Consider the expression

$$\lim_{x \to 4} \frac{x^2}{x+4} \, .$$

x:	4.1	4.01	4.001	4.0001	3.9	3.99	3.999	3.9999
$\frac{x^2}{x+4}$:	2.1	2.01	2.001	2.0001	1.9	1.99	1.999	1.9999

We see that the values of $f(x) = \dfrac{x^2}{x+4}$ are getting closer and closer to 2 as x approaches 4. We write this as

$$\lim_{x \to 4} \frac{x^2}{x+4} = 2 \, .$$

Notice that $f(4) = 2$.

Now consider the limit

$$\lim_{x \to \infty} \frac{1}{x} \, .$$

As x increases to infinity, its reciprocal $\dfrac{1}{x}$ decreases to 0. Hence we write

$$\lim_{x \to \infty} \frac{1}{x} = 0 \, .$$

Definition of limit

Suppose that $f(x)$ becomes arbitrarily close to the number L (written as $f(x) \to L$) as x approaches a ($x \to a$). Then we say that the limit of $f(x)$ as x approaches a is L and we write

$$\lim_{x \to a} f(x) = L\,.$$

The number a may be replaced by ∞ or $-\infty$.

Example

Evaluate the following limits:

1. $\lim_{x \to -1} (x^2 - 3)\,.$
2. $\lim_{x \to -2} \dfrac{4 - x^2}{x + 2}\,.$
3. $\lim_{x \to 3} \dfrac{9 - x^2}{x - 3}\,.$

One-sided limits

Right hand limit: When x approaches a from the right, that is, through values larger than a, the limit obtained is called right-hand limit and is written as

$$\lim_{x\to a^+} f(x) = L\,.$$

Left hand limit: When x approaches a from the left, that is, through values smaller than a, the limit obtained is called left-hand limit and is written as

$$\lim_{x\to a^-} f(x) = L\,.$$

Example

Evaluate the following limits:

1. $\lim\limits_{x\to 2^+} \sqrt{x-2}\,.$
2. $\lim\limits_{x\to 3^-} \dfrac{3-x}{\sqrt{3-x}}\,.$

Properties of limits

Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then we have

1. $\lim_{x \to a}[f(x) \pm g(x)] = L \pm M$.
2. $\lim_{x \to a} f(x)g(x) = LM$.
3. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ provided that $M \neq 0$.
4. $\lim_{x \to a} kf(x) = kL$.

Example

Evaluate the limit

$$\lim_{x \to \infty} \frac{3x^2 + x + 1}{2x^2 - x + 2} .$$

Definition of continuity
A function f is continuous at $x = a$ if f is defined at a and

$$\lim_{x \to a} f(x) = f(a) \ .$$

If a function f is continuous at all points in an interval, it is said to be continuous in the interval.

Example Find whether the following functions are continuous at $x = 1$.

1. $f(x) = x^2 + x$.
2. $g(x) = \begin{cases} 2 & x \geq 1 \\ x^3 & x < 1 \end{cases}$.