M16600 Lecture Notes

Section 10.2: Calculus with Parametric Curve

■ **Section 10.2** textbook exercises, *page 695*: #3, 4, 5, 7(a), 17, 11, 13. For #11, 13, only compute $\frac{d^2y}{dx^2}$, don't need to do concavity.

GOALS: Given a parametric curve $x = x(t)$ and $y = y(t)$

- Compute $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$
- Find the **slope of the tangent line** to the given parametric curve at a point.
- Write an **equation of the tangent line** to the given parametric curve at a point.
- Find points on parametric curves such that the tangent line is *horizontal* or *vertical*

Recall:

- Let $y = y(x)$ be a curve in the xy-plane (e.g $y = x^2 + 1$). Then

  the **SLOPE** of the TANGENT LINE to $y = y(x)$ at the point $x = a$ is $y'(a)$.

- The point-slope formula for **an equation of a line** is $y - y_1 = m(x - x_1)$ where (x_1, y_1) is one point on the line and m is the slope of the line.

Given a parametric curve: $x = x(t), y = y(t)$. We can compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$. How do we find $\frac{dy}{dx}$ so that we can compute the slope of a tangent line to this parametric curve?

Note that we can write $y(t)$ as the composite function $y(t) = y\big(x(t)\big)$, where $x(t)$ is the inner function. Then by the Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Therefore,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

Geometrically, $\frac{dy}{dx}$ represents the ***slope formula*** of tangent lines to the parametric curve $x = x(t), y = y(t)$ at any point. To find the ***slope of the tangent line*** at one specific when $t = a$, we evaluate $\frac{dy}{dx}$ at $t = a$. Notation: $\left.\dfrac{dy}{dx}\right|_{t=a}$.

Given parametric equations $x = x(t), y = y(t)$, the second derivative of y with respect to x is

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

Example 1: Let $x = t^2 - 3$ and $y = t^3 - 3t$. Find

(a) $\frac{dx}{dt}$ and $\frac{dy}{dt}$

$$\frac{dx}{dt} = \frac{d}{dt}(t^2-3) = 2t \quad , \quad \frac{dy}{dt} = 3t^2 - 3$$

(b) $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t}$$

(c) the slope of the tangent line to the given parametric curve when $t = -2$

$$t=-2 \Rightarrow \frac{dy}{dx} = \frac{3(-2)^2-3}{2(-2)} = \frac{3(4)-3}{-4} = \frac{12-3}{-4} = \frac{-9}{4}$$

(d) an equation of the tangent line to the given parametric curve when $t = -2$

$$x = t^2-3, \quad y = t^3-3t, \quad t=-2 \Rightarrow x_1 = (-2)^2-3 = 4-3 = 1$$

$$y_1 = (-2)^3 - 3(-2) = -8+6 = -2 \Rightarrow y_1 = -2$$

$$y - y_1 = m(x-x_1) \Rightarrow y-(-2) = \frac{-9}{4}(x-1) \Rightarrow y+2 = \frac{-9}{4}(x-1)$$

$$y+2 = \frac{-9}{4}(x-1) \Rightarrow 4(y+2) = -9(x-1) \Rightarrow 4y+8 = -9x+9 \Rightarrow 9x+4y=1$$

$$9x+4y-1=0$$

(e) an equation of the tangent line to the given parametric curve at the point $(-2, 2)$

$$x(t) = t^2-3, \quad y(t) = t^3-3t$$

$$(-2, 2) \qquad \Rightarrow x'(t) = 2t, \quad y'(t) = 3t^2-3$$

$$\Rightarrow \frac{dy}{dx} = \frac{3t^2-3}{2t}$$

$t = ?$

$$t^2-3 = -2, \quad t^3-3t = 2 \quad (2)$$

$$t^2 = 3-2 = 1 \Rightarrow t = \pm 1$$

$\Rightarrow t=1$ does not satisfy (2)

$\Rightarrow t=-1$ does satisfy.

