M16600 Lecture Notes

Section 11.2: Series

■ **Section 11.2** textbook exercises, *page 755*: #6, 15, 22, 23, 24, 26, 29, 31, 33, 37, 46, 47.

Definition of Series. An ***infinite series*** (or just ***series***) is an infinite SUM of the terms of the sequence $\{a_n\}$

Series Notation:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

Note: n does not have to start from 1.

E.g., $\sum_{n=1}^{\infty} 2^n =$ $2^1 + 2^2 + 2^3 + 2^4 + \cdots\cdots + 2^n + \cdots\cdots \infty$

Here, $a_n =$ 2^n

Partial Sums of a Series. If we have a series $\sum_{n=1}^{\infty} a_n$ then

· the first partial sum $s_1 =$ a_1

$s_4 = a_1 + a_2 + a_3 + a_4$

· the second partial sum $s_2 =$ $a_1 + a_2$

· the 3^{rd} partial sum $s_3 =$ $a_1 + a_2 + a_3$

· the n^{th} partial sum $s_n =$ $a_1 + a_2 + a_3 + \cdots\cdots + a_n$

Example 1: Find the 4^{th} partial sum of $\sum_{n=1}^{\infty} \frac{1}{2^n}$ $\Rightarrow a_n = \frac{1}{2^n}$

$s_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

Definition of Convergent and Divergent Series. Given a series $\sum_{n=1}^{\infty} a_n$, we can establish a sequence of its *partial sums* $\{s_n\} = \{\ s_1,\ s_2,\ s_3, \ldots, s_n, \ldots\ \}$

We can compute $\lim_{n\to\infty} s_n$. If

$\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n$

$$\begin{cases} \lim_{n\to\infty} s_n = \pm\infty, \ \text{| limit does not exist} & \text{then } \sum_{n=1}^{\infty} a_n \text{ is } \textbf{divergent} \\ \lim_{n\to\infty} s_n = S, \text{ a finite number,} & \text{then } \sum_{n=1}^{\infty} a_n \text{ is } \textbf{convergent} \text{ and } \sum_{n=1}^{\infty} a_n = S \end{cases}$$

Remark: By writitng $\sum_{n=1}^{\infty} a_n = S$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number S.

Example 2: Given the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Calculate the first eight terms of the sequence of partial sums correct to the four decimal places. Does it appear that the series is convergent or divergent?

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \qquad \boxed{a_n = \frac{1}{2^n}}$$

$$s_1 = a_1 = \frac{1}{2^1} = \frac{1}{2} = 1 - \frac{1}{2}$$

$$s_2 = \frac{1}{2^1} + \frac{1}{2^2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{4}$$

$$s_3 = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 1 - \frac{1}{8}$$

$$s_4 = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = s_3 + \frac{1}{2^4} = \frac{7}{8} + \frac{1}{16} = \frac{15}{16} = 1 - \frac{1}{16}$$

$$s_5 = s_4 + \frac{1}{2^5} = \frac{15}{16} + \frac{1}{32} = \frac{31}{32} = 1 - \frac{1}{32}$$

$$s_6 = s_5 + \frac{1}{2^6} = \frac{31}{32} + \frac{1}{64} = \frac{63}{64} = 1 - \frac{1}{64}$$

$$s_7 = s_6 + \frac{1}{2^7} = \frac{63}{64} + \frac{1}{128} = \frac{127}{128} = 1 - \frac{1}{128}$$

$$s_8 = s_7 + \frac{1}{2^8} = \frac{127}{128} + \frac{1}{256} = \frac{255}{256} = 1 - \frac{1}{256}$$

$$\vdots$$

appears to be converging to 1.

Series with Names. There are three special series which come up fairly often in Chapter 11.

- ***Geometric Series:***

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

$\frac{ar}{a} = r$, $\frac{ar^2}{ar} = r$, ... , $\frac{ar^n}{ar^{n-1}} = \frac{r^n}{r^{n-1}} = r$

r is called the **common ratio** of the geometric series.

Remark: For a GEOMETRIC series, the first term is always a and the second term is always ar.

E.g., $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n}$ is a geometric series. Find a and r for this geometric series.

$a_1 = \frac{2}{3}$ $\qquad \frac{a_2}{a_1} = \frac{2/3^2}{2/3} = \frac{2}{3^2} \times \frac{3}{2} = \frac{1}{3}$

$a_2 = \frac{2}{3^2}$ $\qquad \frac{a_3}{a_2} = \frac{2/3^3}{2/3^2} = \frac{2}{3^3} \times \frac{3^2}{2} = \frac{1}{3}$

$a_3 = \frac{2}{3^3}$

$\vdots$

$a_n = \frac{2}{3^n}$, $a_{n+1} = \frac{2}{3^{n+1}}$.

