M16600 Lecture Notes

Section 6.8: Indeterminate Forms and L'Hospital's Rule

■ **Section 6.8** exercises, 499 *page* : #9, 15, 19, 21, 27, 35, 37, 43, 47, 52, 53, 57, 59, 65.
Optional: Practice more problems from #8 to #68.

GOALS: Use L'Hospital's Rule to compute the limit of the following ***indeterminate form***

LH Rules

- **Indeterminate Quotient**: $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$
- **Indeterminate Product**: $0 \cdot \infty$
- **Indeterminate Difference**: $\infty - \infty$
- **Indeterminate Power**: 0^0, ∞^0, 1^∞

The Intuition of a Limit Statement: $\lim_{x \to 1}(x^2 + 2) = 3$. This equation states that as x approaches 1 (from the left and the right side of 1), the values of $x^2 + 2$ approaches ________.

Some Notation:

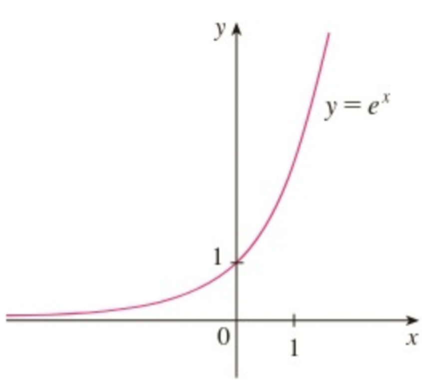
$x \to 1^+$ means x approaches 1 from the RIGHT, i.e., x is slightly BIGGER than 1 (e.g., $x = 1.01, 1.000012$, etc.)

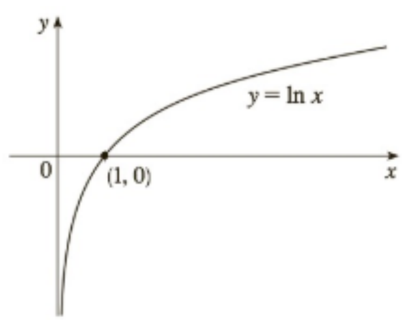
$x \to 1^-$ means x approaches 1 from the LEFT, i.e., x is a little SMALLER than 1 (e.g., $x = 0.99, 0.999999$, etc.)

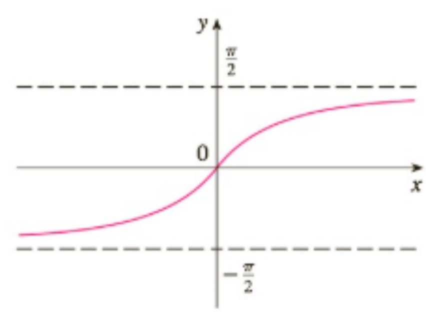
$x \to 1$ means x approaches 1 from both directions, left and right (i.e., x can take any values slightly less than or bigger than 1)

Warning: 1^- does NOT mean -1.

Limit Facts about e^x, $\ln x$, and $\arctan(x)$







$\lim_{x \to \infty} e^x = \infty$

$\lim_{x \to -\infty} e^x = 0$

$\lim_{x \to \infty} \ln x = \infty$

$\lim_{x \to 0^+} \ln x = -\infty$

$\lim_{x \to \infty} \arctan(x) = \frac{\pi}{2}$

$\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}$

Computing Limits: The FIRST step in computing limit is what I call ***"direct substitution"*** (D.S.) Keep in mind, $x \to 1$ means x is very close to 1 but never equal 1.

After we do "direct substitution", we either get a **determinate form** or an **indeterminate form**.

Determinate Forms

- A real number → the limit is this real number
- $\dfrac{\text{a number}}{\pm\infty}$ $= 0$
- $\dfrac{\text{a nonzero number}}{0}$ $= \pm\infty$

$\lim_{x \to 1} (x^2+2)$ ↪ $1^2+2=3$

$\lim_{x \to \infty} \frac{1}{e^x} = \frac{1}{+\infty} = 0$

$\lim_{x \to 1} \frac{1}{(x-1)^2} = \frac{1}{\to 0} = +\infty$

Indeterminate Forms

- $\frac{0}{0}$ → in section 1.6, we learn some algebra techniques to find the limit. In this section, we can apply L'Hospital's rule.
- $\frac{\pm\infty}{\pm\infty}$ → in section 3.4, we learn a technique to solve this case. In this section, we can apply *L'Hospital's Rule* for this indeterminate form.
- $0 \cdot \infty$ → rewrite as indeterminate quotient form then apply *L'Hospital's Rule.*
- $\infty - \infty$ → rewrite as indeterminate quotient form then apply *L'Hospital's Rule.*
- 0^0, ∞^0, 1^∞ → apply the tool of natural log then rewrite into indeterminate quotient form then apply *L'Hospital's Rule.*

L'Hosptital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a).

