

## 1. Lines in Two-Dimensional Space

**Problem 1:** Describe and sketch all possible types of intersections of two lines in  $\mathbb{R}^2$ . Give equations of a pair of lines that illustrate each type of intersection.

*Solution:*

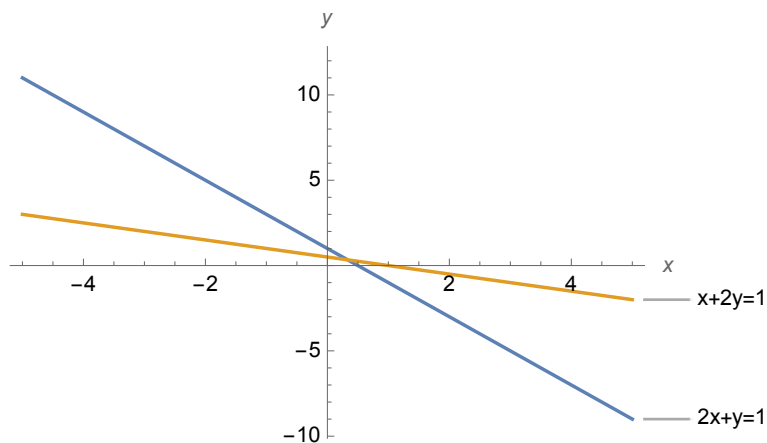


Figure 1: Unique Solution: A pair of line intersecting at a point

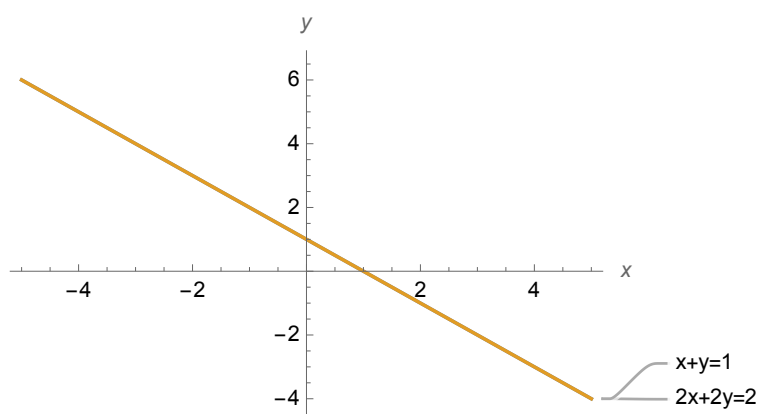


Figure 2: Infinite Solutions: A pair of coinciding lines

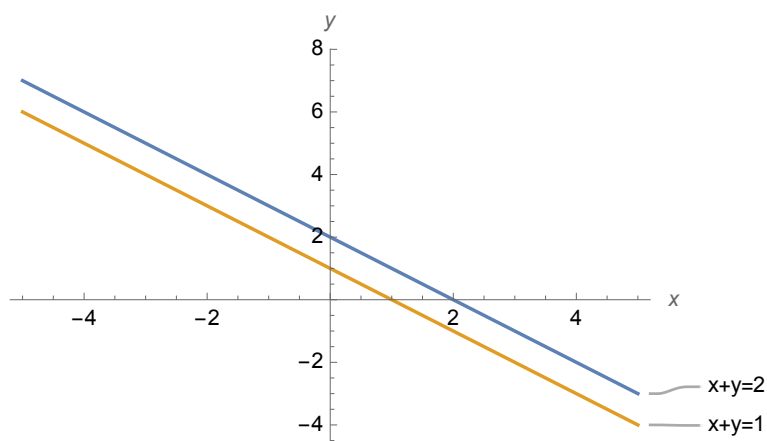
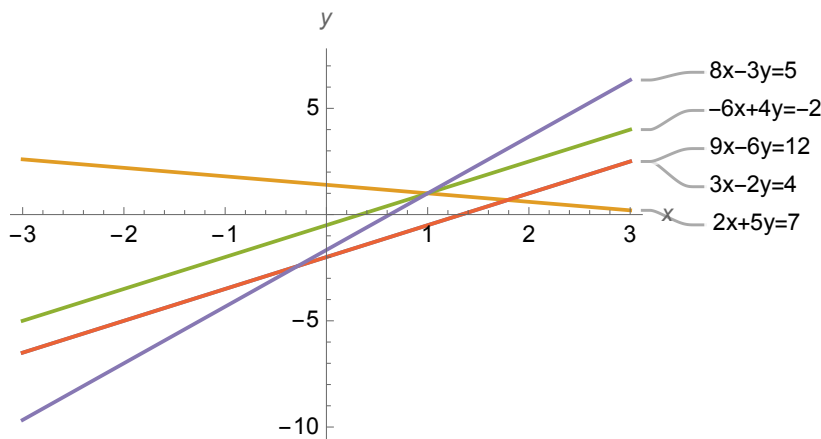


Figure 3: No Solution: A pair of parallel lines

**Problem 2:** On one page graph all of the lines below. Then for each pair in the list eliminate a variable to determine if the pair define the same line, are parallel or intersect at one point. If a pair intersects at one point, find the point.

1.  $3x - 2y = 4$
2.  $2x + 5y = 7$
3.  $-6x + 4y = -2$
4.  $9x - 6y = 12$
5.  $8x - 3y = 5$ .

*Solution:*



Consider the pair  $3x - 2y = 4$  and  $2x + 5y = 7$ :-

$\Rightarrow 6x - 4y = 8$  and  $6x + 15y = 21$ . Subtracting both we get  $y = 13/19 \Rightarrow 3x = 4 + 2(13/19) = 102/19 \Rightarrow x = 34/19$ . Thus, the lines intersect at the point  $(34/19, 13/19)$ .

Consider the pair  $3x - 2y = 4$  and  $-6x + 4y = -2$ :-

$\Rightarrow 6x - 4y = 8$  and  $6x - 4y = 2$ . Subtracting both we get  $0 = 2$  which is not possible. Thus, the lines do not intersect at all and hence are parallel.

Consider the pair  $3x - 2y = 4$  and  $9x - 6y = 12$ :-

$3x - 2y = 4 \Rightarrow 9x - 6y = 12$ . Subtracting both we get  $0 = 0$ . Thus, the lines coincide with each other and hence both the equations have infinitely many solutions.

Consider the pair  $3x - 2y = 4$  and  $8x - 3y = 5$ :-

$\Rightarrow 24x - 16y = 32$  and  $24x - 9y = 15$ . Subtracting both we get  $y = -17/7 \Rightarrow 3x = 4 + 2(-17/7) = -6/7 \Rightarrow x = -2/7$ . Thus, the lines intersect at the point  $(-2/7, -17/7)$ .

Consider the pair  $2x + 5y = 7$  and  $-6x + 4y = -2$ :-

$\Rightarrow 6x + 15y = 21$  and  $6x - 4y = 2$ . Subtracting both we get  $y = 1 \Rightarrow 2x = 7 - 5(1) \Rightarrow x = 1$ . Thus, the lines intersect at the point  $(1, 1)$ .

Consider the pair  $2x + 5y = 7$  and  $9x - 6y = 12$ :-

$9x - 6y = 12$  coincides with  $3x - 2y = 4$ . Thus, the two given lines intersect at  $(34/19, 13/19)$ .

Consider the pair  $2x + 5y = 7$  and  $8x - 3y = 5$ :-

$2x + 5y = 7 \Rightarrow 8x + 20y = 28$ . Subtracting both we get  $23y = 23 \Rightarrow y = 1$ .  $\Rightarrow 2x = 7 - 5(1) \Rightarrow x = 1$ . Thus, the two given lines intersect at the point  $(1, 1)$ .

Consider the pair  $-6x + 4y = -2$  and  $9x - 6y = 12$ :-

$\Rightarrow -18x + 12y = -6$  and  $18x - 12y = 24$ . Adding both we get  $0 = 18$ . Thus, the lines do not intersect at all and hence are parallel.

Consider the pair  $-6x + 4y = -2$  and  $8x - 3y = 5$ :-

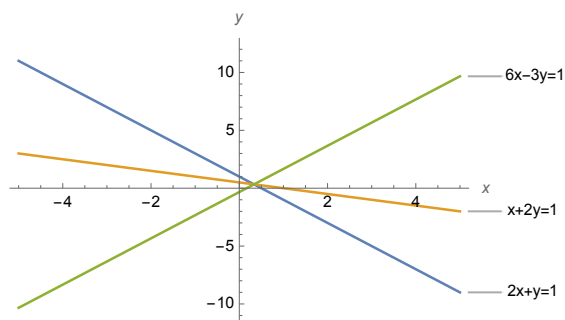
$\Rightarrow -24x + 16y = -8$  and  $24x - 9y = 15$ . Adding both we get  $7y = 7 \Rightarrow y = 1 \Rightarrow 8x = 5 + 3(1) = 8 \Rightarrow x = 1$ . Thus, the given lines intersect at the point  $(1, 1)$ .

Consider the pair  $9x - 6y = 12$  and  $8x - 3y = 5$ :-

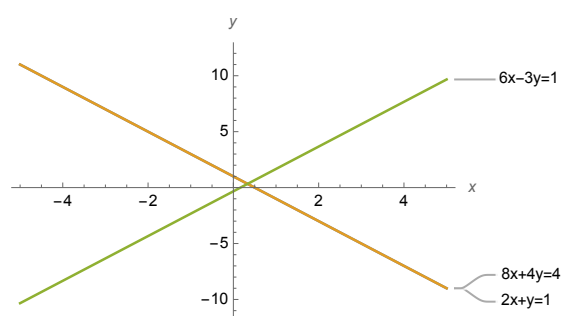
$9x - 6y = 12$  is same as  $3x - 2y = 4$ . Thus, the two given lines intersect at the point  $(-2/7, -17/7)$ .

**Problem 3:** Describe and sketch all possible types of intersections of three lines in  $\mathbb{R}^2$ . Give equations of a triple of lines that illustrate each type of intersection.

