6.1 Inverse Functions

- **One-to-One Functions**: A function f is called a one-to-one function if it never takes the same value twice, that is, $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.
- **Horizontal Line Test**: A function is one-to-one if and only if no horizontal line intersects its graph more than twice.
- **Inverse of a function**: Let f be a function with domain D and range R. Then its inverse function f^{-1} has domain R and range D, and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

  for any y in R.
- **Cancellation Equations**:

$$f^{-1}\big(f(x)\big) = x \text{ for every } x \text{ in } D$$

$$f\big(f^{-1}(x)\big) = x \text{ for every } x \text{ in } R$$

- **How to find the inverse function of a one-to-one function** f:

Step 1 Write $y = f(x)$.
Step 2 Interchange x and y.
Step 3 Solve for y. The resulting equation is $y = f^{-1}(x)$.

- **Graph of** f^{-1}: The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.
- **Continuity**: If f is one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.
- **Derivative**: If f is one-to-one differentiable function with inverse function f^{-1} and $f'\big(f^{-1}(a)\big) \neq 0$, then the inverse function is differentiable at a, and

$$(f^{-1})'(a) = \frac{1}{f'\big(f^{-1}(a)\big)}$$

6.2 Exponential Functions and their Derivatives

- For $b > 0$, $b \neq 1$, $f(x) = b^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$.
- If $0 < b < 1$, then f is a decreasing function. If $b > 1$, then f is an increasing function.
- For any real numbers x, y,

$$b^{x+y} = b^x b^y \quad , \quad b^{x-y} = \frac{b^x}{b^y} \quad , \quad (b^x)^y = b^{xy} \quad , \quad (ab)^x = a^x b^x$$

- If $0 < b < 1$, then

$$\lim_{x\to\infty} b^x = 0 \text{ and } \lim_{x\to-\infty} b^x = \infty \ .$$

  If $b > 1$, then

$$\lim_{x\to\infty} b^x = \infty \text{ and } \lim_{x\to-\infty} b^x = 0 \ .$$

- Derivative of b^x:

$$f'(x) = f'(0)b^x$$

- e is the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

  Note that $e \approx 2.718$.

- Derivative of e^x:

$$\frac{d}{dx}(e^x) = e^x \quad , \quad \frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$

- Integral of e^x:

$$\int e^x \, dx = e^x + c$$

6.2 Logarithmic Functions

- The inverse function to $f(x) = b^x$ is called the logarithmic function. Thus,

$$\log_b x = y \Leftrightarrow b^y = x$$

- Cancellation laws:

$$\log_b (b^x) = x \text{ for every } x \in \mathbb{R} \quad , \quad b^{\log_b x} = x \text{ for every } x > 0$$

- If $b > 1$, the function $f(x) = \log_b x$ is a one-to-one, continuous, increasing function with domain $(0, \infty)$ and range $\mathbb{R}$.

- If $x, y > 0$ and r is any real number then

$$\log_b (xy) = \log_b x + \log_b y \quad , \quad \log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y \quad , \quad \log_b (x^r) = r \log_b x$$

- If $b > 1$, then

$$\lim_{x \to \infty} \log_b x = \infty \quad , \quad \lim_{x \to ^+} \log_b x = -\infty$$

- The natural logarithm is defined to be

$$\ln x = \log_e x$$

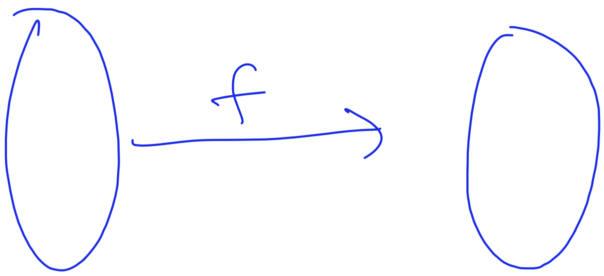
- For any positive number b ($b \neq 1$) we have

