Name:

Problem 1: Evaluate the following definite integrals

1. $\int_0^1 x\,dx$
2. $\int_0^1 x^2\,dx$
3. $\int_0^1 x^3\,dx$

as limit of the right Riemann sums, that is, using the formula

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)$$

① $\int_0^1 x\,dx = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} 0 + i\frac{(1-0)}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n}$

$= \lim_{n\to\infty} \frac{1}{n^2}\left[1+2+\cdots+n\right] = \lim_{n\to\infty} \frac{1}{n^2} \frac{n(n+1)}{2}$

$= \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2} + 0 = \frac{1}{2}$

② $\int_0^1 x^2\,dx = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} \left(0 + i\frac{(1-0)}{n}\right)^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i^2}{n^2}$

$= \lim_{n\to\infty} \frac{1}{n^3}\left[1^2+2^2+\cdots+n^2\right] = \lim_{n\to\infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6}$

$= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) = \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$= \frac{1}{6} \times 1 \times 2 = \frac{1}{3}$

③ $\int_0^1 x^3\,dx = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} \left(0 + i\frac{(1-0)}{n}\right)^3 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i^3}{n^3}$

$= \lim_{n\to\infty} \frac{1}{n^4}\left[1^3+2^3+\cdots+n^3\right] = \lim_{n\to\infty} \frac{1}{n^4} \frac{n^2(n+1)^2}{4}$

$= \lim_{n\to\infty} \frac{1}{4} \frac{(n+1)^2}{n^2} = \lim_{n\to\infty} \frac{1}{4}\left(\frac{n+1}{n}\right)^2 = \lim_{n\to\infty} \frac{1}{4}\left(1+\frac{1}{n}\right)^2$

$= \frac{1}{4}(1+0)^2 = \frac{1}{4}$

Problem 2: Using properties of definite integrals and the results of problem 1, evaluate

$$\int_1^0 (4x^3 - 6x^2 - 2x + 1)\, dx$$

$$\int_1^0 (4x^3-6x^2-2x+1)\,dx = \int_1^0 4x^3\,dx - \int_1^0 6x^2\,dx - \int_1^0 2x\,dx + \int_1^0 dx$$

$$= -4\int_0^1 x^3\,dx + 6\int_0^1 x^2\,dx + 2\int_0^1 x\,dx - \int_0^1 dx$$

$$= -4\left(\frac{1}{4}\right) + 6\left(\frac{1}{3}\right) + 2\left(\frac{1}{2}\right) - 1\times 1$$

$$= -1 + 2 + 1 - 1 = \underline{\underline{1}}$$

Problem 3: Use midpoint rule with $n = 5$ to approximate the integral

$$\int_0^2 \frac{x}{x+1}\, dx.$$

$\underline{n=5}$ $\Rightarrow$ subintervals are $\left[0, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, \frac{6}{5}\right], \left[\frac{6}{5}, \frac{8}{5}\right], \left[\frac{8}{5}, \frac{10}{5}\right]$

The midpoints are $\frac{1}{5}, \frac{3}{5}, \frac{5}{5} = 1, \frac{7}{5}, \frac{9}{5}$

$$\Rightarrow I = \int_0^2 \frac{x}{x+1}\,dx \simeq \frac{2}{5}\left[f\left(\frac{1}{5}\right) + f\left(\frac{3}{5}\right) + f(1) + f\left(\frac{7}{5}\right) + f\left(\frac{9}{5}\right)\right]$$

$$= \frac{2}{5}\left[\frac{\frac{1}{5}}{\frac{1}{5}+1} + \frac{\frac{3}{5}}{\frac{3}{5}+1} + \frac{1}{1+1} + \frac{\frac{7}{5}}{\frac{7}{5}+1} + \frac{\frac{9}{5}}{\frac{9}{5}+1}\right]$$

$$= \frac{2}{5}\left[\frac{1}{6} + \frac{3}{8} + \frac{1}{2} + \frac{7}{12} + \frac{9}{14}\right]$$

$$= \frac{2}{5} \times \frac{28 + 63 + 84 + 98 + 108}{24\times 7} = \frac{2}{5} \times \frac{\cancel{381}^{127}}{\cancel{168}_{56}} = \frac{254}{280}$$

$$\simeq 0.907$$

Problem 4: Use Fundamental Theorem of Calculus to find the following derivatives:-

1. $\int_2^{1/x} \sin^4 u \, du$
2. $\int_{\sin x}^1 \sqrt{1+t^2} \, dt$
3. $\int_{x^2}^{\tan x} \frac{1}{\sqrt{2+u^4}} du$
4. $\int_{\sqrt{x}}^{x^2} \cos(t^2) \, dt$

① $\frac{d}{dx}\left[\int_0^{\frac{1}{x}} \sin^4 u \, du\right] = \sin^4\left(\frac{1}{x}\right) \frac{d}{dx}\left(\frac{1}{x}\right)$

$= \frac{-\sin^4\left(\frac{1}{x}\right)}{x^2}$

② $\frac{d}{dx}\left[\int_{\sin x}^1 \sqrt{1+t^2} \, dt\right] = -\sqrt{1+\sin^2 x} \times \frac{d}{dx}(\sin x)$

