Name:

Problem 1: Find the equation of tangent line to the hyperbola $y = \frac{5}{x}$ at the point $(1, 5)$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{5}{x}\right) = 5\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-5}{x^2}$$

$\Rightarrow$ Equation of tangent : $\frac{y-5}{x-1} = \frac{dy}{dx}\Big|_{(1,5)} = \frac{-5}{x^2}\Big|_{x=1} = \frac{-5}{1^2}$

$$\Rightarrow y-5 = -5(x-1) \Rightarrow y = -5x+5+5 \Rightarrow \boxed{y = -5x+10}$$

Problem 2: Let $f(x) = \begin{cases} -x & \text{if } x \le 0 \\ x^2 & \text{if } 0 < x < 1 \\ 2x-1 & \text{if } x \ge 1 \end{cases}$.

Is f continuous everywhere on $\mathbb{R}$? If not, then find the points where f is discontinuous.

Is f differentiable everywhere on $\mathbb{R}$? If not, then find the points where f is not differentiable.

$\underset{x=0}{LHL} = -0 = 0$, $\underset{x=0}{RHL} = 0^2 = 0$, $f(0) = 0 \Rightarrow$ Continuous at 0.

$\underset{x=1}{LHL} = 1^2 = 1$, $\underset{x=1}{RHL} = 2(1)-1 = 1$, $f(1) = 2(1)-1 = 1 \Rightarrow$ Continuous at 1.

Yes, f is continuous everywhere on $\mathbb{R}$.

$$f'(x) = \begin{cases} -1 & x \le 0 \\ 2x & 0 < x < 1 \\ 2 & x \ge 1 \end{cases}$$

$\Rightarrow \underset{x=0}{LHD} = -1$, $\underset{x=0}{RHD} = 2(0) = 0$

$\Rightarrow \underset{x=1}{LHD} = 2(1) = 2$, $\underset{x=1}{RHD} = 2$

$\Rightarrow f$ is differentiable at $x=1$ but not at $x=0$

Problem 3: Let $f(x) = \begin{cases} \sin x + 2\cos x & \text{if } x \le \frac{\pi}{4} \\ \cos x + 2\sin x & \text{if } x > \frac{\pi}{4} \end{cases}$.

Is f continuous everywhere on $\mathbb{R}$? If not, then find the points where f is discontinuous.

Is f differentiable everywhere on $\mathbb{R}$? If not, then find the points where f is not differentiable.

Note that $\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

$\underset{x=\frac{\pi}{4}}{LHL} = \sin\frac{\pi}{4} + 2\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{3}{\sqrt{2}} = f\left(\frac{\pi}{4}\right)$

$\underset{x=\frac{\pi}{4}}{RHL} = \cos\frac{\pi}{4} + 2\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$ $\Rightarrow f$ is continuous at $\frac{\pi}{4}$.

Since, $\sin x + 2\cos x$ and $\cos x + 2\sin x$ are continuous in their domains, and f is continuous at $\frac{\pi}{4}$, we have f is continuous everywhere on $\mathbb{R}$.

<u>Differentiability</u> : $f'(x) = \begin{cases} \cos x - 2\sin x & \text{if } x \le \frac{\pi}{4} \\ -\sin x + 2\cos x & \text{if } x > \frac{\pi}{4} \end{cases}$

$\underset{x=\frac{\pi}{4}}{LHD} = \cos\frac{\pi}{4} - 2\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$, $\underset{x=\frac{\pi}{4}}{RHD} = -\sin\frac{\pi}{4} + 2\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\Rightarrow \underset{x=\frac{\pi}{4}}{LHD} \neq \underset{x=\frac{\pi}{4}}{RHD}$

Thus, f is differentiable everywhere on $\mathbb{R}$ except at $x=\frac{\pi}{4}$.

Problem 4: A particle starts to move along x-axis at time $t = 0$ with its position varying with time as $x(t) = t^3 - 27t + 7$.

1. Find the velocity of the particle as a function of time.
2. At what time instant was the particle at rest?
3. Find the time interval for which the particle was moving backwards.
4. Find the acceleration of the particle as a function of time.
5. When was the particle speeding up? When was it slowing down?

(1) $v(t) = 3t^2 - 27$

(2) $v(t)=0 \Rightarrow 3t^2-27=0 \Rightarrow 3t^2=27 \Rightarrow t^2=9$

$\Rightarrow t = \pm 3s$ But time cannot be $-$ve.

