MATH 16500 Final Exam, *Spring 2024*

The exam is 10 pages plus the cover page. Follow the instructions for each question. Show your work!

---

1. (18 points) Compute the limit.

(a) $\lim_{x\to 9^+} \frac{\sqrt{x}}{(9-x)^3}$ $\underline{\underline{D.S.}}$ $\frac{\sqrt{9}}{(9-9)^3} = $ undefined $\frac{\text{finite}}{\to 0}$

$$= \lim_{h\to 0} \frac{\sqrt{9+h}}{(9-(9+h))^3}$$

$(x = 9+h)$ $(h>0)$

$$= \lim_{h\to 0} \frac{\sqrt{9+h}}{(\cancel{9}-\cancel{9}-h)^3} = \lim_{h\to 0} \frac{\sqrt{9+h}}{-\underbrace{h^3}_{+ve}} = \frac{\to 3}{(-1)(\to 0)}$$

$$= -\infty$$

(b) $\lim_{x\to 0} \frac{\frac{1}{8+x} - \frac{1}{8}}{x}$ $\underline{\underline{D.S.}}$ $\frac{\frac{1}{8} - \frac{1}{8}}{0} = \frac{0}{0}$

$$= \lim_{x\to 0} \frac{\frac{1}{8+x} - \frac{1}{8}}{x} = \lim_{x\to 0} \frac{\frac{8-(8+x)}{8(x+8)}}{x}$$

$$= \lim_{x\to 0} \frac{8-8-x}{x\cdot 8(x+8)} = \lim_{x\to 0} \frac{-\cancel{x}}{8\cancel{x}(x+8)} = \lim_{x\to 0} \frac{-1}{8(x+8)}$$

$$= \frac{-1}{64}$$

(c) $\lim_{x\to -\infty} \frac{x-3}{\sqrt{11x^2+3x+2}}$

$$= \lim_{x\to -\infty} \frac{\frac{1}{x}(x-3)}{\frac{1}{x}\sqrt{11x^2+3x+2}} = \lim_{x\to -\infty} \frac{\frac{x}{x} - \frac{3}{x}}{-\sqrt{\frac{1}{x^2}}\sqrt{11x^2+3x+2}}$$

$$= \lim_{x\to -\infty} \frac{1-\frac{3}{x}}{-\sqrt{\frac{1}{x^2}(11x^2+3x+2)}} = \lim_{x\to -\infty} \frac{1-\frac{3}{x}}{-\sqrt{11+\frac{3}{x}+\frac{2}{x^2}}} = \frac{-1}{\sqrt{11}}$$

2. (10 points) Use the limit definition of the derivative $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ to compute the derivative of $f(x) = \sqrt{x-2}$.

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h} \qquad \underline{\underline{\text{D.S.}}} \quad \frac{\sqrt{x-2}-\sqrt{x-2}}{0} = \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{x+h-2}-\sqrt{x-2}\right)\left(\sqrt{x+h-2}+\sqrt{x-2}\right)}{h\left(\sqrt{x+h-2}+\sqrt{x-2}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h\cancel{-2} - \cancel{(x-2)}}{h\left(\sqrt{x+h-2}+\sqrt{x-2}\right)} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h-2}+\sqrt{x-2}\right)}$$

$$= \frac{1}{\sqrt{x-2}+\sqrt{x-2}} = \frac{1}{2\sqrt{x-2}}$$

3. (18 points) Compute the derivative of the following function. **Note:** You don't need to simplify your answers.

(a) $P(r) = \dfrac{7r^4 - (r^{12}-r)^3}{(12r+1)^5}$

$$\Rightarrow P'(r) = \frac{(12r+1)^5\left[28r^3 - 3(r^{12}-r)(12r^{11}-1)\right] - \left[7r^4-(r^{12}-r)^3\right]5(12r+1)^4(12)}{(12r+1)^{10}}$$

$$= \frac{\cancel{(12r+1)^4}\left[(12r+1)\{28r^3-3(r^{12}-r)(12r^{11}-1)\} - 60\{7r^4-(r^{12}-r)^3\}\right]}{(12r+1)^{\cancel{10}\ 6}}$$

