■ **Section 6.6** exercises, *page 481*: #1, 2, 3, 4, 5, 7, 12, 13, 22, 23, 25, 27, 31, 33, 59, 61, 65, 64, 67.

GOALS

- Compute the values of the **inverse trigonometric functions**, e.g., $\sin^{-1}(\frac{1}{2})$, $\cos^{-1}(0)$, $\tan^{-1}(\sqrt{3})$, etc.
- Compute or simplify expressions such as $\tan\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$, $\cos\left(\tan^{-1} x\right)$, etc.
- Compute derivatives and integrals involving inverse trigonometric functions.

In this section, we explore the inverse functions of trigonometric functions. The functions $\sin(x)$, $\cos(x)$, $\tan(x)$ are not one-to-one over their domains. However, if we restrict their domains, they will be one-to-one on the restricted domain. We then can find their inverse functions.

◇ **Inverse Sine Function**. *Notation*: $\sin^{-1}(x)$ or $\arcsin(x)$

$\sin\theta$ is one-to-one for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Thus, we have

$$\boxed{\sin^{-1} x = \theta \iff \sin\theta = x \quad \text{for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$$

Note: $\sin^{-1} \neq \frac{1}{\sin x}$

$\frac{1}{\sin x} = \csc(x)$

Example 1: Evaluate (a) $\sin^{-1}(\frac{1}{2})$ (b) $\tan\left(\arcsin\frac{1}{3}\right)$

$\xrightarrow{\sin(x)}$

0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\sqrt{3}/2$
$\frac{\pi}{2}$	1

$\xleftarrow{\sin^{-1}(x)}$

(a) $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

(b) $\tan\left(\arcsin\frac{1}{3}\right) = \tan\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$

Let $\theta = \sin^{-1}\left(\frac{1}{3}\right) \Rightarrow \sin\theta = \frac{1}{3}$

we want to find $\tan\theta$

$\sin\theta = \frac{P}{H}$ and $\tan\theta = \frac{P}{B}$

$\Rightarrow \frac{P}{H} = \frac{1}{3} \Rightarrow$ let $P = 1$, then $H = 3$

$P^2 + B^2 = H^2$

$\Rightarrow 1^2 + B^2 = 3^2 \Rightarrow 1 + B^2 = 9 \Rightarrow B^2 = 9 - 1 = 8 \Rightarrow B = \sqrt{8}$

$\Rightarrow \operatorname{Tan}\theta = \frac{1}{\sqrt{8}} \Rightarrow \operatorname{Tan}\left(\sin^{-1}\left(\frac{1}{3}\right)\right) = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$

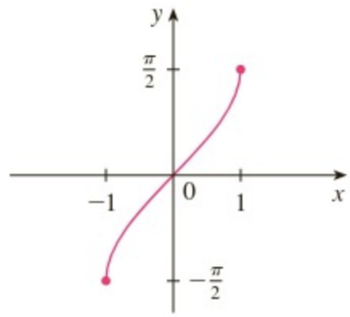


FIGURE 4
$y = \sin^{-1}x = \arcsin x$

$\boxed{\sin^{-1}(-x) = -\sin^{-1}(x)}$

$$\sin^{-1}(\sin x) = x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

$$\sin(\sin^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$

Cancellation Laws

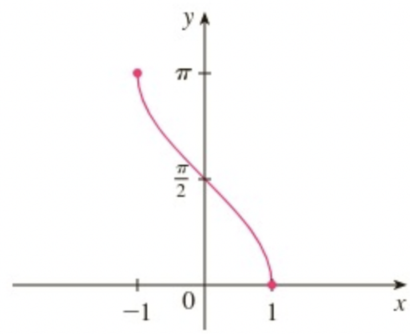
◇ **Inverse Cosine Function**. *Notation*: $\cos^{-1}(x)$ or $\arccos(x)$

$$\boxed{\cos^{-1} x = \theta \iff \cos\theta = x \quad \text{for } 0 \leq \theta \leq \pi}$$

$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

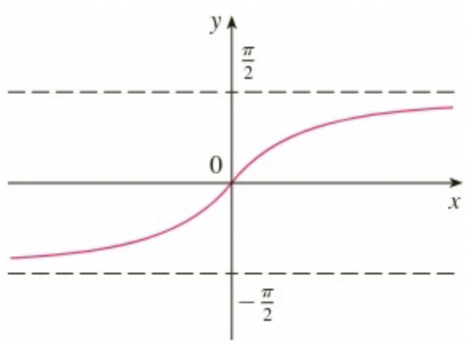
$$\cos^{-1}(\cos x) = x \quad \text{for } 0 \leqslant x \leqslant \pi$$

$$\cos(\cos^{-1}x) = x \quad \text{for } -1 \leqslant x \leqslant 1$$



◇ **Inverse Tangent Function**. *Notation*: tan −1(*x*) or arctan(*x*)

$$\boxed{\tan^{-1} x = \theta \iff \tan\theta = x \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}}$$



$\lim_{x \to \infty} \operatorname{Tan}^{-1}(x) = \frac{\pi}{2}$

$\lim_{x \to -\infty} \operatorname{Tan}^{-1}(x) = \frac{-\pi}{2}$

$\operatorname{Tan}^{-1}(-x) = -\operatorname{Tan}^{-1}(x)$

x	$\operatorname{Tan} x$
0	0
$\frac{\pi}{6}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{2}$	∞

$\xleftarrow{\operatorname{Tan}^{-1}}$

Example 2: Evaluate (a) $\cos^{-1}(-1)$ and (b) $\arctan(\sqrt{3})$.