Suppose that $\lim_{x \to a} \frac{f(x)}{g(x)} \to \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$. Then, by ***L'Hospital's Rule***, we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{1}$$

provide that the limit on the right side of the equation exists or is $\pm\infty$.

Note: L'Hospital's Rule also applies for $x \to a^+$, $x \to a^-$, or $x \to \pm\infty$.

Remark: We can apply L'Hospital more than one times if needed.

Examples: Evaluate the following limits. **Warning:** Don't blindly use L'Hospital's rule for every problem, see if it applies.

(a) $\lim_{x \to 1} \frac{\ln x}{x-1}$ Direct Substitution : $\frac{\ln(1)}{1-1} = \frac{0}{0}$

indeterminate

$= \lim_{x\to 1} \frac{(\ln x)'}{(x-1)'} = \lim_{x \to 1} \frac{1/x}{1} = 1$

Direct Subs. : $\frac{1/1}{1} = 1$ (finite real no.)

$\Rightarrow \lim_{x\to 1} \frac{\ln x}{x-1} = 1$

(b) $\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}}$ D.S. : $\frac{\to\infty}{\to\infty}$ (indeterminate)

$\frac{d}{dx}\left(x^{1/3}\right) = \frac{1}{3} x^{\frac{1}{3}-1}$

$= \lim_{x\to\infty} \frac{(\ln x)'}{(\sqrt[3]{x})'} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3} x^{-2/3}}$

$= \lim_{x\to\infty} 3 \frac{1}{x} x^{2/3} = \lim_{x\to\infty} \frac{3}{x^{1/3}}$

D.S. : $\frac{3}{\to\infty} = 0$

$\Rightarrow \lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = 0$

(c) $\lim_{x \to \pi^-} \frac{\sin x}{1-\cos x}$

$$\boxed{\begin{aligned} &\underset{x=a}{\text{LHL}}\ f(x) = \lim_{h\to 0} f(a-h) \\ &\underset{x=a}{\text{RHL}}\ f(x) = \lim_{h\to 0} f(a+h) \end{aligned}}$$

$= \lim_{h\to 0} \frac{\sin(\pi - h)}{1-\cos(\pi-h)}$

D.S. : $\frac{\sin\pi}{1-\cos\pi} = \frac{0}{1-(-1)} = \frac{0}{2} = 0 \Rightarrow \lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = 0$

(d) $\lim_{x\to\infty} \sqrt{x}e^{-x/2}$ $\quad \underline{\underline{\text{D.S.}}} \quad (\to\infty)(\to 0)$

$$\lim_{x\to\infty} e^{-x/2} = \lim_{x\to\infty} \frac{1}{e^{x/2}} = \frac{1}{\to\infty} = 0$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{e^{x/2}} \Rightarrow \underline{\text{D.S.}} \quad \frac{\to\infty}{\to\infty} \text{ (indeterminate)}$$

$$= \lim_{x\to\infty} \frac{e^{-x/2}}{\left(\frac{1}{\sqrt{x}}\right)} = \lim_{x\to\infty} \frac{(\sqrt{x})'}{(e^{x/2})'} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2}e^{x/2}}$$

$\underline{\text{D.S.}}$: $\frac{\to 0}{\to\infty} = 0$

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = \frac{1}{\to\infty} = 0$$

$$\lim_{x\to\infty} \sqrt{x}\, e^{-x/2} = 0$$

(e) $\lim_{x\to 0^+} x\ln x$

$\underline{\underline{\text{D.S.}}}$ $(\to 0)(-\infty)$

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} \Rightarrow \text{D.S.}: \frac{-\infty}{+\infty} \Rightarrow L = \lim_{x\to 0^+} \frac{(\ln x)'}{(1/x)'}$$

or

$$\lim_{x\to 0^+} \frac{x}{\left(1/\ln x\right)}$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \left(\frac{1}{x}\right)\left(\frac{-x^2}{1}\right)$$

$$= \lim_{x\to 0^+} (-x) = 0$$

$\underline{\underline{\text{D.S.}}}$

(f) $\lim_{x\to(\pi/2)^-} (\sec x - \tan x)$

$\underline{\underline{\text{D.S.}}}$ $(+\infty - +\infty)$

$$\lim_{x\to\frac{\pi}{2}^-} \frac{1}{\cos x} = +\infty$$

$$L = \lim_{x\to\frac{\pi}{2}^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

