M16600 Lecture Notes

Section 7.4: Integration of Rational Functions by Partial Fractions

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■ **Section 7.4** exercises, *page 541*: #9, 12, 19, 23, 24, 10, 11, 20, 25.

Terminologies:

- **Rational Function:** a ratio of polynomials
- **Partial Fractions Decomposition:** is the technique of decomposing rational function into a combination of simpler fractions

  E.g., $\dfrac{x+5}{x^2+x-2} = \dfrac{2}{x-1} - \dfrac{1}{x+2}$

  $\dfrac{2}{(x-1)^2} - \dfrac{1}{x^2+4}$
- **Integration by Partial Fractions:** is a method of integrating certain types of rational functions by first decomposing the rational function into simpler fractions then integrate.

  E.g., $\displaystyle\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx = 2\ln|x-1| - \ln|x+2| + C$

In order to perform the method of Integration by Partial Fractions, we need to be able to do these three processes:

1. Writing out the form of the partial fractions decomposition
2. Finding the values of the coefficients
3. Doing a u-substitution

Example 1 (Process 1): Write out the form of the partial fractions decomposition of the functions

Step 1: If the (highest degree of the numerator) is $\geq$ the (highest degree of the denominator), do long division
Step 2: Factor the denominator completely
Step 3: Treat ***Linear Factor*** (highest degree is 1) and ***Quadratic Factor*** (highest degree is 2) differently
Step 4: Take care of ***the multiplicity*** of each factor accordingly

(a) $\dfrac{x+5}{x^2+x-2}$

(b) $\dfrac{x^3-x+1}{x(x+4)^3(x^2+4)}$

$-2 = 1x - 2$

$= 2x - 1$

$x^2 + x - 2$

$\underbrace{x^2 + 2x} - x - 2$

$x(x+2) - 1(x+2) = (x+2)(x-1)$

$2x^2 + x - 1$ $\quad 2x - 1 = -2$

$\underbrace{2x^2 - x} + \underbrace{2x - 1}$ $\quad = -1 \times 2$ $\quad = -2 \times 1$

$\underbrace{x(2x-1) + 1(2x-1)}$

$(2x-1)(x+1)$

$x^2 - 5x + 6$ $\quad 6 = -1 \times -6$ $\quad -2 \times -3$

$x^2 - 2x - 3x + 6$

$\underbrace{x(x-2)} - 3\underbrace{(x-2)}$

$(x-2)(x-3)$

$$\frac{x+5}{x^2+x-2} = \frac{x+5}{(x+2)(x-1)} = \frac{a}{x+2} + \frac{b}{x-1}$$

Another ex.

$$\frac{x+5}{(x+2)(x-1)(x+6)} = \frac{a}{x+2} + \frac{b}{x-1} + \frac{c}{x+6}$$

$$\Rightarrow x+5 = \frac{a}{(x+2)}(x+2)(x-1) + \frac{b}{(x-1)}(x+2)(x-1)$$

$$\left\{ x+5 = a(x-1) + b(x+2) \right\}$$

$x = -2 \Rightarrow -2+5 = a(-2-1) \Rightarrow 3 = -3a \Rightarrow a = -1$

$x = 1 \Rightarrow 1+5 = b(1+2) \Rightarrow 6 = 3b \Rightarrow b = 2$

$$\Rightarrow \frac{x+5}{x^2+x-2} = \frac{-1}{x+2} + \frac{2}{x-1}$$

$$I = \int \frac{x+5}{x^2+x-2}\,dx = \int \frac{-1}{x+2}\,dx + \int \frac{2}{x-1}\,dx$$

$$\Rightarrow I = -1\int \frac{1}{x+2}\,dx + 2\int \frac{1}{x-1}\,dx$$

$$\Rightarrow I = -\ln|x+2| + 2\ln|x-1| + C$$

$$\boxed{\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + \text{Const}}$$

$$= \ln\frac{1}{|x+2|} + \ln|x-1|^2 = \ln\left(\frac{|x-1|^2}{|x+2|}\right) + \text{Const}$$

(b) $\dfrac{x^3 - x + 1}{x(x+4)^3(x^2+4)} = \dfrac{a}{x} + \dfrac{b}{x+4} + \dfrac{c}{(x+4)^2} + \dfrac{d}{(x+4)^3} + \dfrac{px+q}{x^2+4}$

$$x^3 - x + 1 = a(x+4)^3(x^2+4) + bx(x+4)^2(x^2+4) + cx(x+4)(x^2+4)$$
$$+ dx(x^2+4) + (px+q)x(x+4)^3$$

