Learning objectives:

1. Understand the definition of a definite integral.
2. Evaluating definite integral as limit of a sum.
3. Use areas to compute definite integrals.
4. Approximate definite integrals using the midpoint rule.
5. Learn the properties of definite integrals.

Definition of a definite integral

Let f be a function defined for $a \leq x \leq b$. Divide $[a, b]$ into n subintervals of equal width and choose sample points x_i^* on every subinterval. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x\,,$$

provided the above limit exists.

If the above limit exists we say f is integrable on $[a, b]$.

Theorem

If f is continuous on $[a, b]$, or if f only a finite number of jump discontinuities, then f is integrable on $[a, b]$, that is, the definite integral $\int_a^b f(x)\,dx$ exists.

Theorem

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x\,.$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\,\Delta x$.

Properties of sums

$$\sum_{i=1}^{n} c = nc\,, \qquad \sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i\,.$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\,.$$

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\,.$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\,, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\,, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}\,.$$

Example 1. Evaluate $\displaystyle\int_0^3 (x^3 - 6x)\,dx$.

Example 2. Set up an expression for $\int_2^5 x^4\,dx$ as a limit of a sum.

Example 3. Evaluate the following integrals by interpreting each in terms of areas.

1. $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$.
2. $\displaystyle\int_0^3 (x-1)\,dx$.

The Midpoint rule

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \Delta x\big(f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\big)\ ,$$

where $\Delta x = \frac{b-a}{n}$ and $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ is the midpoint of $[x_{i-1}, x_i]$.

Example 4. Use the midpoint rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}\,dx$.

Properties of definite integral

1. $\displaystyle\int_a^b c\,dx = c(b-a)$, where c is any constant.
2. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.
3. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.
4. $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.
5. $\displaystyle\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$.
6. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, for $a < c < b$.
7. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$.
8. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
9. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $\displaystyle m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Example 5. Use the properties of integrals to evaluate $\int_0^1 (4+3x^2)\,dx$.

Example 6. If $\int_0^5 f(x)\,dx = 7$ and $\int_0^3 f(x)\,dx = 2$, then find $\int_3^5 f(x)\,dx$.

Example 7. Use property 9 to estimate $\int_1^4 \sqrt{x}\,dx$.