*Solution:* We have the following 7 cases:-

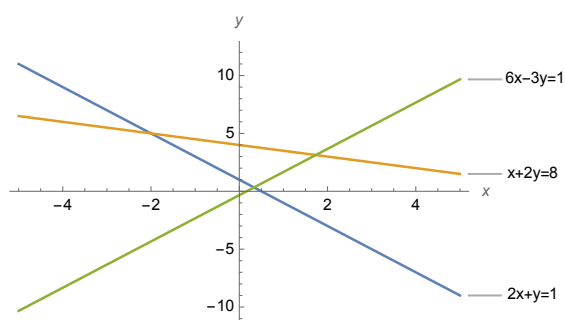


(a) Concurrent Lines

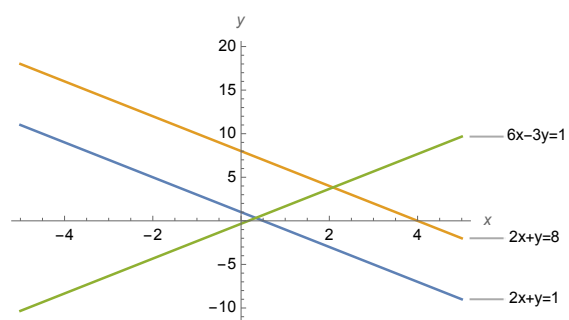


(b) Line intersecting two Coinciding Lines

Figure 4: Unique Solution

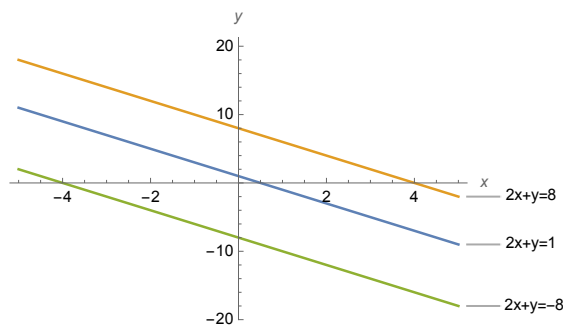


(a) Lines Forming a Triangle

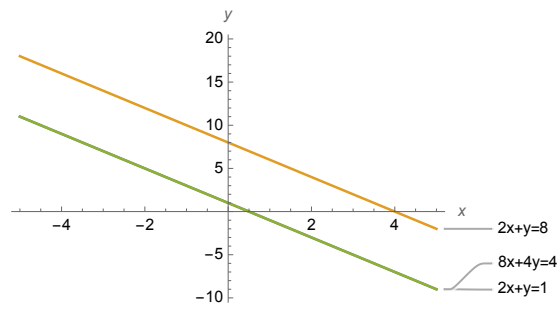


(b) Two Parallel Lines and a Transversal

Figure 5: No Solution



(a) Three Parallel Lines



(b) Two Coinciding and One Parallel

Figure 6: No Solution

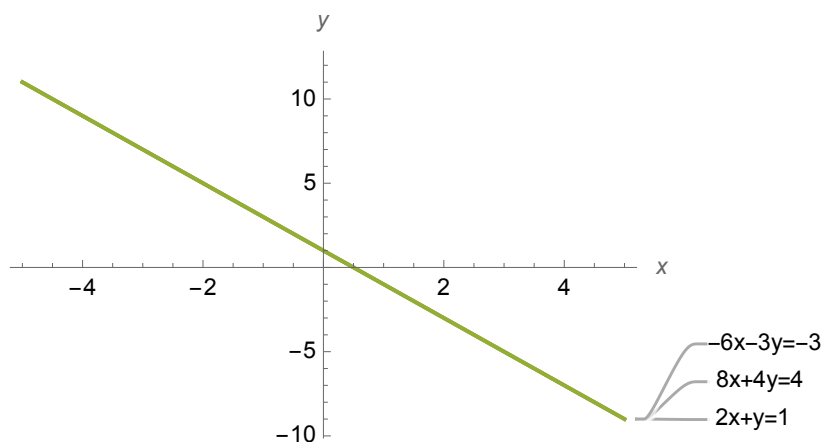


Figure 7: Infinitely Many Solutions: Three Coinciding Lines

**Problem 4:** Eliminate variables to determine if each triple of lines in the list above, intersect or do not intersect. If a triple intersects at one point find the point.

*Solution:*

$3x - 2y = 4, 2x + 5y = 7, -6x + 4y = -2$	—	Do not intersect: Two parallel, one transversal
$3x - 2y = 4, 2x + 5y = 7, 9x - 6y = 12$	—	Line intersecting two coinciding lines at $(34/19, 13/19)$
$3x - 2y = 4, 2x + 5y = 7, 8x - 3y = 5$	—	Do not intersect: Form a triangle
$3x - 2y = 4, -6x + 4y = -2, 9x - 6y = 12$	—	Do not intersect: Two coinciding, one parallel
$3x - 2y = 4, -6x + 4y = -2, 8x - 3y = 5$	—	Do not intersect: Two parallel, one transversal
$3x - 2y = 4, 9x - 6y = 12, 8x - 3y = 5$	—	Line intersecting two coinciding lines at $(-2/7, -17/7)$
$2x + 5y = 7, -6x + 4y = -2, 9x - 6y = 12$	—	Do not intersect: Two parallel, one transversal
$2x + 5y = 7, -6x + 4y = -2, 8x - 3y = 5$	—	Concurrent Lines intersecting at $(1, 1, 1)$
$2x + 5y = 7, 9x - 6y = 12, 8x - 3y = 5$	—	Do not intersect: Form a triangle
$-6x + 4y = -2, 9x - 6y = 12, 8x - 3y = 5$	—	Do not intersect: Two parallel, one transversal

## 2. Planes in Three-Dimensional Space

**Problem 1:** Describe and sketch all possible types of intersections of two planes in  $\mathbb{R}^3$ .

*Solution:*

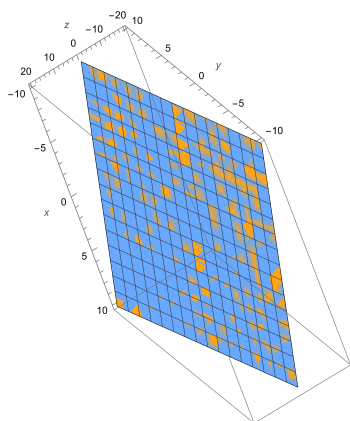


Figure 8: Two Coinciding Planes

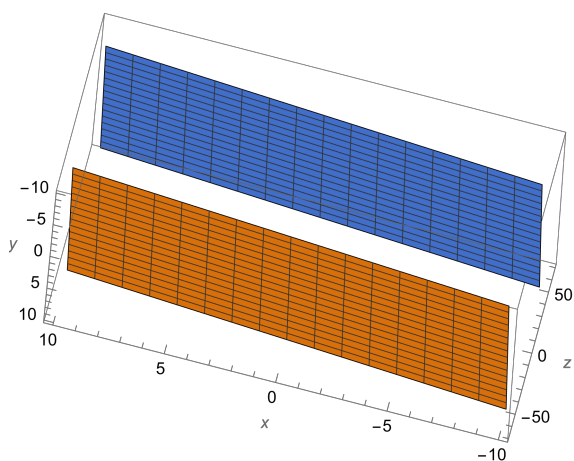


Figure 9: Two Parallel Planes

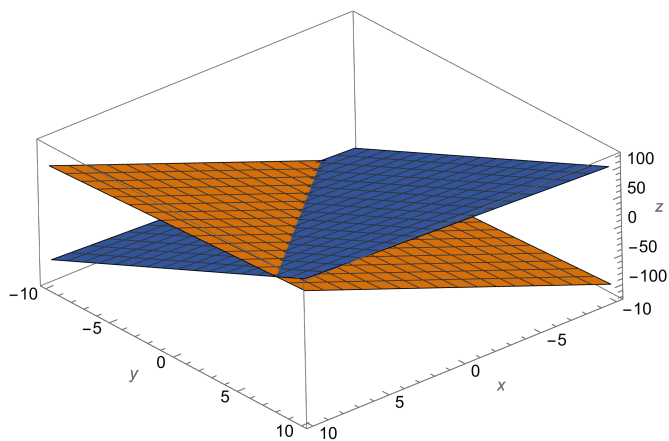


Figure 10: Two Planes Intersecting in a Line

**Problem 2:** Eliminate variables to determine if each pair of planes in the list below, define the same plane, intersect in a line or do not intersect. If the pair intersects in a line find parametric equations for the line of intersection.

1.  $3x - 2y + z = 2$

2.  $2x + y - z = 2$

3.  $-x + 3y + 2z = 4$

4.  $4x + 2y - 2z = 4$

5.  $-6x + 4y - 2z = -2$

*Solution:* For all the pairs, eliminate  $x$ .

Consider the pair  $3x - 2y + z = 2$  and  $2x + y - z = 2$ :-

We have  $6x - 4y + 2z = 4$  and  $6x + 3y - 3z = 6$ . Subtracting we get  $-7y + 5z = -2$ .

Let  $z = t$ . Then  $y = (2 + 5t)/7 \Rightarrow 2x = 2 - (2 + 5t)/7 + t \Rightarrow x = (6 + t)/7$ . Therefore, the parametric equation of the line of intersection is  $x = (6 + t)/7, y = (2 + 5t)/7, z = t$ .

Consider the pair  $3x - 2y + z = 2$  and  $-x + 3y + 2z = 4$ :-

We have  $3x - 2y + z = 2$  and  $-3x + 9y + 6z = 12$ . Adding we get  $7y + 7z = 14$ .

Let  $z = t$ . Then  $y = 2 - t \Rightarrow x = 3(2 - t) + 2t - 4 \Rightarrow x = 2 - t$ . Therefore, the parametric equation of the line of intersection is  $x = 2 - t, y = 2 - t, z = t$ .

Consider the pair  $3x - 2y + z = 2$  and  $4x + 2y - 2z = 4$ :-

We have  $12x - 8y + 4z = 8$  and  $12x + 6y - 6z = 12$ . Subtracting we get  $-14y + 10z = -4$ .

Let  $z = t$ . Then  $y = (2 + 5t)/7 \Rightarrow 4x = 4 - 2(2 + 5t)/7 + 2t \Rightarrow x = (6 + t)/7$ . Therefore, the parametric equation of the line of intersection is  $x = (6 + t)/7, y = (2 + 5t)/7, z = t$ .

Consider the pair  $3x - 2y + z = 2$  and  $-6x + 4y - 2z = -2$ :-

We have  $6x - 4y + 2z = 4$  and  $-6x + 4y - 2z = -2$ . Adding we get  $0 = 2$  which is not possible. Therefore, the given planes **do not intersect** at all and hence are parallel.

Consider the pair  $2x + y - z = 2$  and  $-x + 3y + 2z = 4$ :-

We have  $2x + y - z = 2$  and  $-2x + 6y + 4z = 8$ . Adding we get  $7y + 3z = 10$ .

Let  $z = t$ . Then  $y = (10 - 3t)/7 \Rightarrow x = -4 + 3(10 - 3t)/7 + 2t \Rightarrow x = (2 + 5t)/7$ . Therefore, the parametric equation of the line of intersection is  $x = (2 + 5t)/7, y = (10 - 3t)/7, z = t$ .

Consider the pair  $2x + y - z = 2$  and  $4x + 2y - 2z = 4$ :-

We have  $4x + 2y - 2z = 4$  and  $4x + 2y - 2z = 4$ . Subtracting we get  $0 = 0$ . Thus, the given two planes **coincide** with each other.

Consider the pair  $2x + y - z = 2$  and  $-6x + 4y - 2z = -2$ :-

We have  $6x + 3y - 3z = 6$  and  $-6x + 4y - 2z = -2$ . Adding we get  $7y - 5z = 4$ .

Let  $z = t$ . Then  $y = (4 + 5t)/7 \Rightarrow 2x = 2 - (4 + 5t)/7 + t \Rightarrow x = (5 + t)/7$ . Therefore, the parametric equation of the line of intersection is  $x = (5 + t)/7, y = (4 + 5t)/7, z = t$ .

Consider the pair  $-x + 3y + 2z = 4$  and  $4x + 2y - 2z = 4$ :-

We have  $-4x + 12y + 8z = 16$  and  $4x + 2y - 2z = 4$ . Adding we get  $14y + 6z = 20$ .

Let  $z = t$ . Then  $y = (10 - 3t)/7 \Rightarrow x = 3(10 - 3t)/7 + 2t - 4 \Rightarrow x = (2 + 5t)/7$ . Therefore, the parametric equation of the line of intersection is  $x = (2 + 5t)/7, y = (10 - 3t)/7, z = t$ .

