M16600 Lecture Notes

Section 6.7: Hyperbolic Functions

■ **Section 6.7** exercises, *page 489*: #1, 3, 7, 8, 9, 30, 31, 32, 33, 36, 37, 38, 59, 60, 61, 62, 63, 64.

SUMMARY

- Definitions of Hyperbolic Functions and their graphs
- Some indentities
- Derivatives of Hyperbolic Functions. Hence, we get some more integral formulas.

Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. These are the ***Hyperbolic Functions***. In many ways, the hyperbolic functions are analogous to the trigonometric functions.

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

Graphs of Hyperbolic Functions

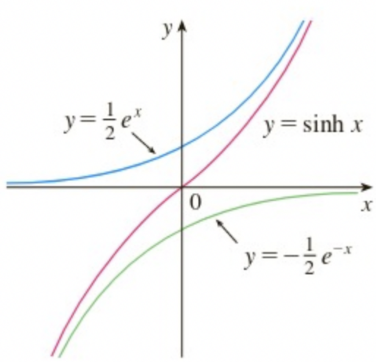

FIGURE 1
$y = \sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

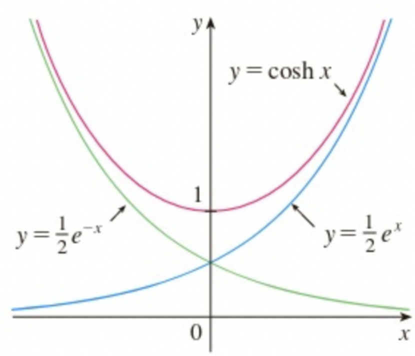

FIGURE 2
$y = \cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$

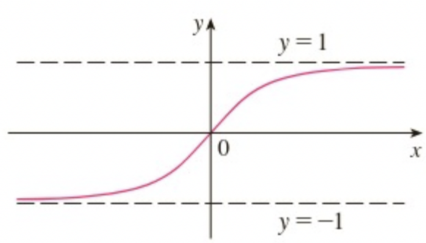

FIGURE 3
$y = \tanh x$

The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities.

Hyperbolic Identities

$$\sinh(-x) = -\sinh(x) \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

Here are the derivative formulas of Hyperbolic Functions. Note that from these formulas, we also obtain integral formulas.

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2}$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) \longrightarrow \text{H.W.}$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Inverse Hyperbolic Functions: See textbook, page 486. NOT IN SYLLABUS

Example 1: Compute the derivative of $y = \tanh^5(x^5)$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\operatorname{Tanh}^5(x^5)\right) = \underbrace{\frac{d}{du}\left(\operatorname{Tanh}^5(u)\right)}\frac{du}{dx}$$

$u = x^5$

$$z = \operatorname{Tanh}(u) \qquad = \frac{d}{du}(z^5)\frac{du}{dx} = \frac{d}{dz}(z^5)\frac{dz}{du}\frac{du}{dx}$$

$$\Rightarrow \frac{dz}{du} = \operatorname{sech}^2(u) \quad (1) \qquad = 5z^4 \operatorname{sech}^2(u)\, 5x^4$$

$$u = x^5 \Rightarrow \frac{du}{dx} = 5x^4 \quad (11) \qquad = 25x^4 \operatorname{Tanh}^4(x^5)\operatorname{sech}^2(x^5)$$

Example 2: Evaluate the integral

$$\boxed{\int \sinh(x)\,dx = \cosh(x) + C}$$

(a) $\displaystyle\int \frac{\sinh(\ln x)}{x}\,dx$

$$= \int \sinh(\ln x)\,\frac{1}{x}\,dx$$

$u = \ln x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x}$

$\Rightarrow du = \dfrac{1}{x}\,dx$

$$= \int \sinh(u)\,du$$

$$= \cosh(u) + c$$

$$= \cosh(\ln x) + c$$

$u = a + f(x)$

$$\int \frac{f'(x)}{a + f(x)}\,dx = \ln|a + f(x)| + c$$

(b) $\displaystyle\int \frac{\sinh x}{1 + \cosh x}\,dx$

$u = 1 + \cosh x$, $\dfrac{du}{dx} = \sinh x$

$\Rightarrow du = (\sinh x)\,dx$

$$I = \int \frac{\sinh x\,dx}{1 + \cosh x}$$

$$= \int \frac{du}{u} = \ln|u| + c$$

$$= \ln|1 + \cosh x| + c$$

$$\int \frac{f'(x)}{1 + [f(x)]^2}\,dx = \tan^{-1}(f(x)) + C$$

$\Rightarrow u = f(x)$

(c) **What about** $\displaystyle\int \frac{\sinh x}{1 + \cosh^2 x}\,dx$?

$u = \cosh x$

$$\Rightarrow \frac{du}{dx} = \sinh x \Rightarrow du = (\sinh x)\,dx$$

$$I = \int \frac{\sinh x \, dx}{1+\cosh^2 x} = \int \frac{du}{1+u^2}$$

$$= \tan^{-1}(u) + c$$

$$= \tan^{-1}(\cosh x) + c$$