# Indiana University, Indianapolis

Spring 2025 Math-I 165

FINAL EXAM (May 01, 2025)

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**Instructions:**

- No cell phones, calculators, watches, technology, hats - stow all in your bags.
- **Write your name** on this cover page.
- This test is **closed book and closed notes.**
- All work must be clearly shown for partial credit.
- If you wish for something not to be graded, please strike it out neatly.
- Box, circle, or otherwise clearly indicate your final answer.
- When you finish, return your test to the proctor, and leave the classroom.
- There are a total of **18 problems including 2 bonus problems.**
  - Problems 1-8 are each worth 10 points.
  - Problems 9-16 are each worth 15 points.
  - The bonus problems are each worth 8 points.
- You can score a **maximum of 216 points out of 200.**
- There are a total of **14 pages** including the cover page.

**Problem 1**. Differentiate the function $f(x) = \dfrac{x-1}{x^2+1}$. [10 pts]

$$f'(x) = \frac{(x^2+1) - (x-1)(2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{x^2+1-(2x^2-2x)}{(x^2+1)^2} = \boxed{\frac{-x^2+2x+1}{(x^2+1)^2}}$$

10

**Problem 2**. Differentiate the function $f(x) = x^2 \cos(x^2)$. [10 pts]

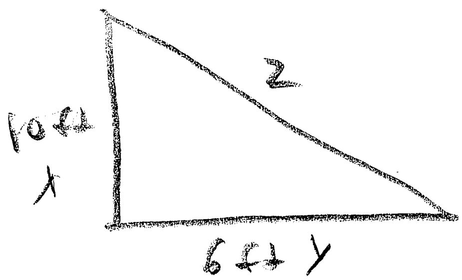
$$f'(x) = 2x\cos(x^2) - x^2 \sin x^2 (2x)$$

$$\boxed{= 2x\cos(x^2) - 2x^3 \sin x^2}$$

10

**Problem 3**. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down when the bottom of the ladder is 6 ft from the wall. [10 pts]

$10^2 + 6^2 = 0$




$x^2 + y^2 = O_2$

$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

$10\frac{dx}{dt} - 6(1) = 0$

$10\frac{dx}{dt} = 6$

$\boxed{\frac{dx}{dt} = \frac{3}{5}\text{ ft/s}}$

8

**Problem 4**. A particle moves in a straight line with the position function $s(t) = \sin\pi t + \cos\pi t$. Find the velocity and acceleration of the particle at $t = 0$ seconds and $t = 1$ seconds. [10 pts]

0, 1

(-1, 0)

$s'(t) = \cos\pi t - \sin\pi t$

$s'(0) = 1 - 0 = 1$

$\boxed{\text{Velocity} = 1}$

$s''(t) = -\sin\pi t - \cos\pi t$

$s''(1) = -0 - -1 = 1$

6

$\boxed{\text{Acceleration} = 1}$

**Problem 5**. The relative error in the radius of a sphere is 0.2%. Find the relative error in the volume of the sphere. [10 pts]

$V = 2\pi r^2 h$

0.2

$100 \overline{)0.200}$ = 0.002

$V = 2\pi (0.004) h$

$V' = 2\pi$

2

**Problem 6**. An objects moves along the $x$-axis with a velocity of $v(t) = t - 2$. Find the distance travelled in the first four seconds, that is, from $t = 0$ to $t = 4$. [10 pts]