Consider the pair  $-x + 3y + 2z = 4$  and  $-6x + 4y - 2z = -2$ :-

We have  $-6x + 18y + 12z = 24$  and  $-6x + 4y - 2z = -2$ . Subtracting we get  $14y + 14z = 26$ . Let  $z = t$ . Then  $y = (13 - 7t)/7 \Rightarrow x = 3(13 - 7t)/7 + 2t - 4 \Rightarrow x = (11 - 7t)/7$ . Therefore, the parametric equation of the line of intersection is  $\boxed{x = (11 - 7t)/7, y = (13 - 7t)/7, z = t}$ .

Consider the pair  $4x + 2y - 2z = 4$  and  $-6x + 4y - 2z = -2$ :-

We have  $12x + 6y - 6z = 12$  and  $-12x + 8y - 4z = -4$ . Adding we get  $14y - 10z = 8$ . Let  $z = t$ . Then  $y = (4 + 5t)/7 \Rightarrow 4x = 4 - 2(4 + 5t)/7 + 2t \Rightarrow x = (5 + t)/7$ . Therefore, the parametric equation of the line of intersection is  $\boxed{x = (5 + t)/7, y = (4 + 5t)/7, z = t}$ .

**Problem 3:** Eliminate variables to determine if each triple of planes in problem 2, define the same plane, intersect in a line, intersect in one point or do not intersect. If the triple intersects in a line find parametric equations for the line of intersection. If the triple intersect at one point find the point.

*Solution:* Consider the triple  $3x - 2y + z = 2$ ,  $2x + y - z = 2$ ,  $-x + 3y + 2z = 4$ :-

Eliminating  $x$  we get  $-7y + 5z = -2$  and  $7y + 3z = 10 \Rightarrow z = 1, y = 1$

$\Rightarrow -x = 4 - 3(1) - 2(1) = -1 \Rightarrow x = 1$ . Therefore, the given triple of planes intersect at a unique point  $\boxed{(1, 1, 1)}$ .

Consider the triple  $3x - 2y + z = 2$ ,  $2x + y - z = 2$ ,  $4x + 2y - 2z = 4$ :-

Eliminating  $x$  we get  $-7y + 5z = -2$  and  $0 = 0$ . Letting  $z = t$ ,  $y = (2 + 5t)/7 \Rightarrow x = (6 + t)/7$ . Therefore, the given triple intersects in a line whose parametric equation is  $\boxed{x = (6 + t)/7, y = (5 + 2t)/7, z = t}$ .

Consider the triple  $3x - 2y + z = 2$ ,  $2x + y - z = 2$ ,  $-6x + 4y - 2z = -2$ :-

Eliminating  $x$  we get  $-7y + 5z = -2$  and  $7y - 5z = 4 \Rightarrow 0 = 2$ . Therefore, the given triple  $\boxed{\text{do not intersect}}$ .

Consider the triple  $3x - 2y + z = 2$ ,  $-x + 3y + 2z = 4$ ,  $4x + 2y - 2z = 4$ :-

Eliminating  $x$  we get  $7y + 7z = 14$  and  $14y + 6z = 20 \Rightarrow z = 1 \Rightarrow y = 1 \Rightarrow x = 1$ . Therefore, the given planes intersect at a unique point  $\boxed{(1, 1, 1)}$ .

Consider the triple  $3x - 2y + z = 2$ ,  $-x + 3y + 2z = 4$ ,  $-6x + 4y - 2z = -2$ :-

Eliminating  $x$  we get  $7y + 7z = 14$  and  $14y + 14z = 26 \Rightarrow 0 = 2$ . Therefore, the given planes  $\boxed{\text{do not intersect}}$ .

Consider the triple  $3x - 2y + z = 2$ ,  $4x + 2y - 2z = 4$ ,  $-6x + 4y - 2z = -2$ :-

Eliminating  $x$  we get  $-14y + 10z = -4$  and  $14y - 10z = 8 \Rightarrow 0 = 4$ . Therefore, the given planes  $\boxed{\text{do not intersect}}$ .

Consider the triple  $2x + y - z = 2$ ,  $-x + 3y + 2z = 4$ ,  $4x + 2y - 2z = 4$ :-

Eliminating  $x$  we get  $7y + 3z = 10$  and  $14y + 6z = 20 \Rightarrow 0 = 0$ . Letting  $z = t$  we have  $y = (10 - 3t)/7 \Rightarrow x = (2 + 5t)/7$ . Therefore, the given planes intersect in a line  $\boxed{x = (2 + 5t)/7, y = (10 - 3t)/7, z = t}$ .

Consider the triple  $2x + y - z = 2$ ,  $-x + 3y + 2z = 4$ ,  $-6x + 4y - 2z = -2$ :-

Eliminating  $x$  we get  $7y + 3z = 10$  and  $14y + 14z = 26 \Rightarrow z = 3/4 \Rightarrow y = 31/28 \Rightarrow x = 23/28$ . Therefore, the given planes intersect at the point  $\boxed{(23/28, 31/28, 3/4)}$ .

Consider the triple  $2x + y - z = 2$ ,  $4x + 2y - 2z = 4$ ,  $-6x + 4y - 2z = -2$ :-

Eliminating  $x$  we get  $0 = 0$  and  $14y - 10z = 8$ . Letting  $z = t$ , we have  $y = (4 + 5t)/7 \Rightarrow x = (5 + t)/7$ . Therefore, the given planes intersect in a line whose parametric equation is  $\boxed{x = (5 + t)/7, y = (4 + 5t)/7, z = t}$ .

Consider the triple  $-x + 3y + 2z = 4$ ,  $4x + 2y - 2z = 4$ ,  $-6x + 4y - 2z = -2$ :-

Eliminating  $x$  we get  $14y + 6z = 20$  and  $14y - 10z = 8 \Rightarrow z = 3/4 \Rightarrow y = 31/28 \Rightarrow x = 23/28$ .

Therefore, the given planes intersect at the point  $\boxed{(23/28, 31/28, 3/4)}$ .

**Problem 4:** Use the parametric equations for the line of intersection of the planes (1) and (2) in problem 2 to find the intersection of this line with the planes defined by (3), (4) and (5). Compare this with the answers in problem 3 you obtained by eliminating variables.

*Solution:* The parametric equation for the line of intersection of planes (1) and (2) is  $x = (6 + t)/7$ ,  $y = (2 + 5t)/7$ ,  $z = t$ .

Intersection with plane (3)  $-x + 3y + 2z = 4$ :-

$-(6 + t)/7 + 3(2 + 5t)/7 + 2t = 4 \Rightarrow 28t = 28 \Rightarrow t = 1 \Rightarrow x = (6 + 1)/7 = 1$ ,  $y = (2 + 5)/7 = 1$ ,  $z = 1$ . Thus, planes (1), (2), (3) intersect at the point  $(1, 1, 1)$ .

Intersection with plane (4)  $4x + 2y - 2z = 4$ :-

$4(6 + t)/7 + 2(2 + 5t)/7 - 2t = 4 \Rightarrow 0 = 0$ . Thus, planes (1), (2), (4) intersect in the line  $x = (6 + t)/7$ ,  $y = (2 + 5t)/7$ ,  $z = t$ .

Intersection with plane (5)  $-6x + 4y - 2z = -2$ :-

$-6(6 + t)/7 + 4(2 + 5t)/7 - 2t = -2 \Rightarrow 0 = 2$ . Thus, the planes (1), (2), (5) do not intersect.



### 3. Matrices and Elementary Row Operations

**Problem 1:** For each pair of equations in problem 2, section 1 (Lines in two-dimensional space), write down the associated augmented matrix, the coefficient matrix and the constant vector.

*Solution:* For the pair  $3x - 2y = 4$ ,  $2x + 5y = 7$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{cc|c} 3 & -2 & 4 \\ 2 & 5 & 7 \end{array} \right] \quad , \quad \left[ \begin{array}{cc} 3 & -2 \\ 2 & 5 \end{array} \right] \quad , \quad \left[ \begin{array}{c} 4 \\ 7 \end{array} \right]$$

For the pair  $3x - 2y = 4$  and  $-6x + 4y = -2$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{cc|c} 3 & -2 & 4 \\ -6 & 4 & -2 \end{array} \right] \quad , \quad \left[ \begin{array}{cc} 3 & -2 \\ -6 & 4 \end{array} \right] \quad , \quad \left[ \begin{array}{c} 4 \\ -2 \end{array} \right]$$

For the pair  $3x - 2y = 4$  and  $9x - 6y = 12$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{cc|c} 3 & -2 & 4 \\ 9 & -6 & 12 \end{array} \right] \quad , \quad \left[ \begin{array}{cc} 3 & -2 \\ 9 & -6 \end{array} \right] \quad , \quad \left[ \begin{array}{c} 4 \\ 12 \end{array} \right]$$

For the pair  $3x - 2y = 4$  and  $8x - 3y = 5$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{cc|c} 3 & -2 & 4 \\ 8 & -3 & 5 \end{array} \right] \quad , \quad \left[ \begin{array}{cc} 3 & -2 \\ 8 & -3 \end{array} \right] \quad , \quad \left[ \begin{array}{c} 4 \\ 5 \end{array} \right]$$

For the pair  $2x + 5y = 7$  and  $-6x + 4y = -2$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{cc|c} 2 & 5 & 7 \\ -6 & 4 & -2 \end{array} \right] \quad , \quad \left[ \begin{array}{cc} 2 & 5 \\ -6 & 4 \end{array} \right] \quad , \quad \left[ \begin{array}{c} 7 \\ -2 \end{array} \right]$$

For the pair  $2x + 5y = 7$  and  $9x - 6y = 12$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{cc|c} 2 & 5 & 7 \\ 9 & -6 & 12 \end{array} \right] \quad , \quad \left[ \begin{array}{cc} 2 & 5 \\ 9 & -6 \end{array} \right] \quad , \quad \left[ \begin{array}{c} 7 \\ 12 \end{array} \right]$$

For the pair  $2x + 5y = 7$  and  $8x - 3y = 5$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{cc|c} 2 & 5 & 7 \\ 8 & -3 & 5 \end{array} \right] \quad , \quad \left[ \begin{array}{cc} 2 & 5 \\ 8 & -3 \end{array} \right] \quad , \quad \left[ \begin{array}{c} 7 \\ 5 \end{array} \right]$$

For the pair  $-6x + 4y = -2$  and  $9x - 6y = 12$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{cc|c} -6 & 4 & -2 \\ 9 & -6 & 12 \end{array} \right] \quad , \quad \left[ \begin{array}{cc} -6 & 4 \\ 9 & -6 \end{array} \right] \quad , \quad \begin{bmatrix} -2 \\ 12 \end{bmatrix}$$

For the pair  $-6x + 4y = -2$  and  $8x - 3y = 5$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{cc|c} -6 & 4 & -2 \\ 8 & -3 & 5 \end{array} \right] \quad , \quad \left[ \begin{array}{cc} -6 & 4 \\ 8 & -3 \end{array} \right] \quad , \quad \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

For the pair  $9x - 6y = 12$  and  $8x - 3y = 5$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{cc|c} 9 & -6 & 12 \\ 8 & -3 & 5 \end{array} \right] \quad , \quad \left[ \begin{array}{cc} 9 & -6 \\ 8 & -3 \end{array} \right] \quad , \quad \begin{bmatrix} 12 \\ 5 \end{bmatrix}$$

**Problem 2:** For each triple of equations in problem 2, section 2 (Planes in three-dimensional space), write down the associated augmented matrix, the coefficient matrix and the constant vector.

