Chapter 3 Introduction to Logic

Our goal in this chapter is use logic to determine the validity of an argument.

What is Logic?

- **Logic** is the analysis, without regard to meaning or context, of the patterns of reasoning by which conclusions are validly derived from a set of premises. *Your gut instinct might not always agree*

Section 3.1 Statements and Quantifiers

Statements

- A **statement** is a declarative statement that is known to be either true or false (but not both).
- **Logic** will be used to determine the **truth value** of a statement.

Ex. Which of these are statements? Determine the truth value of each statement.

These are statements -- the truth values can be determined

- Today is Saturday. False
- I will sleep in this morning False
- The sky is full of clouds False
- $2 + 3 = 6$ False
- The cube root of 8 is 2. True

These are not statements -- they are either commands, questions, paradoxes or statements for which the truth value is undetermined

- $2 + 3 = x$
- Will it rain tomorrow?
- Do your homework!
- $x < 7$ → don't know what x is.
- This sentence is false → Paradox

Logical Connectives and Compound Statements

Negations (aka "not", or "is not the case that")

- Negating a statement involves rewriting a statement in a way that changes its truth value.
- Symbolically, "not p" is written $\sim p$
- If statement p is true, then $\sim p$ is false.
- If p is false, then $\sim p$ is true.

Writing negations carefully…

Ex. Negate the statements

a. $P =$ The moon is made of cheese

$\sim P =$ The moon is not made of Cheese.

b. $P =$ There is not a test next Wednesday

$\sim P =$ There is a test next wednesday.

c. $P = x > 5$

$\sim P = x \not> 5$ or $x \leq 5$

Assume you know what x, y are.

d. $P = y \leq 0$

$\sim P = y \not\leq 0$ or $y > 0$

e. $P =$ I have at least \$10 in my ~~wallet~~

$\sim P =$ I have less than \$10 in my wallet.

f. $P =$ The student score on the test was above 80 percent.

$\sim P =$ The student score on the test was less than or equal to 80 percent.

$\sim p$ = The student score on the test was atmost 80 Percent.

Compound Statements

Conjunctions (aka "and")

- When two statements are joined by ***and***, we call it a **conjunction**
- Symbolically, ***p and q*** is written… $p \wedge q$
- $\boldsymbol{p \wedge q}$ is **true** only if **both** simple statements are true. It is false if at least one statement is false

Disjunctions (aka "or")

- When the two statements are joined by ***or***, we call it a **disjunction.**
- Symbolically, ***p or q*** is written… $p \vee q$
- $\boldsymbol{p \vee q}$ is **true** as long as **at least one** of the simple statements are true. It is false if both statements are false

Ex. Let p represent "It is raining," and let q represent "It is March." Write each symbolic statement in words.

a. $p \vee q$

It is raining or it is March.

b. $\sim p \wedge q$

It is not raining and it is March.

c. $\sim(p \wedge q)$ $p \wedge q$ = It is raining and it is march.

It is not the case that, it is raining and it is march.

Ex. Write each statement in symbolic form.

Let p: Today is Saturday and q: I will sleep in this morning.

- *Today is not Saturday*

  $\sim p$

- *Today is Saturday and I will sleep in this morning.*

  $p \wedge q$

- *Today is Saturday or I will not sleep in this morning..*

  $p \vee \sim q$ or $p \vee (\sim q)$

Quantifiers

- The words *all*, *each*, *every*, and *no*(*ne*) are called **universal quantifiers**
- Words and phrases such as *some*, *there exists*, and (*for*) *at least one* are called **existential quantifiers.**
- Quantifiers are used extensively in mathematics to indicate *how many* cases of a particular situation exist.

Negations of Quantified Statements

negation

1) For all x, $P(x)$ is true

Statement	Negation
All do.	Some do not.
Some do.	None do.

There exists an x for which $P(x)$ is false

2) There exists an x for which $P(x)$ is true.

For all x, $P(x)$ is false

same

negation

$P(x)$ is true for none x

Ex. Negate the statements

a. P= All students do their homework.

$\sim P$ = Some students do not do their homework.

b. P= No one answered all of the questions correctly.

rewrite. For all, all questions were not answered correctly.

$\sim P$= Some answered ~~some of~~ all questions ~~incorrectly~~ correctly.

c. Some cats like to take baths.

No cat like to take baths.