$t = 2$

− +

0 2, 4

$$\int_0^2 t-2 + \int_2^4 t-2 = (0 - -2) + (2 - 0)$$

$$\boxed{= 4}$$

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**Problem 7**. Find the work done when a force of magnitude $F(x) = \cos(\pi x/2)$ newtons is applied to move an object from $x = 1$ to $x = 2$ meters. [10 pts]

$$\int_1^2 \cos\left(\frac{\pi x}{2}\right) = \int_1^2 \sin\left(\frac{\pi x}{2}\right)\left(\frac{2}{\pi}\right)$$

$$= 0 - - 1\left(\frac{2}{\pi}\right) = \boxed{\frac{2}{\pi} \text{ lbm}^2}$$

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**Problem 8**. Find a number $c$ such that the average value of the function $f(x) = \sqrt{x}$ on the interval $[0,4]$ equals $f(c)$. [10 pts]

$$f_{av} = \int_0^4 (\sqrt{x})^2 dx = \frac{x^2}{2}\Big|_0^4 \quad \times$$

$$\frac{16}{2} = 8$$

$$f(c) = \sqrt{x} = 8$$

$$\boxed{x = 64}$$

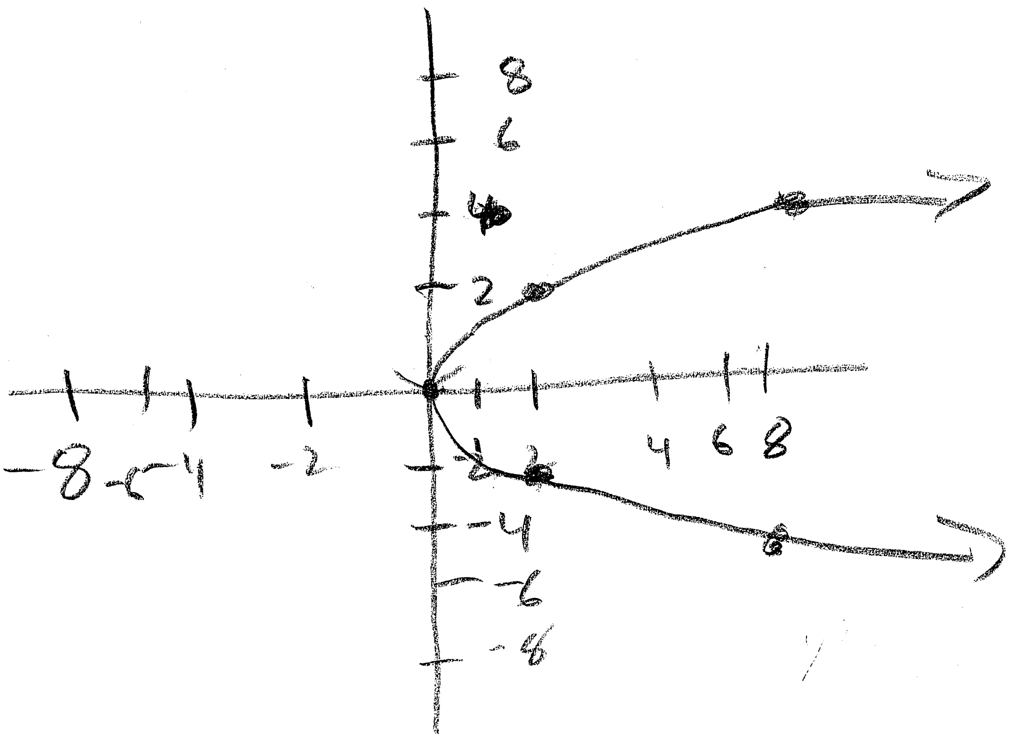
3

**Problem 9**. Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$. [15 pts]

$x\ y$

$\frac{y^2}{2} = x$

| y | x |
|---|---|
| 0 | 0 |
| ±2 | 2 |
| ±4 | 8 |




$x'$ $\frac{2y}{2} = y$

$y = \sqrt{2x} = (2x)^{\frac{1}{2}}$

$x = \frac{y^2}{2} = \frac{16}{2} = 8$

$y' = \frac{1}{\sqrt{2x}}$

$x = y$

$y = 1$

$4 = \frac{1}{\sqrt{2x}}$

$4\sqrt{2x} = 1$

$\sqrt{2x} = \frac{1}{4}$

$2x = \frac{1}{16}$

$x = \frac{1}{32}$

$\boxed{\left(\frac{1}{32}, 1\right)}$

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**Problem 10**. Let $x^4 + y^4 = 16$. Find $\frac{d^2y}{dx^2}$. [15 pts]

$$4x^3 + 4y^3 \frac{dy}{dx} = 0$$

$$12x^2 + 12y^2 \frac{d^2y}{dx^2} = 0$$

$$-12y^2 \frac{d^2y}{dx^2} = 12x^2$$

$$\frac{d^2y}{dx^2} = \boxed{-\frac{x^2}{y^2}}$$

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**Problem 11**. Evaluate the following limits:

