M16600 Lecture Notes
Section 11.3: The Integral Test

■ **Section 11.3** textbook exercises, *page 765*: #3, 5, 7, 21, 23, 22. **Note:** For # 21, 23, 22, show that the conditions of the Integral Test are true.

THE INTEGRAL TEST. Suppose f is a *continuous, positive, decreasing* function on $[1, \infty)$ and let $a_n = f(n)$. Then

(i) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \quad \text{we use} \quad \int_4^\infty \frac{1}{(\cancel{n}\,x - 3)^2}\,dx$$

Also, it is not necessary that f be *always* decreasing. What is important is that f be *ultimately* decreasing, that is decreasing for x larger than some number N.

Example 1: Use the Integral Test to test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence. Show that the conditions of the Integral Test are true for this problem.

$$a_n = \frac{1}{n^2+1} \xrightarrow[\text{with } x]{\text{replace } n} f(x) = \frac{1}{x^2+1}$$

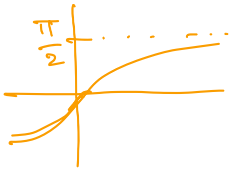
✓ • f is continuous on $[1,\infty)$ because x^2+1 is not zero for $x \geq 1$

✓ • f is positive on $[1,\infty)$ because x^2+1 is positive

✓ • $f'(x) = \frac{d}{dx}\left(\frac{1}{x^2+1}\right) = \frac{-1}{(x^2+1)^2} \cdot 2x = \frac{-2x}{(x^2+1)^2}$

For x in $[1,\infty)$, $x > 0 \Rightarrow \frac{-2x}{(x^2+1)^2} < 0 \Rightarrow f'(x) < 0$
$\Rightarrow f$ is decreasing

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx$$



$$= \lim_{t\to\infty} \tan^{-1}x \Big|_1^t = \lim_{t\to\infty} \left(\tan^{-1}t - \tan^{-1}1\right)$$

$$= \lim_{t\to\infty} \tan^{-1}t - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty \Rightarrow \text{the integral converges}$$

$\Rightarrow$ the series converges

Example 2: Use the Integral Test to test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence or divergence. Show that the conditions of the Integral Test are true for this problem.

$$a_n = \frac{\ln n}{n} \xrightarrow[\text{with } x]{\text{replace } n} f(x) = \frac{\ln x}{x}$$

For the interval $[1, \infty)$,

- f is continuous.
- for $x \geq 1$, $\ln x \geq 0 \Rightarrow \dfrac{\ln x}{x} \geq 0 \Rightarrow f$ is nonnegative.
- $f'(x) = \dfrac{x(\ln x)' - (x)' \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$

when is $f'(x) < 0 \Rightarrow \dfrac{1-\ln x}{x^2} < 0 \Rightarrow 1 - \ln x < 0$

$\Rightarrow 1 < \ln x \Rightarrow \ln x > 1$

$\Rightarrow x > e^1$

For (e, ∞), $f'(x) < 0$

$\Rightarrow f$ is decreasing.

$\Rightarrow f$ is ultimately decreasing.

$$\Rightarrow \int_1^{\infty} \frac{\ln x}{x}\, dx = \int_{\ln 1}^{\ln \infty} u\, du = \int_0^{\infty} u\, du = \lim_{t\to\infty} \int_0^t u\, du$$

$u = \ln x \Rightarrow du = \frac{1}{x}\, dx$

$$= \lim_{t\to\infty} \frac{u^2}{2}\Big|_0^t$$

$$= \lim_{t\to\infty} \frac{t^2}{2} = \infty$$

$\Rightarrow$ The integral diverges

$\Rightarrow$ the series diverges.