$x=0 \Rightarrow 1 = a(0+4)^3(0^2+4) \Rightarrow 1 = 256a \Rightarrow \boxed{a = \frac{1}{256}}$

$x=-4 \Rightarrow (-4)^3 - (-4) + 1 = d(-4)((-4)^2+4) \Rightarrow -59 = -80d$

$\Rightarrow \boxed{d = \frac{59}{80}}$

Coeff. of x

$-1 = 4a \times 48 + 64b + 16c + 4d + 64q$

$-1 = 192a + 64b + 16c + 4d + 64q$ —— (I)

Coeff. of $x^2 \rightarrow$ (II)

Coeff. of $x^3 \rightarrow$ (III)

Coeff. of $x^4 \rightarrow$ (IV)

4 eqns. b, c, p, q

$a = ✓$

$d = ✓$

$$\int \frac{x^3-x+1}{x(x+4)^3(x^2+4)}\,dx = \int \frac{a}{x}\,dx + \int \frac{b}{x+4}\,dx + \int \frac{c}{(x+4)^2}\,dx + \int \frac{d}{(x+4)^3}\,dx + \int \frac{px+q}{x^2+4}\,dx$$

$$= a\int \frac{1}{x}\,dx + b\int \frac{1}{x+4}\,dx + c\int \frac{1}{(x+4)^2}\,dx + d\int \frac{1}{(x+4)^3}\,dx$$

$$+ \int \frac{px+q}{x^2+4}\,dx$$

$$= a\ln|x| + b\ln|x+4| + c\int \underbrace{(x+4)^{-2}\,dx}_{= \frac{(x+4)^{-2+1}}{-2+1}} + d\int \underbrace{(x+4)^{-3}\,dx}_{= \frac{(x+4)^{-3+1}}{-3+1}}$$

$$+ \int \frac{px+q}{x^2+4}\,dx$$

$$= a\ln|x| + b\ln|x+4| + c\,\frac{(x+4)^{-1}}{-1} + d\,\frac{(x+4)^{-2}}{-2}$$

$$+ \underbrace{\int \frac{px+q}{x^2+4}\,dx}$$

$$= p\underbrace{\int \frac{x}{x^2+4}\,dx} + q\underbrace{\int \frac{dx}{x^2+4}}$$

$$\int \frac{x}{x^2+4}\,dx = \int \frac{1}{2}\frac{1}{u}\,du = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$$

$$u = x^2+4 \Rightarrow du = 2x\,dx \qquad = \frac{1}{2}\ln|x^2+4|$$

$$\int \frac{1}{x^2+4}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$$

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\,\text{Tan}^{-1}\left(\frac{x}{a}\right)} \qquad \boxed{\int \frac{1}{x^2+1}\,dx = \text{Tan}^{-1}x}$$

$$\int \frac{dx}{a^2\left(\frac{x^2}{a^2}+1\right)} = \frac{1}{a^2}\int \frac{dx}{\left(\frac{x}{a}\right)^2+1}$$

$$u = \frac{x}{a} \Rightarrow x = a\,u \Rightarrow dx = a\,du$$

$$= \frac{1}{a^2}\int \frac{a\,du}{u^2+1} = \frac{a}{a^2}\int \frac{du}{u^2+1} = \frac{1}{a}\,\text{Tan}^{-1}(u)$$

$$= \frac{1}{a}\,\text{Tan}^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{x^3-x+1}{x(x+4)^3(x^2+4)}\,dx = a\ln|x| + b\ln|x+4| - \frac{c}{x+4} - \frac{d}{2(x+4)^2} + \frac{p}{2}\ln|x^2+4| + \frac{q}{2}\,\text{Tan}^{-1}\left(\frac{x}{2}\right) + \underline{\text{Const}}$$

(c) $\dfrac{x^3+x^2+1}{x^2(x-1)(x^2+x+1)(x^2+1)^2}$

$$= \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1} + \frac{dx+e}{x^2+x+1} + \frac{fx+g}{x^2+1} + \frac{hx+i}{(x^2+1)^2}$$

Example 2 (Processes 1 and 2): Write out the form of the partial fraction decomposition of the functions then find the values of the coefficients

(a) $\dfrac{x+5}{(x-1)(x+2)}$ $\rightarrow$ completed

(b) $\dfrac{x^2+2x-1}{x(2x-1)(x+2)} = \dfrac{a}{x} + \dfrac{b}{2x-1} + \dfrac{c}{x+2}$