*Solution:* For the triple  $3x - 2y + z = 2$ ,  $2x + y - z = 2$ ,  $-x + 3y + 2z = 4$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 2 & 1 & -1 & 2 \\ -1 & 3 & 2 & 4 \end{array} \right] \quad , \quad \left[ \begin{array}{ccc} 3 & -2 & 1 \\ 2 & 1 & -1 \\ -1 & 3 & 2 \end{array} \right] \quad , \quad \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

For the triple  $3x - 2y + z = 2$ ,  $2x + y - z = 2$ ,  $4x + 2y - 2z = 4$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 2 & 1 & -1 & 2 \\ 4 & 2 & -2 & 4 \end{array} \right] \quad , \quad \left[ \begin{array}{ccc} 3 & -2 & 1 \\ 2 & 1 & -1 \\ 4 & 2 & -2 \end{array} \right] \quad , \quad \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

For the triple  $3x - 2y + z = 2$ ,  $2x + y - z = 2$ ,  $-6x + 4y - 2z = -2$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 2 & 1 & -1 & 2 \\ -6 & 4 & -2 & -2 \end{array} \right] \quad , \quad \left[ \begin{array}{ccc} 3 & -2 & 1 \\ 2 & 1 & -1 \\ -6 & 4 & -2 \end{array} \right] \quad , \quad \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

For the triple  $3x - 2y + z = 2$ ,  $-x + 3y + 2z = 4$ ,  $4x + 2y - 2z = 4$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ -1 & 3 & 2 & 4 \\ 4 & 2 & -2 & 4 \end{array} \right] \quad , \quad \left[ \begin{array}{ccc} 3 & -2 & 1 \\ -1 & 3 & 2 \\ 4 & 2 & -2 \end{array} \right] \quad , \quad \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

For the triple  $3x - 2y + z = 2$ ,  $-x + 3y + 2z = 4$ ,  $-6x + 4y - 2z = -2$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ -1 & 3 & 2 & 4 \\ -6 & 4 & -2 & -2 \end{array} \right], \quad \left[ \begin{array}{ccc} 3 & -2 & 1 \\ -1 & 3 & 2 \\ -6 & 4 & -2 \end{array} \right], \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

For the triple  $3x - 2y + z = 2$ ,  $4x + 2y - 2z = 4$ ,  $-6x + 4y - 2z = -2$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 4 & 2 & -2 & 4 \\ -6 & 4 & -2 & -2 \end{array} \right], \quad \left[ \begin{array}{ccc} 3 & -2 & 1 \\ 4 & 2 & -2 \\ -6 & 4 & -2 \end{array} \right], \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

For the triple  $2x + y - z = 2$ ,  $-x + 3y + 2z = 4$ ,  $4x + 2y - 2z = 4$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ -1 & 3 & 2 & 4 \\ 4 & 2 & -2 & 4 \end{array} \right], \quad \left[ \begin{array}{ccc} 2 & 1 & -1 \\ -1 & 3 & 2 \\ 4 & 2 & 2 \end{array} \right], \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

For the triple  $2x + y - z = 2$ ,  $-x + 3y + 2z = 4$ ,  $-6x + 4y - 2z = -2$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ -1 & 3 & 2 & 4 \\ -6 & 4 & -2 & -2 \end{array} \right], \quad \left[ \begin{array}{ccc} 2 & 1 & -1 \\ -1 & 3 & 2 \\ -6 & 4 & -2 \end{array} \right], \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

For the triple  $2x + y - z = 2$ ,  $4x + 2y - 2z = 4$ ,  $-6x + 4y - 2z = -2$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 4 & 2 & -2 & 4 \\ -6 & 4 & -2 & -2 \end{array} \right], \quad \left[ \begin{array}{ccc} 2 & 1 & -1 \\ 4 & 2 & -2 \\ -6 & 4 & -2 \end{array} \right], \quad \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

For the triple  $-x + 3y + 2z = 4$ ,  $4x + 2y - 2z = 4$ ,  $-6x + 4y - 2z = -2$  the augmented matrix, the coefficient matrix and the constant vector are respectively:-

$$\left[ \begin{array}{ccc|c} -1 & 3 & 2 & 4 \\ 4 & 2 & -2 & 4 \\ -6 & 4 & -2 & -2 \end{array} \right], \quad \left[ \begin{array}{ccc} -1 & 3 & 2 \\ 4 & 2 & -2 \\ -6 & 4 & -2 \end{array} \right], \quad \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$$

**Problem 3:** Determine which of the following matrices are in reduced row-echelon form.

*Solution:*

1.

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \rightarrow \text{No} \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \rightarrow \text{No} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \text{Yes} \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \rightarrow \text{No}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \text{Yes} \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \text{No} \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \rightarrow \text{Yes} \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \text{No}$$

2.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \text{Yes} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \text{No} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Yes}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \rightarrow \text{No} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Yes} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \text{No}$$

3.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{No} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{No} \quad \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{No}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Yes} \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \text{No} \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Yes}$$

4.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{No} \quad \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Yes} \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{No}$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Yes} \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix} \rightarrow \text{No}$$

**Problem 4:** There are 15 leading 1 patterns for  $3 \times 4$  matrices in reduced row-echelon form. Write down all of them.

*Solution:*

$$\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

## 4. Gaussian Elimination

**Problem 1:** Use Gaussian elimination to put the following matrices in reduced row-echelon form.

$$1. \left[ \begin{array}{cccc|c} 3 & 3 & -4 & -2 & 1 \\ 2 & 2 & -3 & 1 & 3 \\ 1 & 1 & -2 & 4 & 5 \end{array} \right]$$

*Solution:*

$$\left[ \begin{array}{cccc|c} 3 & 3 & -4 & -2 & 1 \\ 2 & 2 & -3 & 1 & 3 \\ 1 & 1 & -2 & 4 & 5 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[ \begin{array}{cccc|c} \boxed{1} & 1 & -2 & 4 & 5 \\ 2 & 2 & -3 & 1 & 3 \\ 3 & 3 & -4 & -2 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left[ \begin{array}{cccc|c} \boxed{1} & 1 & -2 & 4 & 5 \\ 0 & 0 & 1 & -7 & -7 \\ 3 & 3 & -4 & -2 & 1 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow R_3 - 3R_1} \left[ \begin{array}{cccc|c} \boxed{1} & 1 & -2 & 4 & 5 \\ 0 & 0 & \boxed{1} & -7 & -7 \\ 0 & 0 & 2 & -14 & -14 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - 2R_2]{R_1 \rightarrow R_1 + 2R_2} \left[ \begin{array}{cccc|c} \boxed{1} & 1 & 0 & -10 & -9 \\ 0 & 0 & \boxed{1} & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$2. \left[ \begin{array}{cccc|c} 2 & 3 & 3 & -1 & 3 \\ 1 & 1 & -2 & 3 & 4 \\ 5 & 7 & 4 & 1 & 5 \end{array} \right]$$

*Solution:*

$$\left[ \begin{array}{cccc|c} 2 & 3 & 3 & -1 & 3 \\ 1 & 1 & -2 & 3 & 4 \\ 5 & 7 & 4 & 1 & 5 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[ \begin{array}{cccc|c} \boxed{1} & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - 5R_1]{R_2 \rightarrow R_2 - 2R_1} \left[ \begin{array}{cccc|c} \boxed{1} & 1 & -2 & 3 & 4 \\ 0 & \boxed{1} & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{array} \right]$$

$$\xrightarrow[R_3 \rightarrow R_3 - 2R_2]{R_1 \rightarrow R_1 - R_2} \left[ \begin{array}{cccc|c} \boxed{1} & 0 & -9 & 10 & 9 \\ 0 & \boxed{1} & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{array} \right] \xrightarrow{R_3 \rightarrow (-1/5)R_3} \left[ \begin{array}{cccc|c} \boxed{1} & 0 & -9 & 10 & 9 \\ 0 & \boxed{1} & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array} \right]$$

$$\xrightarrow[R_2 \rightarrow R_2 + 5R_3]{R_1 \rightarrow R_1 - 9R_3} \left[ \begin{array}{cccc|c} \boxed{1} & 0 & -9 & 10 & 0 \\ 0 & \boxed{1} & 7 & -7 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array} \right]$$

$$3. \left[ \begin{array}{cc|c} 2 & -4 & 6 \\ -2 & 3 & -3 \\ 3 & 7 & 5 \end{array} \right]$$

*Solution:*

$$\left[ \begin{array}{cc|c} 2 & -4 & 6 \\ -2 & 3 & -3 \\ 3 & 7 & 5 \end{array} \right] \xrightarrow{R_1 \rightarrow (1/2)R_1} \left[ \begin{array}{cc|c} \boxed{1} & -2 & 3 \\ -2 & 3 & -3 \\ 3 & 7 & 5 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - 3R_1]{R_2 \rightarrow R_2 + 2R_1} \left[ \begin{array}{cc|c} \boxed{1} & -2 & 3 \\ 0 & -1 & 3 \\ 0 & 13 & -4 \end{array} \right] \xrightarrow{R_2 \rightarrow -R_2} \left[ \begin{array}{cc|c} \boxed{1} & -2 & 3 \\ 0 & \boxed{1} & -3 \\ 0 & 13 & -4 \end{array} \right]$$

$$\xrightarrow[R_3 \rightarrow R_3 - 13R_2]{R_1 \rightarrow R_1 + 2R_2} \left[ \begin{array}{cc|c} \boxed{1} & 0 & -3 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 35 \end{array} \right] \xrightarrow{R_3 \rightarrow (1/35)R_3} \left[ \begin{array}{cc|c} \boxed{1} & 0 & -3 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & \boxed{1} \end{array} \right] \xrightarrow[R_2 \rightarrow R_2 + 3R_3]{R_1 \rightarrow R_1 + 3R_3} \left[ \begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{array} \right]$$

$$4. \left[ \begin{array}{ccc|c} 2 & -4 & 6 & 8 \\ 3 & 7 & 5 & 3 \\ -1 & -1 & 17 & 19 \end{array} \right]$$

