M16600 Lecture Notes

Section 11.7: Strategy for Testing Series/Review Series for Exam 3

■ **Section 11.7** textbook exercises, *page 786*: # 1, 5, 7, 13, 18, 10, 12, 16, 21, <u>19</u>, <u>22</u>.

If we have a sequence $\{a_n\}$, to determine whether the sequence is convergent or divergent, all we need to do is to compute the $\lim_{n\to\infty} a_n$. For a series $\sum a_n$, the process is rather complicated.

We have a definition for convergence or divergence of a series (section 11.2); but we don't usually use this definition to determine whether a series is convergent or divergent because it is difficult to apply for most series (except for telescoping series). On the other hand, mathematicians used this definition to prove several results or tests for convergence or divergence of different types of series. These tests are what we want to focus on.

Write down the statements of these material and memorize them so that you can use them effectively in testing for convergence or divergence of a series. I am listing them in the order that you should think of when testing for series. Note that we will not focus on the Root Test and telescoping series.

- Test for geometric series and finding sum of a convergent geometric series (11.2)
- Test for p-series (11.2)
- Test of Divergence (11.2)
- The Limit Comparison Test (11.4)
- The Ratio Test (11.6)
- The Integral Test
- Is the series absolutely convergent? Because if it is, then it is convergent. If the series is not absolutely convergent, then we still don't know whether it is convergent or divergent. This is a theorem in section 11.6
- The (standard) Comparison Test (11.4). This test is useful especially when the series has positive terms with the expression of sine or cosine.
- The Test for Alternating Series (11.5). This only applies to alternating series

Some Remarks on the Techniques We Use in Testing Series.

We often compute the limit as $n \to \infty$. Before you take the next step, do the "direct substitution" first. Note that

- $\dfrac{\text{a nonzero number}}{\infty} \to 0$
- $\dfrac{\text{a nonzero number}}{0} \to \pm\infty$ depending on whether the numerator, and the denominator, is positive or negative
- $\dfrac{\infty}{\infty}$ is an indeterminate form. One quick method to solve the limit of this form when testing for series is to drop the slower terms of the numerators and of the denominators. *Note that we can only drop slower terms of each factor. We cannot drop the entire factor itself.*

  For example, if we have $\dfrac{n-1}{n^3(e^n+n)}$, then there is **one factor** in the numerator, that is $(n-1)$; but there are **two terms** in the numerator: n and -1. We can drop the -1 on top. In the denominator, we can separate **two factors**: n^3 and (e^n+n). We cannot drop the factor n^3 nor the factor (e^n+n). Nevertheless, the factor (e^n+n) contains a slower term, that is n. Therefore, we can drop the term n inside the factor (e^n+n). As a result of dropping slower terms, we get $\dfrac{n}{n^3 \cdot e^n}$.

Comparing the Growth Rate of Different Class Functions: Slower functions are on the left.

$$\text{logarithmic functions} \; << \; \text{algebra} \; << \; \text{exponential functions} \; << \; \text{factorial}$$

Comparing the Growth Rate within a Class of Function:

- To compare the growth rate within **Logarithmic Functions**, for examples, $\log_2(n), \ln(n)$, $\log_5(n)$, etc., where $b > 1$, we compare the base. The smaller the base, the faster the logarithmic function. In this class, we usually work with $\ln n$ in Chapter 11.
- To compare the growth rate within **Algebra Functions**, for examples, $n^{1/2}, n^2, n^5$, etc., where the exponents are positive, we compare the exponents. The bigger the exponent, the faster the algebra function.
- To compare the growth rate within **Exponential Functions** for examples, $2^n, e^n, 5^n$, etc., where the base is ≥ 1, we compare the base. The bigger the base, the faster the exponential function.

Example 1: Test the series for convergence or divergence

$\cancel{\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n}$ $\ge$

we are not Comparison

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n(n-1)}{n^3+1}$: $a_n = \dfrac{n(n-1)}{n^3+1} \sim \dfrac{n(n)}{n^3} = \dfrac{n^2}{n^3} = \dfrac{1}{n} \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{n(n-1)}{n^3+1} \sim \sum_{n=1}^{\infty} \frac{1}{n}$??

diverges   diverges

Limit Comparison Test

If $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ = finite +ve no.

then $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$: Alternating series.

