Learning objectives:

1. Understand the concept of limits at infinity.
2. Find horizontal asymptotes to a curve.

Intuitive definition of a limit at infinity.

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

means the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Horizontal asymptote.

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if

$$\text{either} \quad \lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L\,.$$

Example 1. Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$

Theorem

If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0 .$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0 .$$

The limit laws are valid for limits at infinity as well (with the exception of direct substitution).

Example 2. Evaluate

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} .$$

Example 3. Find the horizontal and vertical asymptotes to the graph of the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5} .$$

Example 4. Compute $\lim_{x\to\infty} (\sqrt{x^2+1} - x)$.

Example 5. Evaluate $\lim_{x\to\infty} \sin\frac{1}{x}$.

Example 6. Evaluate $\lim_{x\to\infty} \sin x$.

Infinite Limits at Infinity

We write

$$\lim_{x\to\infty} f(x) = \infty$$

when values of $f(x)$ become arbitrarily large as x becomes large.

Similarly, we can define

$$\lim_{x\to-\infty} f(x) = \infty\ , \quad \lim_{x\to\infty} f(x) = -\infty\ , \quad \lim_{x\to-\infty} f(x) = -\infty\ .$$

Example 7. Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

Example 8. Find $\lim_{x \to \infty} (x^2 - x)$.

Example 9. Find $\lim_{x \to \infty} \frac{x^2 + x}{3 - x}$.

Example 10. Find $\lim_{x \to \infty} \frac{x}{x^2 + 1}$.