M16600 Lecture Notes
Sections 6.3: Logarithmic Functions

OBJECTIVES

- Switch back and forth between exponential equations and logarithmic equations
- Know the notation for the natural logarithmic function and how to use it
- Memorize and utilize the properties of logarithm
- Know the graph of logarithmic functions and their limit equations
- Know the change of base formula

The *exponential function* $f(x) = b^x$ has an inverse function $f^{-1}(x)$, which is called ***logarithmic function with base b***. Recalling the relationship

Range of f $=$ Domain of f^{-1}

$$f^{-1}(x) = y. \longrightarrow x = f(y)$$

Then we have

$$y = \log_b x \iff x = b^y$$

The ***natural logarithm*** is the logarithm with base e and has a special notation: $\boxed{\log_e x = \ln x}$. Then

$$\boxed{\ln x = y \iff e^y = x}$$

$\log_2 4 = y \Rightarrow 4 = 2^y$
$\Rightarrow y = 2$
$\Rightarrow \log_2 4 = 2$

Logarithm base 10 also has a special notation: $\boxed{\log_{10} = \log}$.

Example 1: Change the exponential equations to logarithmic equations

(a) $5^x = 35$ $\rightarrow$ identify the base b

$b = 5 \Rightarrow x = \log_5 35$

(b) $\frac{1}{2} = e^{-0.016t}$

$-0.016t = \ln\left(\frac{1}{2}\right)$

Example 2: Change the logarithmic equations to exponential equations

(a) $\log_3 81 = 4$

$\Rightarrow 81 = 3^4$

(b) $\ln(2x-1) = 3$

$\Rightarrow 2x - 1 = e^3$

$\ln e^2 = y$
$\Rightarrow e^2 = e^y$
$\Rightarrow y = 2$
$\Rightarrow \ln e^2 = 2$

Example 3: Evaluate (a) $\log_3 9$ (b) $\log_{25} 5$.

(a) $\log_3 9 = y \Rightarrow 9 = 3^y \Rightarrow y = 2$

(b) $\log_{25} 5 = y \Rightarrow 5 = 25^y \Rightarrow y = \frac{1}{2}$

$\log(1000) = \log 10^3 = y$
$\Rightarrow 10^3 = 10^y \Rightarrow y = 3$

• **Properties of Logarithmic Functions**: If $b > 1$, the function $f(x) = \log_b x$ is one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$. If x, $y > 0$ and p is any real number, then we have the ***Laws of Logarithms*** as follows:

(i) $\log_b(xy) = \log_b x + \log_b y$

(ii) $\log_b \frac{x}{y} = \log_b x - \log_b y$

(iii) $\log_b x^p = p \log_b x$

$e^{x+y} = e^x e^y$

- $\ln\left(\frac{1}{2}\right) = \ln(1) - \ln(2)$
- $\ln(6) = \ln(2 \cdot 3) = \ln(2) + \ln(3)$
- $\ln(8) = \ln(2^3) = 3\ln(2)$
- $\ln(2^p 3^q) = \ln 2^p + \ln 3^q = p \ln 2 + q \ln 3$

Cancellation Equations:

$f(f^{-1}(x)) = x$ $\qquad$ $\log_b(b^x) = x$ $\quad$ for $x \in \mathbb{R}$

$f^{-1}(f(x)) = x$ $\qquad$ $b^{log_b x} = x$ $\quad$ for every $x > 0$

- $\ln(e^x) = x$
- $e^{\ln x} = x$

Example 4: Expand $\quad \ln\sqrt{\dfrac{x+1}{x^2 y}}$

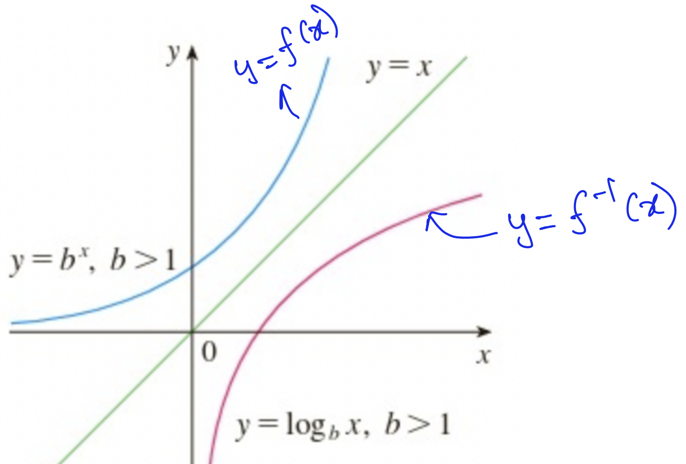
$$= \ln\left(\frac{x+1}{x^2 y}\right)^{1/2}$$

$$= \frac{1}{2}\ln\left(\frac{x+1}{x^2 y}\right) = \frac{1}{2}\left[\ln(x+1) - \ln(x^2 y)\right] = \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x^2 y)$$

$$= \frac{1}{2}\ln(x+1) - \frac{1}{2}\left[\ln(x^2) + \ln y\right] = \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x^2) - \frac{1}{2}\ln y$$

