M16600 Lecture Notes

Sections 6.4: Derivatives of Logarithmic Functions

SUMMARY

$\frac{d}{dx}(e^x) = e^x$

New Differentiation Formulas

- $\frac{d}{dx}(\ln x) = \frac{1}{x}$
- $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$
- $\frac{d}{dx}(b^x) = b^x \ln b$

New Integral Formulas

- $\int \frac{1}{x}\, dx = \ln|x| + C$
- $\int b^x\, dx = \frac{b^x}{\ln b} + C$, where $b \neq 1$.

$\frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right)$

$= \frac{1}{\ln b}\frac{d}{dx}(\ln x)$

$= \frac{1}{x \ln b}$

New Differentiation Technique: Logarithmic Differentiation

- **The Derivative and Integral Formula of** $\ln x$

$\int \frac{1}{x} dx = \ln x + C$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}, \ \underline{\underline{x > 0}}$$

$$\boxed{\int \frac{1}{x}\, dx = \ln|x| + C}$$

$\frac{d}{dx}(b^x) = b^x \ln b$

$b^x = \int b^x \ln b \; dx$

$b^x = (\ln b)\int b^x\, dx$

$\int b^x\, dx = \frac{b^x}{\ln b} + C$

Example 1: Differentiate

(a) $f(x) = \sqrt{\ln x}$

$\Rightarrow \frac{d}{dx}\left(\sqrt{\ln x}\right) = \frac{d}{dx}\left(\sqrt{z}\right) = \frac{d}{dz}\left(\sqrt{z}\right)\frac{dz}{dx}$

Let $z = \ln x$

$\Rightarrow \frac{dz}{dx} = \frac{1}{x}$

$= \left(\frac{1}{2} z^{\frac{1}{2}-1}\right)\left(\frac{1}{x}\right)$

$= \frac{1}{2}(\ln x)^{-\frac{1}{2}}\left(\frac{1}{x}\right)$

$= \frac{1}{2x\sqrt{\ln x}}$

(b) $g(x) = \ln(\sin x)$

$\downarrow$

$\frac{d}{dx}\left(\ln(\sin x)\right) = \frac{d}{dx}(\ln z) = \frac{d}{dz}(\ln z)\frac{dz}{dx} = \frac{1}{z}\cos x = \frac{\cos x}{\sin x}$

Let $z = \sin x$

$\Rightarrow \frac{dz}{dx} = \cos x$

$= \cot x$

Let us consider the function $f(x) = \ln|x|$

$\ln x$ has domain $x > 0$

At $x = 0$, $|x| = 0$

$\Rightarrow \ln|x|$ is not defined.

$\Rightarrow x = 0$ is not in the domain of $\ln|x|$

$\Rightarrow$ Domain of $\ln|x|$ is all real numbers except 0.

$(-\infty, 0) \cup (0, \infty)$

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

$x = -4 \Rightarrow |x| = 4 = -(-4)$
$= -x$

$$\ln|x| = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

$$\begin{cases} \dfrac{d}{dx}(\ln x) = \dfrac{1}{x} & (x > 0) \\ \dfrac{d}{dx}(\ln(-x)) = \dfrac{1}{x} & (x < 0) \end{cases}$$

, $\dfrac{d}{dx}(\ln(-x)) = \dfrac{d}{dx}(\ln z)$

Let $z = -x$

$\Rightarrow \dfrac{dz}{dx} = -1$

$= \dfrac{d}{dz}(\ln z)\dfrac{dz}{dx}$

$= \left(\dfrac{1}{z}\right)(-1)$

$= \left(\dfrac{1}{-x}\right)(-1) = \dfrac{1}{x}$

$$\Rightarrow \frac{d}{dx}\left(\ln|x|\right) = \begin{cases} \dfrac{d}{dx}(\ln x) & \text{if } x > 0 \\ \dfrac{d}{dx}(\ln(-x)) & \text{if } x < 0 \end{cases} = \frac{1}{x}$$

$$\boxed{\Rightarrow \frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x} \quad \text{for every } x \neq 0}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Example 2: Evaluate

(a) $\displaystyle\int \frac{x}{x^2+1}\,dx$ $\Rightarrow$ $\displaystyle I = \int \frac{x}{x^2+1}\,dx = \int \frac{1}{x^2+1}\,x\,dx = \int \frac{1}{u}\,\frac{du}{2}$

