Indiana University - Purdue University, Indianapolis

Math16600

Practice Test (Chapter 6)

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Name:____ [2 pts]

Instructions:

- No cell phones, calculators, watches, technology, hats - stow all in your bags.
- **Write your name** on this cover page. It carries 2 points.
- This test is closed book and closed notes.
- All work must be clearly shown for partial credit.
- If you wish for something not to be graded, please strike it out neatly.
- Box, circle, or otherwise clearly indicate your final answer.
- When you finish, return your test to the proctor, and leave the classroom.
- There are a total of **16 problems** including bonus problem.
  - Problems 1-10 are each worth 6 points.
  - Problems 11-15 are each worth 8 points.
  - The bonus problem is worth 8 points.
- You can score a **maximum of 110 points out of 100**.
- There are a total of **9 pages** including the cover page.

Problem 1: Given a one-to-one function $f(x) = (x+2)^3$, $-\infty < x < \infty$.
Find $f^{-1}(8)$ and $(f^{-1})'(8)$. [6 pts]

let $x = f^{-1}(8) \Rightarrow f(x) = 8 \Rightarrow (x+2)^3 = 8 \Rightarrow x+2 = 2 \Rightarrow x = 0$

$\Rightarrow f^{-1}(8) = 0$

$$(f^{-1})'(8) = \frac{1}{f'(f^{-1}(8))} = \frac{1}{f'(0)}$$

$f(x) = (x+2)^3 \Rightarrow f'(x) = 3(x+2)^2 \Rightarrow f'(0) = 3(0+2)^2 = 12$

$$\Rightarrow (f^{-1})'(8) = \frac{1}{12}$$

Problem 2: Simplify the expression $\tan(\sin^{-1} x)$. [6 pts]

let $\theta = \sin^{-1} x \Rightarrow \sin\theta = x = \frac{P}{H}$

let $P = x \Rightarrow H = 1 \Rightarrow x^2 + B^2 = 1^2 \Rightarrow B^2 = 1 - x^2$

$\Rightarrow B = \sqrt{1-x^2}$

$$\Rightarrow \tan\theta = \frac{P}{B} = \frac{x}{\sqrt{1-x^2}}$$

Problem 3: Compute the derivative

$$y = \ln\big(\cosh(8x)\big)$$

[6 pts]

$$\frac{dy}{dx} = \frac{1}{\cosh(8x)}\big(\sinh(8x)\big)(8) = 8\tanh(8x)$$

Problem 4: Compute the derivative

$$H(t) = \frac{\ln(1+t^2)}{e^{5t}}$$

[6 pts]

$$H'(t) = \frac{\left[\ln(1+t^2)\right]' e^{5t} - \ln(1+t^2)\left[e^{5t}\right]'}{(e^{5t})^2}$$

$$= \frac{\left(\frac{2t}{1+t^2}\right)e^{5t} - 5e^{5t}\ln(1+t^2)}{(e^{5t})^2}$$

$$= \frac{\cancel{e^{5t}}\left[\frac{2t}{1+t^2} - 5\ln(1+t^2)\right]}{(e^{5t})^{\cancel{2}}}$$

$$\Rightarrow \quad H'(t) = \frac{2t - 5(1+t^2)\ln(1+t^2)}{e^{5t}(1+t^2)}$$

Problem 5: Compute the derivative

$$f(x) = \ln\left(x\, e^x\right)$$

[6 pts]

$\Rightarrow f(x) = \ln x + \ln(e^x)$

$\Rightarrow f(x) = \ln x + x$

$\Rightarrow f'(x) = \frac{1}{x} + 1$

Problem 6: Compute the derivative

$$g(x) = \tan^{-1}\left(\sqrt{x}\right) e^{2x^5}$$

[6 pts]

$g'(x) = \left(\tan^{-1}\sqrt{x}\right)' e^{2x^5} + \left(\tan^{-1}\sqrt{x}\right)\left(e^{2x^5}\right)'$

$= \frac{1}{1+(\sqrt{x})^2}\left(\frac{1}{2\sqrt{x}}\right) e^{2x^5} + \left(\tan^{-1}\sqrt{x}\right)\left(e^{2x^5}\right)\left(10x^4\right)$

