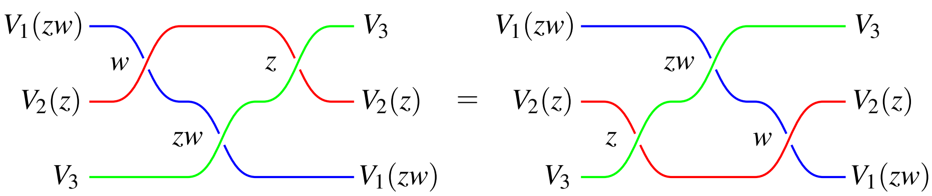
Research Statement

Keshav Dahiya

The quantum Yang-Baxter equation (QYBE) plays a central role in statistical mechanics, knot theory, integrable systems, stochastic vertex models, and mathematical physics in general. As part of my Ph.D. thesis, I have been working to find new solutions of the QYBE with a spectral parameter. A rich source of solutions comes from the intertwiners of representations of quantum affine algebras and Yangians. A quantum affine algebra $U_q\hat{\mathfrak{g}}$ comes with a Hopf algebra structure. Given a $U_q\hat{\mathfrak{g}}$-module V, one can obtain a one-parameter family of modules $V(z)$, $z \in \mathbb{C}$, using a Hopf algebra automorphism of $U_q\hat{\mathfrak{g}}$, called the shift of spectral parameter. An intertwiner $\check{R}(z)$ is a homomorphism of $U_q\hat{\mathfrak{g}}$-modules $\check{R}(z) : V(z) \otimes V \to V \otimes V(z)$, which is an isomorphism except for some finitely many values of z. After a choice of basis, $\check{R}(z)$ is a matrix whose entries are rational functions in the spectral parameter z and the quantization parameter q. The matrices $\check{R}(z)$ satisfy the QYBE in the following (trigonometric) form:

$$\left(\check{R}(z)\otimes I\right)\left(I\otimes\check{R}(zw)\right)\left(\check{R}(w)\otimes I\right)=\left(I\otimes\check{R}(w)\right)\left(\check{R}(zw)\otimes I\right)\left(I\otimes\check{R}(z)\right)\,,$$

where I is the identity matrix. In the form above, QYBE is the braid equation, as in the following picture.



Intertwiners of representations of quantum affine algebras and Yangians

My first two projects focused on deriving explicit expressions of the intertwiners $\check{R}(z)$ for the first fundamental modules of all types of untwisted, respectively twisted, quantum affine algebras. The joint work with Evgeny Mukhin, which also improved our understanding of these intertwiners, can be found here: [DM25a], [DM25b]. This work is significantly important in mathematical physics. Both these papers were published in the **Journal of Mathematical Physics** within a month and both were selected as **Editor's Pick**.

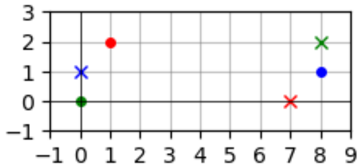
The main result of the first paper was the expression of matrix $\check{R}(z)$ for the first fundamental representation of the quantum affine algebra $U_q\mathrm{E}_8^{(1)}$, in terms of $U_q\mathrm{E}_8$-idempotents. This was a 62001×62001 matrix with nearly two billion entries, and posed some computational challenges, addressed by leveraging the sparsity of the matrix. Our method is based on q-characters, computed using the Frenkel-Mukhin algorithm. The methods developed in [DM25a] were then applied to the first fundamental representations of the twisted quantum affine algebras of all types in [DM25b]. The main result for the twisted types was the case of $U_q\mathrm{E}_6^{(2)}$.