(b) $f(x) = \pi x \sin(1+x^2)$

$$f'(x) = [\pi x]' \sin(1+x^2) + \pi x\,[\sin(1+x^2)]'$$

$$= \pi \sin(1+x^2) + \pi x \cos(1+x^2)(2x)$$

$$= \pi\left[\sin(1+x^2) + 2x^2\cos(1+x^2)\right]$$

(c) $y = \tan^5(x^5)$

$$= [\tan(x^5)]^5$$

$$\frac{dy}{dx} = 5\tan^4(x^5)\,\frac{d}{dx}(\tan(x^5)) = 5\tan^4(x^5)\sec^2(x^5)\,\frac{d}{dx}(x^5)$$

$$= 25x^4\tan^4(x^5)\sec^2(x^5)$$

4. (10 points) Let $f(x) = \sqrt[3]{6x^2 - x^3}$ and given that $f'(x) = \dfrac{4-x}{\sqrt[3]{x(6-x)^2}}$. Find critical number(s) of f.

$f' = 0$ or f' is undefined.

$f' = 0$ $\quad 4 - x = 0$

$\Rightarrow x = 4$

f' undefined $\quad \sqrt[3]{x(6-x)^2} = 0 \Rightarrow x(6-x)^2 = 0$

$\Rightarrow x = 0$ or $(6-x)^2 = 0$

$\Rightarrow 6 - x = 0$

$\Rightarrow x = 6$

$\Rightarrow$ The critical numbers are $x = 0, 4, 6$

5. (14 points) Use Implicit Differentiation to find $\dfrac{dy}{dx}$: $4xy = x^4 - y^4$.

$4xy = x^4 - y^4$

$\rightarrow$ Diff. both sides w.r.t. x :-

$\frac{d}{dx}(4xy) = \frac{d}{dx}(x^4) - \frac{d}{dx}(y^4)$

$\frac{d}{dy}(y^4)\,\frac{dy}{dx}$

$\Rightarrow 4\,\frac{d}{dx}(xy) = 4x^3 - 4y^3\,\frac{dy}{dx}$

$\Rightarrow 4\left[y + x\,\frac{dy}{dx}\right] = 4x^3 - 4y^3\,\frac{dy}{dx}$

$\Rightarrow 4x\,\frac{dy}{dx} + 4y^3\,\frac{dy}{dx} = 4x^3 - 4y$

$\Rightarrow \frac{dy}{dx}(4x + 4y^3) = 4x^3 - 4y \Rightarrow \frac{dy}{dx} = \frac{4x^3 - 4y}{4x + 4y^3}$

$\Rightarrow \frac{dy}{dx} = \frac{4(x^3 - y)}{4(x + y^3)} \Rightarrow \frac{dy}{dx} = \frac{x^3 - y}{x + y^3}$

6. (17 points) Let $f(x) = \dfrac{x}{x^2+1}$ and given that $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$.

(a) Find the intervals of increase or decrease of f.

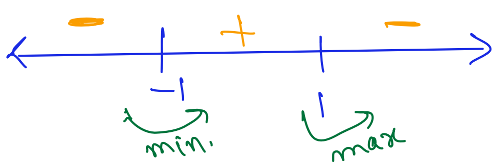
(b) Find the local maximum and minimum values of f.

(c) Find the absolute maximum and minimum values of f on the interval $[0, 2]$.

(a) $f'(x) > 0$ or $f'(x) < 0$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

Critical numbers $\Rightarrow$ $1 - x^2 = 0 \Rightarrow 1 = x^2 \Rightarrow x = \pm 1$



Intervals of increase are $(-1, 1)$

Intervals of decrease are $(-\infty, -1) \cup (1, \infty)$

(b) $x = -1$ is pt. of l. min
$x = 1$ is pt. of l. max

$f(x) = \dfrac{x}{x^2+1}$

$f(-1) = \dfrac{-1}{(-1)^2+1} = \dfrac{-1}{2}$ (minimum value)

$f(1) = \dfrac{1}{1^2+1} = \dfrac{1}{2}$ (maximum value)

(c) Absolute max/min values on $[0, 2]$

Critical numbers in $[0, 2]$: $f(1) = \frac{1}{2} = 0.5$

Endpoints : $f(0) = 0$

$f(2) = \dfrac{2}{2^2+1} = \dfrac{2}{5} = 0.4$

Absolute max value $= \frac{1}{2}$

Absolute min value $= 0$

7. (14 points) The area of a triangle is decreasing at a rate of 10 cm^2/min while its base is increasing at a rate of 4 cm/min. How fast is the height of the triangle changing when the base is 8 cm and the area is 12 cm^2?

