**M16600 Lecture Notes**

*Section 11.5: Alternating Series*

■ **Section 11.5** textbook exercises, *page 776*: # 4, 5, 7, 9, 6, 14.

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**Definition.** An ***alternating series*** is a series whose terms are alternately positive and negative.

E.g., $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ $\rightarrow \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \pm \cdots\cdots\infty$

$\rightarrow -b_1 + b_2 - b_3 + b_4 - b_5 \pm \cdots\cdots$

As a convention, we write an alternating series as $\sum(-1)^n\, b_n$, where $b_n > 0$ for all $n$.

For the example above, $b_n =$ $\displaystyle\sum_n (-1)^{n+1} b_n = \sum_n (-1)^{n-1} b_n = b_1 - b_2 + b_3 \mp \cdots\infty$

**Convergence/Divergence for Alternating Series** $\sum(-1)^n\, b_n$

- **Alternating Series Test (AST):** The alternating series $\sum(-1)^n\, b_n$ **converges** if these two conditions are satisfied:
  (i) $\lim_{n\to\infty} b_n = 0$
  (ii) $b_{n+1} \le b_n$ (the terms $b_n$ are decreasing)
- The alternating series $\sum(-1)^n\, b_n$ **diverges** if $\lim_{n\to\infty} b_n \neq 0$.

*Example 1:* Use the Alternating Series Test to show that the alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges.

$b_n = \frac{1}{n}$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = \frac{1}{\infty} = 0 \Rightarrow$ (i) is true ✓

$n+1 \geq n \Rightarrow \frac{1}{n+1} \le \frac{1}{n} \Rightarrow b_{n+1} \le b_n \Rightarrow$ (ii) is true ✓

$\Rightarrow$ By AST, the given series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges.

*Example 2:* Test the series for convergence or divergence

**Hint:** The first step in determining convergence or divergence for an **alternating series** is to compute $\lim_{n\to\infty} b_n = 0$.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}+5}$

$$b_n = \frac{1}{2\sqrt{n}+5}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2\sqrt{n}+5} = \frac{1}{\infty} = 0 \Rightarrow (i) \text{ is true}$$

Now want to check whether $b_{n+1} \le b_n$ or not.

$$b_{n+1} = \frac{1}{2\sqrt{n+1}+5} \le \frac{1}{2\sqrt{n}+5} = b_n \Rightarrow b_{n+1} \le b_n \Rightarrow (ii) \text{ is true}$$

By AST, the series converges.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3n^4+n}{4n^4-n^3+n^2+2}$

$$b_n = \frac{3n^4+n}{4n^4-n^3+n^2+2}$$

$$\Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n^4+n}{4n^4-n^3+n^2+2} = \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{3\cancel{n^4}}{4\cancel{n^4}} = \frac{3}{4} \neq 0$$

By AST, series diverges.

$$\sum_{n=1}^{\infty} (-1)^{2n+1} \frac{3n^4+n}{4n^4+2}$$

$$(-1)^{2n+1} = (-1)^{2n}(-1)^1$$
$$= [(-1)^2]^n (-1)^1$$
$$= [1]^n (-1)^1$$
$$= 1(-1) = -1$$

Not Alternative series

$$(-1)^{3n+1} = (-1)^{3n}(-1)$$
$$= [(-1)^3]^n (-1)$$

Alternating $= (-1)^n(-1) = (-1)^{n+1}$

$$\sum_{n=1}^{\infty} (-1)^{cn+d}\, b_n$$

is alternating if $c$ is odd and is not alternating if $c$ is even