Let $u = x^2+1$

$\Rightarrow \dfrac{du}{dx} = 2x$

$\Rightarrow du = 2x\,dx$

$\Rightarrow \dfrac{du}{2x} = dx$

$\dfrac{du}{2} = x\,dx$

$$= \int \frac{\cancel{x}}{u}\,\frac{du}{2\cancel{x}} = \int \frac{1}{2u}\,du$$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+1| + C$$

(b) $\displaystyle\int_1^e \frac{\ln x}{x}\,dx = \int_1^e \ln x \,\frac{1}{x}\,dx = \int_{\ln 1}^{\ln e} u\,du$

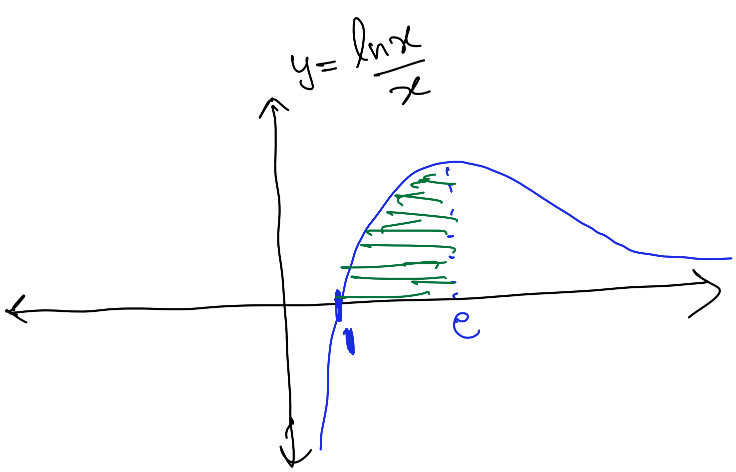
$u = \ln x$

$\Rightarrow \dfrac{du}{dx} = \dfrac{1}{x}$

$\Rightarrow du = \dfrac{1}{x}\,dx$

$$= \int_0^1 u\,du = \left.\frac{u^2}{2}\right|_0^1$$

$$= \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2} - 0 = \frac{1}{2}$$



- **Logarithmic Differentiation**

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. This method is called ***Logarithmic Differentiation***

Example 3: Use **Logarithmic Differentiation** to find the derivative of

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

Step 1: Take logarithm on both sides.

$$\Rightarrow \ln y = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)$$

Step 2: Simplify the RHS using properties of logarithm. (expand)

$$\Rightarrow \ln y = \ln\left(x^{3/4}\sqrt{x^2+1}\right) - \ln\left((3x+2)^5\right)$$

$$= \ln\left(x^{3/4}\right) + \ln\left(\sqrt{x^2+1}\right) - \ln(3x+2)^5$$

$$= \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Step 3: Diff. both the sides w.r.t. x.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left[\frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)\right]$$

$$\Rightarrow \frac{d}{dy}(\ln y)\frac{dy}{dx} = \frac{3}{4}\frac{d}{dx}(\ln x) + \frac{1}{2}\frac{d}{dx}\left(\ln(x^2+1)\right) - 5\frac{d}{dx}\left(\ln(3x+2)\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\left(\frac{1}{x}\right) + \frac{1}{2}\frac{(x^2+1)'}{x^2+1} - 5\frac{(3x+2)'}{3x+2}$$

$$= \frac{3}{4x} + \frac{1}{\cancel{2}}\frac{\cancel{2}x}{x^2+1} - 5\frac{3}{3x+2} = \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2} \Rightarrow \frac{dy}{dx} = y\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$$

Step 4: multiply both sides with y

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{3x+2\ ^5}\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$$

Example Diff. $y = \dfrac{x^2+1}{(\sqrt{x+1})\,x^{5/4}}$

$$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{f'(x)}{f(x)}$$

Step 1: $\ln y = \ln\left(\dfrac{x^2+1}{\sqrt{x+1}\;x^{5/4}}\right)$

Step 2: $\ln y = \ln(x^2+1) - \ln\left(\sqrt{x+1}\;x^{5/4}\right)$

$$= \ln(x^2+1) - \ln\left(\sqrt{x+1}\right) - \ln x^{5/4}$$

$$\Rightarrow \ln y = \ln(x^2+1) - \frac{1}{2}\ln(x+1) - \frac{5}{4}\ln(x)$$

Step 3: $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{d}{dx}\left[\ln(x^2+1)\right] - \dfrac{1}{2}\dfrac{d}{dx}\left[\ln(x+1)\right] - \dfrac{5}{4}\dfrac{d}{dx}\left[\ln(x)\right]$

$$= \frac{(x^2+1)'}{x^2+1} - \frac{1}{2}\frac{(x+1)'}{x+1} - \frac{5}{4}\frac{(x)'}{x}$$

$$= \frac{2x}{x^2+1} - \frac{1}{2}\frac{1}{x+1} - \frac{5}{4}\frac{1}{x} = \frac{2x}{x^2+1} - \frac{1}{2(x+1)} - \frac{5}{4x}$$