$$A = \frac{1}{2} b h$$

rate of change of A with time is $\frac{dA}{dt}$

Diff. both sides w.r.t. t :—

$$\frac{dA}{dt} = \frac{d}{dt}\left(\frac{1}{2} bh\right) = \frac{1}{2} \frac{d}{dt}(bh)$$

$$\underbrace{\frac{dA}{dt}}_{-10\ cm^2/min} = \frac{1}{2}\left[\underbrace{\left(\frac{db}{dt}\right)}_{+4\ cm/min} h + b \left(\frac{dh}{dt}\right)\right]$$

8 cm

* A is dereasing $\Rightarrow \frac{dA}{dt} < 0$

* b is increasing $\frac{db}{dt} > 0$

$12 = \frac{1}{2}(8) h \Rightarrow h = \frac{12}{4} = 3$

$$\Rightarrow -10 = \frac{1}{2}\left[4(3) + 8 \frac{dh}{dt}\right]$$

$$\Rightarrow -20 = 12 + 8 \frac{dh}{dt} \Rightarrow -\frac{32}{8} = \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -4 \text{ cm/min}$$

8. (10 points) Let $f(x)$ be a function that satisfies the following conditions. Sketch the graph of f.

(i) $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 0$. $\Longrightarrow$ $y=0$ is a horizontal asymptote.

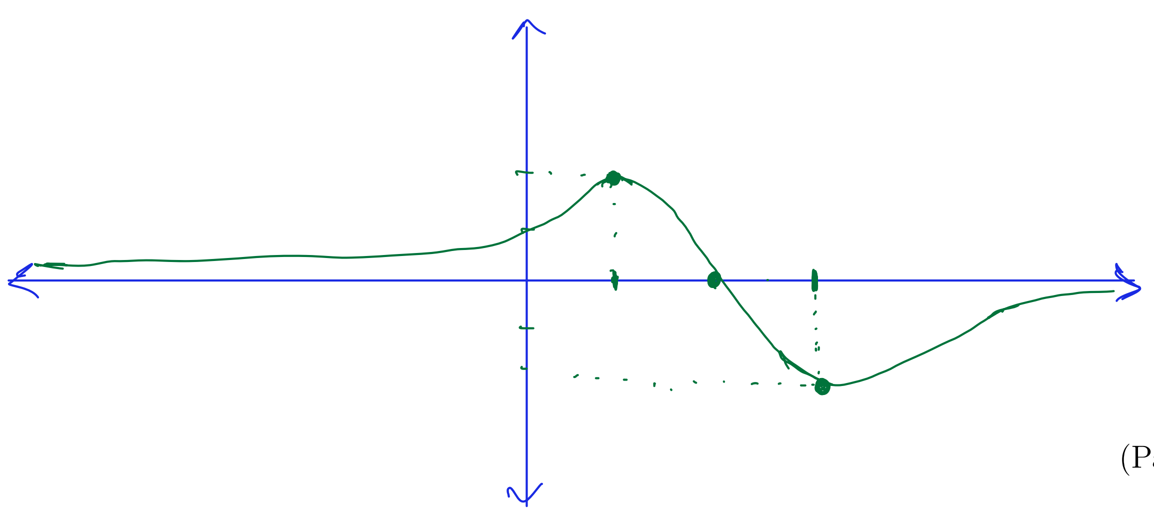
(ii) $f'(1) = 0$ and $f'(3) = 0$. $\Longrightarrow$ $x=1, x=3$ are critical numbers.

(iii) $f(1) = 2$, $f(2) = 0$, and $f(3) = -2$

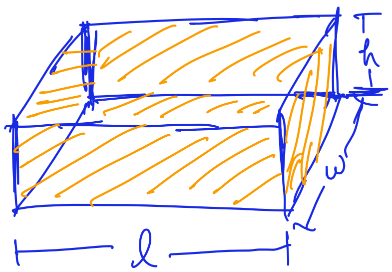
(iv) $f'(x) > 0$ on $(-\infty, 1) \cup (3, \infty)$.

(v) $f'(x) < 0$ on $(1, 3)$.