*Solution:*

$$\left[ \begin{array}{ccc|c} 2 & -4 & 6 & 8 \\ 3 & 7 & 5 & 3 \\ -1 & -1 & 17 & 19 \end{array} \right] \xrightarrow{R_1 \rightarrow (1/2)R_1} \left[ \begin{array}{ccc|c} \boxed{1} & -2 & 3 & 4 \\ 3 & 7 & 5 & 3 \\ -1 & -1 & 17 & 19 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 + R_1]{R_2 \rightarrow R_2 - 3R_1} \left[ \begin{array}{ccc|c} \boxed{1} & -2 & 3 & 4 \\ 0 & 13 & -4 & -9 \\ 0 & -3 & 20 & 23 \end{array} \right]$$

$$\xrightarrow{R_2 \rightarrow (1/13)R_2} \left[ \begin{array}{ccc|c} \boxed{1} & -2 & 3 & 4 \\ 0 & \boxed{1} & -4/13 & -9/13 \\ 0 & -3 & 20 & 23 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 + 3R_2]{R_1 \rightarrow R_1 + 2R_2} \left[ \begin{array}{ccc|c} \boxed{1} & 0 & 31/13 & 34/13 \\ 0 & \boxed{1} & -4/13 & -9/13 \\ 0 & 0 & 248/13 & 272/13 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow (13/248)R_3} \left[ \begin{array}{ccc|c} \boxed{1} & 0 & 31/13 & 34/13 \\ 0 & \boxed{1} & -4/13 & -9/13 \\ 0 & 0 & \boxed{1} & 34/31 \end{array} \right] \xrightarrow[R_2 \rightarrow R_2 + (4/13)R_3]{R_1 \rightarrow R_1 - (31/13)R_3} \left[ \begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & -11/13 \\ 0 & 0 & \boxed{1} & 34/31 \end{array} \right]$$

$$5. \left[ \begin{array}{ccccc|c} -3 & -6 & 9 & 6 & -12 & -3 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ 1 & 2 & -3 & -2 & 4 & 1 \end{array} \right]$$

*Solution:*

$$\left[ \begin{array}{ccccc|c} -3 & -6 & 9 & 6 & -12 & -3 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ 1 & 2 & -3 & -2 & 4 & 1 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[ \begin{array}{ccccc|c} \boxed{1} & 2 & -3 & -2 & 4 & 1 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ -3 & -6 & 9 & 6 & -12 & -3 \end{array} \right]$$

$$\xrightarrow[R_2 \rightarrow R_2 - 2R_1]{R_3 \rightarrow R_3 + 3R_1} \left[ \begin{array}{ccccc|c} \boxed{1} & 2 & -3 & -2 & 4 & 1 \\ 0 & \boxed{1} & -2 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 - 2R_2} \left[ \begin{array}{ccccc|c} \boxed{1} & 0 & 1 & -8 & 8 & -3 \\ 0 & \boxed{1} & -2 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

### 5. Reduced Row Echelon Matrices and Solution Sets

**Problem 1:** Use elementary row operations to put each matrix from problem 1, section 3 (Matrices and elementary row operations), in reduced row-echelon form. Use this to write down the solution set and compare the answer to the ones you obtained in problem 2, section 1 (Lines in two-dimensional space).

*Solution:* For the pair  $3x - 2y = 4$ ,  $2x + 5y = 7$  we have

$$\begin{aligned}
 \left[ \begin{array}{cc|c} 3 & -2 & 4 \\ 2 & 5 & 7 \end{array} \right] &\xrightarrow{R_1 \rightarrow (1/3)R_1} \left[ \begin{array}{cc|c} \boxed{1} & -2/3 & 4/3 \\ 2 & 5 & 7 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left[ \begin{array}{cc|c} \boxed{1} & -2/3 & 4/3 \\ 0 & 19/3 & 13/3 \end{array} \right] \\
 &\xrightarrow{R_2 \rightarrow (3/19)R_2} \left[ \begin{array}{cc|c} \boxed{1} & -2/3 & 4/3 \\ 0 & \boxed{1} & 13/19 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 + (2/3)R_2} \left[ \begin{array}{cc|c} \boxed{1} & 0 & 34/19 \\ 0 & \boxed{1} & 13/19 \end{array} \right] \\
 &\qquad\qquad\qquad \uparrow \quad \uparrow \\
 &\qquad\qquad\qquad x \quad y
 \end{aligned}$$

Both  $x$  and  $y$  are basic variables. The solution set is thus the single point  $\left(\frac{34}{19}, \frac{13}{19}\right)$

**Problem 2:** Use elementary row operations to put each matrix from problem 2, section 3 (Matrices and elementary row operations), in reduced row-echelon form. Use this to write down the solution set and compare the answer to the ones you obtained in problem 2, section 2 (Planes in two-dimensional space).

*Solution:* For the triple  $3x - 2y + z = 2$ ,  $2x + y - z = 2$ ,  $-x + 3y + 2z = 4$  we have

$$\begin{aligned}
 \left[ \begin{array}{ccc|c} 3 & -2 & 1 & 2 \\ 2 & 1 & -1 & 2 \\ -1 & 3 & 2 & 4 \end{array} \right] &\xrightarrow{R_1 \rightarrow (1/3)R_1} \left[ \begin{array}{ccc|c} \boxed{1} & -2/3 & 1/3 & 2/3 \\ 2 & 1 & -1 & 2 \\ -1 & 3 & 2 & 4 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 + R_1]{R_2 \rightarrow R_2 - 2R_1} \left[ \begin{array}{ccc|c} \boxed{1} & -2/3 & 1/3 & 2/3 \\ 0 & 7/3 & -5/3 & 2/3 \\ 0 & 7/3 & 7/3 & 14/3 \end{array} \right] \\
 &\xrightarrow{R_2 \rightarrow (3/7)R_2} \left[ \begin{array}{ccc|c} \boxed{1} & -2/3 & 1/3 & 2/3 \\ 0 & \boxed{1} & -5/7 & 2/7 \\ 0 & 7/3 & 7/3 & 14/3 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - (7/3)R_2]{R_1 \rightarrow R_1 + (2/3)R_2} \left[ \begin{array}{ccc|c} \boxed{1} & 0 & -1/7 & 6/7 \\ 0 & \boxed{1} & -5/7 & 2/7 \\ 0 & 0 & 4 & 4 \end{array} \right] \\
 &\xrightarrow{R_3 \rightarrow (1/4)R_3} \left[ \begin{array}{ccc|c} \boxed{1} & 0 & -1/7 & 6/7 \\ 0 & \boxed{1} & -5/7 & 2/7 \\ 0 & 0 & \boxed{1} & 1 \end{array} \right] \xrightarrow[R_2 \rightarrow R_2 + (5/7)R_3]{R_1 \rightarrow R_1 + (1/7)R_3} \left[ \begin{array}{ccc|c} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 1 \end{array} \right] \\
 &\qquad\qquad\qquad \uparrow \quad \uparrow \quad \uparrow \\
 &\qquad\qquad\qquad x \quad y \quad z
 \end{aligned}$$

All the variables  $x$ ,  $y$  and  $z$  are basic. The solution set is thus the single point  $(1, 1, 1)$ .

**Problem 3:** For each reduced row-echelon matrix obtained in problem 1, section 4 (Gaussian elimination), write down the solution set for the system of linear equations associated to the original matrix.

*Solution:* (1) The reduced row-echelon form obtained was

$$\left[ \begin{array}{cccc|c} \boxed{1} & 1 & 0 & -10 & -9 \\ 0 & 0 & \boxed{1} & -7 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & x_4 \end{array}$$

The variables  $x_1$  and  $x_3$  are basic while the variables  $x_2$  and  $x_4$  are free.

So we let  $x_2 = s$ ,  $x_4 = t$  where  $s, t \in \mathbb{R}$ .

The second row gives  $x_3 - 7x_4 = -7 \Rightarrow x_3 - 7t = -7 \Rightarrow x_3 = -7 + 7t$ .

The first row gives  $x_1 + x_2 - 10x_4 = -9 \Rightarrow x_1 + s - 10t = -9 \Rightarrow x_1 = -9 - s + 10t$ .

Thus the solution set is given by

$$\boxed{x_1 = -9 - s + 10t \quad , \quad x_2 = s \quad , \quad x_3 = -7 + 7t \quad , \quad x_4 = t \quad , \quad s, t \in \mathbb{R}}$$

which is a plane.

(2) The reduced row-echelon form obtained was

$$\left[ \begin{array}{cccc|c} \boxed{1} & 0 & -9 & 10 & 0 \\ 0 & \boxed{1} & 7 & -7 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array} \right]$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 & x_4 \end{array}$$

Since we have a pivot position in the column corresponding to the constant vector, the solution set is empty. Thus, no solutions exist for the given system.

(3) The reduced row-echelon form obtained was

$$\left[ \begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{array} \right]$$

$$\begin{array}{cc} \uparrow & \uparrow \\ x_1 & x_2 \end{array}$$

Since we have a pivot position in the column corresponding to the constant vector, the solution set is empty. Thus, no solutions exist for the given system.

(4) The reduced row-echelon form obtained was

$$\left[ \begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & -11/13 \\ 0 & 0 & \boxed{1} & 34/31 \end{array} \right]$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x_1 & x_2 & x_3 \end{array}$$

All the variables  $x_1, x_2, x_3$  are basic. The solution set is thus the single point  $\left(0, -\frac{11}{13}, \frac{34}{31}\right)$ .



(5) The reduced row-echelon form obtained was

$$\left[ \begin{array}{ccccc|c} \boxed{1} & 0 & 1 & -8 & 8 & -3 \\ 0 & \boxed{1} & -2 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$   
 $x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$

The variables  $x_1$  and  $x_2$  are basic while the variables  $x_3$ ,  $x_4$  and  $x_5$  are free.

So we let  $x_3 = s$ ,  $x_4 = t$ ,  $x_5 = u$  where  $s, t, u \in \mathbb{R}$ .

The second row gives  $x_2 - 2x_3 + 3x_4 - 2x_5 = 2 \Rightarrow x_2 - 2s + 3t - 2u = 2 \Rightarrow x_2 = 2 + 2s - 3t + 2u$ .

The first row gives  $x_1 + x_3 - 8x_4 + 8x_5 = -3 \Rightarrow x_1 + s - 8t + 8u = -3 \Rightarrow x_1 = -3 - s + 8t - 8u$ .

Thus the solution set is given by

$$\boxed{x_1 = -3 - s + 8t - 8u, \quad x_2 = 2 + 2s - 3t + 2u, \quad x_3 = s, \quad x_4 = t, \quad x_5 = u, \quad s, t, u \in \mathbb{R}}$$

which is a 3-dimensional linear space.

**Problem 4:** Each of the following matrices is in reduced row-echelon form and represents a system of linear equations. Write down the solution set for the system of equations associated to each matrix.

1.  $\left[ \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$

*Solution:* Since there is pivot in the column vector of constant terms, there is no solution.

2.  $\left[ \begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$

*Solution:* The variable  $x_2$  is free. So we let  $x_2 = t$ . The second row gives  $x_3 = -3$ . The first row gives  $x_1 + 2x_2 = 0 \Rightarrow x_1 = -2t$ . Thus, the solution set is  $\{(-2t, t, 0) : t \in \mathbb{R}\}$

3.  $\left[ \begin{array}{ccc|c} 1 & 0 & -2 & 3 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{array} \right]$

*Solution:* The variable  $x_3$  is free. So we let  $x_3 = t$ . The second row gives  $x_2 + 4x_3 = -5 \Rightarrow x_2 = -5 - 4t$ . The first row gives  $x_1 - 2x_3 = 3 \Rightarrow x_1 = 3 + 2t$ .

Thus, the solution set is  $\{(3 + 2t, -5 - 4t, t) : t \in \mathbb{R}\}$

4.  $\left[ \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{array} \right]$

*Solution:* All variables are basic. The solution set is the point  $(2, -3, 0)$

5.  $\left[ \begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$

*Solution:* The variables  $x_2$  and  $x_3$  are free. So we let  $x_2 = s$ ,  $x_3 = t$ .

The first row gives  $x_1 + 2x_2 - 3x_3 = 4 \Rightarrow x_1 = 4 - 2s + 3t$ .

