Logarithms of Products

ESSENTIALS

The Product Rule for Logarithms

For any positive numbers M, N, and a ($a \neq 1$),

$$\log_a(MN) = \log_a M + \log_a N.$$

(The logarithm of a product is the sum of the logarithms of the factors.)

$\log_2 4 + \log_2 5 = \log_2 20$

$\log_3 4 + \log_2 5 \neq \log_2 20$ or $\log_3 20$

Examples

- Express $\log_4(12 \cdot 3)$ as an equivalent expression that is a sum of logarithms.

  $$\log_4(12 \cdot 3) = \log_4 12 + \log_4 3$$

- Express $\log_a 3 + \log_a 5$ as an equivalent expression that is a single logarithm.

  $$\log_a 3 + \log_a 5 = \log_a(3 \cdot 5) = \log_a 15$$

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EXAMPLE 1	YOUR TURN 1
Express $\log_2(4 \cdot 16)$ as an equivalent expression that is a sum of logarithms. $\log_2(4 \cdot 16) = \boxed{\log_2 4} + \log_2 16$	Express $\log_4(16 \cdot 64)$ as an equivalent expression that is a sum of logarithms. $\log_4(16 \cdot 64) = \log_4 16 + \log_4 64$
EXAMPLE 2	**YOUR TURN 2**
Express $\log_t(5 \cdot 6)$ as an equivalent expression that is a sum of logarithms. $\log_t(5 \cdot 6) = \log_t 5 + \boxed{\log_t 6}$	Express $\log_x(12 \cdot 14)$ as an equivalent expression that is a sum of logarithms. $\log_x(12 \cdot 14) = \log_x 12 + \log_x 14$
EXAMPLE 3	**YOUR TURN 3**
Express $\log_b 2 + \log_b 18$ as an equivalent expression that is a single logarithm. $\log_b 2 + \log_b 18 = \log_b(\boxed{2 \times 18})$ $= \log_b 36$	Express $\log_a 25 + \log_a 16$ as an euivalent expression that is a single logarithm. $\log_a 25 + \log_a 16 = \log_a(25 \times 16)$ $= \log_a 400$

EXAMPLE 4	YOUR TURN 4
Express $\log_t A + \log_t B$ as an equivalent expression that is a single logarithm. $\log_t A + \log_t B = \log_t\left(\boxed{AB}\right)$	Express $\log_a R + \log_a T$ as an equivalent expression that is a single logarithm. $\log_a R + \log_a T = \log_a(RT)$

YOUR NOTES Write your questions and additional notes.

Logarithms of Powers

ESSENTIALS

The Power Rule for Logarithms

For any positive numbers M and a ($a \neq 1$), and any real number p,

$$\log_a M^p = p \cdot \log_a M.$$

(The logarithm of a power of M is the exponent times the logarithm of M.)

$\log_a x$ is defined only for positive x

Example

- Express $\log_a 10^4$ as an equivalent expression that is a product.

$$\log_a 10^4 = 4\log_a 10$$

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EXAMPLE 1	YOUR TURN 1
Express $\log_b t^{-6}$ as an equivalent expression that is a product. $\log_b t^{-6} = \boxed{-6}\log_b t$	Express $\log_a c^{-12}$ as an equivalent expression that is a product. $\log_a c^{-12} = -12\log_a c$
EXAMPLE 2	**YOUR TURN 2**
Express $\log_4 c^{1/2}$ as an equivalent expression that is a product. $\log_4 c^{1/2} = \boxed{\frac{1}{2}}\log_4 c$	Express $\log_e M^{2/3}$ as an equivalent expression that is a product. $\log_a M^{2/3} = \frac{2}{3}\log_a M$
EXAMPLE 3	**YOUR TURN 3**
Express $\log_9 \sqrt[4]{x}$ as an equivalent expression that is a product. $\log_9 \sqrt[4]{x} = \log_9 x^{\boxed{1/4}}$ Writing exponential notation $= \boxed{\frac{1}{4}}\log_9 x$	Express $\log_5 \sqrt[6]{y}$ as an equivalent expression that is a product. $\log_5 \sqrt[6]{y} = \log_5 y^{1/6} = \frac{1}{6}\log_5 y$

YOUR NOTES Write your questions and additional notes.

Logarithms of Quotients

ESSENTIALS

The Quotient Rule for Logarithms

For any positive numbers M, N, and a ($a \neq 1$),

$$\log_a \frac{M}{N} = \log_a M - \log_a N.$$

(The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.)