A short outline of the method is as follows. One can restrict the $U_q\hat{\mathfrak{g}}$-module $V(z)\otimes V$ to the non-affine subalgebra $U_q\mathfrak{g} \subset U_q\hat{\mathfrak{g}}$. In non-affine case, the category of representations of $U_q\mathfrak{g}$ is semi-simple, so that one can decompose $V(z)\otimes V$ (same as $V \otimes V$ in the non-affine case) as a direct sum $\oplus_k\left(M_k \otimes L_{\lambda_k}\right)$ of irreducibles L_{λ_k} with M_k being multiplicity spaces. The intertwiner $\check{R}(z)$ can then be expressed in terms of P_k, the $U_q\mathfrak{g}$-projectors onto L_{λ_k}, as $\check{R}(z) = \sum_k f_k(z)P_k$, where $f_k(z)$ are operators acting in multiplicity spaces. The easy case is when the multiplicity space is trivial or one-dimensional, and therefore $f_k(z)$ is a rational function. This is the case for the first fundamental modules in all types except for untwisted type $\mathrm{E}_8^{(1)}$ and twisted types $\mathrm{D}_r^{(2)}$, $\mathrm{E}_6^{(2)}$ and $\mathrm{D}_4^{(3)}$, where two of the multiplicity spaces are non-trivial. We show that the properties of $\check{R}(z)$, such as $\check{R}(1)$ and $\check{R}(z)\check{R}(z^{-1})$ both being identity matrices, the knowledge of $\check{R}(0)$ using Casimir values in the spaces L_{λ_k}, along with the information obtained from q-characters, are enough to determine $f_k(z)$, completely in the trivial multiplicity case, and assuming poles are simple in the non-trivial multiplicity cases up to a $\pm$ sign in one of the matrix entries.

Representations of quantum affine superalgebras

In my current project, in preparation, the goal is to better understand the finite dimensional representations of quantum affine superalgebras. A sub-goal is to find the explicit expressions for the intertwiners of their representations. Of particular interest is the case of $U_q\mathrm{D}^{(1)}_{2|1;x}$, where the algebra itself depends on an additional parameter x. For this case, we have computed the expression of normalized $\check{R}(z;q,x)$. Unlike the even ("bosonic") cases in [DM25a], [DM25b], here when restricted to finite type $U_q\mathrm{D}_{2|1;x}$ the tensor square $V\otimes V$ is not completely reducible and we have indecomposable summands.

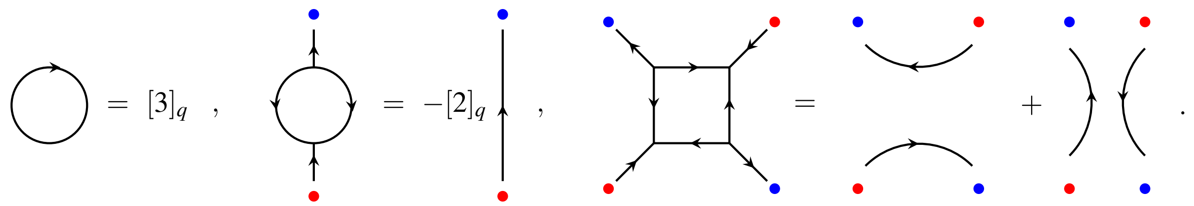
Much less is known about the representations in the supersymmetric case. For example, the isomorphism theorem between Drinfeld-Jimbo and Drinfeld's loop realization is not proved yet for $U_q\mathfrak{osp}(m|n)^{(1)}$ series for all m, n. We do not have a classification of the finite dimensional irreducible representations for all $U_q\mathfrak{osp}(m|n)^{(1)}$ type series, or for $U_q\mathrm{D}^{(1)}_{2|1;x}$. In the case of $U_q\mathrm{D}^{(1)}_{2|1;x}$, we already have an important class of finite dimensional representations - those for which the zeros and poles of the eigenfunctions, of the three commuting fermionic operators $K_{\mathrm{blue}}, K_{\mathrm{red}}, K_{\mathrm{green}}$ acting on the singular vector, form particular types of hexagons. For example, one of these hexagons is shown in picture below with dots at (a,b) representing a zero at $z=q^{-a-bx}$ and a cross at (a,b) representing a pole at $z=q^{-a-bx}$.



We also have conjectural formula for the dimension of these representations: $16n^2+2$, where n depends on the length of the hexagon. For the picture above, $n=4$. One ambitious goal of this project is to have completely combinatorial descriptions (as above) for all the finite dimensional irreducible representations of $U_q\mathrm{D}^{(1)}_{2|1;x}$.