Thus, the solution set is  $\boxed{\{(4 - 2s + 3t, s, t) : s, t \in \mathbb{R}\}}$

$$6. \left[ \begin{array}{ccc|c} 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

*Solution:* The variable  $x_1$  is free. So we let  $x_1 = t$ . The second row gives  $x_3 = 4$  and the first row gives  $x_2 = -3$ . Thus, the solution set is  $\boxed{\{(t, -3, 4) : t \in \mathbb{R}\}}$

$$7. \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right]$$

*Solution:* All variables are basic. Thus, the solution set is the single point  $\boxed{(2, 7, -3, 1)}$

$$8. \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 3 & 0 & 7 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

*Solution:* The variable  $x_3$  is free. So we let  $x_3 = t$ . The third row gives  $x_4 = -3$ .

The second row gives  $x_2 + 3x_3 = 7 \Rightarrow x_2 = 7 - 3t$ . The first row gives  $x_1 = 2$ .

Thus, the solution set is  $\boxed{\{(2, 7 - 3t, t, -3) : t \in \mathbb{R}\}}$

$$9. \left[ \begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

*Solution:* The variable  $x_2$  is free. So we let  $x_2 = t$ . The third row gives  $x_4 = -3$ .

The second row gives  $x_3 = 7$ . The first row gives  $x_1 + 2x_2 = 2 \Rightarrow x_1 = 2 - 2t$ .

Thus, the solution set is  $\boxed{\{(2 - 2t, t, 7, -3) : t \in \mathbb{R}\}}$

$$10. \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

*Solution:* The variables  $x_3$  and  $x_4$  are free. So we let  $x_3 = s$  and  $x_4 = t$ .

The second row gives  $x_2 = 7$  and the first row gives  $x_1 = 2$ .

Thus, the solution set is  $\boxed{(2, 7, s, t) : s, t \in \mathbb{R}}$

**Problem 5:** Suppose we start with a system of three linear equations in three variables, write down the associated augmented matrix, put it in reduced row-echelon form and find the solution set. For each solution set below, find the reduced row-echelon matrix it came from.

1. The solution set is the point  $(-4, 3, 1)$ .

*Solution:* 
$$\left[ \begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

2. The solution set is the line  $x = 1 - 4t, y = 3 + 3t, z = 2 - t$ .

*Solution:* Let  $z = 2 - t = s$ . Then  $t = 2 - s \Rightarrow y = 3 + 3t = 3 + 3(2 - s) = 9 - 3s$   
 $\Rightarrow x = 1 - 4t = 1 - 4(2 - s) = -7 + 4s$ . Thus the given line can also be represented by the parametric equation  $x = -7 + 4s, y = 9 - 3s$  and  $z = s$ . So the variable  $z$  is free and the corresponding reduced row-echelon matrix would be

$$\left[ \begin{array}{ccc|c} 1 & 0 & -4 & -7 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

3. The solution set is the line  $x = 3 + 4t, y = 1 + 4t, z = 5$ .

*Solution:* Let  $y = 1 + 4t = s$ . Then  $4t = s - 1 \Rightarrow x = 3 + 4t = 3 + s - 1 = 2 + s$ . Thus, the given line can also be represented by the parametric equation  $x = 2 + s, y = s, z = 5$ . So the variable  $y$  is free and the corresponding reduced row-echelon matrix would be

$$\left[ \begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

4. The solution set is the plane  $3x - 4y + 6z = 9$ .

*Solution:* Choose a parametrization of the given equation of the plane. Let  $y = s, z = t$ . Then  $3x = 9 + 4s - 6t \Rightarrow x = 3 + (4/3)s - 2t$ . Thus, the corresponding reduced row-echelon matrix would be

$$\left[ \begin{array}{ccc|c} 1 & -4/3 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

## 6. Matrix Arithmetic

**Problem 1:** Given the matrices

$$E = \begin{bmatrix} -1 & 3 & 5 \\ -4 & 2 & 2 \end{bmatrix} \text{ and } F = \begin{bmatrix} 3 & -5 & 1 \\ 6 & -2 & -1 \end{bmatrix}$$

compute  $E + F$ ,  $3E$  and  $4E - 2F$ .

*Solution:*

$$E + F = \begin{bmatrix} -1+3 & 3-5 & 5+1 \\ -4+6 & 2-2 & 2-1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 6 \\ 2 & 0 & 1 \end{bmatrix}$$

$$3E = \begin{bmatrix} 3(-1) & 3(3) & 3(5) \\ 3(-4) & 3(2) & 3(2) \end{bmatrix} = \begin{bmatrix} -3 & 9 & 15 \\ -12 & 6 & 6 \end{bmatrix}$$

$$4E - 2F = \begin{bmatrix} 4(-1) - 2(3) & 4(3) - 2(-5) & 4(5) - 2(1) \\ 4(-4) - 2(6) & 4(2) - 2(-2) & 4(2) - 2(-1) \end{bmatrix} = \begin{bmatrix} -10 & 22 & 18 \\ -28 & 12 & 10 \end{bmatrix}$$

**Problem 2:** Given the matrices

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 5 \\ -4 & 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -5 \\ 6 & -2 \\ 0 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

compute all possible products of pairs of matrices.

*Solution:* The possible products are  $A^2$ ,  $AB$ ,  $BC$ ,  $BD$ ,  $CA$ ,  $CB$ ,  $DC$  and  $D^2$ .

$$A^2 = \begin{bmatrix} 2(2) + 1(-1) & 2(1) + 1(3) \\ -1(2) + 3(-1) & -1(1) + 3(3) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2(-1) + 1(-4) & 2(3) + 1(2) & 2(5) + 1(2) \\ -1(-1) + 3(-4) & -1(3) + 3(2) & -1(5) + 3(2) \end{bmatrix} = \begin{bmatrix} -6 & 8 & 12 \\ -11 & 3 & 1 \end{bmatrix}$$

$$BC = \begin{bmatrix} -1(3) + 3(6) + 5(0) & -1(-5) + 3(-2) + 5(4) \\ -4(3) + 2(6) + 2(0) & -4(-5) + 2(-2) + 2(4) \end{bmatrix} = \begin{bmatrix} 15 & 19 \\ 0 & 24 \end{bmatrix}$$

$$BD = \begin{bmatrix} -1(2) + 3(-1) + 5(0) & -1(3) + 3(2) + 5(2) & -1(-2) + 3(3) + 5(4) \\ -4(2) + 2(-1) + 2(0) & -4(3) + 2(2) + 2(2) & -4(-2) + 2(3) + 2(4) \end{bmatrix} = \begin{bmatrix} -5 & 13 & 31 \\ -10 & -4 & 22 \end{bmatrix}$$

$$CA = \begin{bmatrix} 3(2) - 5(-1) & 3(1) - 5(3) \\ 6(2) - 2(-1) & 6(1) - 2(3) \\ 0(2) + 4(-1) & 0(1) + 4(3) \end{bmatrix} = \begin{bmatrix} 11 & -12 \\ 14 & 0 \\ -4 & 12 \end{bmatrix}$$

$$CB = \begin{bmatrix} 3(-1) - 5(-4) & 3(3) - 5(2) & 3(5) - 5(2) \\ 6(-1) - 2(-4) & 6(3) - 2(2) & 6(5) - 2(2) \\ 0(-1) + 4(-4) & 0(3) + 4(2) & 0(5) + 4(2) \end{bmatrix} = \begin{bmatrix} 17 & -1 & 5 \\ 2 & 14 & 26 \\ -16 & 8 & 8 \end{bmatrix}$$

$$DC = \begin{bmatrix} 2(3) + 3(6) - 2(0) & 2(-5) + 3(-2) - 2(4) \\ -1(3) + 2(6) + 3(0) & -1(-5) + 2(-2) + 3(4) \\ 0(3) + 2(6) + 4(0) & 0(-5) + 2(-2) + 4(4) \end{bmatrix} = \begin{bmatrix} 24 & -24 \\ 9 & 13 \\ 12 & 12 \end{bmatrix}$$

$$D^2 = \begin{bmatrix} 2(2) + 3(-1) - 2(0) & 2(3) + 3(2) - 2(2) & 2(-2) + 3(3) - 2(4) \\ -1(2) + 2(-1) + 3(0) & -1(3) + 2(2) + 3(2) & -1(-2) + 2(3) + 3(4) \\ 0(2) + 2(-1) + 4(0) & 0(3) + 2(2) + 4(2) & 0(-2) + 2(3) + 4(4) \end{bmatrix} = \begin{bmatrix} 1 & 8 & -3 \\ -4 & 7 & 20 \\ -2 & 12 & 22 \end{bmatrix}$$

**Problem 3:** Given the matrices  $A$  and  $D$  from the previous problem compute  $A^2 - 5A$  and  $D^3 + 2D$ .

*Solution:* As computed before  $A^2 = \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix}$ . Thus,

$$A^2 - 5A = \begin{bmatrix} 3 - 5(2) & 5 - 5(1) \\ -5 - 5(-1) & 8 - 5(3) \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}$$

As computed before  $D^2 = \begin{bmatrix} 1 & 8 & -3 \\ -4 & 7 & 20 \\ -2 & 12 & 22 \end{bmatrix}$ . Then we have

$$\begin{aligned} D^3 = D^2.D &= \begin{bmatrix} 1(2) + 8(-1) - 3(0) & 1(3) + 8(2) - 3(2) & 1(-2) + 8(3) - 3(4) \\ -4(2) + 7(-1) + 20(0) & -4(3) + 7(2) + 20(2) & -4(-2) + 7(3) + 20(4) \\ -2(2) + 12(-1) + 22(0) & -2(3) + 12(2) + 22(2) & -2(-2) + 12(3) + 22(4) \end{bmatrix} \\ &= \begin{bmatrix} -6 & 13 & 10 \\ -15 & 42 & 109 \\ -16 & 62 & 126 \end{bmatrix} \end{aligned}$$

Thus,

$$D^3 + 2D = \begin{bmatrix} -6 + 2(2) & 13 + 2(3) & 10 + 2(-2) \\ -15 + 2(-1) & 42 + 2(2) & 109 + 2(3) \\ -16 + 2(0) & 62 + 2(2) & 126 + 2(4) \end{bmatrix} = \begin{bmatrix} -2 & 19 & 6 \\ -17 & 46 & 115 \\ -16 & 66 & 134 \end{bmatrix}$$

## 7. The Multiplicative Identity and Inverse of Matrices

**Problem 1:** Determine whether each matrix is invertible or not and find the inverse matrix for each invertible matrix. Check your answer by multiplying the matrices.