Examples

- Express $\log_3 \frac{6}{7}$ as an equivalent expression that is the difference of logarithms.

  $$\log_3 \frac{6}{7} = \log_3 6 - \log_3 7$$

- Express $\log_a 12 - \log_a 13$ as an equivalent expression that is a single logarithm.

  $$\log_a 12 - \log_a 13 = \log_a \frac{12}{13}$$

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EXAMPLE 1	YOUR TURN 1
Express $\log_a \frac{b}{c}$ as an equivalent expression that is the difference of logarithms. $\log_a \frac{b}{c} = \log_a b - \log_a \boxed{c}$	Express $\log_t \frac{R}{S}$ as an equivalent expression that is the difference of logarithms. $\log_t \frac{R}{S} = \log_t R - \log_t S$
EXAMPLE 2	**YOUR TURN 2**
Express $\log_a x - \log_a y$ as an equivalent expression that is a single logarithm. $\log_a x - \log_a y = \log_a \boxed{\frac{x}{y}}$	Express $\log_t Q - \log_t R$ as an equivalent expression that is a single logarithm. $\log_t Q - \log_t R = \log_t \frac{Q}{R}$

YOUR NOTES Write your questions and additional notes.

Using the Properties Together

ESSENTIALS

The Logarithm of the Base to an Exponent

For any base a, $\log_a a^k = k$.

Examples

- Express $\log_b \dfrac{x^2}{y^3 z}$ as an equivalent expression, using the individual logarithms of x, y, and z.

$$\begin{aligned}
\log_b \frac{x^2}{y^3 z} &= \log_b x^2 - \log_b y^3 z && \text{Quotient rule}\\
&= 2\log_b x - \log_b y^3 z && \text{Power rule}\\
&= 2\log_b x - \left(\log_b y^3 + \log_b z\right) && \text{Product rule}\\
&= 2\log_b x - \left(3\log_b y + \log_b z\right) && \text{Power rule}\\
&= 2\log_b x - 3\log_b y - \log_b z
\end{aligned}$$

- Express $\frac{1}{2}\log_b x - 3\log_b y + \log_b z$ as an equivalent expression that is a single logarithm.

$$\begin{aligned}
\frac{1}{2}\log_b x - 3\log_b y + \log_b z &= \log_b x^{1/2} - \log_b y^3 + \log_b z && \text{Power rule}\\
&= \left(\log_b \sqrt{x} - \log_b y^3\right) + \log_b z && x^{1/2} = \sqrt{x}\\
&= \log_b \frac{\sqrt{x}}{y^3} + \log_b z && \text{Quotient rule}\\
&= \log_b \frac{z\sqrt{x}}{y^3} && \text{Product rule}
\end{aligned}$$

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EXAMPLE 1

Express $3\log_a x + \frac{1}{2}\log_a y$ as an equivalent expression that is a single logarithm.

$$\begin{aligned}
3\log_a x + \frac{1}{2}\log_a y &= \log_a x^3 + \log_a y^{\boxed{1/2}} && \text{Power rule}\\
&= \log_a x^3 + \log_a \sqrt{y}\\
&= \log_a \left(\boxed{x^3}\sqrt{y}\right) && \text{Product rule}
\end{aligned}$$

YOUR TURN 1

Express $\frac{1}{3}\log_a x - \frac{2}{3}\log_a y$ as an equivalent expression that is a single logarithm.

$\log_a x^{1/3} - \log_a y^{2/3}$

$= \log_a \dfrac{x^{1/3}}{y^{2/3}}$

$\| $

$\log_a \dfrac{\sqrt[3]{x}}{\sqrt[3]{y^2}}$

$\frac{1}{2}\log_a y = \log_a y^{1/2}$

$3\log_a x = \log_a x^3$

$\log_a M + \log_a N = \log_a MN$

x^3 $\sqrt{y}$

EXAMPLE 2	YOUR TURN 2
Express $\log_a \sqrt[5]{\frac{x^4y^2}{z}}$ as an equivalent expression, using the individual logarithms x, y, and z. $\log_a \sqrt[5]{\frac{x^4y^2}{z}} = \log_a\left(\frac{x^4y^2}{z}\right)^{1/5}$ Exponential notation $= \frac{1}{5}\cdot\log_a \frac{x^4y^2}{z}$ Power rule $= \frac{1}{5}\left(\log_a(x^4y^2) - \boxed{\log_a z}\right)$ Quotient rule $= \frac{1}{5}\left(\log_a x^4 + \boxed{\log_a y^2} - \log_a z\right)$ Product rule $= \frac{1}{5}\left(4\boxed{\log_a x} + 2\boxed{\log_a y} - \log_a z\right)$ Power rule	Express $\log_a \sqrt[3]{\frac{b}{c^6d^7}}$ as an equivalent expression, using the individual logarithms b, c, and d. $\log_a\left(\frac{b}{c^6d^7}\right)^{1/3} = \frac{1}{3}\log_a \frac{b}{c^6d^7}$ $= \frac{1}{3}\left[\log_a b - \log_a c^6d^7\right]$ $= \frac{1}{3}\left[\log_a b - (\log_a c^6 + \log_a d^7)\right]$ $= \frac{1}{3}\left[\log_a b - \log_a c^6 - \log_a d^7\right]$ $= \frac{1}{3}\left[\log_a b - 6\log_a c - 7\log_a d\right]$ $= \frac{1}{3}\log_a b - 2\log_a c - \frac{7}{3}\log_a d$
EXAMPLE 3	**YOUR TURN 3**
Given $\log_a 4 = 0.602$ and $\log_a 3 = 0.477$, find $\log_a 12$. $\log_a 12 = \log_a(4\cdot3)$ $= \boxed{\log_a 4} + \log_a 3$ $= 0.602 + 0.477$ $= \boxed{1.079}$	Given $\log_a 2 = 0.301$ and $\log_a 5 = 0.699$, find $\log_a \frac{2}{5}$. $= \log_a 2 - \log_a 5$ $= 0.301 - 0.699$ $= -0.398$
EXAMPLE 4	**YOUR TURN 4**
Simplify: $\log_4 4^{-6}$. $= -6\underbrace{\log_4 4}_{m}$ $\quad \log_4 4 = m$, $4^m = 4^1$, $m = 1$ $\log_4 4^{-6} = \boxed{-6}$	Simplify: $\log_6 6^{1.5}$. $\log_6 6^{1.5} = 1.5\log_6 6 = 1.5$

