M16600 Lecture Notes

Section 11.1: Sequences

■ **Section 11.1** textbook exercises, *page 744*: #3, 5, 13, 15, 23, 25, 27, 29, 31, 33, 35, 39, 41, 50.

Definition of a Sequence. A ***sequence*** is a Collection of numbers written in a definite order.

E.g., $\{2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14, \ldots, 2n, \ldots\}$ is a sequence.

$2\times1,\ 2\times2, \ldots\ldots,\ 2\times n, \ldots\ldots$

n^{th} term

Find the 27^{th}-term of the above sequence. $2\times27 = 54$

Notation: A sequence $\{a_1,\ a_2,\ a_3,\ a_4, \ldots, a_n, \ldots\}$ could be written as $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Note: n does not have to start from 1.

For the above sequence $\{2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14, \ldots, 2n, \ldots\}$,

$a_n =$ $2n$. Therefore, we could write this sequence as $\{2n\}$, $\{2n\}_{n=1}^{\infty}$

Here are more examples of a sequences

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \longrightarrow \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\ldots\right\}$$

$$\left\{\frac{(-1)^n}{n^2}\right\} \longrightarrow \left\{\frac{-1}{1}, \frac{(-1)^2}{2^2}, \frac{(-1)^3}{3^2}, \frac{(-1)^4}{4^2}, \ldots\right.$$

$$= \left\{-1, \frac{1}{4}, \frac{-1}{9}, \frac{1}{16}, \frac{-1}{25}, \ldots\right\}$$

$$\left\{a_n = \frac{3^n}{(n+1)!}\right\} \longrightarrow \left\{\frac{3}{2}, \frac{9}{6}, \frac{27}{24}, \frac{81}{120}, \ldots\right\}$$

Here, for any positive integer k, $k! = 1\cdot2\cdot3\cdot\cdots\cdot k$.

$k!$ is read "k factorial"

$2! = 1\cdot2 = 2$

$3! = 1\cdot2\cdot3 = 6$

$4! = 4\ 3! = 24$

$5! = 5\ 4! = 120$

$\vdots$

Example 1: Find a formula for the general term a_n of the sequence

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16} \cdots\right\}$$

Numerators : 1, 2, 3, 4, 5, ...

at n^{th} Place, numerator would be n

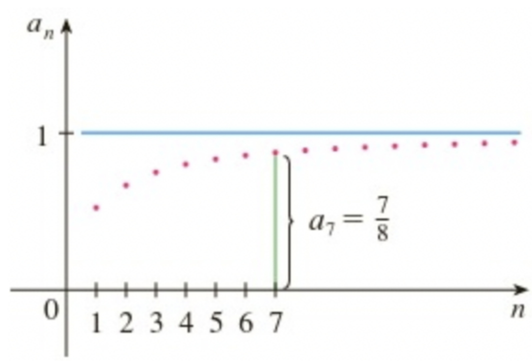
Denominators : $2^1, 2^2, 2^3, 2^4, \ldots\ldots$

at n^{th} Place, denominator would be 2^n

$$a_n = \frac{n}{2^n}$$

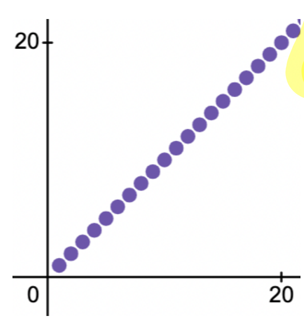
Limit of a Sequence. We write $\lim\limits_{n\to\infty} a_n = L$ if we can make the terms a_n as close to L as we like by taking n sufficiently large.

For example, given the sequence $a_n = \dfrac{n}{n+1}$, we have $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$ because the terms $a_n = \dfrac{n}{n+1}$ approaches 1 as n gets large. Below is the plot of some terms of this sequence.



Convergent or Divergent Sequence.

- If $\lim\limits_{n\to\infty} a_n =$ (a finite number), then the sequence a_n is said to be ***convergent***.
- If $\lim\limits_{n\to\infty} a_n = \pm\infty$ or $\lim\limits_{n\to\infty} a_n$ does not exists , then the sequence a_n is said to be ***divergent***.



Example 2: Determine whether the sequence converges or diverges. If it converges, find the limit

$$\lim_{n\to\infty} \frac{1}{n^k} = 0 \quad k>0$$

(a) $a_n = \dfrac{4n^2+2}{n+n^2}$. To answer this question, we want to compute $\lim_{n\to\infty} a_n$.

