Indiana University, Indianapolis

Spring 2025 Math-I 165

Practice Test 3b

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Name:____

Instructions:

- No cell phones, calculators, watches, technology, hats - stow all in your bags.
- **Write your name** on this cover page.
- This test is **closed book and closed notes**.
- All work must be clearly shown for partial credit.
- If you wish for something not to be graded, please strike it out neatly.
- Box, circle, or otherwise clearly indicate your final answer.
- When you finish, return your test to the proctor, and leave the classroom.
- There are a total of **12 problems including 2 bonus problems**.
  - Problems 1-10 are each worth 10 points.
  - The bonus problems are each worth 5 points.
- You can score a **maximum of 110 points out of 100**.
- There are a total of **7 pages** including the cover page.

Problem 1. Use midpoint rule with $n = 4$ to compute the integral $\int_0^8 \sqrt{x+1}\,dx$. [10 pts]

$[0, 8]$, $n=4$ $\Rightarrow$ $\Delta x = \frac{8-0}{4} = 2$

$x_i = a + i\,\Delta x$, $a=0$

$\Rightarrow x_0 = 0$, $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, $x_4 = 8$

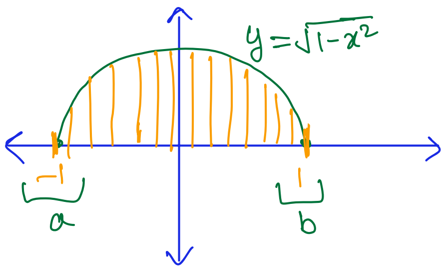
The midpoints $\left(\overline{x_i} = \frac{x_{i-1}+x_i}{2}\right)$: $\overline{x_1} = \frac{0+2}{2} = 1$, $\overline{x_2} = 3$, $\overline{x_3} = 5$, $\overline{x_4} = 7$

$$\int_0^8 \underbrace{f(x)}_{\sqrt{x+1}}\,dx = \Delta x\left[f(\overline{x_1}) + f(\overline{x_2}) + f(\overline{x_3}) + f(\overline{x_4})\right]$$

$$\Rightarrow \int_0^8 \sqrt{x+1}\,dx = 2\left[\sqrt{1+1} + \sqrt{1+3} + \sqrt{1+5} + \sqrt{1+7}\right]$$

$$= 2\left[\sqrt{2} + 2 + \sqrt{6} + 2\sqrt{2}\right] = 6\sqrt{2} + 4 + 2\sqrt{6}$$

Problem 2. Express the area under the semicircle $y = \sqrt{1-x^2}$ as limit of a sum. [10 pts]



$a=-1$, $b=1$

$$\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{n} = \frac{2}{n}$$

$$A = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i)$$

$$x_i = a + i\,\Delta x = -1 + i\left(\frac{2}{n}\right) = \frac{2i}{n} - 1$$

$$\Rightarrow f(x_i) = \sqrt{1 - x_i^2} = \sqrt{1 - \left(\frac{2i}{n} - 1\right)^2}$$

$$= \sqrt{1 - \left[\frac{4i^2}{n^2} - \frac{4i}{n} + 1\right]} = \sqrt{\frac{4i}{n} - \frac{4i^2}{n}}$$

$$= \sqrt{\frac{4}{n}\left(i - i^2\right)} = \frac{2}{\sqrt{n}}\sqrt{i-i^2}$$

$$\Rightarrow A = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \frac{2}{\sqrt{n}}\sqrt{i-i^2} = \lim_{n\to\infty} \frac{4}{n\sqrt{n}} \sum_{i=1}^{n} \sqrt{i-i^2}$$

Problem 3. Find the derivative of the function $f(x) = \int_{\tan x}^{1} (\theta^2 + 1)\, d\theta$. [10 pts]

$$\frac{d}{dx}\left(f(x)\right) = \frac{d}{dx}\int_{v(x)}^{u(x)} g(\theta)\, d\theta$$

$$= g(u(x))\, \underbrace{u'(x)}_{0} - g(v(x))\, v'(x)$$

$u(x) = 1 \Rightarrow u'(x) = 0$

$v(x) = \tan x \Rightarrow v'(x) = \sec^2 x$

$$\frac{d}{dx}\left(f(x)\right) = -\left(\tan^2 x + 1\right)\sec^2 x$$

$$= -\sec^4 x$$

Problem 4. A particle moves in a straight line with velocity varying as a function of time such that $v(t) = t + 1$. Find the distance travelled from $t = 0$ to $t = 2$ seconds. [10 pts]

$$\text{Distance} = \int_{a}^{b} |v(t)|\, dt$$

$b \to 2$, $a \to 0$, $v(t) \to t+1$

$$\Rightarrow \text{distance} = \int_0^2 |t+1|\, dt$$

$t+1 > 0$ for $0 \le t \le 2 \Rightarrow |t+1| = t+1$

$$\Rightarrow \text{distance} = \int_0^2 (t+1)\, dt = \left(\frac{t^2}{2} + t\right)\Big|_0^2$$

$$= \frac{2^2}{2} + 2 = 4$$

Problem 5. Evaluate the indefinite integral $\int (\sin x + \cos x)^2\, dx$. [10 pts]

$$I = \int (\sin x + \cos x)^2 \, dx$$

$$= \int \Big(\underbrace{\sin^2 x + \cos^2 x}_{=1} + \underbrace{2 \sin x \cos x}_{= \sin 2x} \Big) \, dx$$

$$= \int (1 + \sin 2x)\, dx = \underbrace{\int dx}_{=x} + \int \sin \underbrace{2x}_{u=2x} \, dx$$