$= \frac{e^{2x^5}}{2\sqrt{x}\,(1+x)} + 10x^4\, e^{2x^5}\, \tan^{-1}(\sqrt{x})$

Problem 7: Evaluate the integral

$$\int \frac{x}{x^2+4}\,dx$$

[6 pts]

Let $u = x^2+4 \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$$\Rightarrow I = \int \frac{1}{u}\,\frac{1}{2}du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$

$$\Rightarrow I = \frac{1}{2}\ln|x^2+4| + C$$

Problem 8: The mass of a radio-active material is reduced to 75% of the original quantity in 10 years. What is the half-life? [6 pts]

$$m(0) \xrightarrow{10\text{ yrs}} 75\% \text{ of } m(0) = \frac{3}{4}m(0)$$

$m(t) = m(0)e^{kt}$.

At $t=10$, $m(t) = \frac{3}{4}m(0)$

$\Downarrow$ Put $t=10$

$\Rightarrow m(10) = \frac{3}{4}m(0)$

$$m(10) = m(0)e^{10k}$$

$$\Rightarrow \frac{3}{4}\cancel{m(0)} = \cancel{m(0)}e^{10k} \Rightarrow \frac{3}{4} = e^{10k} \Rightarrow \ln\left(\frac{3}{4}\right) = \underbrace{\ln\left(e^{10k}\right) = 10k}_{1}$$

Let half-life be t_0 $\Rightarrow$ After t_0 yrs, $m(t)$ will be $\frac{1}{2}m(0)$

$\Rightarrow m(t_0) = \frac{1}{2}m(0)$

$$\Rightarrow \frac{1}{2}\cancel{m(0)} = \cancel{m(0)}e^{kt_0}$$

$t = t_0$

$m(t) = m(0)e^{kt}$

$$\Rightarrow \frac{1}{2} = e^{kt_0}$$

From 1 $\Rightarrow 10k = \ln\frac{3}{4}$

$$\Rightarrow \ln\left(\frac{1}{2}\right) = \ln\left(e^{kt_0}\right) = kt_0$$

$\Rightarrow k = \frac{1}{10}\ln\frac{3}{4}$

$$\Rightarrow \ln\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{10}\ln\tfrac{3}{4}\right)t_0 \Rightarrow \frac{10\ln\left(\tfrac{1}{2}\right)}{\ln\tfrac{3}{4}} = t_0$$

$$\Rightarrow t_0 = \frac{-10\ln 2}{\ln 3 - \ln 4} = \frac{10\ln 2}{\ln 4 - \ln 3}$$

$$\Rightarrow t_0 = \frac{10\ln 2}{\ln 4 - \ln 3} \text{ years}$$

$$\boxed{t_{1/2} = \frac{-\ln 2}{K}} = \frac{-\ln 2}{\tfrac{1}{10}\ln\tfrac{3}{4}} = -\frac{10\ln 2}{\ln 3 - \ln 4}$$

Problem 9: Compute the limit

$$L = \lim_{x\to\infty} x \sin^{-1}\left(\frac{1}{x}\right) \overset{D.S.}{=} \infty \sin^{-1}\left(\frac{1}{\infty}\right)$$

[6 pts]

$$= \infty \sin^{-1}(0)$$

$$= \infty \cdot 0 \text{ (indeterminate)}$$

$$\Rightarrow L = \lim_{x\to\infty} \frac{\sin^{-1}\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\overset{D.S.}{=} \frac{\sin^{-1}\left(\frac{1}{\infty}\right)}{\frac{1}{\infty}} = \frac{\sin^{-1} 0}{0} = \frac{0}{0}$$

$$\Rightarrow L = \lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\right)\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}$$

$$\overset{D.S.}{=} \frac{1}{\sqrt{1-\left(\frac{1}{\infty}\right)^2}} = \frac{1}{\sqrt{1-0^2}} = 1$$