$$\log_b x = \frac{\ln x}{\ln b}$$

6.1 → 6.3 Recap

6.1 → Inverse functions

(*) Given f, find f^{-1}

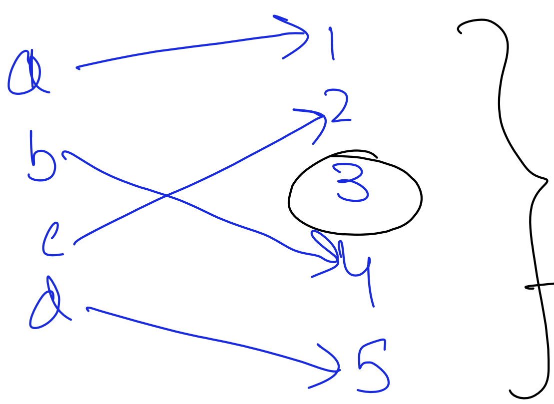


A — Domain

B — Codomain of f

f is one-one

$f(x) = f(y) \Rightarrow x = y$



→ Range of f = $\{1, 2, 4, 5\}$

$f^{-1} : \{1, 2, 4, 5\} \longrightarrow \{a, b, c, d\}$

$f^{-1}(1) = a$, $f^{-1}(2) = c$, $f^{-1}(4) = b$, $f^{-1}(5) = d$

Range of f = Codomain of f

then f is called onto/surjective.

(*) f has an inverse if and only if f is one-one and onto.

Example

$$f(x) = \frac{2+x}{3-x}$$

Step 1 $\quad y = \frac{2+x}{3-x}$

Step 2 Interchange x and y.

$$x = \frac{2+y}{3-y}$$

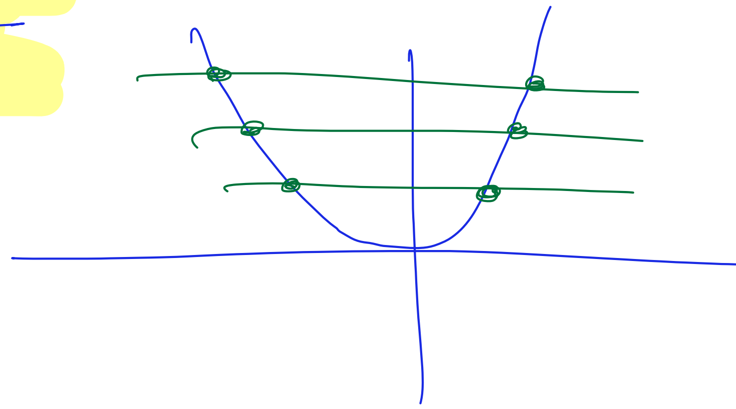
Step 3 Solve for y

$$x(3-y) = 2+y \Rightarrow 3x - xy = 2+y$$

$$\Rightarrow -y - xy = 2-3x \Rightarrow y(-1-x) = 2-3x$$

$$\Rightarrow y = \frac{2-3x}{-1-x}$$

$$\Rightarrow f^{-1}(x) = \frac{2-3x}{-1-x}$$



Given f which is invertible,

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

$$\boxed{(f^{-1})'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}}$$

Examples

$$f(x) = \frac{2+x}{3-x}$$

$$(f^{-1})'(0) = \frac{1}{f'\left(f^{-1}(0)\right)} = \frac{1}{f'(-2)}$$

$y = -2$

$$y = f^{-1}(0) \Rightarrow f(y) = 0 \Rightarrow \frac{2+y}{3-y} = 0$$

$$f'(x) = \frac{(3-x)\frac{d}{dx}(2+x) - (2+x)\frac{d}{dx}(3-x)}{(3-x)^2}$$

$$= \frac{(3-x)\times 1 - (2+x)(-1)}{(3-x)^2}$$

$$= \frac{3-x+2+x}{(3-x)^2} = \frac{5}{(3-x)^2}$$

$$f'(-2) = \frac{5}{(3-(-2))^2} = \frac{5}{5^2} = \frac{1}{5}$$

$$(f^{-1})'(0) = \frac{1}{1/5} = 5$$

6.2 Exponential Functions

$f(x) = b^x$

$b \longrightarrow$ constant.