Web categories and diagrammatic calculus

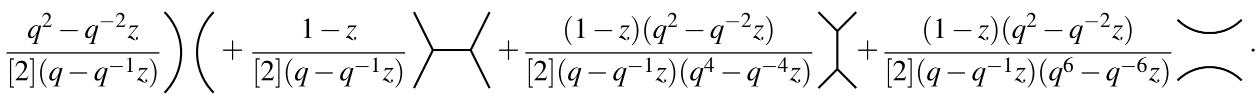
In [K96], Greg Kuperberg described a diagrammatic category via generators and relations encoding the representations of $U_q\mathfrak{g}$ for all rank 2 Lie algebras $\mathfrak{g}$. For example, in the case of $U_q\mathfrak{sl}_3$, the objects of this diagrammatic/web category correspond to strings (tensor products) of $\{\bullet, \bullet\}$, with $\bullet$ corresponding to first and $\bullet$ corresponding to second fundamental representation, while the morphisms are generated by trivalent oriented graphs where every arrow originates from $\bullet$ and sinks into $\bullet$, with every allowed vertex having in-degree/out-degree of 3, modulo the following relations (allowing one to remove circles, bigons and squares):



Formally, the above correspondence is an equivalence of the web category as described above and the pivotal tensor category $\mathrm{Rep}(U_q\mathfrak{g})$ of representations of $U_q\mathfrak{g}$. The above description was extended for $U_q\mathfrak{sl}_n$ in [CKM14] and then for $U_q\mathfrak{sp}_{2n}$ in [BERT21]. I would like to pursue this goal to its end: namely giving diagrammatic presentations as above for all types of simple finite dimensional Lie algebras $\mathfrak{g}$. It seems the problem poses significant challenges even for type B. It would also be interesting to extend such a description to the case of supersymmetric quantum algebras.

Quantum link invariants

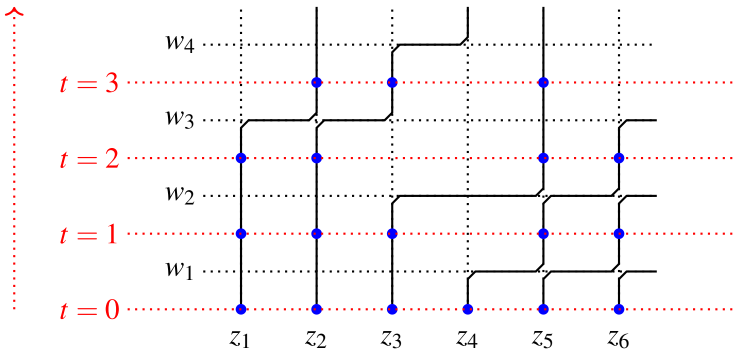
In [Z20], Paul Zinn-Justin gave a description of the $U_q\mathrm{E}^{(1)}_8$ trigonometric R-matrix using diagrammatic calculus of $U_q\mathrm{E}_8$-invariants. In the same way, for instance the $U_q\mathrm{G}^{(1)}_2$ intertwiner $\check{R}(z)$ in [DM25a] can be written as:

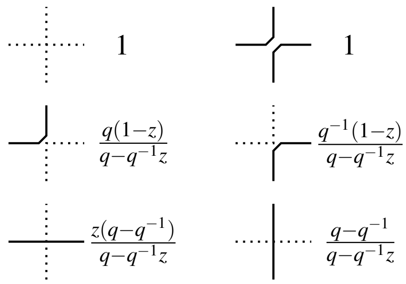


At $z=0$, the expression above reduces to the formula of crossing given in [K96]. In general, for every value of z except $q^{\pm 2}, q^{\pm 8}, q^{\pm 12}$, where $\check{R}(z)$ has zeros/poles, we have an expression of a crossing for finite type quantum algebra $U_q\mathrm{G}_2$. At $z=1$, the above expression reduces to identity operator, while $\check{R}(q^4)$ is a well-defined intertwiner which reduces to the second morphism in the expression above. These are all solutions of the braid equation and lead to link invariants. Motivated by such applications, it seems desirable to study diagrammatic descriptions in the affine case, that is, for the category $\mathrm{Rep}(U_q\hat{\mathfrak{g}})$. One of the difficulties here is that the category $\mathrm{Rep}(U_q\hat{\mathfrak{g}})$ is no longer semi-simple. Ideally, one should have some kind of deformation of the web categories, as in [K96] for instance.