↳ AST

$b_n = \dfrac{1}{n}$

(i) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

(ii) $\dfrac{1}{n+1} < \dfrac{1}{n} \Rightarrow b_{n+1} < b_n$

By AST, the series in (b) converges.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2-2k+5}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^5}$

(e) $\displaystyle\sum_{n=1}^{\infty} n e^{-n^2}$

(f) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{7^n}{2^n+3^n}$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n}$

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n n^{10}}{n!}$

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^n}$

(k) $\displaystyle\sum_{n=1}^{\infty} (-1)^n n$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2-2k+5}$: $a_k = \dfrac{1}{k^2-2k+5} \sim \dfrac{1}{k^2}$

$$\sum_{k=1}^{\infty} \frac{1}{k^2-2k+5} \sim \underbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}_{\text{converges}}$$

→ P-series with $p = 2 > 1$

By LCT, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2-2k+5}$ also converges.

Example 2: Determine whether the series is absolutely convergent, or conditionally convergent, or divergent. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[8]{n+8}}$

(d) $\sum_{n=1}^{\infty} \frac{5^n}{n^5}$: Use Test for divergence. by showing that $\lim_{n\to\infty} a_n \neq 0$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5^n}{n^5} = \frac{\text{faster growing}}{\text{slower growing}} = \infty \neq 0$$

$\Rightarrow$ By TD, $\sum_{n=1}^{\infty} \frac{5^n}{n^5}$ diverges.

(e) $\sum_{n=1}^{\infty} ne^{-n^2}$: use integral test.

$\hookrightarrow f(x) = xe^{-x^2}$ $\rightarrow$ f is cont, Positive, ultimately dec

$\frac{x}{e^{x^2}}$ denominator grows faster than numerator $\Rightarrow$ fraction should dec.

$$f'(x) = \frac{d}{dx}\left[x e^{-x^2}\right] = e^{-x^2} + x(-2x)e^{-x^2}$$

$$= e^{-x^2} - 2x^2 e^{-x^2}$$

$$= \underbrace{e^{-x^2}}_{\text{Positive}} (1-2x^2) \rightarrow 1-2x^2 = 0 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$$

$x^2 > \frac{1}{2}$

$\Rightarrow 1-2x^2 < 0$

$-$, $-\frac{1}{\sqrt{2}}$, $+$, $\frac{1}{\sqrt{2}}$, $-$

$f'(x) < 0 \quad \forall \ x > \frac{1}{\sqrt{2}} \quad \Rightarrow$ f is ultimately decreasing

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} x e^{-x^2}\,dx$$

$u = -x^2 \Rightarrow du = -2x\,dx \Rightarrow \frac{-1}{2}du = x\,dx$

$u = -x^2$, $x = \infty$: $-\infty^2$

$u = -x^2$, $x = 1$: -1^2

$$= \int_{-1^2}^{-\infty^2} \underbrace{e^u}_{e^{-x^2}} \underbrace{\left(\frac{-1}{2}du\right)}_{x\,dx} = \frac{-1}{2}\int_{-1}^{-\infty} e^u\,du = \frac{1}{2}\int_{-\infty}^{-1} e^u\,du$$

$$= \frac{1}{2} e^u \Big|_{-\infty}^{-1} = \frac{1}{2}\left[e^{-1} - e^{-\infty}\right] = \frac{1}{2}\left[e^{-1} - 0\right] = \frac{1}{2e} < \infty$$

$\Rightarrow \int_1^{\infty} f(x)\,dx$ converges $\Rightarrow \sum_{n=1}^{\infty} n e^{-n^2}$ Converges. (by integral test)

(f) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^4}{4^n}$ $\Rightarrow b_n = \dfrac{n^4}{4^n}$

Alternating series.

1) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n^4}{4^n} = \frac{\text{slower growth}}{\text{faster growth}} = 0$

use LH Rules 4 times

2) $b_{n+1} = \dfrac{(n+1)^4}{4^{n+1}} < \dfrac{n^4}{4^n} = b_n$

$\dfrac{(n+1)^4}{n^4} < \dfrac{4^{n+1}}{4^n} \Rightarrow \dfrac{(n+1)^4}{n^4} < 4^{n+1-n}$

$\Rightarrow \dfrac{(n+1)^4}{n^4} < 4 \Rightarrow (n+1)^4 < 4n^4 \Rightarrow$ true.

$\Rightarrow 4n^4 - \underbrace{(n+1)^4}_{1\ n^4 \text{ term.}} > 0$ → Positive $\forall n \geq n_0$

can be found out

$3n^4 - (\text{slower growing terms})$

dominates.

$b_{n+1} < b_n$ eventually.
$\forall n \geq n_0$

By AST, $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^4}{4^n}$ Converges.

(g) $\displaystyle\sum_{n=1}^{\infty}\frac{7^n}{2^n+3^n}$ $\sim$ similar $\displaystyle\sum_{n=1}^{\infty}\frac{7^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{7}{3}\right)^n$

2^n grows slower than 3^n

$r = \dfrac{a_{n+1}}{a_n} = \dfrac{\left(\frac{7}{3}\right)^{n+1}}{\left(\frac{7}{3}\right)^n} = \left(\dfrac{7}{3}\right)^{n+1-n} = \dfrac{7}{3}$

$\Rightarrow$ geometric series with $r = \dfrac{7}{3}$

$\Rightarrow |r| = \dfrac{7}{3} > 1 \Rightarrow$ the geometric series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{7}{3}\right)^n$ diverges. $\Rightarrow \displaystyle\sum_{n=1}^{\infty}\frac{7^n}{2^n+3^n}$ diverges

(h) $\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n}$ $\qquad a_n = \frac{\sin(2n)}{1+2^n}$, $\quad -1 \le \sin(2n) \le 1$

not an alternating series

takes −ve values

$\Downarrow$

Cannot use Comparison tests directly

(*) Absolute Convergence $\Rightarrow$ Convergence.

$\left\{ \sum_{n=1}^{\infty} |a_n| \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ Converges.} \right.$

Check the convergence of $\sum_{n=1}^{\infty} \left| \frac{\sin(2n)}{1+2^n} \right|$

$$= \sum_{n=1}^{\infty} \frac{|\sin(2n)|}{1+2^n}$$

$$\underbrace{0 \le |\sin(2n)| \le 1}_{} \Rightarrow \underbrace{\sum_{n=1}^{\infty} \frac{|\sin(2n)|}{1+2^n}}_{} \le \sum_{n=1}^{\infty} \frac{1}{1+2^n}$$

Now check convergence of the dominating series $\sum_{n=1}^{\infty} \frac{1}{1+2^n}$

$$\sum_{n=1}^{\infty} \frac{1}{1+2^n} \sim \text{similar} \quad \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-\frac{1}{2}} = 1$$

geometric series.

$r = \frac{1}{2} < 1 \Rightarrow$ Converges.

By LCT, $\sum_{n=1}^{\infty} \frac{1}{1+2^n}$ Converges. $\Rightarrow$ dominating series converges.

By Comparison test, $\sum_{n=1}^{\infty} \frac{|\sin(2n)|}{1+2^n}$ converges.

$$\boxed{\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} \\ &r = \frac{a_{n+1}}{a_n} = \frac{\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)^n} = \left(\frac{1}{2}\right)^{n+1-n} = \frac{1}{2} \\ &a = a_1 = \frac{1}{2^1} = \frac{1}{2} \end{aligned}}$$

Because absolute convergence $\Rightarrow$ convergence

so we have $\sum_{n=1}^{\infty} \frac{\sin(2n)}{1+2^n}$ converges.

