M16600 Lecture Notes
Section 11.8: Power Series

■ **Section 11.8** textbook exercises: # 3, 4, 6, 7, 9, 11, 12, 15 (these will take some time to do).

DEFINITION OF POWER SERIES. The ***power series centered at a*** is a series of the form

depends on x

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where x is the variable and c_n's are constants called the **coefficients** of the series. Here, a is a fixed number called the **center**.

Example 1: Here are some examples of power series

(a) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ $\rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$ $\Rightarrow$ $c_n = \frac{1}{n}$

$a=3$

diff. values of x give us diff series

The center $a = 3$. The coefficients $c_n = \frac{1}{n}$

(b) $\sum_{n=0}^{\infty}(-1)^n x^n$ $\rightarrow$ $\sum_{n=0}^{\infty}(-1)^n(x-0)^n$ $\Rightarrow$ $c_n = (-1)^n$

$a=0$

The center $a = 0$. The coefficients $c_n = (-1)^n$.

Note: A power series is a **function** in the variable x, where the domain is the set of all values of x such that the series converges. The outputs are series.

For example, let $f(x) = \sum_{n=0}^{\infty} x^n$, i.e., $f(x) =$ $\rightarrow$ $\sum_{n=0}^{\infty} x^n$ $\rightarrow$ $\boxed{a_n = x^n \quad a_{n+1} = x^{n+1}}$

$c_n = 1$, $\sum_{n=0}^{\infty}(x-0)^n \Rightarrow a=0$

$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{x^n} = x^{n+1-n} = x$

independent of n.

Geometric series with the common ration being x

$\frac{a}{1-r}$ $\rightarrow$ first term

$a = x^0 = 1$

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$$ $\Rightarrow$ true as long as $|x|<1$

$f(3) = \sum_{n=0}^{\infty} 3^n$

$r=3$

$\rightarrow$ does not converge

$f(\frac{1}{2}) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$

$r = \frac{1}{2} < 1$

$\rightarrow$ series converges

$\frac{1}{2}$ is in the domain but 3 is not in the domain.

Example 2: For what values of x is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

$a_n = \frac{(x-3)^n}{n}$ ← n^{th} Power

⇒ Ratio test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r \begin{cases} <1 \Rightarrow \text{series converges} \\ 1 \quad \text{can't say} \\ >1 \Rightarrow \text{series diverges.} \end{cases}$$

$$a_n = \frac{(x-3)^n}{n} \Rightarrow a_{n+1} = \frac{(x-3)^{n+1}}{n+1} \Rightarrow \frac{a_{n+1}}{a_n} = \underbrace{\frac{(x-3)^{n+1}}{n+1} \times \frac{n}{(x-3)^n}}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = (x-3)^{n+1-n} \frac{n}{n+1} = (x-3)\frac{n}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|(x-3)\frac{n}{n+1}\right| = \lim_{n\to\infty} |x-3| \frac{n}{n+1} = |x-3| \lim_{n\to\infty}\frac{n}{n+1}$$

$$= |x-3| \frac{\infty}{\infty} \rightarrow \text{indeterminate}$$

$$= |x-3| \lim_{n\to\infty}\frac{n}{n}$$

$$= |x-3|$$

$n+1 \sim n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-3|$$

If $|x-3| < 1$ then the series converges.

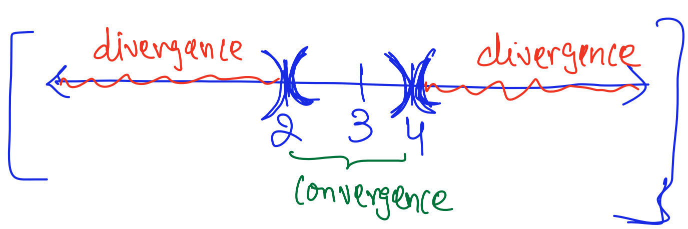
If $|x-3| > 1$ then the series diverges.