Find the ratio $\frac{a_{n+1}}{a_n}$

$= \frac{2/3^{n+1}}{2/3^n} = \frac{\cancel{2}}{3^{n+1}} \times \frac{3^n}{\cancel{2}}$

$= \frac{1}{3}$

Convergence/Divergence Test for a Geometric Series.

$$\begin{cases} \text{The geometric series } \displaystyle\sum_{n=1}^{\infty} ar^{n-1} \text{ is } \textbf{divergent} \text{ if } & |r| \geq 1 \\ \text{The geometric series } \displaystyle\sum_{n=1}^{\infty} ar^{n-1} \text{ is } \textbf{convergent} \text{ if } & |r| < 1 \quad \text{and} \quad \displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \end{cases}$$

Example 3: Is the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ convergent or divergent? If it converges, find its sum

$a_n = \frac{1}{2^n}$, $a_{n+1} = \frac{1}{2^{n+1}}$ $\Rightarrow \frac{a_{n+1}}{a_n} = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{1}{2^{n+1}} \times \frac{2^n}{1}$

$= \frac{1}{2}$

$r = \frac{a_{n+1}}{a_n} = \frac{1}{2} < 1 \Rightarrow$ the given series converges.

$\Rightarrow a = \frac{1}{2^1} = \frac{1}{2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{a}{1-r} = \frac{1/2}{1-\frac{1}{2}} = \frac{1/2}{1/2} = 1$

$\sum_{n=3}^{\infty} \frac{1}{2^n} \Rightarrow a = \frac{1}{2^3} = \frac{1}{8}$

$= \frac{1/8}{1-1/2} = \frac{1/8}{1/2} = \frac{1}{4}$

$\sum_{n=1}^{\infty} a_n = 4$

$\sum_{n=3}^{\infty} a_n = 4 - a_1 - a_2$

- ***The p-Series:*** $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a real number. (section 11.3)

Convergence/Divergence Test for a p-Series.

$$\begin{cases} \text{The } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is \textbf{divergent} if} & p \leq 1 \\ \text{The } p\text{-series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is \textbf{convergent} if} & p > 1. \end{cases}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \to p = \frac{1}{2}$ divergent

$\sum_{n=1}^{\infty} \frac{1}{n^{4/3}} \to p = 4/3$ convergent

p-series.

$\sum_{n=1}^{\infty} \frac{1}{n^{100}} \to p = 100$ convergent

$\sum_{n=1}^{\infty} \frac{1}{n^{-1}} \to p = -1$ divergent

Here are examples of p-series.

$\sum_{n=n_0}^{\infty} \frac{b}{n^p}$

$\to$ b can be any real number.

$\to n_0$ can be any natural number

answer to convergence/divergence depends only on p.

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ $\quad p = 3 > 1$

converges

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $\quad p = \frac{1}{2} < 1$

diverges

$\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad p = 1 \leq 1$

diverges

- ***Telescoping Series:***

An example of a telescoping series is $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

There is no quick test of convergence/divergence of telescoping series. To test the **Convergence/Divergence for Telescoping Series**, we must use the **definition of convergent and divergent series** on page 1.

$$s_1 = \frac{1}{1} - \frac{1}{2}$$

$$s_2 = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - \frac{1}{3}$$

$$s_3 = s_2 + a_3 = \left(1 - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4}$$

$$s_4 = s_3 + a_4 = \left(1 - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right) = 1 - \frac{1}{5}$$

$$\vdots$$

$$s_n = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - \frac{1}{\infty} = 1 - 0 = 1$$

The given series is <u>convergent</u>

Here is a very useful tool to see whether a series is **divergent**

TEST FOR DIVERGENCE (TD). Given a series $\sum a_n$. If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$ then the series is divergent.

Example 4: Show that $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$a_n = \frac{n^2}{5n^2+4}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cancel{n^2}}{5\cancel{n^2}} = \frac{1}{5} \neq 0$

$\Rightarrow$ series diverges

(*) when $\lim_{n\to\infty} a_n = 0$, we cannot say anything.

$\sum_{n=1}^{\infty} \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ For both $\lim_{n\to\infty} a_n = 0$ but one diverges and the other converges.

Warning: If $\lim_{n\to\infty} a_n = 0$, the series $\sum a_n$ could be convergent or divergent. We don't know! **Never** conclude that a series is convergent if you use the Test for Divergence.

Example 5: Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

Note: We know a series is a geometric series if the term a_n can be rewritten as $(\text{constant})(r)^{\text{exponent in terms of } n}$.

