Learning objectives:

1. Learn the concept of **absolute** maximum and minimum points/values of a function.
2. Learn the concept of **local** maximum and minimum points/values of a function.
3. The Extreme value theorem and the Fermat's theorem.
4. Critical numbers of a function.
5. The closed interval method.

Absolute maximum and minimum

Let c be a number in the domain D of a function f. Then $f(c)$ is the

1. absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.
2. absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

Local maximum and minimum

Let c be a number in the domain D of a function f. Then $f(c)$ is the

1. local maximum value of f on D if $f(c) \geq f(x)$ when x is near c.
2. local minimum value of f on D if $f(c) \leq f(x)$ when x is near c.

Example 1.

1. $y = \cos x$.
2. $y = x^2$.
3. $y = x^3$.

The Extreme value theorem.

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Example 2.

1. $y = \cos x$.
2. $y = x^2$.
3. $y = x^3$.
4. $y = |x|$.

Critical number

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 3. Find the critical numbers of the following functions.

1. $f(x) = x^{3/5}(4 - x)$.
2. $f(x) = 2x^3 - 3x^2 - 36x$.
3. $g(t) = |3t - 4|$.

Fermat's theorem rephrased

If f has a local maximum or minimum at c, then c is a critical number of f.

The closed interval method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the critical numbers of f in the open interval (a, b).
2. Find the values of f at the critical numbers of f in (a, b).
3. Find the values of f at the endpoints, that is, find $f(a)$ and $f(b)$.
4. The largest of the values from steps 2 and 3 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 4. Find the absolute maximum and minimum values of the given function on the given interval.

$$f(x) = x^3 - 3x^2 + 1\ , \quad -\frac{1}{2} \leq x \leq 4\ .$$

Example 5. Find the absolute maximum and minimum values of the given function on the given interval.

$$f(x) = x - 2\sin x\,, \quad 0 \le x \le 2\pi\,.$$

Example 6. Find the absolute maximum and minimum values of the given function on the given interval.

$$f(x) = x + \frac{1}{x}\,, \quad [-1.5, -0.5] \cup [0.5, 1.5]\,.$$

Example 7. Find the absolute maximum and minimum values of the given function on the given interval.

$$f(\theta) = 2\cos\theta + \sin 2\theta\,, \quad [0, \pi/2]\,.$$