$\Rightarrow x^2+2x-1 = a(2x-1)(x+2) + bx(x+2) + cx(2x-1)$

$x=0 \Rightarrow -1 = a(-1)(2) \Rightarrow -1 = -2a \Rightarrow a = \frac{1}{2}$

$2x-1=0 \Rightarrow x=\frac{1}{2} \Rightarrow \left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 1 = b\cdot\frac{1}{2}\cdot\left(\frac{1}{2}+2\right)$

$\Rightarrow \frac{1}{4} + 1 - 1 = \frac{b}{2}\cdot\frac{5}{2} \Rightarrow \frac{1}{4} = \frac{5b}{4} \Rightarrow b = \frac{1}{5}$

$x+2=0 \Rightarrow x=-2 \Rightarrow (-2)^2 + 2(-2) - 1 = c(-2)(2(-2)-1)$

$4-4-1 = -2c(-4-1)$

$-1 = 10c \Rightarrow c = \frac{-1}{10}$

$$\int \frac{x^2+2x-1}{x(2x-1)(x+2)}\,dx = \int \frac{a}{x}\,dx + \int \frac{b}{2x-1}\,dx + \int \frac{c}{x+2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x}\,dx + \frac{1}{5}\int \frac{1}{2x-1}\,dx - \frac{1}{10}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{5}\cdot\frac{1}{2}\ln|2x-1| - \frac{1}{10}\ln|x+2|$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + \text{Const}$$

Example 3 (Process 3): Evaluate

1. $\int \frac{1}{x+2}\,dx = \ln|x+2| + C$

2. $\int \frac{2}{x-1}\,dx = 2\ln|x-1| + C$

3. $\int \frac{1}{5}\frac{1}{2x-1}\,dx = \frac{1}{5}\frac{1}{2}\ln|2x-1| + C$

$$\boxed{\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)}}$$

$$\int \frac{2}{(2x-1)^3}\,dx = 2\int (2x-1)^{-3}\,dx = \frac{2}{2}\frac{(2x-1)^{-3+1}}{-3+1}$$

4. $\int \frac{2}{(x-1)^2}\,dx = 2\int (x-1)^{-2}\,dx = 2\,\frac{(x-1)^{-2+1}}{-2+1} + C$

$$= \frac{2(x-1)^{-1}}{-1} + C$$

$$= \frac{-2}{x-1} + C$$

Example 4: Evaluate $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx$

$$\left[\frac{5x+1}{(2x+1)(x-1)} = \frac{a}{2x+1} + \frac{b}{x-1}\right] \times (2x+1)(x-1)$$

$$5x+1 = a(x-1) + b(2x+1)$$

$$x = \frac{-1}{2} \Rightarrow 5\left(\frac{-1}{2}\right)+1 = a\left(\frac{-1}{2}-1\right)+0 \Rightarrow \frac{-5}{2}+1 = \frac{-3}{2}a$$

$$\Rightarrow \frac{-3}{2} = \frac{-3}{2}a$$

$$\Rightarrow a = 1$$

$$x = 1 \Rightarrow 5(1)+1 = 0 + b(2(1)+1) \Rightarrow 6 = 3b$$

$$\Rightarrow b = 2$$

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \frac{1}{2x+1}\,dx + \int \frac{2}{x-1}\,dx$$

$$= \frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$$

Example 5: Evaluate $\displaystyle\int \frac{4x}{(x-1)^2(x+1)}\,dx$

$$\left[\frac{4x}{(x-1)^2(x+1)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1}\right] \times (x-1)^2(x+1)$$

$$\Rightarrow 4x = a(x-1)(x+1) + b(x+1) + c(x-1)^2$$

$$x=1 \Rightarrow 4 = 0 + b(1+1) + 0 \Rightarrow 4 = 2b \Rightarrow b=2$$

$$x=-1 \Rightarrow -4 = 0 + 0 + c(-1-1)^2 \Rightarrow -4 = 4c \Rightarrow c=-1$$

$$\underbrace{x=0}_{\text{choose any value of } x \text{ different from } 1 \text{ and } -1} \Rightarrow 0 = a(0-1)(0+1) + b(0+1) + c(0-1)^2$$

$$\Rightarrow 0 = -a + b + c$$

$$\Rightarrow 0 = -a + 2 - 1 \Rightarrow a = 2-1 = 1$$

$$\Rightarrow a=1$$

$$\int \frac{4x}{(x-1)^2(x+1)}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{2}{(x-1)^2}\,dx - \int \frac{1}{x+1}\,dx$$

$$= \ln|x-1| + 2\,\frac{|x-1|^{-2+1}}{-2+1} - \ln|x+1| + C$$

$$= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

It is useful to remember this integral formula

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \tfrac{1}{a}\tan^{-1}\left(\tfrac{x}{a}\right) + C}$$

When $a = 1$, the above formula becomes one we already know $\displaystyle\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C$.