Method 1 (an algebra approach): Factor as many ~~x~~ n's as we can on the numerator and on the denominator then simplify. Then compute the limit.

$$a_n = \frac{\cancel{n^2}\left(\frac{4n^2}{n^2}+\frac{2}{n^2}\right)}{\cancel{n^2}\left(\frac{n}{n^2}+\frac{n^2}{n^2}\right)} = \frac{4+\frac{2}{n^2}}{\frac{1}{n}+1} \Rightarrow \lim_{n\to\infty} a_n = \frac{4+2(0)}{0+1} = \frac{4}{1} = 4$$

Method 2 (a calculus approach): Use L'Hospital's Rule if applicable

$$\underline{\underline{\text{D.S}}} \quad \frac{\infty}{\infty} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n^2+2}{n+n^2} = \lim_{n\to\infty} \frac{8n}{1+2n} \overset{\text{D.S.}}{=} \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{8}{2} = 4$$

Method 3 (the dropping-slower-terms approach): Keep the term with the largest growth rate of the numerator. Do the same for the denominator. Then simplify if possible. Then compute the limit.

Limit Facts:

divergent

convergent

$$\lim_{n\to\infty} \frac{\text{faster growth rate function}}{\text{slower growth rate function}} = \infty \qquad \lim_{n\to\infty} \frac{\text{slower growth rate function}}{\text{faster growth rate function}} = 0$$

$$\lim_{n\to\infty} \frac{4n^2+2}{n+n^2} = \lim_{n\to\infty} \frac{4\cancel{n^2}}{\cancel{n^2}} = \lim_{n\to\infty} 4 = 4$$

$$\lim_{n\to\infty} \frac{n^{100}+4n^{50}+9n^{10}+n^9+n^5+1}{n^{99}-2n^{100}+3n^{98}-5n^{50}+6n^{-2}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^{100}}}{-2\cancel{n^{100}}} = \frac{-1}{2}$$

(b) $\left\{\dfrac{3\sqrt{n}}{n+\sqrt[3]{n^2}-5}\right\}$

$n^{1/2}$ $\sqrt{n^2+5} \sim \sqrt{n^2}$

$$= \lim_{n\to\infty} \frac{3\sqrt{n}}{n+\sqrt[3]{n^2}-5} = \lim_{n\to\infty} \frac{3\sqrt{n}}{n} = \lim_{n\to\infty} \frac{\text{slower}}{\text{faster}} = 0$$

n^1 $n^{2/3}$ n^0 n^1

$$\Rightarrow \lim_{n\to\infty} \frac{5n^2+3\sqrt{n}}{n+\sqrt[3]{n^2}-5} = \lim_{n\to\infty} \frac{5n^2}{n} = \lim_{n\to\infty} \frac{\text{faster}}{\text{slower}} = \infty$$

$\Rightarrow$ divergent

(c) $a_n = \dfrac{\sqrt{10+n+3n^2+4n^5}}{6n^2+2n}$

$$\lim_{n\to\infty} \frac{\sqrt{10+n+3n^2+4n^5}}{6n^2+2n} = \lim_{n\to\infty} \frac{\sqrt{4n^5}}{6n^2} = \lim_{n\to\infty} \frac{2n^{5/2}}{6n^2}$$

$$= \lim_{n\to\infty} \frac{\text{faster}}{\text{slower}} = \infty$$

$\Rightarrow$ divergent

$$\lim_{n\to\infty} e^{-\frac{2}{n^2}} = e^{\lim_{n\to\infty} \frac{-2}{n^2}} = e^{-2\lim_{n\to\infty}\frac{1}{n^2}} = e^{-2(0)} = e^0 = 1$$

||

(d) $a_n = e^{-2/n^2}$

increasing

$\ln n >> \frac{1}{n^2}$ $\to$ dec.

$e^{n^2} << (n^2)!$

$\ln(n) << \sqrt[5]{n}$ $\frac{1}{n^2}$ $\boxed{\cancel{e^{1/n^2}}}$ $e^n << e^{n^2}$

$e^n << n!$

constant functions

THE GROWTH RATE ORDER OF DIFFERENT TYPES OF FUNCTIONS.

$\ll$ logarithmic functions $<<$ algebra $<<$ exponential functions $<<$ factorial

Example 3: Determine whether the sequence converges or diverges. If it converges, find the limit

(a) $\left\{\dfrac{\ln n}{n}\right\}$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\text{slower}}{\text{faster}} = 0$$

(convergent)

(b) $\left\{\dfrac{2^n}{5^n+4}\right\}$

$$\lim_{n\to\infty} \frac{2^n}{5^n+4} = \lim_{n\to\infty} \frac{2^n}{5^n} = \lim_{n\to\infty} \frac{\text{slower}}{\text{faster}}$$

$$= 0$$

$a^n << b^n$

$(a<b)$

$2^n << 5^n$

$3^n \ll 7^n$

(c) $a_n = n!e^{-2n}$

$$\lim_{n\to\infty} n!\, e^{-2n} = \lim_{n\to\infty} \frac{n!}{e^{2n}} = \lim_{n\to\infty} \frac{\text{faster}}{\text{slower}}$$

$$= \infty$$

(divergent)