Intervals of increase and decrease



9. (16 points) A rectangular storage container with an open top is to have a volume of 36 m^3. The length of its base is twice the width. Find the dimensions of the storage container that minimize the amount of material used.



open top

$V = lwh = 36 \text{ m}^3$

$l = 2w$

$$\underbrace{A}_{\text{minimize}} = lw + 2lh + 2wh \xrightarrow{l=2w}$$

$$= (2w)w + 2(2w)h + 2wh$$

$$= 2w^2 + 4wh + 2wh$$

$$= 2w^2 + 6wh$$

$$= 2w^2 + 6\cancel{w}\left(\frac{18}{w^{\cancel{2}}}\right) \qquad h = \frac{18}{w^2}$$

$$\Rightarrow A(w) = \underbrace{2w^2 + \frac{108}{w}}_{\text{minimize}}$$

$lwh = 36$

$\downarrow l = 2w$

$(2w)wh = 36$

$\Rightarrow 2w^2h = 36 \Rightarrow h = \dfrac{36}{2w^2}$

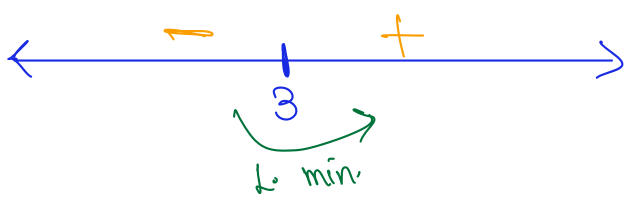
$\Rightarrow h = \dfrac{18}{w^2}$

$$A'(w) = 4w - \frac{108}{w^2}$$

Critical numbers : $A'(w) = 0 \Rightarrow 4w - \dfrac{108}{w^2} = 0 \Rightarrow \dfrac{4w^3 - 108}{w^2} = 0$

$\Rightarrow 4w^3 = 108 \Rightarrow w^3 = \dfrac{108}{4} \Rightarrow w^3 = 27 \Rightarrow w = 3$

Check whether $w=3$ gives a min or a max



Dimensions that minimize the area

$w = 3$

$l = 2w = 6$

$h = \dfrac{18}{w^2} = \dfrac{18}{9} = 2$

10. (30 points) Evaluate the integral.

(a) $\displaystyle\int \frac{x+2}{\sqrt{x^2+4x}}\,dx.$

$u = x^2+4x \Rightarrow \frac{du}{dx} = 2x+4 \Rightarrow du = (2x+4)dx$

$\Rightarrow du = 2(x+2)dx$

$\Rightarrow \frac{1}{2}du = (x+2)dx$

$\parallel$

$\displaystyle\int \frac{1}{\sqrt{\underbrace{x^2+4x}_{u}}}(x+2)\,dx$

$$= \int \frac{1}{\sqrt{u}}\frac{1}{2}du = \frac{1}{2}\int\frac{1}{\sqrt{u}}du = \frac{1}{2}\int\frac{1}{u^{1/2}}du = \frac{1}{2}\int u^{-1/2}du$$

$$= \frac{1}{2}\frac{u^{-1/2+1}}{-1/2+1} + C = \frac{1}{2}\frac{u^{1/2}}{1/2} + C = \frac{1}{2}\cdot 2\cdot u^{1/2} + C$$

$$= u^{1/2} + C = \sqrt{u} + C$$

$$= \sqrt{x^2+4x} + C$$

(b) $\displaystyle\int \frac{\sqrt[3]{x} - x^5 + 3x}{x^3}\,dx.$

$$= \int\left(\frac{\sqrt[3]{x}}{x^3} - \frac{x^5}{x^3} + \frac{3x}{x^3}\right)dx = \int\frac{\sqrt[3]{x}}{x^3}dx - \int\frac{x^5}{x^3}dx + \int\frac{3x}{x^3}dx$$

$$= \int x^{1/3-3}\,dx - \int x^{5-3}\,dx + 3\int x^{1-3}\,dx$$

$$= \int x^{-8/3}\,dx - \int x^2\,dx + 3\int x^{-2}\,dx$$

$$= \frac{x^{-8/3+1}}{-8/3+1} - \frac{x^{2+1}}{2+1} + 3\frac{x^{-2+1}}{-2+1} + C = \frac{x^{-5/3}}{-5/3} - \frac{x^3}{3} + 3\frac{x^{-1}}{-1} + C$$