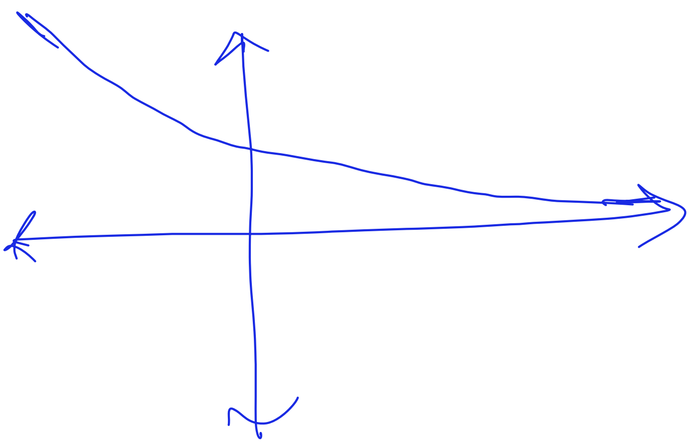
$b \neq 1$

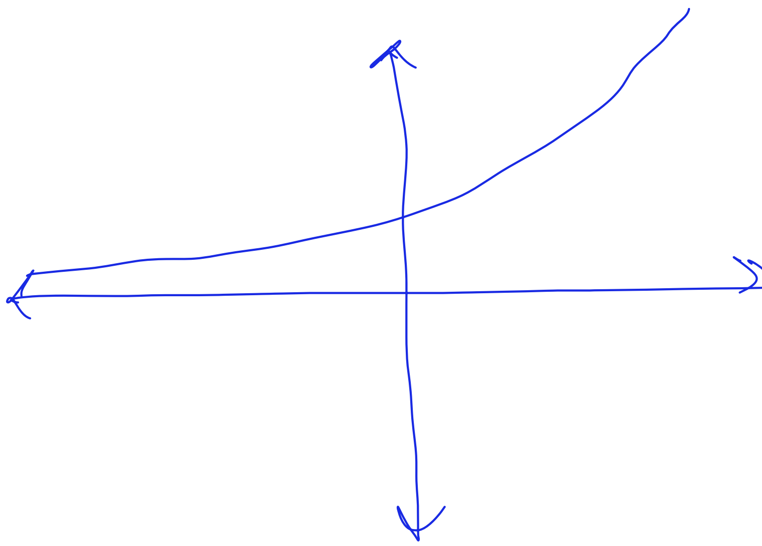
$b > 0$

$0 < b < 1$, $b > 1$

b^x is a decreasing function

b^x is an increasing function.





$$b^{x+y} = b^x \, b^y$$

Euler's Number

$b = e \approx 2.71$

Irrational number

$\frac{d}{dx}(e^x) = e^x$, $\int e^x \, dx = e^x + C$

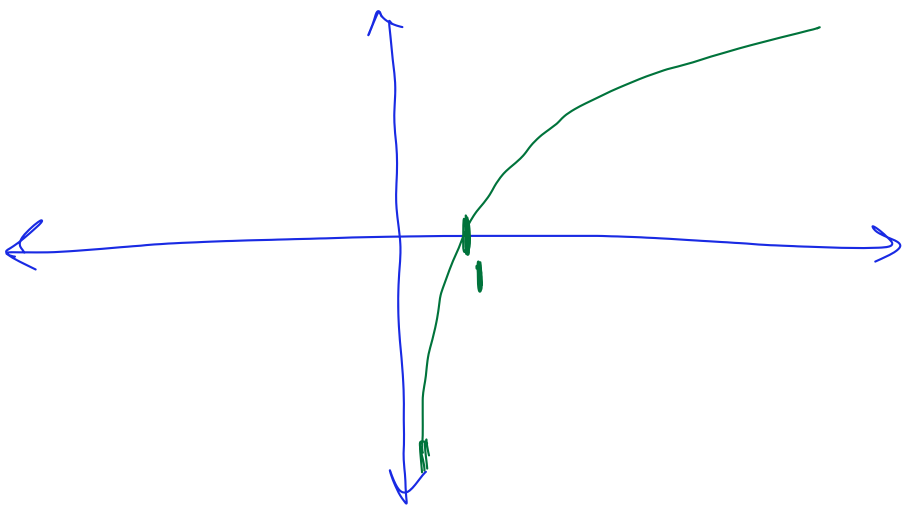
6.3 Logarithmic Functions

$y = \ln x \iff x = e^y$

→ Take $y = e^x$.