$\Rightarrow \underline{t=3s}$

(3) $v(t)<0 \Rightarrow 3t^2-27<0$

$\Rightarrow 3(t^2-9)<0 \Rightarrow (t-3)(t+3)<0$

$+ \quad -3 \quad - \quad 3 \quad +$

$\Rightarrow \underline{0<t<3}$

(4) $a(t) = v'(t) = 6t$

(5) Since $t \geq 0$, $a(t) = 6t \geq 0$. $\Rightarrow$ Particle speeds up if $v(t)>0$

$\Rightarrow \underline{t>3}$

and

slows down if $v(t)<0$

$\Rightarrow \underline{0<t<3}$

Problem 5: Let $y = \sin x + \cos x$. Find $\frac{d^2y}{dx^2}$ and $\frac{d^4y}{dx^4}$.

$y = \sin x + \cos x \Rightarrow \frac{dy}{dx} = \cos x - \sin x$

$\Rightarrow \boxed{\frac{d^2y}{dx^2} = -\sin x - \cos x}$ $\Rightarrow \frac{d^3y}{dx^3} = -\cos x + \sin x$

$\Rightarrow \boxed{\frac{d^4y}{dx^4} = \sin x + \cos x}$

Problem 6: If f is a differentiable function, find an expression for the derivative of

$$y = \frac{1 + xf(x)}{\sqrt{x}}$$

in terms of $f'(x)$.

$$\frac{dy}{dx} = \frac{\sqrt{x}\,\frac{d}{dx}\left(1+xf(x)\right) - \left(1+xf(x)\right)\frac{d}{dx}\left(\sqrt{x}\right)}{(\sqrt{x})^2}$$

Product rule

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x}\left[0 + xf'(x) + f(x)\right] - \left(1+xf(x)\right)\frac{1}{2\sqrt{x}}}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x} \times 2\sqrt{x}\left(xf'(x)+f(x)\right) - \left(1+xf(x)\right)}{2x\sqrt{x}} = \frac{2x^2f'(x) + 2xf(x) - 1 - xf(x)}{2x\sqrt{x}}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2x^2 f'(x) + x f(x) - 1}{2x\sqrt{x}}}$$

Problem 7: Suppose $x \sin y + y \sin x = 1$. Find $\dfrac{dy}{dx}$ by implicit differentiation.

$$\frac{d}{dx}(x \sin y) + \frac{d}{dx}(y \sin x) = \frac{d}{dx}(1)$$

Product rule   Product rule

$$\frac{d}{dx}(x)\sin y + x\frac{d}{dx}(\sin y) + \frac{dy}{dx}\sin x + y\frac{d}{dx}(\sin x) = 0$$

$$\Rightarrow \sin y + x\frac{d}{dy}(\sin y)\frac{dy}{dx} + \sin x\frac{dy}{dx} + y\cos x = 0$$

$$\Rightarrow (x\cos y + \sin x)\frac{dy}{dx} = -\sin y - y\cos x \Rightarrow \boxed{\frac{dy}{dx} = \frac{-\sin y - y\cos x}{x\cos y + \sin x}}$$

Problem 8: Let $\sin y + \cos x = 1$. Find $\dfrac{d^2y}{dx^2}$ by implicit differentiation.

$$\Rightarrow \frac{d}{dx}(\sin y) + \frac{d}{dx}(\cos x) = \frac{d}{dx}(1) \Rightarrow \frac{d}{dy}(\sin y)\frac{dy}{dx} + (-\sin x) = 0$$

$$\Rightarrow \cos y\frac{dy}{dx} - \sin x = 0 \Rightarrow \frac{dy}{dx} = \frac{\sin x}{\cos y}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\sin x}{\cos y}\right) = \frac{\cos y\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(\cos y)}{\cos^2 y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cos y\cos x - \sin x\frac{d}{dy}(\cos y)\frac{dy}{dx}}{\cos^2 y} = \frac{\cos y\cos x - \sin x(-\sin y)\frac{\sin x}{\cos y}}{\cos^2 y}$$

Problem 9: Find the equation of tangent line to the "devil's curve" $y^2(y^2-4) = x^2(x^2-5)$ at the point $(0,-2)$.