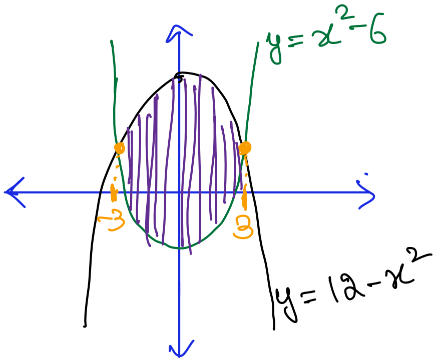
$$= -\frac{3}{5}x^{-5/3} - \frac{1}{3}x^3 - 3\frac{1}{x} + C$$

(c) $\displaystyle\int \csc x\,(\cot x - \csc x)\,dx.$

$$= \int \csc x\cot x\,dx - \int \csc^2 x\,dx$$

$$= -\csc x + \cot x + C$$

11. (14 points) Sketch the region enclosed by the given curves and find the area. $y = 12 - x^2$ and $y = x^2 - 6$.



Pts of intersections

$y = 12 - x^2$, $y = x^2 - 6$

$\Rightarrow 12 - x^2 = x^2 - 6 \Rightarrow 12 + 6 = 2x^2$

$\Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

$$A = \int_{-3}^{3} \left[(12 - x^2) - (x^2 - 6) \right] dx$$

$$= \int_{-3}^{3} (18 - 2x^2)\, dx = 2 \int_{0}^{3} (18 - 2x^2)\, dx$$

$$= 2\left[18x - 2\frac{x^3}{3} \right]\Bigg|_0^3 = 2\left[54 - \cancel{\frac{54}{3}}^{18} \right] = \boxed{72}$$

12. (14 points) A force of $\cos\left(\frac{\pi}{2}x\right)$ pounds moves an object along a straight line when it is x feet from the origin. Compute the work done in moving this object from $x = 1$ to $x = 2$.

$$W = \int_a^b F(x)\, dx \quad \text{moving from } x=a \text{ to } x=b$$

$$= \int_1^2 \cos\left(\frac{\pi}{2}x\right) dx \longrightarrow$$

Alternatively, substitute $u = \frac{\pi}{2}x$

$$= \frac{1}{\pi/2} \sin\left(\frac{\pi}{2}x\right)\Bigg|_1^2 = \frac{2}{\pi}\left[\sin\left(\frac{\pi}{2}\cdot 2\right) - \sin\left(\frac{\pi}{2}\cdot 1\right) \right]$$

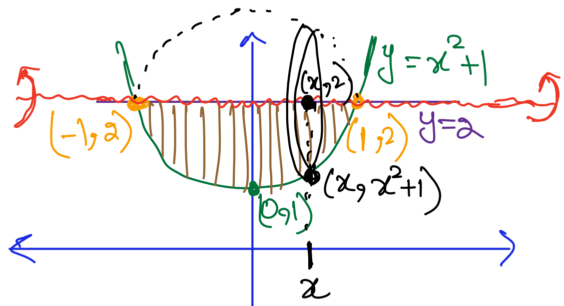
$$= \frac{2}{\pi}\left[\sin(\pi) - \sin(\pi/2) \right]$$

$$= \boxed{-\frac{2}{\pi}} \text{ lb-ft.}$$

13. (15 points) Let $\mathcal{S}$ be the region bounded by the parabola $y = x^2 + 1$ and the horizontal line $y = 2$.

Set up (don't need to evaluate) the definite integral that computes the volume of the solid obtained by rotating $\mathcal{S}$

(a) about the line $y = 2$ (Disk/Washer Method).



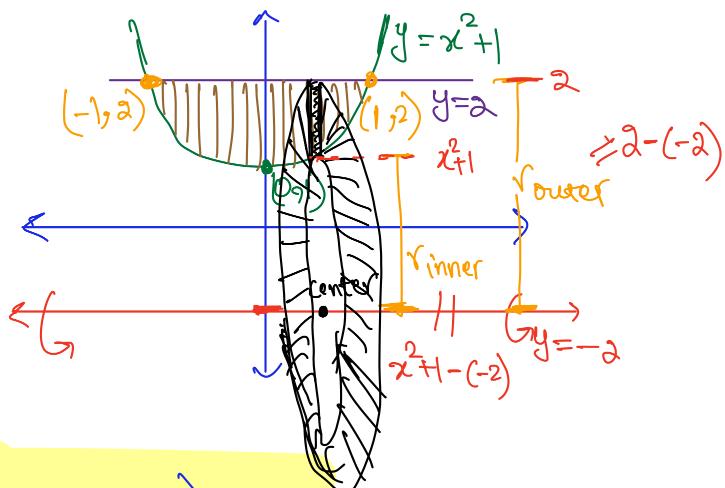
$$r(x) = \underset{\text{curve}}{y_{upper}}(x) - \underset{\text{curve}}{y_{lower}}(x)$$