Deciding Whether the Quantified Statements are True or False

Ex. Decide of the statements are true or false

a. All irrational numbers are real numbers. True

b. No square roots are rational numbers. False

c. Every integer is a natural number. False

d. Some rational numbers are integers. True

e. There exists a whole number that is not a natural number. True

Section 3.2 Truth Tables and Equivalent Statements

Negations (aka "not", or "is not the case that")

- $\sim p$ is **false** when p is **true** and $\sim p$ is **true** when p is **false**

Conjunctions (aka "and")

- $p \wedge q$ is **true** only if **both** simple statements are true. It is **false** if **at least one statement** is **false**

Disjunctions (aka "or")

- $p \vee q$ is **true** as long as **at least one** of the simple statements are true. It is **false** if **both statements** are **false**

Using truth tables to determine the validity (true or false) of a statement

- A **truth table** is an organized approach to determining truth values of compound statements
- Use **letters to represent** simple **statements.**
- For a compound statement having n **simple statements**, the truth table will have 2^n **rows**

Ex. *When working with statements p, q and r, there will be* $2^3 = 8$ *rows.*

- The number of columns will vary.

Ex. Complete the truth table for the given statements.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \vee q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	F	F

Ex. Find the truth table for $p \wedge \sim q$ and $\sim(p \wedge \sim q)$

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

$\sim(\sim q) = q$

Logically Equivalent Statements

- Two statements are **logically equivalent** if their truth values are identical under identical truth conditions of the simple statements.

Identical

Ex. Are $\sim(p \wedge q)$ and $\sim p \vee \sim q$ logically equivalent? YES

p	**q**	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

$\boxed{\sim(p \wedge q) \equiv \sim p \vee \sim q}$

Identical

Ex. Are $\sim(p \vee q)$ and $\sim p \wedge \sim q$ logically equivalent? YES

p	**q**	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

$\boxed{\sim(p \vee q) \equiv \sim p \wedge \sim q}$

DeMorgan's Laws

- Negation of a Conjunction: $\sim(p \wedge q) \equiv \sim p \vee \sim q$
- Negation of a Disjunction: $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Note: The symbol ≡ means "is equivalent to"

Ex. Negate the statement: *We have a test next week or will cancel class.*

We do not have a test next week but and I will not cancel class.

Ex. Negate the statement: *I like the opera, but I do not like to ski.* and

I do not like the opera or I like to ski.

Section 3.3 Conditional Statements (aka "If..., then...")

Conditional Statement

- When two statements are written as an "If…, then…" statement, we call it a **conditional statement**. We say p is the **premise** and q is the **conclusion**.
- Symbolically, **if p, then q** is written $p \to q$.
- $p \to q$, is **true** when either **p is true and q is true** or **p is false**. It is **false if p is true and q is false.**

Ex. Find the truth table for $p \to q$, $q \to p$, $\sim p \to q$ and $p \to \sim q$

p	q	~p	~q	$p \to q$	$q \to p$	$\sim p \to q$	$p \to \sim q$
T	T	F	F	T	T	T	F
T	F	F	T	F	T	T	T
F	T	T	F	T	F	T	T
F	F	T	T	T	T	F	T

Ex. Construct the truth table for $(p \vee \sim q) \to q$

p	q	~q	$p \vee \sim q$	$(p \vee \sim q) \to q$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

p	q	or $p \vee q$	and $p \wedge q$	implies $p \to q$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	T
F	F	F	F	T

$\vee$ or
$\wedge$ and
$\rightarrow$ implies

Conditional Statements, Equivalent Statements and Negations

Ex. Are $p \rightarrow q$ and $\sim p \vee q$ logically equivalent? YES

identical

p	q	$\sim p$	$p \rightarrow q$	$\sim p \vee q$		
T	T	F	T	T		
T	F	F	F	F		
F	T	T	T	T		
F	F	T	T	T		

$\boxed{p \rightarrow q \equiv \sim p \vee q}$

Ex. Are $p \rightarrow q$ and $p \wedge \sim q$ logically equivalent? NO

$p \rightarrow q \equiv \sim(p \wedge \sim q)$

p	q	$\sim q$	$p \rightarrow q$	$p \wedge \sim q$		
T	T	F	T	F		
T	F	T	F	T		
F	T	F	T	F		
F	F	T	T	F		

Tautologies and Contradictions

- A statement is a **tautology** if it **true** under all conditions.
- A statement is a **contradiction** if it **false** under all conditions.

Ex. Determine if the statement is a tautology, contradiction or neither.