$|x-3| = 1 \rightarrow$ ratio test fails

$$|x-3| < 1 \Rightarrow -1 < x-3 < 1 \Rightarrow 3-1 < x-3+3 < 1+3$$

$$\Rightarrow \underbrace{2 < x < 4}_{\text{series converges}}$$

$x > 4$ or $x < 2$ → series diverges.



$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

Put $x=2$

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Alternating series with $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n$$

By AST, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

$\Rightarrow$ The given power series converges for $x=2$

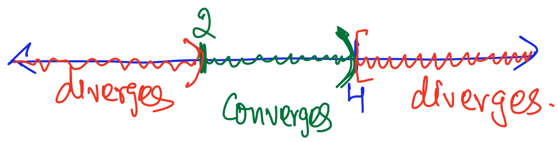
Put $x=4$

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

P-series with $P=1 \leq 1$

$\Rightarrow$ diverges

$\Rightarrow$ The given power series diverges for $x=4$



$2 \leq x < 4 \Rightarrow [2,4)$

Interval of convergence.

The point of focus for this section is to determine for what values of x a power series is convergent. Hence, we have the following concepts.

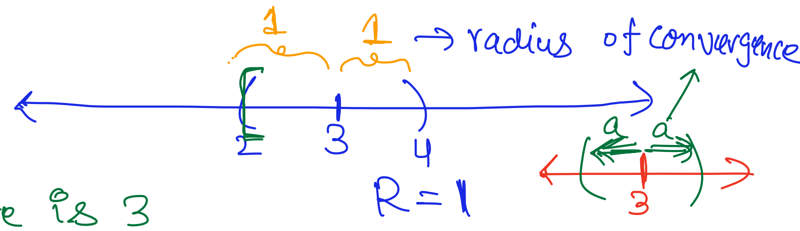
Radius of Convergence and Interval of Convergence.

In example 2, we get $|x-3|<1$. Geometrically, this implies the ***distance*** between x and the center 3 is less than 1.

$|x-3|<1 \rightarrow$ converges

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$$

centre is 3



The ***Radius of Convergence*** of a power series is the greatest distance between x and the center a such that the series is convergent.

If R is the radius of convergence, then the series is convergent for all x such that $|x-a|<R$, where a is the center of the power series.

In example 2, we find the interval $2 \le x < 4$ for which the series is convergent. The interval $[2,4)$ is called the *interval of convergence.*

Note: To find the interval of convergence, we had to test the endpoints $x=2$ and $x=4$ separately to determine whether the series is convergent of divergent. This will be the case in general.

The ***Interval of Convergence*** of a power series is the interval that consists of all values of x for which the series converges.

centres is 0.

Example 3: Find the radius of convergence and the interval of convergence of the series

$(-3)^n = (-1)^n 3^n$

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

$(x-0)^n$

$\rightarrow$ values of x for which given power series converges.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^n}{\sqrt{n+1}} \rightarrow x=-1$$

Not an alternating series for every value of x

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^n (-1)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^{n+n} 3^n}{\sqrt{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{2n} 3^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{3^n}{\sqrt{n+1}}$$

$(-1)^{2n} = ((-1)^2)^n = 1^n = 1$

NOT AN ALTERNATING SERIES

Cannot apply AST as long as there is x.

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+1+1}} \times \frac{\sqrt{n+1}}{(-3)^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-3)^{n+1-n} \, x^{n+1-n} \, \sqrt{n+1}}{\sqrt{n+2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-3)\, x \sqrt{n+1}}{\sqrt{n+2}} \right| = \lim_{n\to\infty} |-3|\,|x| \frac{\sqrt{n+1}}{\sqrt{n+2}}$$

$$= \lim_{n\to\infty} \underbrace{3|x|}\frac{\sqrt{n+1}}{\sqrt{n+2}} = 3|x| \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n+2}} = 3|x| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}}$$

