Math17100 Section 22866 Quiz 10

Spring 2023, April 05

Name: Solutions [1 pt]

Problem 1: Use Gaussian Elimination to solve the following system of linear equations by getting it into Reduced Row Echelon Form.

$$2x + y - 3z = -3 \quad , \quad 3x + 2y - 2z = 2 \quad , \quad x + y + z = 5$$

[10 pts]

Solution:

$$\left[\begin{array}{ccc|c} 2 & 1 & -3 & -3 \\ 3 & 2 & -2 & 2 \\ 1 & 1 & 1 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 3 & 2 & -2 & 2 \\ 2 & 1 & -3 & -3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -1 & -5 & -13 \\ 0 & -1 & -5 & -13 \end{array}\right]$$

$$\xrightarrow{R_2 \to -R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 5 & 13 \\ 0 & -1 & -5 & -13 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_2]{R_1 \to R_1 - R_2} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -4 & -8 \\ 0 & 1 & 5 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x & y & z \end{array} \end{array}$$

The variables x and y are basic while the variable z is free.

So let $z = t$ for some parameter $t \in \mathrm{R}$.

The second row gives $y + 5z = 13 \Rightarrow y = 13 - 5t$

The first row gives $x - 4z = -8 \Rightarrow x = -8 + 4t$

Thus, the solution set of the given linear system is $\boxed{x = -8 + 4t\,,\ y = 13 - 5t\,,\ z = t \text{ where } t \in \mathrm{R}}$

Problem 2: Let $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Compute E^2, F^2 and $EF - FE$.
Use these answers and properties of matrix arithmetic to compute $(E+F)(E-F)$, that is, find this product without computing $E+F$ or $E-F$. [9 pts]

Solution: $E^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$F^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$EF = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$FE = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow EF - FE = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$(E+F)(E-F) = E^2 - EF + FE - F^2 = -EF + FE$ since $E^2 = F^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$\Rightarrow (E+F)(F-E) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Bonus Problem: Let $D_1 = \begin{bmatrix} 199 & 0 \\ 0 & 201 \end{bmatrix}$, $D_2 = \begin{bmatrix} 200 & 0 \\ 0 & 200 \end{bmatrix}$. Compute $(D_1 - D_2)(D_2 - D_1) + I_2$ where I_2 is the 2×2 identity matrix. [2 pts]

Solution: $D_1 - D_2 = \begin{bmatrix} 199 - 200 & 0 \\ 0 & 201 - 200 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$D_2 - D_1 = -(D_1 - D_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\Rightarrow (D_1 - D_2)(D_2 - D_1) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2 \Rightarrow (D_1 - D_2)(D_2 - D_1) + I_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$