→ Interchange x and y : $x = e^y$,

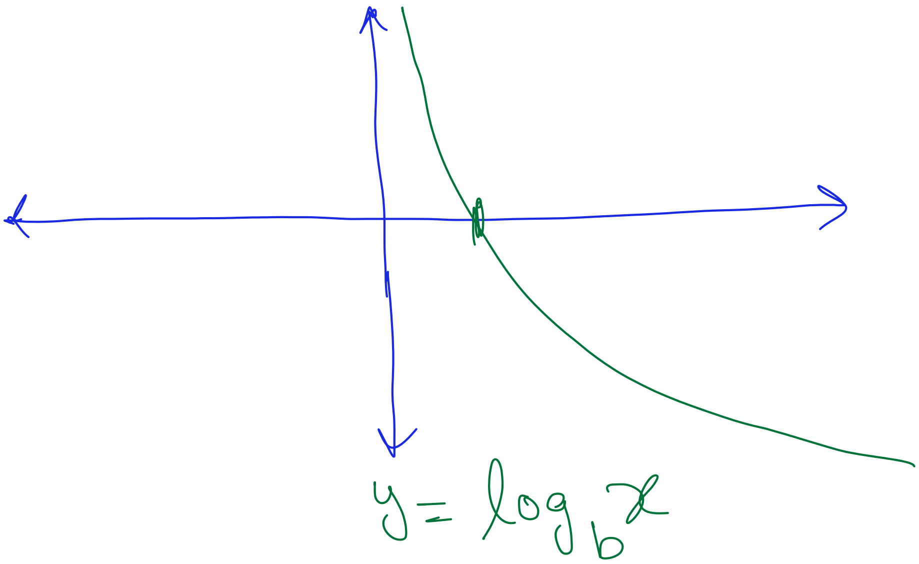
$\ln x : \log_e x$



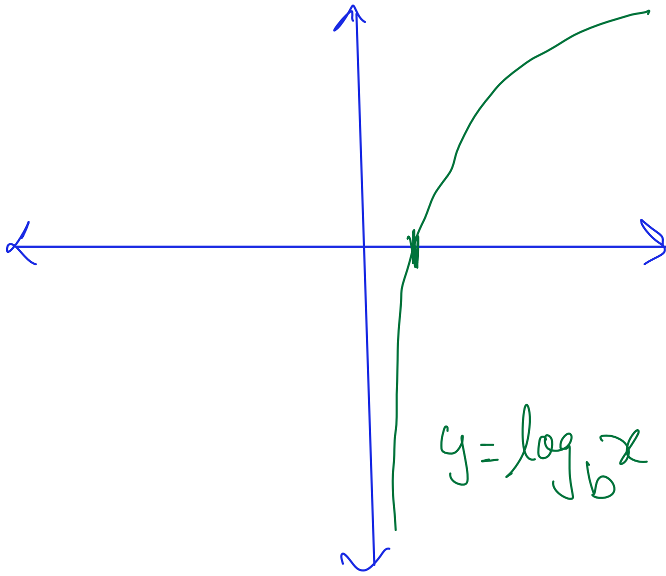
Domain of ln $= (0, \infty)$

$\boxed{y = \log_b x}$, $0 < b < 1$ or $b > 1$

$0 < b < 1$



$b > 1$



$$\log_b x = \frac{\ln x}{\ln b}$$

$$\log_b x = \frac{\log_c x}{\log_c b}$$

where c can be any number such that $0 < c < 1$ or $c > 1$