$\cos^{-1}(-1) = \pi - \cos^{-1}(1) = \pi - 0 = \pi$

$\operatorname{Tan}^{-1}(\sqrt{3}) = \pi/3$

Example 3: Simplify the expression $\cos\left(\tan^{-1}(x)\right)$

try to do as HW.

Let $\theta = \tan^{-1}(x)$

$\Rightarrow \tan\theta = \frac{x}{1} = \frac{P}{B}$

$P = x, \quad B = 1$

want to find $\cos\theta = \frac{B}{H}$

$\Rightarrow H^2 = x^2 + 1^2$

$\Rightarrow H = \sqrt{x^2+1}$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{x^2+1}} \Rightarrow \boxed{\cos\left(\tan^{-1}x\right) = \frac{1}{\sqrt{x^2+1}}}$

Derivative and Integral Formulas Involving Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$
$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$
$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}$$

$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}x + C = \frac{\pi}{2} - \sin^{-1}x + C = -\sin^{-1}x + \frac{\pi}{2} + C$

$-1\int \frac{1}{\sqrt{1-x^2}}\,dx = -\sin^{-1}x + C'$

$\boxed{\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}}$

$\Rightarrow C' = C + \frac{\pi}{2}$

Example 4: Differentiate

(a) $H(x) = 2\tan^{-1}(x) + \arcsin(2x^2) + \cos^{-1}\left(\tan x\right)$

$H'(x) = \frac{d}{dx}\left(2\tan^{-1}(x)\right) + \frac{d}{dx}\left(\sin^{-1}(2x^2)\right) + \frac{d}{dx}\left(\cos^{-1}(\tan x)\right)$

$= \frac{2}{1+x^2} + \frac{d}{dz}\left(\sin^{-1}(z)\right)\frac{dz}{dx} + \frac{d}{dy}\left(\cos^{-1}y\right)\frac{dy}{dx}$

where $z = 2x^2 \Rightarrow \frac{dz}{dx} = 4x$

$y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x$

$\frac{2}{1+x^2} + \frac{1}{\sqrt{1-z^2}}\cdot(4x) + \frac{-1}{\sqrt{1-y^2}}\sec^2 x$

$= \frac{2}{1+x^2} + \frac{4x}{\sqrt{1-4x^4}} - \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}$

(b) $f(x) = x\arctan(\sqrt{x})$

$f(x) = x \tan^{-1}(\sqrt{x}) \Rightarrow f'(x) = (x)' \tan^{-1}\sqrt{x} + x\left(\tan^{-1}\sqrt{x}\right)'$

$= \tan^{-1}\sqrt{x} + x\left(\tan^{-1}\sqrt{x}\right)'$

$\frac{d}{dx}\left(\tan^{-1}(\sqrt{x})\right) = \frac{d}{dz}\left(\tan^{-1}(z)\right)\frac{dz}{dx} = \frac{1}{1+z^2}\;\frac{1}{2\sqrt{x}} = \frac{1}{1+(\sqrt{x})^2}\cdot\frac{1}{2\sqrt{x}}$

Let $z = \sqrt{x} \longrightarrow$ $x^{1/2} \xrightarrow{\frac{d}{dx}} \frac{1}{2}x^{\frac{1}{2}-1}$ $= \left(\frac{1}{1+x}\right)\left(\frac{1}{2\sqrt{x}}\right)$

$\Rightarrow \frac{dz}{dx} = \frac{1}{2\sqrt{x}}$

$\Rightarrow f'(x) = \tan^{-1}\sqrt{x} + x\left(\frac{1}{1+x}\right)\left(\frac{1}{2\sqrt{x}}\right)$

$x = (\sqrt{x})^2$

$= \tan^{-1}\sqrt{x} + \frac{x}{2\sqrt{x}(1+x)}$

$= \tan^{-1}\sqrt{x} + \frac{\sqrt{x}}{2(1+x)}$

Example 5: Evaluate

(a) $\displaystyle\int \frac{1}{15\sqrt{1-x^2}}\,dx$ $\displaystyle = \frac{1}{15}\int \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{15}\sin^{-1}(x) + C$

(b) $\displaystyle\int \frac{3}{1+x^2}\,dx$ $\displaystyle = 3\int \frac{1}{1+x^2}\,dx = 3\,\mathrm{Tan}^{-1}(x) + C$

(c) $\displaystyle\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$ Let $u = \mathrm{Tan}\,x \Rightarrow \dfrac{du}{dx} = \sec^2 x$

$$= \int \frac{1}{\sqrt{1-\mathrm{Tan}^2 x}}\,\underbrace{(\sec^2 x\,dx)} \qquad \Rightarrow du = \underbrace{\sec^2 x\,dx}$$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C = \sin^{-1}(\mathrm{Tan}\,x) + C$$

(d) $\displaystyle\int_0^1 \frac{x}{1+x^4}\,dx$. **Note**: Evaluate all expressions into real numbers for your final answer.

$$= \int_0^1 \frac{x}{1+(x^2)^2}\,dx$$

Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\Rightarrow du = 2\,x\,dx$

$\Rightarrow \frac{1}{2}\,du = \underbrace{x\,dx}$

$$= \int_0^1 \frac{1}{1+(x^2)^2}\,\underbrace{x\,dx}$$

$$= \int_{0^2}^{1^2} \frac{1}{1+u^2}\,\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du = \frac{1}{2}\,\mathrm{Tan}^{-1}(u)\Big|_0^1$$

$$= \frac{1}{2}\left[\mathrm{Tan}^{-1}(1) - \mathrm{Tan}^{-1}(0)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8}$$