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{\cos y\cos x(\cos y) + \sin x\sin y\sin x}{(\cos^2 y)(\cos y)}$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2} = \frac{(\cos x)(\cos^2 y) + (\sin^2 x)(\sin y)}{\cos^3 y}}$$

$$y^2(y^2-4) = x^2(x^2-5) \Rightarrow y^4 - 4y^2 = x^4 - 5x^2$$

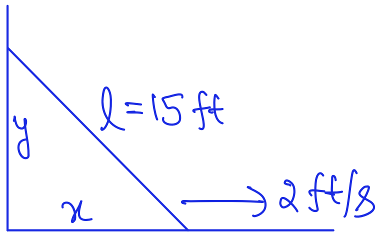
$$\Rightarrow \frac{d}{dx}(y^4) - 4\frac{d}{dx}(y^2) = \frac{d}{dx}(x^4) - 5\frac{d}{dx}(x^2) \Rightarrow 4y^3\frac{dy}{dx} - 8y\frac{dy}{dx} = 4x^3 - 10x$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3-10x}{4y^3-8y} \Rightarrow \left.\frac{dy}{dx}\right|_{(0,-2)} = \frac{4(0)^3 - 10(0)}{4(-2)^3 - 8(-2)} = \frac{0-0}{-32+16} = \frac{0}{-16} = 0$$

Thus, eqn. of tangent line is :-

$$\frac{y-(-2)}{x-0}=0 \Rightarrow \boxed{y+2=0}$$

Problem 10: A ladder 15 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 12 ft from the wall?



$$x^2+y^2=l^2, \quad \frac{dx}{dt}=2, \quad \underline{\text{To find}} \ \frac{dy}{dt} \text{ when } x=12$$

$$\frac{d}{dt}(x^2)+\frac{d}{dt}(y^2)=\frac{d}{dt}(l^2)=0$$

$$\Rightarrow \frac{d}{dx}(x^2)\frac{dx}{dt}+\frac{d}{dy}(y^2)\frac{dy}{dt}=0 \Rightarrow 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

when $x=12$, $12^2+y^2=15^2$

$\Rightarrow 144+y^2=225 \Rightarrow y^2=81 \Rightarrow y=9$ ft. (y is +ve)

$$\Rightarrow 2(12)(2)+2(9)\frac{dy}{dt}=0$$

$$\boxed{\Rightarrow \frac{dy}{dt}=-\frac{8}{3}\text{ ft/s}}$$

Problem 11: Find the linearization $L(x)$ of the function $f(x)=\dfrac{2}{\sqrt{x^2-5}}$ at the point $x=3$.

Note that $L(x)=f(a)+f'(a)(x-a)$ at $x=a$.

$$a=3, \quad f(x)=\frac{2}{\sqrt{x^2-5}} \Rightarrow f'(x)=2\frac{d}{dx}\left(\left(x^2-5\right)^{-1/2}\right)$$

$$f(3)=\frac{2}{\sqrt{3^2-5}}=\frac{2}{2}=1$$

Letting $z=x^2-5 \Rightarrow \dfrac{dz}{dx}=2x$

we have $f'(x)=2\dfrac{d}{dx}\left(z^{-1/2}\right)=2\dfrac{d}{dz}\left(z^{-1/2}\right)\dfrac{dz}{dx}$

$$\Rightarrow f'(x)=\cancel{2}\times\frac{-1}{\cancel{2}}z^{\frac{-1}{2}-1}\times 2x=-\left(x^2-5\right)^{-3/2}(2x)$$

$$\Rightarrow f'(3)=-\left(3^2-5\right)^{-3/2}(2\times 3)=-6\times 4^{-3/2}=-6\times 2^{-3}=\frac{-6}{8}=\frac{-3}{4}$$

$$\Rightarrow L(x)=f(3)+f'(3)(x-3)=1-\frac{3}{4}(x-3)=1+\frac{9}{4}-\frac{3x}{4}=\underline{\underline{\frac{13}{4}-\frac{3x}{4}}}$$

Problem 12: The radius of a sphere was measured and found to be 10 cm with a possible error of 10^{-3} cm. What is the maximum error in using this value of the radius to compute the volume of the sphere? Note that volume V of a sphere of radius r is given by $V=\frac{4}{3}\pi r^3$.

$$\Delta r=10^{-3}\text{ cm}, \quad r=10\text{ cm}$$

$$V=\frac{4\pi}{3}r^3. \qquad \frac{\Delta V}{\Delta r}\simeq\frac{dV}{dr}=\frac{d}{dr}\left(\frac{4\pi}{3}r^3\right)=\frac{4\pi}{3}\times 3r^2=4\pi r^2$$

$$\Rightarrow \Delta V \simeq 4\pi r^2\,\Delta r$$

$$\Rightarrow \Delta V \simeq 4\pi(10)^2\,10^{-3}=4\pi\times\frac{100}{1000}=\frac{4\times 3.14}{10}=1.256$$

$$\Rightarrow \boxed{\Delta V \simeq 1.256\text{ cm}^3}$$