1. $\lim_{x\to 0} \frac{\sin 4x}{\tan 3x}$. [5 pts]
2. $\lim_{x\to 0} \frac{x^2 - x}{x^2 + x}$. [5 pts]
3. $\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x}$. [5 pts]

1. $\lim_{x\to 0} \frac{\sin 4(x)}{\tan 3(x)} = \frac{\sin 4(x)\ \sin 3(x)}{\cos 3(x)} = \boxed{0}$

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2. $\lim_{x\to 0} \frac{\cancel{x}(x-1)}{\cancel{x}(x+1)} = \frac{x-1}{x+1} = \frac{-1}{1} = \boxed{-1}$

3. $\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x} \times \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1} = \frac{x^{\cancel{2}}\cancel{+1-1}}{\cancel{x}(\sqrt{x^2+1}+1)}$

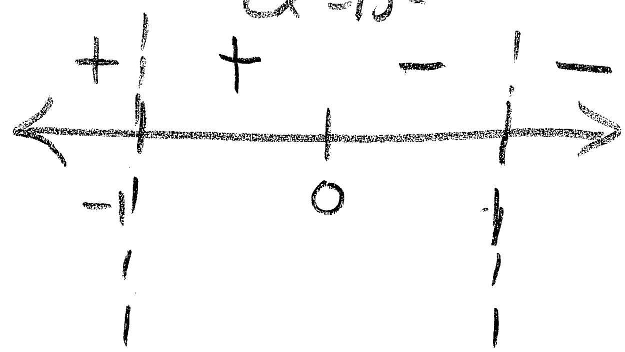
$= \frac{x}{\sqrt{x^2+1}+1} = \boxed{0}$

**Problem 12**. Sketch the graph of the function $f(x) = \dfrac{x^2}{x^2 - 1}$. [15 pts]

$$f'(x) = \frac{2x(x^2-1) - (x^2)(2x)}{(x^2-1)^2} = \frac{2x^3 - 2x - 2x^3}{(x^2-1)^2}$$