Problem 10: Compute the limit

$$\lim_{x\to 0} \frac{x^2}{\tan^2 x}$$

[6pts]

$$L = \lim_{x\to 0} \frac{x^2}{\tan^2 x} \overset{D.S.}{=} \frac{0^2}{\tan^2 0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cancel{2}x}{\cancel{2}\tan x \sec^2 x} \overset{D.S.}{=} \frac{2(0)}{2(\tan 0)\sec^2 0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{1}{(\sec^2 x)\sec^2 x + \tan x(2\sec x)(\sec x \tan x)}$$

$$\overset{D.S.}{=} \frac{1}{(\sec^2 0)(\sec^2 0) + \tan 0\,(2\sec 0)(\sec 0 \tan 0)}$$

$$= \frac{1}{1+0} = 1$$

Problem 11: Use logarithmic differentiation to compute $\dfrac{dy}{dx}$ where

$$y = x^{\sin x}$$

[8 pts]

$$\Rightarrow \ln y = \ln\left(x^{\sin x}\right) = \sin x \ln x$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\sin x)' \ln x + \sin x (\ln x)'$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \frac{\sin x}{x}$$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left[\cos x \ln x + \frac{\sin x}{x}\right]$$

Problem 12: Compute the limit

$$\lim_{x\to 0}(1+x)^{2/x}$$

[8 pts]

$$\Rightarrow L = \lim_{x\to 0}(1+x)^{2/x} \overset{D.S.}{=} (1+0)^{2/0} = 1^{\infty} \text{ (indeterminate)}$$

$$\Rightarrow \ln L = \lim_{x\to 0} \ln(1+x)^{2/x} = \lim_{x\to 0} \frac{2}{x}\ln(1+x)$$

$$= \lim_{x\to 0}\frac{2\ln(1+x)}{x} \overset{D.S.}{=} \frac{2\ln 1}{0} = \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{\frac{2}{1+x}}{1} \overset{D.S.}{=} \frac{2}{1+0} = 2$$

$$\Rightarrow \ln L = 2 \Rightarrow L = e^{2}$$

Problem 13: Evaluate the integral

$$\int \frac{\cos x}{\sqrt{1-\sin^2 x}}\,dx$$

$\neq \int \frac{\cos x}{\cos x}\,dx = \int dx = x + C$ (this would be wrong)

[8 pts]

Let $u = \sin x$

$\Rightarrow du = \cos x\, dx$

$\Rightarrow I = \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C$

$= \sin^{-1}(\sin x) + C$

(*) Note that $\sqrt{1-\sin^2 x} = |\cos x|$ and $\sin^{-1}(\sin x) = x$ only if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

In general $\sin^{-1}(\sin x) \neq x$ (outside the interval $[-\pi/2, \pi/2]$)

Problem 14: Evaluate the integral

$$\int_0^1 \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$$

[8 pts]

$u = \sin^{-1} x$

$\Rightarrow du = \frac{1}{\sqrt{1-x^2}}\,dx$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

$= \int_0^1 (\sin^{-1} x)\, \underbrace{\frac{1}{\sqrt{1-x^2}}\,dx}_{du}$

$= \int_{\sin^{-1} 0}^{\sin^{-1} 1} u\, du = \int_0^{\pi/2} u\, du$

$= \frac{1}{2} u^2 \Big|_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}$

Problem 15: Evaluate the integral

$$\int_0^1 \frac{x^3}{1+x^8}\,dx$$

[8 pts]

Let $u = x^4 \Rightarrow du = 4x^3\,dx \Rightarrow \frac{1}{4}\,du = x^3\,dx$

$$\int_0^1 \frac{1}{1+x^8}\,x^3\,dx = \int_{0^4}^{1^4} \frac{1}{1+u^2}\left(\frac{1}{4}du\right) = \frac{1}{4}\int_0^1 \frac{1}{1+u^2}\,du$$

$$= \frac{1}{4}\,\mathrm{Tan}^{-1}(u)\Big|_0^1$$

$$= \frac{1}{4}\left[\mathrm{Tan}^{-1}(1) - \mathrm{Tan}^{-1}(0)\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{16}$$

Bonus Problem: Evaluate the integral

$$\int (\sec^2 x + e^x)(\tan x + e^x)^4\,dx$$

[8 pts].

Let $u = \mathrm{Tan}\,x + e^x$

$\Rightarrow du = (\sec^2 x + e^x)\,dx$

$$\Rightarrow I = \int u^4\,du = \frac{u^5}{5} + C$$

$$= \frac{(\mathrm{Tan}\,x + e^x)^5}{5} + C$$