1.  $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{aligned} \text{Solution: } [A|I_2] &= \left[ \begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - R_1} \left[ \begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow -R_2} \left[ \begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array} \right] \\ &\xrightarrow{R_1 \rightarrow R_1 - R_2} \left[ \begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{array} \right]. \text{ Thus, } A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

2.  $\begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}$

$$\begin{aligned} \text{Solution: } [A|I_2] &= \left[ \begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \rightarrow -R_1} \left[ \begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 1 & 5 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - R_1} \left[ \begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 7 & 1 & 1 \end{array} \right] \\ &\xrightarrow{R_2 \rightarrow (1/7)R_2} \left[ \begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 1 & 1/7 & 1/7 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 + 2R_2} \left[ \begin{array}{cc|cc} 1 & 0 & -5/7 & 2/7 \\ 0 & 1 & 1/7 & 1/7 \end{array} \right] \\ \text{Thus, } A^{-1} &= \begin{bmatrix} -5/7 & 2/7 \\ 1/7 & 1/7 \end{bmatrix} \end{aligned}$$

3.  $\begin{bmatrix} 3 & -4 \\ 6 & 8 \end{bmatrix}$

$$\begin{aligned} \text{Solution: } [A|I_2] &= \left[ \begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \rightarrow (1/3)R_1} \left[ \begin{array}{cc|cc} 1 & -4/3 & 1/3 & 0 \\ 6 & 8 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - 6R_1} \left[ \begin{array}{cc|cc} 1 & -4/3 & 1/3 & 0 \\ 0 & 16 & -2 & 1 \end{array} \right] \\ &\xrightarrow{R_2 \rightarrow (1/16)R_2} \left[ \begin{array}{cc|cc} 1 & -4/3 & 1/3 & 0 \\ 0 & 1 & -1/8 & 1/16 \end{array} \right] \xrightarrow{R_1 \rightarrow R_1 + (4/3)R_2} \left[ \begin{array}{cc|cc} 1 & 0 & 1/6 & 1/12 \\ 0 & 1 & -1/8 & 1/16 \end{array} \right] \\ \text{Thus, } A^{-1} &= \begin{bmatrix} 1/6 & 1/12 \\ -1/8 & 1/16 \end{bmatrix} \end{aligned}$$

4.  $\begin{bmatrix} 1/2 & -1/3 \\ 1/4 & 1/2 \end{bmatrix}$

$$\begin{aligned} \text{Solution: } [A|I_2] &= \left[ \begin{array}{cc|cc} 1/2 & -1/3 & 1 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \rightarrow 2R_1} \left[ \begin{array}{cc|cc} 1 & -2/3 & 2 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array} \right] \\ &\xrightarrow{R_2 \rightarrow R_2 - (1/4)R_1} \left[ \begin{array}{cc|cc} 1 & -2/3 & 2 & 0 \\ 0 & 2/3 & -1/2 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow (3/2)R_2} \left[ \begin{array}{cc|cc} 1 & -2/3 & 2 & 0 \\ 0 & 1 & -3/4 & 3/2 \end{array} \right] \\ &\xrightarrow{R_1 \rightarrow R_1 + (2/3)R_2} \left[ \begin{array}{cc|cc} 1 & 0 & 3/2 & 1 \\ 0 & 1 & -3/4 & 3/2 \end{array} \right] \text{ Thus, } A^{-1} = \begin{bmatrix} 3/2 & 1 \\ -3/4 & 3/2 \end{bmatrix} \end{aligned}$$

**Problem 2:** Determine whether each matrix is invertible or not and find the inverse matrix for each invertible matrix. Check your answer by multiplying the matrices.

1.  $\begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$

$$\begin{aligned} \text{Solution: } [A|I_3] &= \left[ \begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 - 3R_1]{R_2 \rightarrow R_2 - R_1} \left[ \begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 2 & 12 & -3 & 0 & 1 \end{array} \right] \\ &\xrightarrow[R_3 \rightarrow R_3 - 2R_2]{R_1 \rightarrow R_1 - R_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & -7 & 2 & -1 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & -2 & 1 \end{array} \right] \xrightarrow{R_3 \rightarrow (1/2)R_3} \left[ \begin{array}{ccc|ccc} 1 & 0 & -7 & 2 & -1 & 0 \\ 0 & 1 & 5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & -1 & 1/2 \end{array} \right] \\ &\xrightarrow[R_2 \rightarrow R_2 - 5R_3]{R_1 \rightarrow R_1 + 7R_3} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & -8 & 7/2 \\ 0 & 1 & 0 & 3/2 & 6 & -5/2 \\ 0 & 0 & 1 & -1/2 & -1 & 1/2 \end{array} \right]. \text{ Thus, } A^{-1} = \begin{bmatrix} -3/2 & -8 & 7/2 \\ 3/2 & 6 & -5/2 \\ -1/2 & -1 & 1/2 \end{bmatrix} \end{aligned}$$

2.  $\begin{bmatrix} -2 & 1 & 3 \\ 3 & -1 & 5 \\ 12 & -5 & 1 \end{bmatrix}$

$$\begin{aligned} \text{Solution: } [A|I_3] &= \left[ \begin{array}{ccc|ccc} -2 & 1 & 3 & 1 & 0 & 0 \\ 3 & -1 & 5 & 0 & 1 & 0 \\ 12 & -5 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \rightarrow (-1/2)R_1} \left[ \begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 3 & -1 & 5 & 0 & 1 & 0 \\ 12 & -5 & 1 & 0 & 0 & 1 \end{array} \right] \\ &\xrightarrow[R_3 \rightarrow R_3 - 12R_1]{R_2 \rightarrow R_2 - 3R_1} \left[ \begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1/2 & 19/2 & 3/2 & 1 & 0 \\ 0 & 1 & 19 & 6 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow 2R_2} \left[ \begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1 & 19 & 3 & 2 & 0 \\ 0 & 1 & 19 & 6 & 0 & 1 \end{array} \right] \\ &\xrightarrow[R_3 \rightarrow R_3 - R_2]{R_1 \rightarrow R_1 + (1/2)R_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 8 & 1 & 1 & 0 \\ 0 & 1 & 19 & 3 & 2 & 0 \\ 0 & 0 & 0 & 3 & -2 & 1 \end{array} \right] \end{aligned}$$

Since we have an all zero row, the given matrix is singular.

3.  $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 8 \\ 2 & 5 & 3 \end{bmatrix}$

$$\begin{aligned} \text{Solution: } [A|I_3] &= \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 8 & 0 & 1 & 0 \\ 2 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \\ &\xrightarrow[R_3 \rightarrow R_3 - 2R_1]{R_2 \rightarrow R_2 - R_1} \left[ \begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 1 & 0 \\ 0 & 2 & -5 & 1 & -1 & 0 \\ 0 & 5 & -13 & 0 & -2 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow (1/2)R_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/2 & -1/2 & 0 \\ 0 & 5 & -13 & 0 & -2 & 1 \end{array} \right] \\ &\xrightarrow{R_3 \rightarrow R_3 - 5R_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & -1/2 & -5/2 & 1/2 & 1 \end{array} \right] \xrightarrow{R_3 \rightarrow -2R_3} \left[ \begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 1 & 0 \\ 0 & 1 & -5/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 5 & -1 & -2 \end{array} \right] \\ &\xrightarrow[R_2 \rightarrow R_2 + (5/2)R_3]{R_1 \rightarrow R_1 - 8R_3} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 9 & 16 \\ 0 & 1 & 0 & 13 & -3 & -5 \\ 0 & 0 & 1 & 5 & -1 & -2 \end{array} \right]. \text{ Thus, } A^{-1} = \begin{bmatrix} -40 & 9 & 16 \\ 13 & -3 & -5 \\ 5 & -1 & -2 \end{bmatrix} \end{aligned}$$

$$4. \begin{bmatrix} 1/3 & 1/4 & 0 \\ -1/3 & 1/4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{aligned} \text{Solution: } [A|I_3] &= \left[ \begin{array}{ccc|ccc} 1/3 & 1/4 & 0 & 1 & 0 & 0 \\ -1/3 & 1/4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \rightarrow 3R_1} \left[ \begin{array}{ccc|ccc} 1 & 3/4 & 0 & 3 & 0 & 0 \\ -1/3 & 1/4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array} \right] \\ &\xrightarrow{R_2 \rightarrow R_2 + (1/3)R_1} \left[ \begin{array}{ccc|ccc} 1 & 3/4 & 0 & 3 & 0 & 0 \\ 0 & 1/2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow 2R_2} \left[ \begin{array}{ccc|ccc} 1 & 3/4 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array} \right] \\ &\xrightarrow{R_1 \rightarrow R_1 - (3/4)R_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -3/2 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3 \rightarrow (1/3)R_3} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3/2 & -3/2 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array} \right] \\ \text{Thus, } A^{-1} &= \begin{bmatrix} 3/2 & -3/2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \end{aligned}$$

**Problem 3:** Use an inverse matrix from above to solve the system of linear equations

$$\begin{aligned} -x + 2y &= 3 \\ x + 5y &= -1 \end{aligned}$$

*Solution:* The given system is  $AX = b$  where  $A = \begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}$ ,  $X = \begin{bmatrix} x \\ y \end{bmatrix}$  and  $b = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ .

As computed earlier  $A^{-1} = \begin{bmatrix} -5/7 & 2/7 \\ 1/7 & 1/7 \end{bmatrix}$  and we have  $X = A^{-1}b$

$$\text{Thus, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/7 & 2/7 \\ 1/7 & 1/7 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -17/7 \\ 2/7 \end{bmatrix} \Rightarrow x = -17/7, y = 2/7.$$

**Problem 4:** Use an inverse matrix from above to solve the system of linear equations

$$\begin{aligned} x + y - 2z &= -2 \\ x + 2y + 3z &= 5 \\ 3x + 5y + 6z &= 3 \end{aligned}$$

*Solution:* The given system is  $AX = b$  where  $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ 3 & 5 & 6 \end{bmatrix}$ ,  $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $b = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}$ .

As computed earlier,  $A^{-1} = \begin{bmatrix} -3/2 & -8 & 7/2 \\ 3/2 & 6 & -5/2 \\ -1/2 & -1 & 1/2 \end{bmatrix}$  and we have  $X = A^{-1}b$

$$\text{Thus, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3/2 & -8 & 7/2 \\ 3/2 & 6 & -5/2 \\ -1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -53/2 \\ 39/2 \\ -5/2 \end{bmatrix} \Rightarrow x = -53/2, y = 39/2, z = -5/2.$$



**Problem 5:** Use an inverse matrix from above to solve the system of linear equations

$$\begin{array}{rcl} (1/3)x & + (1/4)y & = 8 \\ (-1/3)x & + (1/4)y & = -3 \\ & 3z & = 4 \end{array}$$

*Solution:* The given system is  $AX = b$  where  $A = \begin{bmatrix} 1/3 & 1/4 & 0 \\ -1/3 & 1/4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ ,  $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $b = \begin{bmatrix} 8 \\ -3 \\ 4 \end{bmatrix}$ .

As computed earlier,  $A^{-1} = \begin{bmatrix} 3/2 & -3/2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$  and we have  $X = A^{-1}b$

$$\text{Thus, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/2 & -3/2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 8 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 33/2 \\ 10 \\ 4/3 \end{bmatrix} \Rightarrow x = 33/2, y = 10, z = 4/3.$$

## 8. Determinants

**Problem 1:** Compute the determinants of the matrices

1.  $\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$

*Solution:*  $\begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = 3(4) - 1(-2) = 14$

2.  $\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$

*Solution:*  $\begin{vmatrix} -4 & 6 \\ 2 & -3 \end{vmatrix} = -4(-3) - 2(6) = 12 - 12 = 0$

**Problem 2:** Compute the determinants of the matrices in two ways, first by expanding along a row and then by expanding down a column.