(i) $\sum_{n=1}^{\infty} \frac{10^n n^{10}}{n!}$ $\Rightarrow a_n = \frac{10^n n^{10}}{n!}$

• Use Ratio test

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ $\begin{cases} <1 \Rightarrow \text{series Converges} \\ >1 \Rightarrow \text{series diverges} \\ =1 \Rightarrow \text{can't say anything} \end{cases}$

$$\frac{a_{n+1}}{a_n} = \frac{10^{n+1}(n+1)^{10}}{(n+1)!} \cdot \frac{n!}{10^n n^{10}} = 10^{n+1-n} \frac{(n+1)^{10}}{n^{10}} \frac{n!}{(n+1)!}$$

$$= 10\left(\frac{n+1}{n}\right)^{10} \frac{\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cdots\cancel{1}}{(n+1)\underbrace{\cancel{n}\cancel{(n-1)}\cdots\cdots\cancel{1}}} = 10\left(\frac{n+1}{n}\right)^{10}\frac{1}{n+1}$$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|10\left(\frac{n+1}{n}\right)^{10}\frac{1}{n+1}\right| = \lim_{n\to\infty} 10\left(\frac{n}{n}+\frac{1}{n}\right)^{10}\frac{1}{n+1}$$

$$= \lim_{n\to\infty} 10\left(1+\frac{1}{n}\right)^{10}\frac{1}{n+1} = 10\left(1+\frac{1}{\infty}\right)^{10}\frac{1}{\infty}$$

$$= 10(1+0)^{10} \times 0 = 10 \times 1 \times 0 = 0 < 1$$

$\Rightarrow L<1 \Rightarrow$ By Ratio test, the series $\sum_{n=1}^{\infty}\frac{10^n n^{10}}{n!}$ Converges.

(j) $\sum_{n=1}^{\infty} \frac{n!}{e^n}$ $a_n = \frac{n!}{e^n}$ $\Rightarrow$ factorials grow faster than exponentials.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n!}{e^n} = \frac{\text{faster growing term}}{\text{slower growing term}} = \infty \neq 0$$

By TD, the series diverges.

Alternatively use ratio test.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{e^{n+1}}\cdot\frac{e^n}{n!} = \lim_{n\to\infty}\frac{(n+1)\cancel{n}\cancel{(n-1)}\cdots\cancel{1}}{e^{n+1-n}\left[\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{1}\right]}$$

$$= \lim_{n\to\infty} \frac{n+1}{e} = \frac{\infty}{e} = \infty > 1$$

$\Rightarrow$ By Ratio test, the series diverges.

(k) $\sum_{n=1}^{\infty}(-1)^n n$. $b_n = n$

Alternating series

use AST,

$$\left.\begin{array}{l} 1)\ \lim_{n\to\infty} b_n = 0 \\ 2)\ b_{n+1} < b_n \end{array}\right\} \Rightarrow \text{convergence}$$

If $\lim_{n\to\infty} b_n \neq 0$, then the series diverges.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} n = \infty \neq 0$$ $\Rightarrow$ By AST, the series diverges.

Example 2: Determine whether the series is absolutely convergent, or conditionally convergent, or divergent. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[8]{n+8}}$

$$a_n = \frac{(-1)^n}{\sqrt[8]{n+8}} \Rightarrow |a_n| = \frac{|(-1)^n|}{|\sqrt[8]{n+8}|} = \frac{1}{\sqrt[8]{n+8}}$$

Absolute Convergence : Check the convergence of $\sum_{n=1}^{\infty} |a_n|$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[8]{n+8}} \underset{\text{similar}}{\sim} \sum_{n=1}^{\infty} \frac{1}{\sqrt[8]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/8}}$$

$\hookrightarrow$ P-series with $P = \frac{1}{8} \leq 1$

$\Rightarrow$ diverges.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt[8]{n+8}}$ also diverges.

$\Rightarrow$ No absolute Convergence.

Conditional Convergence : Check the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[8]{n+8}}$

Alternating series

$$b_n = \frac{1}{\sqrt[8]{n+8}}$$

1) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt[8]{n+8}} = \frac{1}{\infty} = 0$ ✓

2) $b_{n+1} < b_n$

$$b_{n+1} = \frac{1}{\sqrt[8]{n+9}} < \frac{1}{\sqrt[8]{n+8}} = b_n$$ ✓

By the AST, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[8]{n+8}}$ converges.

$\Rightarrow$ The given series is conditionally convergent.