■ **Section 7.2** exercises, *page 524*: #1, 3, 7, 21, 23, 25, 13, 27, 17, 11, 29.

In this section, there are no new methods of integration. We mainly concern about **integrals that involve only trigonometric functions**, which we will call ***Trigonometric Integrals***.

Then main tools we are going to use to solve trigonometric integrals are

- The method of u-substitution
- Trigonometric identities

  $\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad \sin(2x) = 2\sin x\,\cos x$

  $\cos^2 x = \frac{1}{2}[1 + \cos(2x)] \qquad \sin^2 x = \frac{1}{2}[1 - \cos(2x)]$ (Half-angle formulas)
- Sometimes, we will need to do integration by parts

Example 1: Evaluate $\int \sin^5 x \cos^2 x\, dx$

$u = \cos x$

$\Rightarrow du = -\sin x\, dx$

$$\frac{d}{dx}\left(\sin^3 x\right) = 3\sin^2 x \cos x$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\Rightarrow \sin^4 x = \left(1 - \cos^2 x\right)^2 = (1-u^2)^2$$

$$I = \int \underbrace{\sin^4 x}_{(1-u^2)^2}\ \underbrace{\cos^2 x}_{u^2}\ \underbrace{(\sin x\, dx)}_{(-du)}$$

$$= \int (1-u^2)^2\, u^2\, (-du)$$

$$= -\int u^2 (1-u^2)^2\, du$$

$$(1-u^2)^2 = 1^2 + (u^2)^2 - 2(1)(u^2) = 1 + u^4 - 2u^2$$

$$= -\int u^2 (1 + u^4 - 2u^2)\, du$$

$$= -\int (u^2 + u^6 - 2u^4)\, du = -\int u^2\, du - \int u^6\, du + 2\int u^4\, du$$

$$= -\frac{u^3}{3} - \frac{u^7}{7} + 2\frac{u^5}{5} + C = -\frac{\cos^3 x}{3} - \frac{\cos^7 x}{7} + \frac{2\cos^5 x}{5} + C$$

Example 2: Find $\int \cos^3 x\, dx$

$$\int \sin^m x \cos^n x \, dx$$

Substitute $\begin{cases} u = \cos x & \text{if } m \text{ is odd} \\ u = \sin x & \text{if } n \text{ is odd} \end{cases}$

$m = 0, \ n = 3$

$u = \sin x$

$\Rightarrow \frac{du}{dx} = \cos x$

$\Rightarrow du = \cos x \, dx$

$\cos^2 x = 1 - \sin^2 x = 1 - u^2$

$$I = \int \cos^3 x \, dx = \int \cos^2 x \left(\cos x \, dx\right) = \int (1 - u^2)\, du$$

$$= \int 1 \, du - \int u^2 \, du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C$$

Example 3: Evaluate $\int_0^\pi \sin^2 x\, dx$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$= \int_0^\pi \frac{1}{2}(1 - \cos 2x)\, dx = \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \int_0^\pi \frac{1}{2}\, dx - \int_0^\pi \frac{1}{2} \cos 2x \, dx$$

$$= \frac{1}{2}\int_0^\pi dx - \frac{1}{2}\int_0^\pi \cos 2x \, dx$$

$$= \frac{1}{2}\, x \Big|_0^\pi - \frac{1}{2} \left. \frac{\sin 2x}{2} \right|_0^\pi$$

$$= \frac{1}{2}(\pi - 0) - \frac{1}{4}\left(\underset{0}{\sin 2\pi} - \underset{0}{\sin 2(0)}\right) = \frac{\pi}{2}$$

$$\text{If } \int f(x)\,dx = g(x) + C$$

$$\text{then } \int f(ax+b)\,dx = \frac{g(ax+b)}{a} + C$$

Proof :-

$$\int f(ax+b)\,dx$$

$u = ax+b \Rightarrow \frac{du}{dx} = a \Rightarrow du = a\,dx$

or $dx = \frac{1}{a}\,du$

$$\int f(u)\,\frac{1}{a}\,du = \frac{1}{a}\int f(u)\,du = \frac{1}{a}\,g(u) + C$$

$$= \frac{1}{a}\,g(ax+b) + C$$

HW:

$$\int \cos^2 x\,dx$$

(Hint : Use Half-angle formula)