YOUR NOTES Write your questions and additional notes.

Practice Exercises

Readiness Check

Match each expression with an equivalent expression from the column on the right.

1. $\log_3 7^4$

2. $\log_3 \frac{5}{7}$

3. $\log_3 7 + \log_3 4$

4. $\log_3 35$

5. $\log_3 3^7$

A. $\log_3 28$

B. 7

C. $\log_3 5 + \log_3 7$

D. $4\log_3 7$

E. $\log_3 5 - \log_3 7$

Logarithms of Products

Express as an equivalent expression that is a sum of logarithms.

6. $\log_2 (4 \cdot 8)$

7. $\log_a (xyz)$

8. $\log_b (5ac)$

Express as an equivalent expression that is a single logarithm.

9. $\log_a 2 + \log_a 15$

10. $\log_t R + \log_t M$

11. $\log_a 4 + \log_a c$

Logarithms of Powers

Express as an equivalent expression that is a product.

12. $\log_a x^6$

13. $\log_b M^{-3}$

14. $\log_{12} x^{1/3}$

Logarithms of Quotients

Express as an equivalent expression that is a difference of two logarithms.

15. $\log_3 \frac{13}{15}$

16. $\log_5 \frac{3}{4}$

17. $\log_a \frac{t}{s}$

Express as an equivalent expression that is a single logarithm.

18. $\log_b 17 - \log_b 21$ **19.** $\log_a 25 - \log_a 8$ **20.** $\log_b R - \log_b T$

Using the Properties Together

Express as an equivalent expression, using the individual logarithms of w, x, y, and z.

21. $\log_a\left(x^2 z^{-4}\right)$ **22.** $\log_a \frac{w^2}{y^4 z^3}$ **23.** $\log_a \sqrt[3]{\frac{x^6 y^3}{w^4 z^5}}$

(23) $\log_a \sqrt[3]{\frac{x^6y^3}{w^4z^5}} = \frac{1}{3}\log_a \frac{x^6y^3}{w^4z^5} = \frac{1}{3}\log_a x^6y^3 - \frac{1}{3}\log_a w^4z^5$

$= \frac{1}{3}\log_a x^6 + \frac{1}{3}\log_a y^3 - \frac{1}{3}\log_a w^4 - \frac{1}{3}\log_a z^5$

$= 2\log_a x + \log_a y - \frac{4}{3}\log_a w - \frac{5}{3}\log_a z$

Express as an equivalent expression that is a single logarithm and, if possible, simplify.

24. $3\log_a x + \frac{1}{3}\log_a y$ **25.** $3\log_a x - 3\log_a \sqrt[3]{x}$

$a^2 - b^2 = (a-b)(a+b)$

$= \log_a x^3 - \log_a(\sqrt[3]{x})^3$

$= \log_a \frac{x^3}{(\sqrt[3]{x})^3} = \log_a \frac{x^3}{x} = \log_a x^2$

26. $\log_a(4x-16) - \log_a\left(x^2-16\right)$

$= \log_a \frac{4x-16}{x^2-16} = \log_a \frac{4\cancel{(x-4)}}{\cancel{(x-4)}(x+4)} = \log_a \frac{4}{x+4}$

Given $\log_a 2 = 0.301$ *and* $\log_a 7 = 0.845$*, if possible, use the properties of logarithms to calculate values for each of the following.*

27. $\log_a 14$ **28.** $\log_a 8$ **29.** $\log_a 9$

Simplify.

30. $\log_a a^5$ **31.** $\log_t t^{-2}$ **32.** $\log_m m^c$

(31) $\log_t t^{-2} = -2\log_t t = -2$