If it is Saturday, then I will sleep in late, or if I sleep in late, then it is Saturday.

p: It is Saturday

q: I'll sleep in late

If p then q, or if q then p $\equiv p \rightarrow q$, or $q \rightarrow p$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \rightarrow q \vee q \rightarrow p$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

<u>Tautology</u>

Section 3.4 The Conditional and Related Statements

The Converse, Contrapositive and Inverse

Ex. Construct the truth table for $p \to q$, $q \to p$, $\sim p \to \sim q$ and $\sim q \to \sim p$

Identical

p	q	$\sim p$	$\sim q$		$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	F	F		T	T	T	T
T	F	F	T		F	T	T	F
F	T	T	F		T	F	F	T
F	F	T	T		T	T	T	T

Identical

We call $\sim \mathbf{q} \to \sim \mathbf{p}$ *the **contrapositive**; it is logically equivalent to* $p \to q$

We call $\mathbf{q} \to \mathbf{p}$ *the **converse**; in general is it not equivalent to* $p \to q$.

We call $\sim \mathbf{p} \to \sim \mathbf{q}$ *the **inverse**; it is equivalent to* $q \to p$.

Ex. $\underbrace{\text{If the sky is blue}}_{p}$, then $\underbrace{\text{the sun is shining}}_{q}$. Write the

a. contrapositive

If the sun is not shining then the sky is not blue.

b. converse

If the sun is shining then the sky is blue.

c. inverse

If the sky is not blue then the sun is not shining.

Bi-Conditional Statements (aka "...,if and only if,...")

- When two statements are written as an "..., if and only if,..." statement, we call it a **bi-conditional** statement.
- Symbolically, **p, if and only if, q** is written $\boldsymbol{p \leftrightarrow q}$
- $\boldsymbol{p \leftrightarrow q}$ is **true** when p and q are **both true** or **both false.** It is false if the statements differ in value.

Ex. Construct the truth table for $p \leftrightarrow q$, $\sim p \leftrightarrow \sim q$, $\sim p \leftrightarrow q$, and $p \leftrightarrow \sim q$

identical identical

p	q	$\sim p$	$\sim q$		$p \leftrightarrow q$	$\sim p \leftrightarrow \sim q$	$\sim p \leftrightarrow q$	$p \leftrightarrow \sim q$
T	T	F	F		T	T	F	F
T	F	F	T		F	F	T	T
F	T	T	F		F	F	T	T
F	F	T	T		T	T	F	F

$\boxed{p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q}$ $\boxed{\sim p \leftrightarrow q \equiv p \leftrightarrow \sim q}$

Ex. Construct the truth table for $(\sim p \wedge q) \leftrightarrow q$

p	q	$\sim p$	$\sim p \wedge q$	$(\sim p \wedge q) \leftrightarrow q$
T	T	F	F	F
T	F	F	F	T
F	T	T	T	T
F	F	T	F	T

✱ $\sim$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$

		or	and	implies	if and only if
p	q	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	T	F
F	F	F	F	T	T

Section 3.6 Arguments

Analyzing arguments…

- An argument consists of a set of premises (P_1, P_2, ..., P_n) and a conclusion (Q).
- The argument is **valid** if $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow Q$ is a **tautology**; otherwise, the argument is invalid (or a **fallacy**).

Ex. Determine if the argument is valid.

$p \wedge \sim q$
p Premises
Conclusion $\sim q$

$\boxed{((p \wedge \sim q) \wedge p) \rightarrow \sim q}$ → Tautology or not?

5th column → 3rd column

p	q		$\sim q$	$p \wedge \sim q$	$(p \wedge \sim q) \wedge p$	$((p \wedge \sim q) \wedge p) \rightarrow \sim q$
T	T		F	F	F	T
T	F		T	T	T	T
F	T		F	F	F	T
F	F		T	F	F	T

The argument is VALID.

Ex. Determine if the argument is valid.

Premises $p \rightarrow \sim q$
$\sim p$
Conclusion. $\sim q$

$r = (\underbrace{(p \rightarrow \sim q)}_{P_1} \wedge \underbrace{\sim p}_{P_2}) \rightarrow \underbrace{\sim q}_{Q}$ → check tautology or not?

$P_1 \wedge P_2 \rightarrow Q$
||

p	q		$\sim p$	$\sim q$	$p \rightarrow \sim q$	$(p \rightarrow \sim q) \wedge \sim p$	r
T	T		F	F	F	F	T
T	F		F	T	T	F	T
F	T		T	F	T	T	F
F	F		T	T	T	T	T

The argument is INVALID or FALLACY.

