Exercise (Pg 516): 1, 2, 5, 11, 9, 7, 10, 12, 23, 19, 26, 17.

The method of ***Integration by Parts*** corresponds to the Product Rule in differentiation.

There is one formula you need to remember

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$(fg)' = f'g + fg'$

$fg' = f'g - (fg)'$

$\int fg'\,dx = \int f'g\,dx - fg$

$f \mapsto -f$

$\int \underset{u}{f}\,\underbrace{g'\,dx}_{dv} = fg - \int \underset{du}{f'}\,\underbrace{g\,dx}_{v}$

We will learn how this formula works in examples

Example 1: Find $\int \underbrace{x}\,\underbrace{\sin x\,dx}$

Note: u-substitution will not work for this problem.

$u = x$, $dv = \sin x\,dx$

$v = \int dv = \int \sin x\,dx = -\cos x$

diff.

$\frac{du}{dx} = 1 \Rightarrow du = dx$

$$\int u\,dv = uv - \int v\,du$$

$$\int x \sin x\,dx = x(-\cos x) - \int(-\cos x)\,dx$$

$$\Rightarrow \int x\sin x\,dx = -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

$\circledast$ $\int x \sin x\,dx = \int \underbrace{\sin x}_{u}\,\underbrace{x\,dx}_{dv}$

$u = \sin x \Rightarrow du = \cos x\,dx$

$dv = x\,dx \Rightarrow v = \frac{x^2}{2}$

$$\int u\,dv = uv - \int v\,du$$

$$= (\sin x)\frac{x^2}{2} - \int \frac{x^2}{2}\cos x\,dx$$

→ more difficult to evaluate

Example 2: Evaluate $\int \overbrace{3x^3 \underbrace{\ln x}_{u}\, dx}^{dv}$

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u ↓ dv

I → Inverse Trigo

L → Log.

A → Algebraic

T → Trigo.

E → Exponential

$u = \ln x, \quad dv = 3x^3\, dx$

$\Rightarrow \frac{du}{dx} = \frac{1}{x}, \quad v = \int 3x^3\, dx = \frac{3x^4}{4}$

$\int u\, dv = uv - \int v\, du$

$\int 3x^3 \ln x\, dx = \frac{3}{4}x^4 \ln x - \int \frac{3x^4}{4} \times \frac{dx}{x}$

$= \frac{3}{4}x^4 \ln x - \frac{3}{4}\int x^3\, dx = \frac{3}{4}x^4 \ln x - \frac{3}{16}x^4 + C$

Example 3: Find $\int \underbrace{t^2}_{u} \underbrace{e^t\, dt}_{dv}$

$u = t^2, \quad dv = e^t\, dt \Rightarrow du = 2t\, dt, \quad v = \int e^t\, dt = e^t$

$\int u\, dv = uv - \int v\, du \Rightarrow \int t^2 e^t\, dt = t^2 e^t - \int 2t e^t\, dt$

$\int t^2 e^t\, dt = t^2 e^t - 2\underbrace{\int t e^t\, dt}_{I'}$

$I' = \int \underbrace{t}_{u} \underbrace{e^t\, dt}_{dv}$

$u = t, \quad dv = e^t\, dt \Rightarrow du = dt, \quad v = \int e^t\, dt = e^t$

$= t e^t - \int e^t\, dt = t e^t - e^t$

$I = t^2 e^t - 2\left[t e^t - e^t\right] + C$

$\Rightarrow I = (t^2 - 2t + 2)e^t + C$

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$\uparrow$
u

Example 4: Calculate $\int_0^1 \underbrace{\tan^{-1} x}_{u}\, \underbrace{dx}_{dv}$

$u = \tan^{-1} x, \quad dv = dx \Rightarrow du = \frac{1}{1+x^2}\,dx, \quad v = x$

$$I = \underbrace{(\tan^{-1} x)}_{u}\, \underbrace{x}_{v} - \int \underbrace{x}_{v}\, \underbrace{\frac{1}{1+x^2}\,dx}_{du}$$

$$= x\tan^{-1} x - \boxed{\int \frac{x}{1+x^2}\,dx} \leftarrow I'$$

$I' = \int \frac{x\,dx}{1+x^2}$ Let $u = 1+x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow \frac{du}{2} = x\,dx$

$$\Rightarrow I' = \int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|1+x^2|$$

$$\Rightarrow I = x\tan^{-1} x - \frac{1}{2}\ln|1+x^2|$$

$$\int_0^1 \tan^{-1} x\,dx = \left[x\tan^{-1} x - \frac{1}{2}\ln|1+x^2|\right]\Big|_0^1$$

$$= \left[\tan^{-1} 1 - \frac{1}{2}\ln|1+1^2|\right] - \left[0\tan^{-1} 0 - \frac{1}{2}\ln|1+0^2|\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

$$\frac{e^{ax}}{a^2+b^2}\left[\underbrace{a\sin(bx) - b\cos(bx)}_{\text{Check if it is correct.}}\right]$$