Integrable probability and stochastic vertex models

A vertex model related to $U_q\hat{\mathfrak{g}}$ is a 2-D lattice model in which each edge is a state in some $U_q\hat{\mathfrak{g}}$-module V and each vertex has a weight given by the element of $\check{R}(z)$ which maps the pairs of states of the incoming edges to that of the outgoing edges. To any configuration C of the lattice, we assign a weight $\mathrm{wt}(C)$ which is the product of all vertex weights. The sum of the weights of all possible configurations is called the partition function Z of the model. The fact that the vertex weights are given by $\check{R}(z)$, a solution of the QYBE, enables one to use ideas of integrability. When the matrix $\check{R}(z)$ is stochastic, the ratio $\mathrm{wt}(C)/Z$ is interpreted as the probability of occurrence of configuration C, and the resulting model is called a stochastic vertex model. The most studied is the stochastic six-vertex model [BCG16]. The following picture shows one of the configurations of the stochastic six-vertex model with "domain-wall boundary conditions". The blue dots are interpreted as particles on a line and as time progresses (vertically) the particles keep moving to the right. This interpretation relates the stochastic six-vertex model to the interacting particle system - asymmetric simple exclusion process (ASEP).





The six allowed vertices and their weights

There has been extensive research in recent years on the higher-spin $U_q\hat{\mathfrak{sl}}_2$ vertex models [CP16], colored stochastic vertex models related to $U_q\hat{\mathfrak{sl}}_n$ [AB24], and colored fermionic vertex models related to $U_q\hat{\mathfrak{sl}}_{m|n}$ [ABW23]. They have many interesting connections to random matrices, symmetric functions, spin q-Whittaker polynomials, Markov dynamics, random walks and interacting particle systems. For other quantum affine algebra types we do not yet have a stochastic matrix $\check{R}(z)$. I could find a stochastic R-matrix for $U_q\hat{\mathfrak{so}}_n$, and would like to study the resulting vertex model, specifically the probability distributions arising from it and their relations to random matrix theory.

References

[AB24] A. Aggarwal, A. Borodin, *Colored line ensembles for stochastic vertex models*, Selecta Math. (N.S.) **30** (2024), no. 5, Paper No. 105, 111 pp.

[ABW23] A. Aggarwal, A. Borodin, M. Wheeler, *Colored fermionic vertex models and symmetric functions*, Commun. Am. Math. Soc. **3** (2023), 400-630.

[BCG16] A. Borodin, I. Corwin, and V. Gorin, *Stochastic six-vertex model*, Duke Math. J. **165** (2016), no. 3, 563-624.

[BERT21] E. Bodish, B. Elias, D.E.V. Rose, L. Tantham, *Type C Webs*, arXiv:2103.14997 [math.RT].

[CKM14] S. Cautis, J. Kamnitzer, S. Morrison, *Webs and quantum skew Howe duality*, Math. Ann. **360** (2014), no. 1-2, 351-390.

[CP16] I. Corwin, L. Petrov, *Stochastic higher spin vertex models on the line*, Comm. Math. Phys. **343** (2016), no. 2, 651-700.

[DM25a] K. Dahiya and E. Mukhin, *Intertwiners of representations of untwisted quantum affine algebras and Yangians revisited*, J. Math. Phys. **66** (2025), no. 5, 051701 (DOI).

[DM25b] K. Dahiya and E. Mukhin, *Intertwiners of representations of twisted quantum affine algebras*, J. Math. Phys. **66** (2025), no. 5, 051703 (DOI).

[K96] G. Kuperberg, *Spiders for rank 2 Lie algebras*, Comm. Math. Phys. **180** (1996), no. 1, 109-151.

[Z20] P. Zinn-Justin, *The trigonometric E_8 R-matrix*, Lett. Math. Phys. **110** (2020), no. 12, 3279-3305.