$$\frac{-2x}{(x^2-1)^2}$$

$x = 0, \pm 1$




$f(0) = 0$

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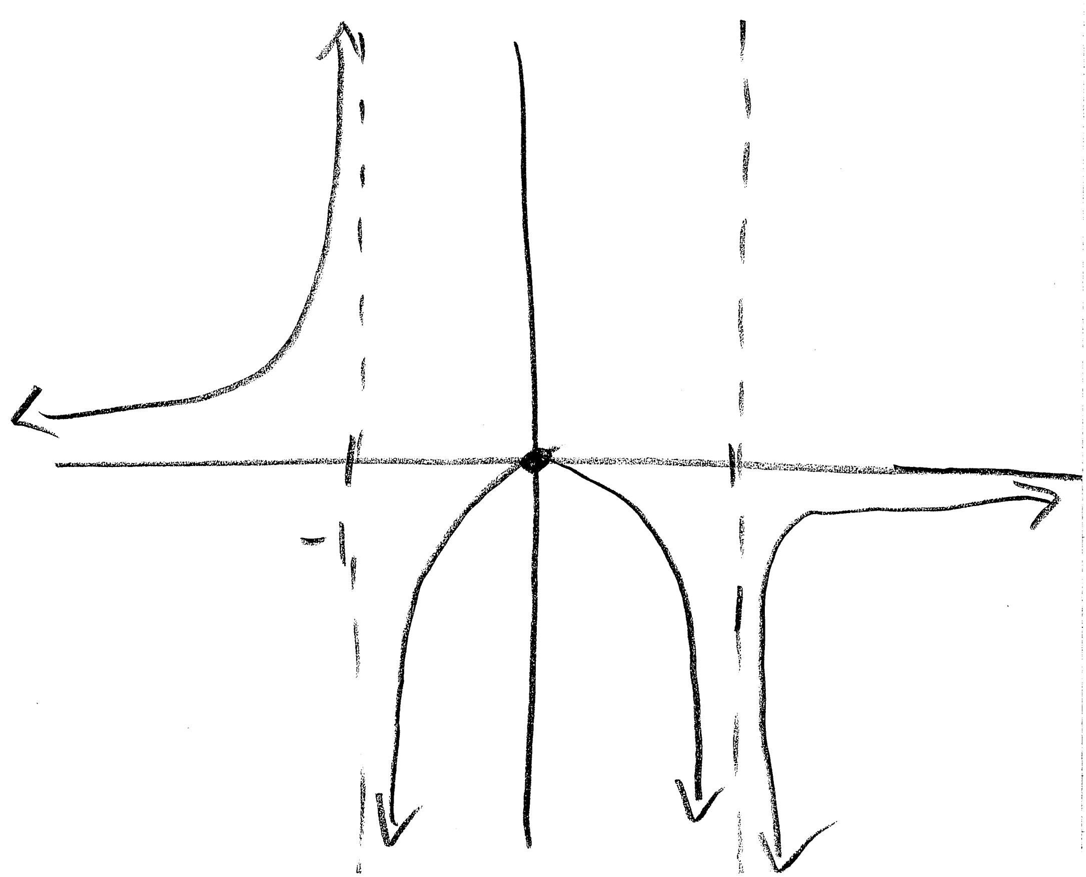

**Problem 13**. Find the equation of tangent and normal lines to the curve $x^4 + y^4 + 2xy = 4$ at $(1, 1)$. [15 pts]

$$4x^3 + 4y^3 \frac{dy}{dx} + 2y + 2x\frac{dy}{dx} = 0$$

$$4x^3 + 2y = \frac{dy}{dx}(-4y^3 - 2x)$$

$$\frac{dy}{dx} = \frac{4x^3+2y}{-4y^3-2x} = \frac{4+2}{-4-2} = \frac{6}{-6} = -1$$

Tangent: $y - 1 = -1(x-1)$

$$y - 1 = -x + 1$$

$$\boxed{y = -x + 2}$$

Normal: $\boxed{y = -x + 2}$

11

**Problem 14**. Evaluate the integral $\int \sqrt{\sin x}\cos^3 x\,dx$. [15 pts]

let $u = \sin^4 x$

$\frac{du}{dx} = 4\cos^3 x$

$(\sin x)^{1/2}$

$(u^{1/4})^{\frac{1}{2}} = \frac{1}{8}$

$u \int \sqrt{u^{1/4}}\,du = \left(\frac{1}{4}\right) u^{1/8} \Rightarrow \frac{8}{9} u^{\frac{9}{8}} + c$

$$\boxed{\int \frac{2}{9} (\sin^4 x)^{9/8} + c}$$

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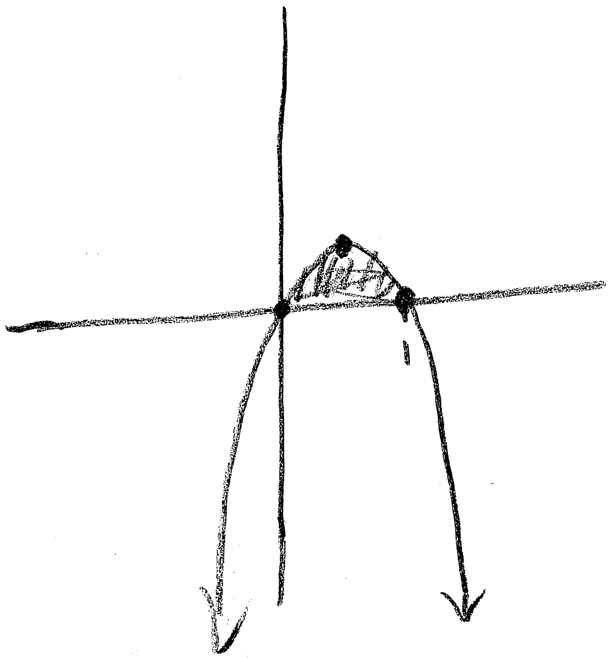
**Problem 15**. Find the area of the following regions:

