{4n^2+2}{n+n^2} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(4+\frac{2}{n^2}\right)}{\cancel{n^2}\left(\frac{n}{n^2}+1\right)} = \lim_{n\to\infty} \frac{4+\frac{2}{n^2}}{\frac{1}{n}+1}$$

$$= \frac{\lim_{n\to\infty}\left(4+\frac{2}{n^2}\right)}{\lim_{n\to\infty}\left(\frac{1}{n}+1\right)} = \frac{4+0}{0+1} = 4 \Rightarrow \text{sequence converges to 4.}$$

Method 2 (a calculus approach): Use L'Hospital's Rule if applicable

$$\lim_{n\to\infty} a_n : DS \to \frac{4(\infty)^2+2}{\infty+\infty^2} = \frac{\infty}{\infty} \to \text{Indeterminate.}$$

$$L = \lim_{n\to\infty} \frac{4(2n)+0}{1+2n} = \lim_{n\to\infty} \frac{8n}{1+2n} \to DS: \frac{8(\infty)}{1+2(\infty)} = \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{8}{0+2} = \frac{8}{2} = 4$$

Method 3 (the dropping-slower-terms approach): Keep the term with the largest growth rate of the numerator. Do the same for the denominator. Then simplify if possible. Then compute the limit.

→ One with highest Power | exponent of n.

Limit Facts:

$$\lim_{n\to\infty} \frac{\text{faster growth rate function}}{\text{slower growth rate function}} = \infty, \qquad \lim_{n\to\infty} \frac{\text{slower growth rate function}}{\text{faster growth rate function}} = 0$$

$$a_n = \frac{4n^2+2}{n+n^2}. \qquad L = \lim_{n\to\infty} \frac{4\cancel{n^2}}{\cancel{n^2}} = 4$$

$$\lim_{n\to\infty} \frac{\text{finite number}}{n^K} = 0, \quad \lim_{n\to\infty} n^K = \infty$$

(*) If we cannot cancel n entirely, we try to bring n^K in either the numerator | denominator where K is Positive

(b) $\left\{\dfrac{3\sqrt{n}}{n+\sqrt[3]{n^2}-5}\right\}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{3\sqrt{n}}{n}$

$\downarrow$

$n^1 + n^{2/3} - 5n^0$

$1 > \frac{2}{3} > 0$

$$= \lim_{n\to\infty} \frac{3n^{1/2}}{n^1} = \lim_{n\to\infty} \frac{3}{n^{1-1/2}}$$

$$= \lim_{n\to\infty} \frac{3}{n^{1/2}} = 3\cdot 0 = 0$$

$$= \frac{3}{\infty} = 0$$

(sequence converges to 0)

(c) $a_n = \dfrac{\sqrt{10+n+3n^2+4n^5}}{6n^2+2n} \longrightarrow 10n^0 + n^1 + 3n^2 + 4n^5$

$0 < 1 < 2 < 5$

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt{4n^5}}{6n^2} = \lim_{n\to\infty} \frac{\sqrt{4}\, n^{5/2}}{6n^2}$$

$$= \lim_{n\to\infty} \frac{2}{6} n^{5/2-2} \qquad \left(\frac{5}{2} > 2\right)$$

$$= \lim_{n\to\infty} \frac{1}{3} n^{1/2} = \infty$$

$\Rightarrow$ The sequence diverges.

(d) $a_n = e^{-2/n^2}$

D.S.: $\quad \lim_{n\to\infty} e^{-2/n^2} = e^{-2/\infty^2} = e^{\frac{-2}{\infty}} = e^0 = 1$

OR

$L = \lim_{n\to\infty} e^{-2/n^2} \Rightarrow \ln L = \lim_{n\to\infty} \frac{-2}{n^2} = 0 \Rightarrow \ln L = 0 \Rightarrow L = e^0 = 1$

The growth rate order of different types of functions.

logarithmic functions $<<$ algebra $<<$ exponential functions $<<$ factorial
(exponent positive)

Example 3: Determine whether the sequence converges or diverges. If it converges, find the limit

(a) $\left\{\frac{\ln n}{n}\right\}$ $\Rightarrow \lim_{n\to\infty} \frac{\ln n}{n} = \frac{\text{smaller growth rate}}{\text{larger growth rate}} = 0$

(b) $\left\{\frac{2^n}{5^n+4}\right\}$ $\Rightarrow \lim_{n\to\infty} \frac{2^n}{5^n+4} = \lim_{n\to\infty} \frac{2^n}{5^n} = \frac{\text{smaller growth rate}}{\text{larger growth rate}}$

exponentials with smaller base $<<$ exponentials with larger base

$= 0$

(c) $a_n = n!e^{-2n}$ $\Rightarrow \lim_{n\to\infty} n!\,e^{-2n} = \lim_{n\to\infty} \frac{n!}{e^{2n}}$

$= \frac{\text{larger growth rate}}{\text{smaller growth rate}} = \infty$

diverges