$\Rightarrow du = 2\, dx$

$\Rightarrow dx = \frac{1}{2} du$

$$\int \sin 2x \, dx = \int \sin u \, \tfrac{1}{2} du$$

$$= \frac{1}{2} \int \sin u \, du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2} \cos(2x) + C$$

$$\Rightarrow I = x - \frac{1}{2} \cos(2x) + C$$

Problem 6. Evaluate definite integral $\int_0^1 x(1 + x^2)^{99}\, dx$. [10 pts]

Use substitution. Let $u = 1 + x^2$

$\Rightarrow du = 2x\, dx \Rightarrow \frac{1}{2} du = x\, dx$

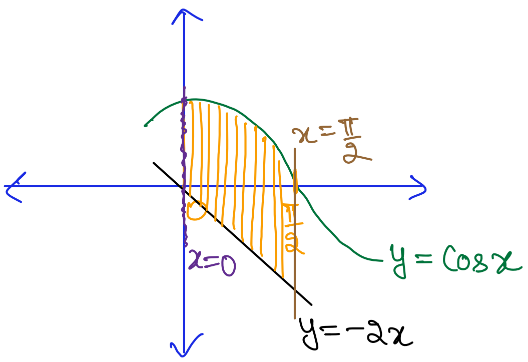
$$I = \int_0^1 x(1+x^2)^{99} dx = \int_0^1 (\underbrace{1+x^2}_{u})^{99} \underbrace{x\, dx}_{\frac{1}{2} du}$$

$$= \int_{1+0^2}^{1+1^2} u^{99} \, \tfrac{1}{2} du = \frac{1}{2} \int_1^2 u^{99} \, u$$

$$= \frac{1}{2} \left. \frac{u^{100}}{100} \right|_1^2 = \frac{1}{2} \left[\frac{2^{100}}{100} - \frac{1^{100}}{100} \right]$$

$$= \frac{2^{100} - 1}{200}$$

Problem 7. Find area of the region bounded by the curves $y = \cos x$ and $y = -2x$, $x = 0$, $x = \pi/2$. [10 pts]



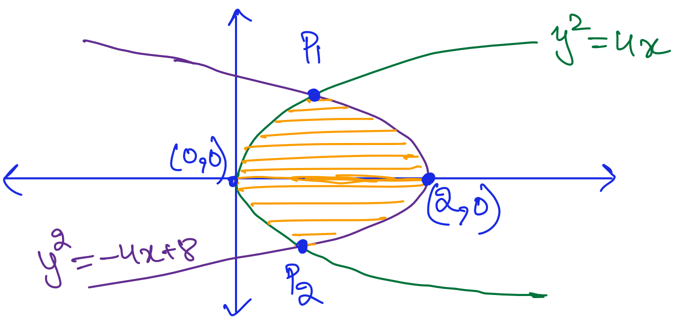
$$A = \int_0^{\pi/2} \underbrace{[\cos x - (-2x)]}_{\text{upper curve} - \text{lower curve}} dx$$

$$\Rightarrow A = \int_0^{\pi/2} (\cos x + 2x)\, dx$$

$$= \int_0^{\pi/2} \cos x \, dx + \int_0^{\pi/2} 2x \, dx$$

$$= \sin x \Big|_0^{\pi/2} + x^2 \Big|_0^{\pi/2} = \left(\sin\frac{\pi}{2} - \sin 0\right) + \left(\left(\frac{\pi}{2}\right)^2 - 0^2\right) = 1 + \frac{\pi^2}{4}$$

Problem 8. Find area of the region bounded by parabolas $y^2 = 4x$ and $y^2 = -4x + 8$. [10 pts]