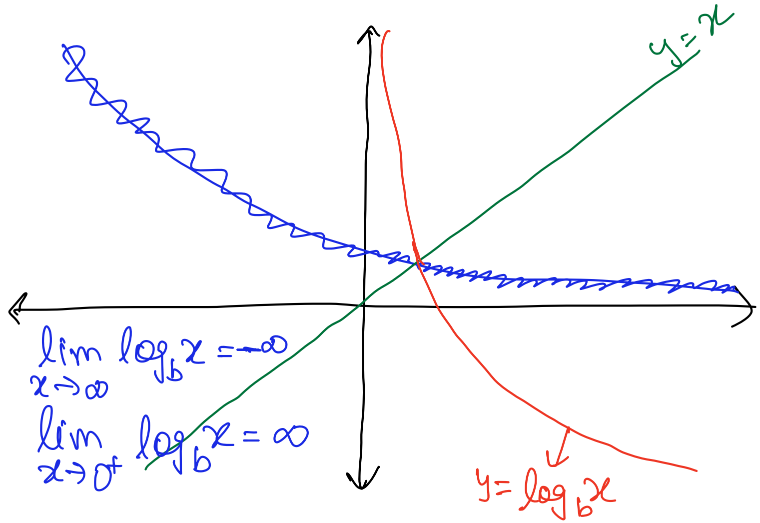
$$= \frac{1}{2}\ln(x+1) - \frac{1}{\cancel{2}} \times \cancel{2}\ln(x) - \frac{1}{2}\ln y = \frac{1}{2}\ln(x+1) - \ln(x) - \frac{1}{2}\ln(y)$$

• **The Graph of the Logarithm Function and the Exponential Function on the Same xy-plane, $b > 1$**



$\lim\limits_{x\to\infty} \log_b x = \infty$, $\lim\limits_{x\to 0^+} \log_b x = -\infty$

$0 < b < 1$



$\lim\limits_{x\to\infty} \log_b x = -\infty$

$\lim\limits_{x\to 0^+} \log_b x = \infty$

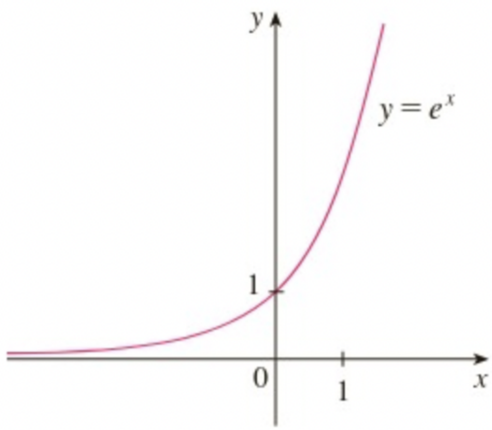
Next, we focus on **the natural logarithm** $\ln(x)$, which is log base e. All of what we know about the general logarithmic function are applied to $\ln x$.

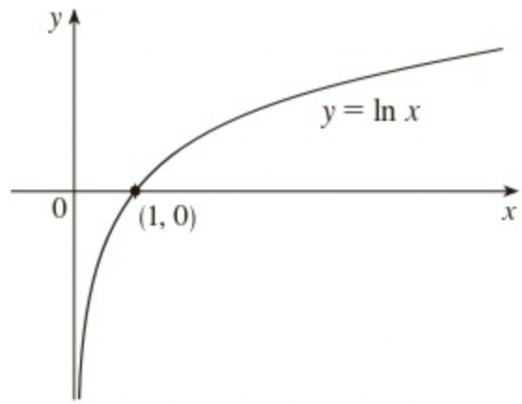
$\ln 1 = 0$ because $1 = e^0$

$\ln e = 1$ because $e = e^1$

$\ln(e^x) = x$

$e^{\ln x} = x$





Example 5: Find x if $\ln x = 5$

$$\boxed{\ln x = y \iff x = e^y}$$

$\Rightarrow x = e^5$

calculate using a calculator

$e = 2.718 \ldots\ldots$

Example 6: Solve the equation $e^{5-3x} + 4 = 14$

$$\Rightarrow e^{5-3x} = 14 - 4 \Rightarrow e^{5-3x} = 10$$

$$\Rightarrow \ln e^{5-3x} = \ln(10)$$

$$\Rightarrow (5-3x)\underbrace{\ln e}_{=1} = \ln(10)$$

$$\Rightarrow 5 - 3x = \ln 10 \Rightarrow 5 = 3x + \ln 10$$

$$\Rightarrow 5 - \ln 10 = 3x$$

$$\Rightarrow x = \frac{1}{3}(5 - \ln 10)$$

Example 7: Express $\ln a + \frac{1}{5}\ln b - \ln(a+b)$ as a single logarithm.

$= \ln(?)$

$$\ln a + \frac{1}{5}\ln b - \ln(a+b) = \ln a + \ln b^{1/5} - \ln(a+b)$$

$$= \ln(a\,b^{1/5}) - \ln(a+b) = \ln\left(\frac{a\,b^{1/5}}{a+b}\right)$$

Change of Base Formula: For any positive number b ($b \neq 1$), we have

$$\log_b x = \frac{\ln x}{\ln b}$$

Example 8: Evaluate $\log_8 5$

$$\log_8 5 = \frac{\ln 5}{\ln 8} \longrightarrow \text{use Calculator to find}$$

$\ln 5$ and $\ln 8$

$$= \frac{\ln 5}{3\,\underbrace{\ln 2}_{\approx 0.69}}$$

$= \ln 2^3 = 3\ln 2$

Section 6.3 exercises, *page 426*, #3, 4, 5, 9, 11, 13, <u>15</u>, 17, <u>19</u>, 27, 29, 31, <u>47</u>, <u>51</u>.