(a) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$ $\Rightarrow a_n = \frac{(-3)^{n-1}}{4^n}$ $\Rightarrow a_{n+1} = \frac{(-3)^{n+1-1}}{4^{n+1}} = \frac{(-3)^n}{4^{n+1}}$

$r = \frac{a_{n+1}}{a_n} = \frac{\frac{(-3)^n}{4^{n+1}}}{\frac{(-3)^{n-1}}{4^n}} = \frac{\cancel{(-3)^n}^{\,-3}}{\cancel{4^{n+1}}_{\,4}} \times \frac{\cancel{4^n}}{\cancel{(-3)^{n-1}}} = \frac{-3}{4}$

$|r| = \left|\frac{-3}{4}\right| = \frac{3}{4} < 1$ $\Rightarrow$ The series converges.

$\Rightarrow a = \frac{(-3)^{1-1}}{4^1} = \frac{1}{4}$

$S = \frac{a}{1-r} = \frac{\frac{1}{4}}{1-\left(\frac{-3}{4}\right)} = \frac{\frac{1}{4}}{1+\frac{3}{4}} = \frac{1/4}{7/4} = \frac{1}{4} \times \frac{4}{7} = \frac{1}{7}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{2n+1}}{(-2)^n}$ $\Rightarrow a_n = \dfrac{3^{2n+1}}{(-2)^n}$ $\Rightarrow a_{n+1} = \dfrac{3^{2(n+1)+1}}{(-2)^{n+1}} = \dfrac{3^{2n+3}}{(-2)^{n+1}}$

$$\Rightarrow r = \frac{a_{n+1}}{a_n} = \frac{\dfrac{3^{2n+3}}{(-2)^{n+1}}}{\dfrac{3^{2n+1}}{(-2)^n}} = \frac{3^{2n+3}}{(-2)^{n+1}} \times \frac{(-2)^n}{3^{2n+1}} = \frac{3^{2n+3-2n-1}}{(-2)^{n+1-n}}$$

$$= \frac{3^2}{(-2)^1} = \frac{9}{-2} = -\frac{9}{2}$$

$\Rightarrow r = \dfrac{-9}{2}$ $\Rightarrow$ geometric series.

$|r| = \left|\dfrac{-9}{2}\right| = \dfrac{9}{2} \geq 1$ $\Rightarrow$ the series diverges.

Example 6: Determine whether the series is convergent or divergent.

Hint: Determine whether each series is a geometric series or a p-series first. If a series is neither one of those, think about using the Test of Divergence.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3+1}{k^2+2k+5}$ $\qquad a_k = \dfrac{k^3+1}{k^2+2k+5}$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^3}{k^2} = \lim_{k\to\infty} k = \infty \neq 0$$

So, by the Test of Divergence, the series diverges.

(b) $\sum_{n=1}^{\infty} 4^{-n}3^{n+1}$

$a_n = \underbrace{4^{-n}\,3^{n+1}}$, $a_{n+1} = 4^{-(n+1)}\,3^{(n+1)+1} = \underbrace{4^{-n-1}\,3^{n+2}}$

$r = \frac{a_{n+1}}{a_n} = \frac{4^{-n-1}\,3^{n+2}}{4^{-n}\,3^{n+1}} = \frac{3^{n+2-n-1}}{4^{-n-(-n-1)}} = \frac{3}{4^{-n+n+1}} = \frac{3}{4}$

$\Rightarrow r = \frac{3}{4}$

$\Rightarrow |r| = \frac{3}{4} < 1 \Rightarrow$ the series converges.

$\Rightarrow$ geometric series

$S = \frac{a}{1-r}$ $\qquad a = a_1 = 4^{-1}\,3^{1+1} = 4^{-1}\,3^2 = \frac{9}{4}$

$= \frac{9/4}{1-\frac{3}{4}} = \frac{9/4}{1/4} = \frac{9}{4} \times \frac{4}{1} = 9$

(c) $\sum_{n=1}^{\infty} \frac{1}{e^{-n}+2}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{e^{-n}+2} = \frac{1}{e^{-\infty}+2} = \frac{1}{0+2} = \frac{1}{2}$

$\left(\lim_{n\to\infty} e^{-n} = 0\right)$

Alternatively

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\frac{1}{e^n}+2} = \lim_{n\to\infty} \frac{1}{\frac{1+2e^n}{e^n}} = \lim_{n\to\infty} \frac{e^n}{1+2e^n}$

$= \lim_{n\to\infty} \frac{e^n}{2e^n} = \frac{1}{2}$

$\lim_{n\to\infty} a_n = \frac{1}{2} \neq 0 \Rightarrow$ By TD, the series diverges

(d) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\downarrow$

It's a P-series with $P = 2 > 1$

$\Rightarrow$ the series converges