$$= 2 - (x^2+1) = 1 - x^2$$

$$V = \int_{-1}^{1} \pi (1-x^2)^2 \, dx$$

$x^2 + 1 = 2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

(b) about the line $y = -2$ (Disk/Washer Method).



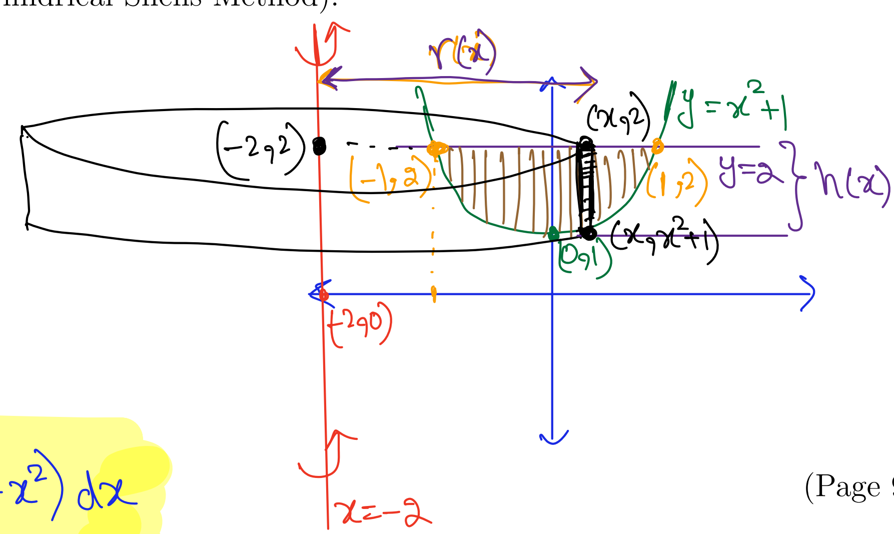
$$r_{inner} = x^2+1-(-2) = x^2+3$$

$$r_{outer} = 2-(-2) = 4$$

$$V = \int_{-1}^{1} \pi \left[(4)^2 - (x^2+3)^2 \right] dx$$

$$= \int_{-1}^{1} \pi \left(16 - (x^2+3)^2 \right) dx$$

(c) about the line $x = -2$ (Cylindrical Shells Method).



$$r(x) = x - (-2)$$
$$= x + 2$$

$$h(x) = 2 - (x^2+1)$$
$$= 1 - x^2$$

$$V = \int_{-1}^{1} 2\pi (x+2)(1-x^2) \, dx$$

14. **(Bonus, 8 points)** Express the integral as a limit of Riemann sums. Do not evaluate the limit. $\int_{-1}^{1} (9+x^9)\ dx$.

$$\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{n} = \frac{2}{n}$$

$$x_i = -1 + i\frac{2}{n}$$

$$\int_{-1}^{1} (9+x^9)\,dx = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} f(x_i) = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} (9+x_i^9)$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left[9 + \left(-1+\frac{2i}{n}\right)^9\right]$$

15. **(Bonus, 8 points)** Evaluate $\int x^2 (x+1)^7\, dx$.

$x = u-1 \Leftarrow$ $u = x+1 \Rightarrow du = dx$

$$I = \int x^2\, u^7\, du = \int (u-1)^2 u^7\, du$$

$x^2 \downarrow (u-1)^2$

$$= \int (u^2 - 2u + 1)\, u^7\, du$$

$$= \int (u^9 - 2u^8 + u^7)\, du$$

$$= \frac{u^{10}}{10} - 2\frac{u^9}{9} + \frac{u^8}{8} + C$$

$$= \frac{(x+1)^{10}}{10} - 2\frac{(x+1)^9}{9} + \frac{(x+1)^8}{8} + C$$