DS: $\frac{\infty}{\infty}$

$= 3|x|$

$r = 3|x|$

$r < 1 \rightarrow$ the series converges

$r > 1 \rightarrow$ the series diverges

$r = 1 \Rightarrow$ can't say

$3|x|<1 \rightarrow$ series converges $\Rightarrow |x| < \frac{1}{3}$

$$\Rightarrow \underbrace{-\frac{1}{3} < x < \frac{1}{3}}_{\text{Convergence}}$$

$$\boxed{\begin{array}{c} |x|<a \\ \Rightarrow -a<x<a \end{array}}$$

$$|x| > \frac{1}{3} \Rightarrow \underbrace{x > \frac{1}{3} \text{ or } x < \frac{-1}{3}}_{\text{divergence}}$$

endpoints.

$$3|x| = 1 \Rightarrow |x| = \frac{1}{3} \Rightarrow x = \pm\frac{1}{3}$$

$$\Rightarrow \boxed{a^n b^n = (ab)^n}$$

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Put $x = \frac{-1}{3}$

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{-1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{\left(-3 \times \frac{-1}{3}\right)^n}{\sqrt{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{1^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

limit comparison test.

similar to $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$

P-series with $P = \frac{1}{2} < 1$

diverges $\Rightarrow \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges.

The Power series diverges at $x = \frac{-1}{3}$

Put $x = \frac{1}{3}$

$$\sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{\left(-3 \times \frac{1}{3}\right)^n}{\sqrt{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

Alternating series

$$b_n = \frac{1}{\sqrt{n+1}}$$

$$\lim_{n\to\infty} b_n = \frac{1}{\infty} = 0$$

$$b_{n+1} = \frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n+1}} = b_n$$

By AST, $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges.

The Power series converges at $x = 1/3$

$\Rightarrow$ Interval of convergence $= \left(-\frac{1}{3}, \frac{1}{3}\right]$ $\Rightarrow R = \left|\frac{1}{3} - 0\right| = \frac{1}{3}$

Example 4: Find the radius of convergence and the interval of convergence of the series $\sum_{n=0}^{\infty} n!(x+1)^n$ $\longrightarrow a_n = n!\,(x+1)^n$

$c_n = n!$

$(x+1)^n = (x-(-1))^n$

Centre $= -1$

-1

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(x+1)^{n+1}}{n!\;x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}(x+1)^{n+1-n}\right|$$

$$\boxed{\begin{array}{l}\infty > 1 \Rightarrow \text{diverges}\\ 0 < 1 \Rightarrow \text{converges}\end{array}}$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}(x+1)\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cancel{n}\cancel{(n-1)}\cancel{\cdots}\cancel{1}}{\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{1}}(x+1)\right|$$

$$= \lim_{n\to\infty}\left|(n+1)(x+1)\right| = \lim_{n\to\infty} |x+1|\,(n+1) = \infty$$

as long as $|x+1| \neq 0$

$x+1=0 \Rightarrow x=-1$

If $|x+1| = 0$ then $\underbrace{0\cdot\infty}$ indeterminate.

$$r = \begin{cases} 0 & \text{if } x=-1 \\ \infty & \text{otherwise}\end{cases}$$

$\sum_{n=0}^{\infty} n!\,(x+1)^n = \underline{0}$ every term is 0.

the Power series converges for $x=-1$

radius of convergence $= 0$

the interval of convergence $= \{-1\}$

Example 5: Find the radius of convergence and the interval of convergence of the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$\sum_{n=0}^{\infty}\frac{1}{n!}(x-0)^n \longrightarrow a_n = \frac{x^n}{n!} \Rightarrow a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

Centre $= 0$

$c_n = \frac{1}{n!}$

Apply ration test.

$$\Rightarrow r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\times\frac{n!}{x^n}\right| = \lim_{n\to\infty}\left|x^{n+1-n}\frac{n!}{(n+1)!}\right|$$