$\overset{\shortparallel}{-4(x-2)}$

$\Rightarrow$ shift graph of $y^2=-4x$ 2 units to the right

Find pts of intersection P_1 and P_2

$y^2 = \underbrace{4x}$ and $y^2 = \underbrace{-4x+8}$ equate

$\Rightarrow 4x = -4x + 8 \Rightarrow 8x = 8$

$\Rightarrow x = 1$

$\Rightarrow y^2 = 4(1) = 4 \Rightarrow y = \pm 2$

$P_1\ (1, 2)$

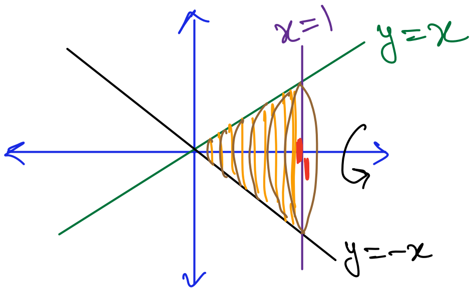
$P_2\ (1, -2)$

$$A = \int_{y=-2}^{y=2} (\underbrace{\text{right curve}}_{y^2=-4x+8} - \underbrace{\text{left curve}}_{y^2=4x \Rightarrow x=\frac{1}{4}y^2})\, dy$$

$\Rightarrow 4x = 8 - y^2 \Rightarrow x = 2 - \frac{1}{4}y^2$

$$\Rightarrow A = \int_{-2}^{2} \left(2 - \frac{1}{4}y^2 - \frac{1}{4}y^2\right) dy = \int_{-2}^{2} \left(2 - \frac{1}{2}y^2\right) dy = \left(2y - \frac{y^3}{6}\right)\Big|_{-2}^{2} = \frac{16}{3}$$

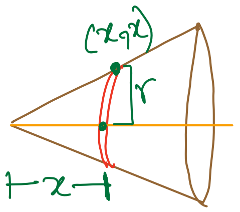
Problem 9. Find the volume of the solid obtained by rotating the region bounded by $y = x$, $x = 1$, and $y = -x$ about the x-axis. [10 pts]



Use disk method

$r = x$

$\Rightarrow A(x) = \pi r^2 = \pi x^2$

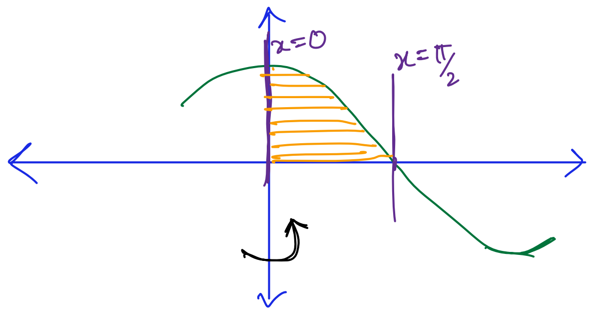


$dV = \pi x^2 \, dx$

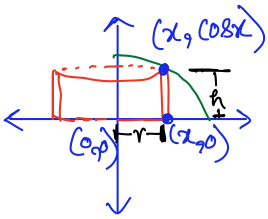
$$\Rightarrow V = \int_0^1 \pi x^2 \, dx$$

$$= \pi \int_0^1 x^2 \, dx = \pi \left. \frac{x^3}{3} \right|_0^1 = \frac{\pi}{3}$$

Problem 10. Set up an integral for the volume of the solid obtained by rotating the region bounded by $y = \cos x$, $y = 0$, $x = 0$, $x = \pi/2$, about the y-axis. [10 pts]



Use shell method



$\Rightarrow r = x$

$h = \cos x$

$\Rightarrow dV = 2\pi x \cos x \, dx$

$$\Rightarrow V = \int_0^{\pi/2} 2\pi x \cos x \, dx$$

Bonus Problem 1. Evaluate $\int_{-1}^{1} \frac{\tan x}{x^4+1}\,dx$. [5 pts]

Notice that $f(-x) = \frac{\tan(-x)}{(-x)^4+1} = \frac{-\tan x}{x^4+1} = -f(x)$

$\Rightarrow f$ is an odd function.

$$\Rightarrow \int_{-1}^{1} \frac{\tan x}{x^4+1}\,dx = 0$$

Bonus Problem 2. A particle moves in a straight line with acceleration $a(t) = 1 - 2t$. Find the position of the particle at $t = 3$ seconds if at $t = 0$ the particle was at rest at 5 m away from origin. [5 pts]

$\Rightarrow v(0)=0$ $\quad s(0)=5$

$a(t) = v'(t)$

$$\Rightarrow v(t) - v(0) = \int_0^t a(s)\,ds$$

$$\Rightarrow v(t) - 0 = \int_0^t (1-2s)\,ds = \left(s - s^2\right)\Big|_0^t = t - t^2$$

$$\Rightarrow v(t) = t - t^2$$

$v(t) = s'(t)$

$$\Rightarrow s(3) - s(0) = \int_0^3 v(t)\,dt = \int_0^3 (t - t^2)\,dt$$

$$= \left(\frac{t^2}{2} - \frac{t^3}{3}\right)\Big|_0^3 = \frac{9}{2} - \frac{27}{3}$$

$$\Rightarrow s(3) - 5 = \frac{9}{2} - 9 = -\frac{9}{2} \quad \Rightarrow s(3) = 5 - \frac{9}{2} = \frac{1}{2}\text{ m.}$$

away from origin