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{3(-1)^2-3}{2(-1)} = \frac{3-3}{-2} = 0$$

$$m = 0, \quad (x_1, y_1) = (-2, 2)$$

$$y - y_1 = m(x - x_1) \Rightarrow y - 2 = 0(x+2) = 0 \Rightarrow y - 2 = 0$$

Example 2: Find an equation of the tangent line to the parametric curve

$$x = t - \sin t, \qquad y = 1 - \cos t$$

at $t = \pi/3$.

$$x'(t) = 1 - \cos t, \quad y'(t) = 0 + \sin t$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\sin t}{1-\cos t} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{\sin \pi/3}{1 - \cos \pi/3}$$

$$= \frac{\sqrt{3}/2}{1 - 1/2} = \frac{\sqrt{3}/2}{1/2}$$

$$x_1 = x(\pi/3), \quad y_1 = y(\pi/3)$$

$$m = \sqrt{3}$$

$$\Rightarrow x_1 = \frac{\pi}{3} - \sin\frac{\pi}{3}, \quad y_1 = 1 - \cos\frac{\pi}{3}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2}, \quad y_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$y - y_1 = m(x - x_1) \Rightarrow y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow y - \frac{1}{2} = \sqrt{3}x - \frac{\pi\sqrt{3}}{3} + \frac{3}{2}$$

$$\Rightarrow y = \sqrt{3}x - \frac{\pi\sqrt{3}}{3} + \frac{3}{2} + \frac{1}{2} \Rightarrow y = \sqrt{3}x - \frac{\pi\sqrt{3}}{3} + 2$$

$$\Rightarrow \sqrt{3}x - y - \frac{\pi\sqrt{3}}{3} + 2 = 0$$

Facts:

- The tangent line is ***horizontal*** at the values of t where $\frac{dy}{dx} = 0$.
- The tangent line is ***vertical*** at the values of t where $\frac{dy}{dx}$ is undefined.

Example 3: Let $\mathcal{C}$ be the parametric curve given by $x = t^3 - 3t$ and $y = t^3 - 3t^2$. Find

(a) Find the points on the curve $\mathcal{C}$ where the tangent line is horizontal.

$x'(t) = 3t^2 - 3$ and $y'(t) = 3t^2 - 6t$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = 0 \Rightarrow \boxed{y'(t) = 0} \Rightarrow 3t^2 - 6t = 0$$

$$\Rightarrow 3t(t-2) = 0$$

$$\Rightarrow t = 0 \text{ or } t - 2 = 0$$

$$\Rightarrow t = 0 \text{ or } t = 2$$

$(x(0), y(0))$ and $(x(2), y(2))$

$(0, 0)$ and $(2^3 - 3(2), (2)^3 - 3(2)^2)$

$(0, 0)$ and $(2, -4)$

(b) Find the points on the curve $\mathcal{C}$ where the tangent line is vertical.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \infty \Rightarrow \frac{x'(t)}{y'(t)} = \frac{1}{\infty} = 0$$

$$\Rightarrow \boxed{x'(t) = 0} \Rightarrow 3t^2 - 3 = 0$$

$$\Rightarrow 3t^2 = 3 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$$

$(x(1), y(1))$ and $(x(-1), y(-1))$

$x(t) = t^3 - 3t$, $y(t) = t^3 - 3t^2$

$t = -1$

$-1 - 3(1) = -4$

$(-2, -2)$, $(2, -4)$

Example Find eqn. of tangent to $x = \cos t$, $y = \sin t$ at $t = 0$.

$(x_1, y_1) = (\cos 0, \sin 0) = (1, 0)$

$x = 1 \Leftarrow$ vertical

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos t}{-\sin t} \Rightarrow m = \left.\frac{dy}{dx}\right|_{t=0} = \frac{\cos 0}{-\sin 0} = \frac{1}{0} = \infty$$

$$\boxed{\begin{aligned} y - y_1 &= \infty (x - x_1) \\ \Rightarrow \frac{1}{\infty}(y - y_1) &= (x - x_1) \end{aligned}}$$

$$\Rightarrow 0 = x - x_1$$

$$\Rightarrow x = 1$$

$$y - y_1 = m(x - x_1)$$

$$\frac{1}{m}(y - y_1) = (x - x_1)$$

Example 4: Let $x = 2t^3$ and $y = 2 + t^2$, find $\dfrac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{6t^2} = \frac{1}{3t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{1}{3t}\right)}{6t^2} = \frac{\frac{1}{3}\frac{d}{dt}\left(t^{-1}\right)}{6t^2} = \frac{\frac{1}{3}(-1)t^{-2}}{6t^2} = \frac{-1}{18t^4}$$

$$\frac{d^2y}{dx^2}$$