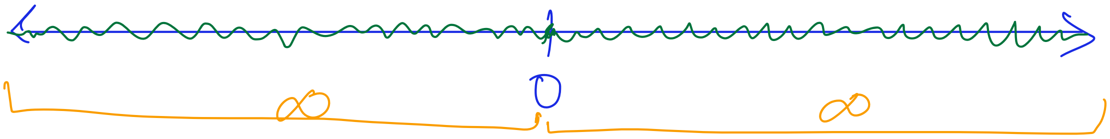
$$= \lim_{n\to\infty}\left|x\,\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}|x|\,\frac{n!}{(n+1)!} = \lim_{n\to\infty}|x|\,\frac{\cancel{n}\cancel{(n-1)}\cdots\cancel{1}}{(n+1)\cancel{n}\cancel{(n-1)}\cdots\cancel{1}}$$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0 \quad \text{unless } \underbrace{|x| = \infty}_{}$$

not Possible since x is a real number

$r = 0 < 1$

$\Rightarrow$ the power series converges for every real number x.



Interval of Convergence $= (-\infty, \infty)$

Radius of Convergence $= \infty$

$$\boxed{\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|}$$

$\rightarrow$ For $\sum_{n=0}^{\infty} \underbrace{c_n (x-a)^n}_{}$

$a_n = c_n \underbrace{(x-a)^n}_{}$

a is the Centre of the power series

$\Rightarrow$ the power series converges for $(a-R, a+R)$

$$\underbrace{\text{Interval of Convergence}}_{} = \begin{cases} (a-R, a+R) \\ [a-R, a+R) \\ (a-R, a+R] \\ [a-R, a+R] \end{cases}$$

Have to find convergence/divergence at the endpoints, that is, $x = a-R$ and $x = a+R$.

Example $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n} (x-1)^n$ → Find radius of convergence
→ Find interval of convergence.

$c_n = \dfrac{(-1)^n}{(2n-1)2^n}$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{\underbrace{(2(n+1)-1)}_{2n+2-1=2n+1}2^{n+1}} \cdot \frac{(2n-1)2^n}{(-1)^n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1-n}(2n-1)}{(2n+1)\;2^{n+1-n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(2n-1)}{(2n+1)2^1} \right| = \lim_{n\to\infty} \frac{2n-1}{2(2n+1)} = \frac{\infty}{\infty}$$
(Not determinate)

$$= \lim_{n\to\infty} \frac{\cancel{2n}}{2(\cancel{2n})} = \frac{1}{2}$$

$\dfrac{1}{R} = \dfrac{1}{2} \Rightarrow \boxed{R=2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} (x+1)^n$
$= (x-(-1))^n$
$\Rightarrow a=-1$

- $(a-R, a+R)$; $a=1$
  $= (1-2, 1+2) = (-1, 3)$

- Now have to check at endpoints.

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n} (x-1)^n$

$\boxed{x=-1}$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n} (-1-1)^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{(2n-1)2^n} = \sum_{n=1}^{\infty} \frac{\cancel{2^n}}{(2n-1)\cancel{2^n}}$$

$\boxed{a^n b^n = (ab)^n}$

$$= \sum_{n=1}^{\infty} \frac{1}{2n-1}$$ similar $$\sum_{n=1} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$$

P-series with P=1 ⇒ diverges

$\boxed{x=3}$ $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n} (3-1)^n$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{2^n}}{(2n-1)\cancel{2^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$$

Alternating series $\cancel{\displaystyle\sum_{n=1} \frac{(-1)^n}{2n}}$

- $b_n = \dfrac{1}{2n-1} \Rightarrow \displaystyle\lim_{n\to\infty} b_n = 0$
- $b_{n+1} = \dfrac{1}{2n+1} < \dfrac{1}{2n-1} = b_n$

By AST, the series converges

- The given Power series converges at $x=3$
  diverges at $x=-1$

$\Rightarrow$ Interval of Convergence $= (-1, 3]$

$R = 2$
$I = (-1, 3]$