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\int \tan^m x \sec^n x\,dx = \begin{cases} u = \tan x, & n \text{ is even} \\ u = \sec x, & m \text{ is odd} \end{cases}$$

Example 4: Find $\int \tan^6 x \sec^4 x \, dx$

Power of sec is even.

$$\int \tan^6 x \sec^2 x \underbrace{\sec^2 x \, dx}$$

$\Rightarrow u = \tan x$

$\Rightarrow du = \sec^2 x \, dx$

$$I = \int u^6 \sec^2 x \, du$$

$$\boxed{\sec^2 x = 1 + \tan^2 x}$$

$$= \int u^6 (1 + \tan^2 x) \, du$$

$$= \int u^6 (1 + u^2) \, du = \int (u^6 + u^8) \, du = \int u^6 \, du + \int u^8 \, du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

Example 5: Find $\int \tan^5 \theta \sec^7 \theta \, d\theta$

$u = \sec\theta \Rightarrow du = \sec\theta \tan\theta \, d\theta$

$$\int \tan^5 \theta \sec^7 \theta \, d\theta = \int (\tan^4 \theta \sec^6 \theta) \underbrace{(\sec\theta \tan\theta \, d\theta)}_{du}$$

$$= \int (\tan^4 \theta) \, u^6 \, du$$

$\sec^2 \theta = \tan^2 \theta + 1$

$\Rightarrow \tan^2 \theta = \sec^2 \theta - 1$

$\Rightarrow \tan^4 \theta = (\sec^2 \theta - 1)^2$

$$= \int (\sec^2 \theta - 1)^2 u^6 \, du$$

$$= \int (u^2 - 1)^2 u^6 \, du$$

$$= \int (u^4 - 2u^2 + 1) u^6 \, du = \int (u^{10} - 2u^8 + u^6) \, du$$

$$= \int u^{10} \, du - 2 \int u^8 \, du + \int u^6 \, du$$

$$= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C = \frac{\sec^{11}\theta}{11} - 2\frac{\sec^9\theta}{9} + \frac{\sec^7\theta}{7} + C$$

Extra Examples:

- $\int \tan^3 x\, dx$ (Example 7, textbook, page 523).
- $\int \sec^3 x\, dx$ (Example 8, textbook, page 523).
- $\int \sin(4x)\cos(5x)\, dx$ (Example 9, textbook, page 524)

Example 6: Compute $\int \sin(2x)\cos^2 x\, dx$.

$$\sin 2x = 2\sin x \cos x$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \int (2\sin x \cos x)\cos^2 x\, dx$$

$$= 2\int \sin x \cos^3 x\, dx$$

$u = \sin x$ <u>HW</u>

$u = \cos x$ <u>HW.</u>

<u>Alternatively</u>

$$\boxed{\begin{aligned} \sin 4x &= \sin(2(2x)) \\ &= 2\sin(2x)\cos(2x) \\ \Rightarrow \sin(2x)\cos(2x) &= \tfrac{1}{2}\sin 4x \end{aligned}}$$

$$I = \int (\sin 2x)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int (\sin 2x)(1+\cos 2x)\, dx$$

$$= \frac{1}{2}\int (\sin 2x + \sin 2x \cos 2x)\, dx$$

$$= \frac{1}{2}\int \sin 2x\, dx + \frac{1}{2}\int \sin 2x \cos 2x\, dx$$

$$= \frac{1}{2}\int \sin 2x\, dx + \frac{1}{2}\int \frac{1}{2}\sin(4x)\, dx$$

$$= \frac{1}{2}\left(\frac{-\cos 2x}{2}\right) + \frac{1}{4}\left(\frac{-\cos(4x)}{4}\right) + C$$

$$= -\frac{1}{4}\cos 2x - \frac{1}{16}\cos 4x + C$$