1. The region enclosed by $y = x - x^2$ and the $x$-axis. [6 pts]
2. The region enclosed by $y = \sin x$, $y = \cos x$, $x = -\pi/4$, $x = \pi/2$. [9 pts]



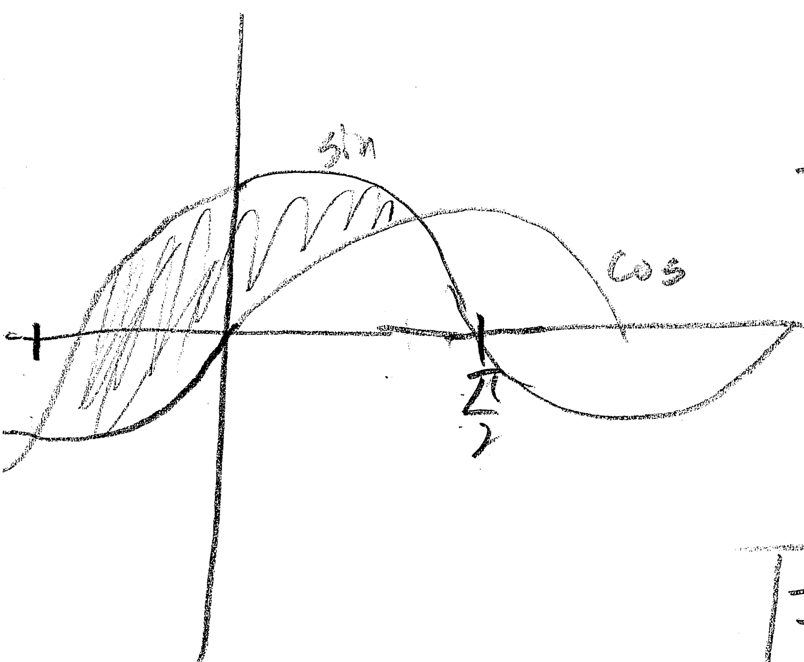

$x(1-x)$

$x = 0, 1$

1. $\int_0^1 (x - x^3)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1$

$\frac{3}{6} - \frac{2}{6} = \boxed{\frac{1}{6}}$

15




$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\,dx$

$= \cos x + \sin x \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= 1 + 0 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)$