(take common denominator)

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin x}{\cos x}$$

$\underline{\text{DS}}$: $\frac{1-\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}} = \frac{0}{0}$

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{-\cos x}{-\sin x} = \frac{-\cos\frac{\pi}{2}}{-\sin\pi/2} = \frac{0}{-1} = 0$$

D.S.

$$\Rightarrow \lim_{x\to\pi/2^-} (\sec x - \tan x) = 0$$

(g) $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$ $\underline{\underline{DS}}$: $(1+\sin 0)^{\cot 0} = (\to 1)^{+\infty}$

$L = \lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$

$\Rightarrow \ln L = \ln\left[\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}\right] =$

↳ cont. function $\Rightarrow$ can interchange $\lim_{x\to 0^+}$ and $\ln$.

$= \lim_{x\to 0^+} \ln(1+\sin 4x)^{\cot x} = \lim_{x\to 0^+} \cot x \ln(1+\sin 4x)$

$\underline{\underline{DS}}$: $(\cot 0^+)\ln(1+\sin 0) = (+\infty)(\to 0)$

$\Rightarrow \ln L = \lim_{x\to 0^+} \dfrac{\ln(1+\sin 4x)}{\tan x}$ bring $\cot x$ in denominator.

$\underline{\underline{DS}}$: $\dfrac{\ln(1+\sin 0)}{\tan 0} = \dfrac{0}{0}$ (indeterminate)

$\lim_{x\to 0^+} \dfrac{\cot x}{\cancel{1/}\ln(1+\sin 4x)}$

Not Preferable

$\Rightarrow \ln L = \lim_{x\to 0^+} \dfrac{[\ln(1+\sin 4x)]'}{(\tan x)'} = \lim_{x\to 0^+} \dfrac{\left(\frac{1}{1+\sin 4x}\right)(4\cos 4x)}{\sec^2 x}$

$\underline{\underline{DS}}$: $\dfrac{\left(\frac{1}{1+\sin 0}\right)(4\cos 0)}{\sec^2(0)} = \dfrac{(1)(4)}{1^2} = 4$

$\Rightarrow \ln L = 4 \Rightarrow L = e^4$

(h) $\lim_{x\to 0^+} x^x$

$(\to 0)^{\to 0}$

$L = \lim_{x\to 0^+} x^x$

$\Rightarrow \ln L = \lim_{x\to 0^+} \ln(x^x) = \lim_{x\to 0^+} x\ln x$ ↳ DS: $0(-\infty)$

$= \lim_{x\to 0^+} \dfrac{\ln x}{1/x} \Rightarrow$ DS. $\dfrac{-\infty}{\infty}$

$= \lim_{x\to 0^+} \dfrac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0 \Rightarrow \ln L = 0$

$\Rightarrow L = e^0 = 1$

Exercise 65 (6.8) $L = \lim_{x\to 0^+} (4x+1)^{\cot x}$

$(+\infty)(\to 0)$

$\uparrow$ D.S.

<u>DS</u>: $(1+0)^{\cot 0} = (\to 1)^{+\infty}$

$\Rightarrow \ln L = \lim_{x\to 0^+} \ln (4x+1)^{\cot x} = \lim_{x\to 0^+} \cot x \ln(4x+1)$

$= \lim_{x\to 0^+} \frac{\ln(4x+1)}{\tan x} \Rightarrow$ DS: $\frac{\to 0}{\to 0}$

$= \lim_{x\to 0^+} \frac{\left(\frac{1}{4x+1}\right)4}{\sec^2 x} = \frac{\left(\frac{1}{0+1}\right)4}{\sec^2 0} = 4$

$\Rightarrow \ln L = 4 \Rightarrow L = e^4$

Exercise 63: $\lim_{x\to\infty} x^{1/x}$ <u>DS</u>: $(+\infty)^{\to 0}$

$L = \lim_{x\to\infty} x^{1/x} \Rightarrow \ln L = \lim_{x\to\infty} \ln(x^{1/x})$

$\Rightarrow \ln L = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x}$

<u>DS</u>: $\frac{+\infty}{+\infty}$ (indeterminate)

$\Rightarrow \ln L = \lim_{x\to\infty} \frac{(\ln x)'}{(x)'} = \lim_{x\to\infty} \frac{1}{x} = 0$

$\Rightarrow \ln L = 0 \Rightarrow L = e^0 = 1$