$= -\cos x \sqrt{1+\sin^2 x}$

③ $\frac{d}{dx}\left[\int_{x^2}^{\tan x} \frac{1}{\sqrt{2+u^4}} du\right] = \frac{1}{\sqrt{2+\tan^4 x}} \frac{d}{dx}(\tan x)$

$- \frac{1}{\sqrt{2+(x^2)^4}} \frac{d}{dx}(x^2)$

$= \frac{\sec^2 x}{\sqrt{2+\tan^4 x}} - \frac{2x}{\sqrt{2+x^8}}$

④ $\int_{\sqrt{x}}^{x^2} \cos(t^2) \, dt = \left[\cos (x^2)^2\right] \times \frac{d}{dx}(x^2) - \left[\cos(\sqrt{x})^2\right] \frac{d}{dx}(\sqrt{x})$

$= 2x \cos x^4 - \frac{\cos x}{2\sqrt{x}}$

Problem 5: Evaluate the following indefinite integrals (use substitution if needed):-

1. $\displaystyle\int \frac{1-\sin^3 t}{\sin^2 t}\,dt$
2. $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$
3. $\displaystyle\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz$
4. $\displaystyle\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}\,\cancel{dx}$

(1) $$I = \int\left(\frac{1}{\sin^2 t} - \frac{\sin^3 t}{\sin^2 t}\right) dt$$
$$= \int \frac{1}{\sin^2 t}\,dt - \int \sin t\,dt$$
$$= \int \csc^2 t\,dt - \int \sin t\,dt$$
$$= -\cot(t) + \cos(t) + C$$

(2) $I = \displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$. Substitute $u = \sqrt{x} \Rightarrow \dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}} \Rightarrow du = \dfrac{dx}{2\sqrt{x}}$

$\Rightarrow 2\,du = \dfrac{dx}{\sqrt{x}}$

$$= \int \sin(\sqrt{x})\,\frac{dx}{\sqrt{x}} = \int \sin u\,(2\,du)$$
$$= 2\int \sin u\,du = -2\cos u + C = -2\cos(\sqrt{x}) + C$$

(3) $I = \displaystyle\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz$. Substitute $u = 1+z^3 \Rightarrow du = 3z^2\,dz$

$\Rightarrow \dfrac{du}{3} = z^2\,dz$

$$= \int \frac{1}{\sqrt[3]{u}}\,\frac{du}{3} = \frac{1}{3}\int u^{-\frac{1}{3}}\,du = \frac{1}{3}\,\frac{u^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C = \frac{1}{3}\times\frac{u^{2/3}}{2/3} + C$$
$$= \frac{1}{3}\times\frac{3}{2}u^{2/3} + C = \frac{1}{2}u^{2/3} + C = \frac{1}{2}(1+z^3)^{2/3} + C$$

(4) $I = \displaystyle\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}}$. Substitute $y = 1+\tan t$

$\Rightarrow dy = \sec^2 t\,dt$

$$= \int \frac{\sec^2 t\,dt}{\sqrt{1+\tan t}}$$
$$= \int \frac{dy}{\sqrt{y}} = \int y^{-1/2}\,dy = \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2\sqrt{y} + C$$
$$= 2\sqrt{1+\tan t} + C$$

Problem 6: Evaluate the following definite integrals:

1. $\int_1^8 \frac{2+t}{\sqrt[3]{t^2}}\,dt$
2. $\int_0^{3\pi/2} |\sin x|\,dx$
3. $\int_{-1}^{2} (x - 2|x|)\,dx$
4. $\int_0^{\pi} f(x)\,dx$ where $f(x) = \begin{cases} \sin x & 0 \le x \le \pi/2 \\ \cos x & \pi/2 \le x \le \pi \end{cases}$.

① $I = \int_1^8 \frac{2+t}{\sqrt[3]{t^2}}\,dt = \int_1^8 \left[\frac{2}{t^{2/3}} + \frac{t}{t^{2/3}}\right] dt$

$= 2\int_1^8 t^{-2/3}\,dt + \int_1^8 t^{1/3}\,dt$

$= 2\left.\frac{t^{-2/3+1}}{-\frac{2}{3}+1}\right|_1^8 + \left.\frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right|_1^8$

$= 2\times 3\left[t^{1/3}\Big|_1^8\right] + \frac{3}{4}\left[t^{4/3}\Big|_1^8\right]$

$= 6\left[8^{1/3} - 1^{1/3}\right] + \frac{3}{4}\left[8^{4/3} - 1^{4/3}\right]$

$= 6\left[2-1\right] + \frac{3}{4}\left[16-1\right]$

$= 6 + \frac{45}{4} = \underline{\underline{\frac{69}{4}}}$

② $I = \int_0^{\frac{3\pi}{2}} |\sin x|\,dx$

$= \int_0^{\pi} |\sin x|\,dx + \int_{\pi}^{\frac{3\pi}{2}} |\sin x|\,dx$

$= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{\frac{3\pi}{2}} -\sin x\,dx$