Example 6: Evaluate $\displaystyle\int \frac{2x^2-x+1}{x^3+x}\,dx$

$$\frac{2x^2-x+1}{x^3+x} = \underbrace{\frac{2x^2-x+1}{x(x^2+1)} = \frac{a}{x} + \frac{bx+c}{x^2+1}}_{\times\, x(x^2+1)}$$

$$\Rightarrow 2x^2-x+1 = a(x^2+1) + (bx+c)x$$

$x=0 \Rightarrow 2(0)^2-0+1 = a(0^2+1) + 0 \Rightarrow 1 = a$

$x=1 \Rightarrow 2(1)^2-1+1 = a(1^2+1) + (b+c)1$

$\Rightarrow 2 = 2a+b+c \Rightarrow 2 = 2(1)+b+c$

$\Rightarrow 2 = 2+b+c \Rightarrow b+c=0$ — (I)

$x=-1 \Rightarrow 2(-1)^2-(-1)+1 = a((-1)^2+1) + (b(-1)+c)(-1)$

$\Rightarrow 2+1+1 = 2a + (-b+c)(-1)$

$\Rightarrow 4 = 2a+b-c \Rightarrow 4 = 2+b-c$

$\Rightarrow b-c=2$ — (II)

$\left.\begin{array}{l} b+c=0 \\ b-c=2 \end{array}\right] \Rightarrow$ (I) gives $b=-c$

Put value of b in (II) $\Rightarrow -c-c=2$

$\Rightarrow -2c = 2$

$\Rightarrow c = -1$

$$b = -c = -(-1) \Rightarrow \boxed{b = 1}$$

$$\int \frac{2x^2 - x + 1}{x^3 + x}\,dx = \int \frac{1}{x}\,dx + \int \frac{x - 1}{x^2+1}\,dx$$

$$= \ln|x| + \int \left(\frac{x}{x^2+1} - \frac{1}{x^2+1} \right) dx$$

$$= \ln|x| + \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx$$

$$u = x^2+1 \Rightarrow du = 2x\,dx$$

$$\int \frac{x}{x^2+1}\,dx = \int \frac{1}{2}\,\frac{du}{u} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|$$

$$= \frac{1}{2}\ln|x^2+1|$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x)$$

$$\int \frac{2x^2 - x + 1}{x^3 + x}\,dx = \ln|x| + \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + C$$

Exercise 24

$$\int \frac{x^2+x+1}{(x^2+1)^2}\,dx$$

$$\frac{x^2+x+1}{(x^2+1)^2} = \frac{ax+b}{x^2+1} + \frac{cx+d}{(x^2+1)^2} \quad \times (x^2+1)^2$$

$$x^2+x+1 = (ax+b)(x^2+1) + cx+d$$

$x=0$
$x=1$
$x=-1$
$x=2$
→ Four eqns. Solve for a, b, c, d

Alternatively,

$$x^2+x+1 = ax^3+bx^2+ax+b+cx+d$$

$+0x^3$

$$= ax^3+bx^2+(a+c)x+(b+d)$$

$a=0$ ← comparing coeff. of x^3 on both sides

$b=1$ ← " " " x^2 " " "

$a+c=1$ ← " " " x " " "

$b+d=1$ ← " " " x^0 " " "

$\Rightarrow 0+c=1 \Rightarrow c=1$, $\quad 1+d=1 \Rightarrow d=0$

($a=0$; $b=1$)

$$\int \frac{x^2+x+1}{(x^2+1)^2}\,dx = \underbrace{\int \frac{1}{x^2+1}\,dx}_{=\,\tan^{-1}x} + \underbrace{\int \frac{x}{(x^2+1)^2}\,dx}_{u=x^2+1}$$

$\Rightarrow du = 2x\,dx$

$$\Rightarrow I = \int \frac{x}{(x^2+1)^2}\,dx = \int \frac{1}{2}\,\frac{1}{u^2}\,du = \frac{1}{2}\int u^{-2}\,du$$

$$= \frac{1}{2}\,\frac{u^{-2+1}}{-2+1} + C = \frac{-1}{2u} + C$$

$$= \frac{-1}{2(x^2+1)} + C$$

$$\int \frac{x^2+x+1}{(x^2+1)^2}\,dx = \operatorname{Tan}^{-1} x - \frac{1}{2(x^2+1)} + C$$