$T \rightarrow F \equiv F$
$T \rightarrow T \equiv T$
$F \rightarrow T \equiv T$
$F \rightarrow F \equiv T$

Ex. Determine if the argument is valid

If it rains, then the baseball game will be postponed.
The game was postponed.
Therefore, it rained

$p \to q$
q
$\overline{\quad p \quad}$

Check tautology or not

Write the argument in symbolic form: $((p \to q) \wedge q) \to p$

p: It rains
q: Baseball game is postponed

p	q		$p \to q$	$(p \to q) \wedge q$	$((p \to q) \wedge q) \to p$
T	T		T	T	T
T	F		F	F	T
F	T		T	T	F
F	F		T	F	T

The argument is INVALID or FALLACY.

Ex. Determine if the argument is valid

I will go to the movies, or I will play miniature golf. $\equiv p \vee q$
I'm not going to the movies. $\equiv \sim p$ } premises
Therefore, I will play miniature golf $\equiv q$ conclusion

Write the argument in symbolic form: $((p \vee q) \wedge \sim p) \to q$

p: I'll go to the movies.
q: I'll play miniature golf.

p	q		$\sim p$	$p \vee q$	$(p \vee q) \wedge \sim p$	
T	T		F	T	F	T
T	F		F	T	F	T
F	T		T	T	T	T
F	F		T	F	F	T

The argument is VALID

Quiz 19

① A sample of contaminated water with $pH = 6.3$
Find the Hydrogen ion concentration.

$$pH = -\log [H^+] \Rightarrow 6.3 = -\log [H^+]$$

$$\Rightarrow \log [H^+] = -6.3 \Rightarrow [H^+] = 10^{-6.3} = 5.01 \times 10^{-7}$$

② A radioactive sample of uranium decays exponentially with a rate of $K = 0.1\ s^{-1}$. Find the half life of the sample.

$$P(t) = P_0 e^{-kt} \Rightarrow P(t) = P_0 e^{-0.1t}$$

Find t for which $P(t) = 0.5 P_0$

$$0.5 P_0 = P_0 e^{-0.1t} \Rightarrow 0.5 = e^{-0.1t}$$

$$\Rightarrow \ln 0.5 = \ln e^{-0.1t} \qquad \ln e = \log_e e = 1$$

$$\Rightarrow \ln 0.5 = -0.1t \underbrace{\ln e}_{1} \Rightarrow -0.1t = \ln 0.5$$

$$\Rightarrow t = \frac{\ln 0.5}{-0.1} = \frac{-0.693}{-0.1} \Rightarrow \underline{\underline{t = 6.93\ s}}$$

Math11000 Section 3962 Quiz 20

Summer 2023, June 14

Name: [5 pt]

Problem 1: Let $f(x) = x^2 + 1$, $g(x) = x + 2$. Find $(f \circ g)(x)$. [5 pts]

Problem 2: Let $f(x) = 2x + 1$. Find the formula for inverse of f. [5 pts]

$\hookrightarrow y = 2x+1$. Interchange x and y.

$\Rightarrow x = 2y+1 \Rightarrow x - 1 = 2y \Rightarrow 2y = x-1$

Solve for y

$\Rightarrow y = \frac{x-1}{2} = \frac{1}{2}x - \frac{1}{2}$

$f^{-1}(x) = \frac{x}{2} - \frac{1}{2}$

$\to (f\circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1$

$(a+b)^2 = a^2+b^2+2ab$

$f(g(x)) = (g(x))^2+1 = (x+2)^2+1$

$= x^2 + 2^2 + 2\cdot x \cdot 2 + 1$

$= x^2 + 4 + 4x + 1$

$= x^2 + 4x + 5$

$(g\circ f)(x) = g(f(x)) = g(x^2+1) = x^2+1+2 = x^2+3$

$g(f(x)) = f(x)+2 = x^2+1+2 = x^2+3$

$f(x) = 2x+1, \quad g(x) = x^3$

$(f\circ g)(x) = f(g(x)) = 2g(x)+1 = 2x^3+1$

$f(g(x)) = f(x^3) = 2x^3+1$

$(g\circ f)(x) = g(f(x)) = g(2x+1) = (2x+1)^3$

$(f\circ f)(x) = f(f(x)) = f(2x+1) = 2(2x+1)+1 = 4x+2+1 = 4x+3$

$(g\circ g)(x) = g(g(x)) = g(x^3) = (x^3)^3 = x^9$