$= -\cos x\Big|_0^{\pi} + \cos x\Big|_{\pi}^{\frac{3\pi}{2}} = -\left[\cos\pi - \cos 0\right] + \left[\cos\frac{3\pi}{2} - \cos\pi\right]$

$= -\left[-1-1\right] + \left[0-(-1)\right] = 2+1 = \underline{\underline{3}}$

③ $\int_{-1}^{2} (x - 2|x|)\,dx = \int_{-1}^{0} (x-2|x|)\,dx + \int_0^2 (x-2|x|)\,dx$

$= \int_{-1}^{0} \left(x - 2(-x)\right)dx + \int_0^2 (x-2x)\,dx = \int_{-1}^{0} 3x\,dx + \int_0^2 -x\,dx$

$= 3\left.\frac{x^2}{2}\right|_{-1}^{0} - \left.\frac{x^2}{2}\right|_0^2 = 3\left[\frac{0^2}{2} - \frac{(-1)^2}{2}\right] - \left[\frac{2^2}{2} - \frac{0^2}{2}\right] = -\frac{3}{2} - 2 = \underline{\underline{\frac{-7}{2}}}$

④ $\int_0^{\pi} f(x)\,dx = \int_0^{\frac{\pi}{2}} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx = \int_0^{\frac{\pi}{2}} \sin x\,dx + \int_{\pi/2}^{\pi} \cos x\,dx$

$= -\cos x\Big|_0^{\frac{\pi}{2}} + \sin x\Big|_{\pi/2}^{\pi} = -\left[\cos\frac{\pi}{2} - \cos 0\right] + \left[\sin\pi - \sin\pi/2\right]$

$= -\left[0-1\right] + \left[0-1\right] = 1 + (-1) = \underline{\underline{0}}$

Problem 7: Evaluate the following definite integrals using substitution and/or symmetry:

1. $\displaystyle\int_{-\pi/3}^{\pi/3} x^4 \sin x\, dx$
2. $\displaystyle\int_0^1 x\sqrt{1-x}\, dx$
3. $\displaystyle\int_0^{\pi/2} \cos x \sin(\sin x)\, dx$
4. $\displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

① $f(x) = x^4 \sin x$

$\Rightarrow f(-x) = (-x)^4 \sin(-x) = x^4(-\sin x) = -x^4 \sin x$

$= -f(x) \Rightarrow f$ is an odd function.

$\Rightarrow \displaystyle\int_{-\pi/3}^{\pi/3} x^4 \sin x\, dx = 0.$

② $I = \displaystyle\int_0^1 x\sqrt{1-x}\, dx$. Substitute $y = 1-x \Rightarrow dy = -dx \Rightarrow dx = -dy$

$x = 1-y$

$\Rightarrow I = \displaystyle\int_{1-0}^{1-1} (1-y)\sqrt{y}\,(-dy) = -\int_1^0 (1-y)\sqrt{y}\, dy = \int_0^1 (1-y)\sqrt{y}\, dy$

$= \displaystyle\int_0^1 \sqrt{y}\, dy - \int_0^1 y\sqrt{y}\, dy = \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_0^1 - \frac{y^{3/2+1}}{\frac{3}{2}+1}\Bigg|_0^1$

$= \frac{2}{3}\left[1^{3/2} - 0^{3/2}\right] - \frac{2}{5}\left[1^{5/2} - 0^{5/2}\right] = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$

③ $I = \displaystyle\int_0^{\frac{\pi}{2}} \cos x \sin(\sin x)\, dx$. Substitute $y = \sin x \Rightarrow dy = \cos x\, dx$

$\Rightarrow I = \displaystyle\int_{\sin 0}^{\sin\frac{\pi}{2}} \sin y\, dy = \int_0^1 \sin y\, dy = -\cos y\Big|_0^1 = -\left[\cos 1 - \cos 0\right]$

$= \cos 0 - \cos 1 = 1 - \cos 1$

④ $I = \displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$. Substitute $y = 1+\sqrt{x}$

$\Rightarrow dy = \frac{1}{2\sqrt{x}}\, dx \Rightarrow dx = 2\sqrt{x}\, dy$

$\Rightarrow dx = 2(y-1)\, dy$

$\Rightarrow I = \displaystyle\int_{1+\sqrt{0}}^{1+\sqrt{1}} \frac{2(y-1)}{y^4}\, dy = \int_1^2 \frac{2(y-1)}{y^4}\, dy$

$= 2\displaystyle\int_1^2 \frac{1}{y^3}\, dy - 2\int_1^2 \frac{1}{y^4}\, dy = 2\,\frac{y^{-3+1}}{-3+1}\Bigg|_1^2 - 2\,\frac{y^{-4+1}}{-4+1}\Bigg|_1^2$

$= \frac{2}{-2}\left[2^{-2} - 1^{-2}\right] - \frac{2}{-3}\left[2^{-3} - 1^{-3}\right] = -1\left(\frac{1}{4} - 1\right) + \frac{2}{3}\left(\frac{1}{8} - 1\right) = \frac{3}{4} - \frac{7}{12}$

$= \frac{9-7}{12} = \frac{2}{12} = \frac{1}{6}$