Step 4: $\dfrac{dy}{dx} = y\left[\dfrac{2x}{x^2+1} - \dfrac{1}{2(x+1)} - \dfrac{5}{4x}\right]$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2+1)}{(\sqrt{x+1})(x^{5/4})}\left[\frac{2x}{x^2+1} - \frac{1}{2(x+1)} - \frac{5}{4x}\right]$$

$(\quad)$

Example 4: Differentiate

$$\frac{d}{dx}\left[\ln(f(x))\right] = \left(\frac{d}{dz}\ln z\right)\frac{dz}{dx}$$

where $z = f(x)$ $= \frac{1}{z} f'(x)$

$$\frac{d}{dx}\left[\ln(f(x))\right] = \frac{f'(x)}{f(x)}$$

(a) $y = x^2$

Step 1: $\ln(y) = \ln(x^2)$

Step 2: $\ln(y) = 2\ln x$

Step 3: $\frac{1}{y}\frac{dy}{dx} = 2\left(\frac{1}{x}\right) = \frac{2}{x} \Rightarrow \frac{dy}{dx} = y\left(\frac{2}{x}\right)$

$\Rightarrow \frac{dy}{dx} = x^2 \frac{2}{x} = 2x$

(b) $y = e^x$

Step 1: $\ln y = \ln e^x$

Step 2: $\ln y = x \ln e = x$

Step 3: $\longrightarrow \frac{1}{y}\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = y \Rightarrow \frac{dy}{dx} = e^x$

(c) $y = x^{\sqrt{x}}$

$\downarrow$

$y = x^{\sqrt{x}}$

Step 1: $\ln y = \ln\left(x^{\sqrt{x}}\right)$

Step 2: $\ln y = \sqrt{x}\ln x$

Step 3: $\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\sqrt{x}\ln x\right)$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \sqrt{x}(\ln x)' + (\sqrt{x})'\ln x$$

$$= \sqrt{x}\,\frac{1}{x} + \left(\frac{1}{2}x^{1/2-1}\right)\ln x$$

$$= \frac{x^{1/2}}{x^1} + \left(\frac{1}{2}x^{-1/2}\right)\ln x$$

$$= \frac{1}{x^{1-1/2}} + \frac{1}{2x^{1/2}}\ln x = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{2 + \ln x}{2\sqrt{x}}$$

Step 4: $\frac{dy}{dx} = y\frac{(2+\ln x)}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = x^{\sqrt{x}}\frac{(2+\ln x)}{2\sqrt{x}}$

Example 5: Evaluate

(a) $\dfrac{d}{dx}(\log_2 x)$ $= \dfrac{d}{dx}\left(\dfrac{\ln x}{\ln 2}\right) = \dfrac{1}{\ln 2}\dfrac{d}{dx}(\ln x)$

$= \dfrac{1}{\ln 2}\dfrac{1}{x} = \dfrac{1}{x\ln 2}$

$$\boxed{\dfrac{d}{dz}(b^z) = b^z \ln b}$$

(b) $\dfrac{d}{dx}(2^{2x})$ $= \dfrac{d}{dx}(2^{2x}) = \dfrac{d}{dx}(2^z)$

Let $z = 2x$

$\dfrac{dz}{dx} = 2$

$= \dfrac{d}{dz}(2^z)\dfrac{dz}{dx} = (2^z \ln 2)\, 2$

$= (2^{2x} \ln 2)\, 2$

$= 2^{2x+1} \ln 2$

Alternatively

$\dfrac{d}{dx}\left((2^2)^x\right) = \dfrac{d}{dx}(4^x) = 4^x \ln 4 = 4^x \ln 2^2 = 4^x (2\ln 2)$

$= 2^{2x}(2\ln 2)$

$= 2^{2x+1}\ln 2$

(c) $\displaystyle\int 2^x\,dx$

$= \dfrac{2^x}{\ln 2} + C$

Section 6.4 exercises, *page 436*, #3, <u>5</u>, 7, 9, 10, 11, 15, <u>13</u>, <u>35</u>, 43, 45, <u>47</u>, 49, 51, <u>53</u>, 71, 73, 75, 77, 78, 80. Hint on #13: rewrite $G(y)$ first.