$\parallel$

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HW. $\boxed{\begin{array}{l}\int e^{ax}\sin(bx)\,dx \\ \int e^{ax}\cos(bx)\,dx\end{array}}$

Example 5: Find $\int \underbrace{e^x \underbrace{\sin x}_{u}\, dx}_{dv}$

$u = \sin x, \quad dv = e^x\,dx$

$\Rightarrow du = \cos x\,dx, \quad v = e^x$

$\Rightarrow \int e^x \sin x\,dx = (\sin x)e^x - \underbrace{\int e^x \cos x\,dx}_{I'}$

$I' = \int e^x \underbrace{\cos x}_{u}\,dx$

$u = \cos x, \quad dv = e^x\,dx$

$du = -\sin x\,dx, \quad v = e^x$

$= (\cos x)e^x - \int e^x(-\sin x)\,dx = \cos x\, e^x + \int e^x \sin x\,dx$

$I' = (\cos x)e^x + \int e^x \sin x\,dx$

$\Rightarrow I = (\sin x)e^x - [(\cos x)e^x + I]$

$I = (\sin x)e^x - (\cos x)e^x - I$

$\Rightarrow 2I = (\sin x)e^x - (\cos x)e^x$

$I = \left(\frac{\sin x - \cos x}{2}\right)e^x + C$

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H.W.

$$I = \int e^{ax} \sin(bx)\, dx$$

$$\int u\, dv = uv - \int v\, du$$

$u = \sin(bx)$, $dv = e^{ax}\, dx$

$\frac{du}{dx} = b \cos(bx) \Rightarrow du = b \cos(bx)\, dx$

$$v = \int dv = \int e^{ax}\, dx = \frac{e^{ax}}{a}$$

$$I = \frac{e^{ax}}{a} \sin(bx) - \int \frac{e^{ax}}{a}\, b \cdot \cos(bx)\, dx$$

$$= \frac{e^{ax}}{a} \sin(bx) - \frac{b}{a} \underbrace{\int e^{ax} \cos(bx)\, dx}_{I'}$$

$$I' = \int e^{ax} \cos(bx)\, dx$$

$u = \cos(bx)$, $dv = e^{ax}\, dx \Rightarrow v = \frac{e^{ax}}{a}$

$du = -b \sin(bx)$

$$\Rightarrow I' = \frac{e^{ax}}{a} \cos(bx) - \int \frac{e^{ax}}{a} \left[-b \sin(bx)\right] dx$$

$$= \frac{e^{ax}}{a} \cos(bx) + \frac{b}{a} \int e^{ax} \sin(bx)\, dx$$

$$I' = \frac{e^{ax}}{a} \cos(bx) + \frac{b}{a} I \qquad (1)$$

$$I = \frac{e^{ax}}{a} \sin(bx) - \frac{b}{a} \left[\frac{e^{ax}}{a} \cos(bx) + \frac{b}{a} I\right]$$

$$I = \frac{a e^{ax}}{a^2} \sin(bx) - \frac{b e^{ax}}{a^2} \cos(bx) - \frac{b^2}{a^2} I$$

$$I + \frac{b^2}{a^2} I = \frac{e^{ax}}{a^2} \left[a \sin(bx) - b \cos(bx) \right]$$

$$I \frac{(a^2+b^2)}{\cancel{a^2}} = \frac{e^{ax}}{\cancel{a^2}} \left(a \sin(bx) - b \cos(bx) \right]$$

$$\boxed{\int e^{ax} \sin(bx)\, dx = \frac{e^{ax}}{a^2+b^2} \left[a \sin(bx) - b \cos(bx) \right]}$$

$$a I' = e^{ax} \cos(bx) + b I \rightarrow \text{HW.}$$

$$I = \int e^{2x} \sin(3x)\, dx$$