M16600 Lecture Notes
Section 7.8: Improper Integrals

■ **Section 7.8** textbook exercises, *page 574*: #2, 5, 7, 9, 11, 13, 19, 21, 27, 29, 31, 33.

GOALS

- Compute **improper integrals** of type I. E.g., $\displaystyle\int_1^\infty \frac{1}{x}\,dx$.
- Compute **improper integrals** of type II. E.g., $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

A definite integral $\displaystyle\int_a^b f(x)\,dx$ that we've encountered so far satisfies both of these conditions:

(i) The interval $[a, b]$ is finite and

(ii) The integrand $f(x)$ is continuous on $[a, b]$

If either one of the two conditions above fails, we say the definite integral to be ***improper***. Here are some examples of improper integrals

- ***Improper Integrals of Type I*** (condition (i) fails):

$$\int_1^\infty \frac{1}{x}\,dx, \qquad \int_{-\infty}^0 xe^x\,dx, \qquad \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx.$$

- ***Improper Integrals of Type II*** (condition (ii) fails):

discont. at $x=1$

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx, \qquad \int_0^1 \ln x\,dx, \qquad \int_{-1}^0 \frac{3}{x^3}\,dx, \qquad \int_0^3 \frac{1}{x-1}\,dx.$$

discont. at $x=2$ discont. at $x=0$ discont. $x=0$

How to Compute Improper Integrals of Type I: Rewrite the integrals as follows:

- $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\left[\int_a^t f(x)\,dx\right]$
- $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\left[\int_t^b f(x)\,dx\right]$
- $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$ where c is a constant

our choice.

Compute as you would in case I and II

Definitions:

· The improper integral is ***convergent*** if the limit = a finite number (i.e., the limit exists)

· The improper integral is ***divergent*** if the limit = $\pm\infty$ or the limit does not exist.

Example 1: Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$

$\int_1^t \frac{1}{x}\,dx = \ln|x| \Big|_1^t = \ln t - \ln 1 = \ln t$

$\lim_{t\to\infty} \ln t = +\infty \Rightarrow$ integral is divergent

(b) $\int_{-\infty}^0 xe^x\,dx = \lim_{t\to-\infty} \int_t^0 x e^x\,dx$

$\int \underbrace{x}_{u}\, \underbrace{e^x\,dx}_{dv} = x e^x - \int e^x\,dx = x e^x - e^x$

$u = x \Rightarrow du = dx$

$dv = e^x\,dx \Rightarrow v = e^x$

$\Rightarrow \int_t^0 x e^x\,dx = \left[x e^x - e^x\right]_t^0$

$= \left[0 e^0 - e^0\right] - \left[t e^t - e^t\right]$

$= -1 - t e^t + e^t$

$\int_{-\infty}^0 x e^x\,dx = \lim_{t\to-\infty} \left(-1 - t e^t + e^t\right)$

$= \lim_{t\to-\infty} (-1) - \lim_{t\to-\infty} t e^t + \lim_{t\to-\infty} e^t$

$= -1 - (0) + 0 = -1$

$\lim_{t\to-\infty} t e^t \Rightarrow DS: (-\infty)(\to 0) \Rightarrow L = \lim_{t\to-\infty} \frac{t}{e^{-t}} \Rightarrow DS: \frac{-\infty}{\infty}$

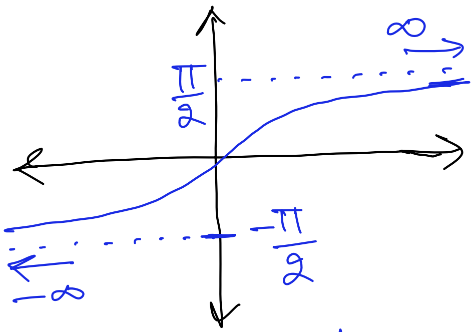
$L = \lim_{t\to-\infty} \frac{1}{-e^{-t}} \Rightarrow DS: \frac{1}{-\infty} = 0 \Rightarrow L = 0$

$\boxed{\Rightarrow \int_{-\infty}^0 x e^x\,dx = -1}$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \underbrace{\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx}_{I_1} + \underbrace{\int_{0}^{\infty} \frac{1}{1+x^2}\,dx}_{I_2}$

$$I_1 = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \operatorname{Tan}^{-1}x \Big|_t^0$$

$$= \lim_{t\to-\infty} \left[\operatorname{Tan}^{-1}0 - \operatorname{Tan}^{-1}t\right]$$



$$= \lim_{t\to-\infty} \left[0 - \operatorname{Tan}^{-1}t\right] = -\lim_{t\to-\infty} \operatorname{Tan}^{-1}t$$

$$= -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$I_1 = \frac{\pi}{2}$$

$$I_2 = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \operatorname{Tan}^{-1}x \Big|_0^t = \lim_{t\to\infty} \operatorname{Tan}^{-1}t = \frac{\pi}{2}$$

$$\Rightarrow I_2 = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = I_1 + I_2 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

How to Compute Improper Integrals of Type II: Rewrite the integrals as follows:

- If f is only discontinuous at $x = b$, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \left[\int_a^t f(x)\,dx\right].$$

- If f is only discontinuous at $x = a$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \left[\int_t^b f(x)\,dx\right].$$

- If f is only discontinuous at $x = c$, where $a < c < b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx \; + \; \int_c^b f(x)\,dx.$$

Example 2: Determine whether the following integrals are convergent or divergent. Evaluate those that are convergent.

(a) $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

$f(x) = \dfrac{1}{\sqrt{x-2}}$ is discontinuous at the left endpoint $x=2$ of $[2,5]$

$[t, 5] \Rightarrow f$ is well-defined and continuous

$$= \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$\int_t^5 (x-2)^{-1/2}\,dx = \left.\frac{(x-2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_t^5 = \left.\frac{(x-2)^{1/2}}{1/2}\right]_t^5$$

$$= 2\sqrt{x-2}\,\Big|_t^5 = 2\sqrt{5-2} - 2\sqrt{t-2} = 2\sqrt{3} - 2\sqrt{t-2}$$

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+}\left(2\sqrt{3} - 2\sqrt{t-2}\right) = 2\sqrt{3} - 2\lim_{t\to 2^+}\sqrt{t-2} \overset{\text{DS}}{=} 0$$

$$\lim_{t\to 2^+}\sqrt{t-2} = \lim_{h\to 0}\sqrt{2+h-2} = \lim_{h\to 0}\sqrt{h} = 0$$

$$= 2\sqrt{3} - 2\times 0$$

$$= 2\sqrt{3}$$

(b) $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$

$f(x) = \dfrac{1}{x-1} \Rightarrow f$ is discontinuous at $x=1$ and 1 lies in $[0,3]$

$$I = \int_0^3 \frac{1}{x-1}\,dx = \underbrace{\int_0^1 \frac{1}{x-1}\,dx}_{I_1} + \underbrace{\int_1^3 \frac{1}{x-1}\,dx}_{I_2}$$

$$I_1 = \int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{x-1}\,dx = \lim_{t\to 1^-}\Big[\ln|x-1|\Big]_0^t$$

$$= \lim_{t\to 1^-}\Big[\ln|t-1| - \ln|0-1|\Big] = \lim_{t\to 1^-}\Big[\ln|t-1| - \cancelto{0}{\ln 1}\Big]$$

$$= \lim_{t\to 1^-}\ln|t-1| = \lim_{h\to 0}\ln|1-h-1| = \lim_{h\to 0}\ln(h) = -\infty$$

$I_1 = -\infty \Rightarrow I_1$ diverges.

$\Rightarrow$ I also diverges (no need to compute I_2)