$\boxed{= 1 + \sqrt{2}}$

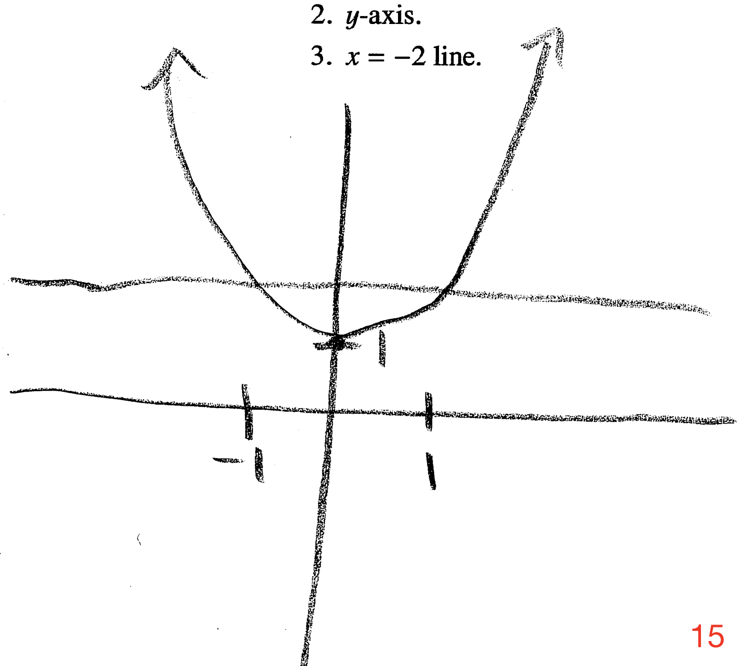
$(x^2+1)(x^2+1)$

$x^4+2x^2+1 \qquad x^4+6x^2+9$



**Problem 16.** Consider the region bounded between the curves $y = x^2 + 1$ and $y = 2$. Set up an integral for the volume of the solid obtained by rotating the given region about the following axis:

1. $x$-axis. [5 pts]
2. $y$-axis. [5 pts]
3. $x = -2$ line. [5 pts]




$y = 2$

$2 = x^2+1$

$x^2 = 1$

$x = \pm 1$

$(-\frac{1}{5}+5) - (\frac{1}{5}-5)$

$-\frac{1}{5}$

$-\frac{2}{5}+10 = \frac{50}{10} = \frac{48}{10} = \frac{24}{5}$

15

1. $\int_{-1}^{1} \pi(2^2-(x^2+1)^2)\,dx = \pi\int_{-1}^{1} 4-x^4-2x-1\,dx = \pi(-\frac{x^5}{5}-x+3x)\Big|_{-1}^{1}$

$2 = \frac{10}{5}$ $\quad \frac{6}{5}, \frac{15}{5}-1$

$\pi(-\frac{1}{5}-1+3)-(\frac{1}{5}+1-3) = \pi(\frac{9}{5}+\frac{9}{5}) = \boxed{\frac{18\pi}{5}}$

2. $\int_{1}^{2} 2\pi x(-2+(x^2+1))\,dx = 2\pi\int_{1}^{2} -2x+x^3+x\,dx = x+x^3$

$\frac{24}{4}-\frac{3}{4}=\frac{21}{4}$

$=(2\pi)\ \frac{x^2}{2}+\frac{x^4}{4}\Big|_1^2 = 2\pi((2+4)-(\frac{3}{4})) = \boxed{\frac{21\pi}{2}}$

3. $\int_{-1}^{1} \pi(4^2-(x^2+3)^2)\,dx = \pi\int_{-1}^{1} -x^4-6x^2+7\,dx = -\frac{x^5}{5}-2x^3+7x\Big|_{-1}^{1}$

$= \pi(-\frac{1}{5}-2+7)-(\frac{1}{5}+2-7) = \boxed{\frac{24\pi}{5}}$

**Bonus Problem 1.** Find the derivate of $f(x) = \dfrac{1}{x+2}$ using the limit definition of derivative. [8 pts]

$$\frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{\frac{1}{(x+h+2)} - \frac{1}{x+2}}{h} = \frac{\frac{\cancel{x}+\cancel{2}-\cancel{x}-h-\cancel{2}}{(x+2)(x+h+2)}}{h}$$

$$= \frac{-\cancel{h}}{(x+2)(x+h+2)} \times \frac{1}{\cancel{h}} = \frac{-1}{(x+2)(x+h+2)} = \boxed{\frac{-1}{(x+2)^2}}$$

8

**Bonus Problem 2.** Evaluate the integral $\int_0^2 (1 - x^2)\,dx$ using the limit definition of integral. [8 pts]

$\Delta x = \frac{2}{n}$ $\quad x_i = \frac{2i}{n}$

$$\boxed{\lim_{n \to \infty} \sum_{i=1}^{n} \left(1 - \left(\frac{2i}{n}\right)^2\right) \frac{2}{n}}$$

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