1.  $\begin{bmatrix} 3 & -2 & -1 \\ 1 & -4 & 2 \\ 3 & 0 & -5 \end{bmatrix}$

*Solution:* Expanding by the third row we have

$$\begin{vmatrix} 3 & -2 & -1 \\ 1 & -4 & 2 \\ 3 & 0 & -5 \end{vmatrix} = 3 \begin{vmatrix} -2 & -1 \\ -4 & 2 \end{vmatrix} - 0 \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} + (-5) \begin{vmatrix} 3 & -2 \\ 1 & -4 \end{vmatrix} = 3(-8) - 0 - 5(-10) = 26$$

Expanding by the second column we have

$$\begin{vmatrix} 3 & -2 & -1 \\ 1 & -4 & 2 \\ 3 & 0 & -5 \end{vmatrix} = -(-2) \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} + (-4) \begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} - 0 \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 2(-11) - 4(-12) - 0 = 26$$

2.  $\begin{bmatrix} -2 & 3 & 1 \\ 5 & -2 & 5 \\ 1 & 4 & 7 \end{bmatrix}$

*Solution:* Expanding by the first row we have

$$\begin{vmatrix} -2 & 3 & 1 \\ 5 & -2 & 5 \\ 1 & 4 & 7 \end{vmatrix} = -2 \begin{vmatrix} -2 & 5 \\ 4 & 7 \end{vmatrix} - 3 \begin{vmatrix} 5 & 5 \\ 1 & 7 \end{vmatrix} + 1 \begin{vmatrix} 5 & -2 \\ 1 & 4 \end{vmatrix} = -2(-34) - 3(30) + 1(22) = 0$$

Expanding by the first column we have

$$\begin{vmatrix} -2 & 3 & 1 \\ 5 & -2 & 5 \\ 1 & 4 & 7 \end{vmatrix} = -2 \begin{vmatrix} -2 & 5 \\ 4 & 7 \end{vmatrix} - 5 \begin{vmatrix} 3 & 1 \\ 4 & 7 \end{vmatrix} + 1 \begin{vmatrix} 3 & 1 \\ -2 & 5 \end{vmatrix} = -2(-34) - 5(17) + 1(17) = 0$$

**Problem 3:** Compute the determinants of the matrices

1. 
$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

*Solution:* Expanding along first row or first column we have

$$\begin{vmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{vmatrix} = 3 \begin{vmatrix} -4 & 0 \\ 0 & -2 \end{vmatrix} = 3(-4 \times -2) = 24$$

2. 
$$\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$$

*Solution:* Expanding along first row or first column we have

$$\begin{vmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{vmatrix} = A_{11} \begin{vmatrix} A_{22} & 0 \\ 0 & A_{33} \end{vmatrix} = A_{11}A_{22}A_{33}$$

**Problem 4:** Compute the determinants of the matrices

1. 
$$\begin{bmatrix} 3 & -2 & 1 \\ 0 & -4 & 9 \\ 0 & 0 & -2 \end{bmatrix}$$

*Solution:* Expanding along first column we have

$$\begin{vmatrix} 3 & -2 & 1 \\ 0 & -4 & 9 \\ 0 & 0 & -2 \end{vmatrix} = 3 \begin{vmatrix} -4 & 9 \\ 0 & -2 \end{vmatrix} = 3(8 - 0) = 24$$

2. 
$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}$$

*Solution:* Expanding along first column we have

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{vmatrix} = A_{11} \begin{vmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{vmatrix} = A_{11}A_{22}A_{33}$$

**Problem 5:** Compute the determinants of the matrices

1. 
$$\begin{bmatrix} 3 & 0 & 0 \\ 5 & -4 & 0 \\ 7 & -9 & -2 \end{bmatrix}$$

*Solution:* Expanding along first row we have

$$\begin{vmatrix} 3 & 0 & 0 \\ 5 & -4 & 0 \\ 7 & -9 & -2 \end{vmatrix} = 3 \begin{vmatrix} -4 & 0 \\ -9 & -2 \end{vmatrix} = 3(8 - 0) = 24$$

2.  $\begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

*Solution:* Expanding along the first row we have

$$\begin{vmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = A_{11} \begin{vmatrix} A_{22} & 0 \\ A_{32} & A_{33} \end{vmatrix} = A_{11}A_{22}A_{33}$$

**Problem 6:** Compute the determinant of the matrix in two ways, first by expanding along a row and then by expanding down a column.

$$\begin{bmatrix} 2 & -3 & 1 & 5 \\ -1 & 4 & 0 & 2 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

*Solution:* Expanding along the fourth row we have

$$\begin{vmatrix} 2 & -3 & 1 & 5 \\ -1 & 4 & 0 & 2 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & 1 \end{vmatrix} = -3 \begin{vmatrix} -3 & 1 & 5 \\ 4 & 0 & 2 \\ 2 & -2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 2 \\ 0 & -2 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & -3 & 5 \\ -1 & 4 & 2 \\ 0 & 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -3 & 1 \\ -1 & 4 & 0 \\ 0 & 2 & -2 \end{vmatrix}$$

$$\begin{vmatrix} -3 & 1 & 5 \\ 4 & 0 & 2 \\ 2 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -3 & 1 & 5 \\ 4 & 0 & 2 \\ -4 & 0 & 11 \end{vmatrix} \quad (R_3 \rightarrow R_3 + 2R_1) \Rightarrow \begin{vmatrix} -3 & 1 & 5 \\ 4 & 0 & 2 \\ 2 & -2 & 1 \end{vmatrix} = -1 \begin{vmatrix} 4 & 2 \\ -4 & 11 \end{vmatrix} = -52$$

$$\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 11 & 5 \\ -1 & 4 & 2 \\ 0 & 0 & 1 \end{vmatrix} \quad (C_2 \rightarrow C_2 + 2C_3) \Rightarrow \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 11 \\ -1 & 4 \end{vmatrix} = 19$$

$$\begin{vmatrix} 2 & -3 & 1 \\ -1 & 4 & 0 \\ 0 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 1 \\ -1 & 4 & 0 \\ 4 & -4 & 0 \end{vmatrix} \quad (R_3 \rightarrow R_3 + 2R_1) \Rightarrow \begin{vmatrix} 2 & -3 & 1 \\ -1 & 4 & 0 \\ 0 & 2 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 4 \\ 4 & -4 \end{vmatrix} = -12$$

Thus,  $\begin{vmatrix} 2 & -3 & 1 & 5 \\ -1 & 4 & 0 & 2 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & 1 \end{vmatrix} = -3(-52) + 1(19) + 1(-12) = 163$

Expanding along the third column we have

$$\begin{vmatrix} 2 & -3 & 1 & 5 \\ -1 & 4 & 0 & 2 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} -1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & -3 & 5 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -3 & 5 \\ -1 & 4 & 2 \\ 3 & 1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & -3 & 5 \\ -1 & 4 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 4 & 2 \\ 0 & 2 & 1 \\ 0 & 13 & 7 \end{vmatrix} \quad (R_3 \rightarrow R_3 + 3R_1) \Rightarrow \begin{vmatrix} -1 & 4 & 2 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{vmatrix} = -1 \begin{vmatrix} 2 & 1 \\ 13 & 7 \end{vmatrix} = -1$$

$$\begin{vmatrix} 2 & -3 & 5 \\ -1 & 4 & 2 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 5 & 9 \\ -1 & 4 & 2 \\ 0 & 13 & 7 \end{vmatrix} \quad \begin{matrix} (R_1 \rightarrow R_1 + 2R_2) \\ (R_3 \rightarrow R_3 + 3R_2) \end{matrix} \Rightarrow \begin{vmatrix} 2 & -3 & 5 \\ -1 & 4 & 2 \\ 3 & 1 & 1 \end{vmatrix} = -(-1) \begin{vmatrix} 5 & 9 \\ 13 & 7 \end{vmatrix} = -82$$

Thus,  $\begin{vmatrix} 2 & -3 & 1 & 5 \\ -1 & 4 & 0 & 2 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & 1 \end{vmatrix} = 1(-1) - 2(-82) = 163$

**Problem 7:** Given a  $2 \times 2$  matrix  $A$  with a nonzero determinant, what is the determinant of its inverse matrix  $A^{-1}$ ?

*Solution:* Using the result of problem 8 below, we have  $\det(AA^{-1}) = \det(A) \det(A^{-1})$

But  $AA^{-1} = I_2 \Rightarrow \det(A) \det(A^{-1}) = \det(I_2) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ . Therefore,  $\det(A^{-1}) = \frac{1}{\det(A)}$ .

ALTERNATIVELY, let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then  $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

$$\Rightarrow \det(A^{-1}) = \frac{1}{(ad-bc)^2} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix} = \frac{ad-bc}{(ad-bc)^2} = \frac{1}{ad-bc} = \frac{1}{\det(A)}.$$

**Problem 8:** Given two  $2 \times 2$  matrices  $A$  and  $B$  show that  $\det(AB) = \det A \det B$

*Solution:* Let  $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$  and  $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ .

$$\text{Then } AB = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}.$$

$$\begin{aligned} \Rightarrow \det(AB) &= (a_1a_2 + b_1c_2)(c_1b_2 + d_1d_2) - (a_1b_2 + b_1d_2)(c_1a_2 + d_1c_2) \\ &= \cancel{a_1c_1a_2b_2} + a_1d_1a_2d_2 + b_1c_1b_2c_2 + \cancel{b_1d_1c_2d_2} - \cancel{a_1c_1a_2b_2} - a_1d_1b_2c_2 - b_1c_1a_2d_2 - \cancel{b_1d_1c_2d_2} \\ &= a_1d_1(a_2d_2 - b_2c_2) - b_1c_1(a_2d_2 - b_2c_2) = (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) = \det(A) \det(B) \end{aligned}$$

**Problem 9:** Suppose  $A$  is a  $3 \times 3$  matrix and  $\det A = 5$ . Using the properties of  $3 \times 3$  determinants find the determinant of:-

1. a matrix obtained by interchanging two rows of  $A$ ,
2. a matrix obtained by multiplying a row by 3,
3. a matrix obtained by multiplying a row by  $-2$ ,
4. the matrix  $3A$ ,
5. the matrix  $(-2)A$ ,

6. the matrix obtained by adding the second row of  $A$  to the first row of  $A$ ,
7. the transpose of  $A$ ,  $A^T$ .

*Solution:*

1.  $-\det A = -5$
2.  $3\det A = 15$
3.  $-2\det A = -10$
4.  $3^3\det A = 135$
5.  $(-2)^3\det A = -40$
6.  $\det A = 5$
7.  $\det A^T = \det A = 5$

**Problem 10:** Given two  $3 \times 3$  matrices  $A$  and  $B$  show that  $\det(AB) = \det A \det B$ .

*Solution:* Each row operation is equivalent to left multiplying with some matrix, called an elementary matrix. The following are the elementary matrices corresponding to each row operation.

$$\begin{aligned}
 E_{12} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (R_1 \leftrightarrow R_2), \quad E_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} (R_2 \leftrightarrow R_3), \quad E_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} (R_3 \leftrightarrow R_1) \\
 E_1^\alpha &= \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (R_1 \rightarrow \alpha R_1), \quad E_2^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} (R_2 \rightarrow \alpha R_2), \quad E_3^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} (R_3 \rightarrow \alpha R_3), \\
 E_{12}^\alpha &= \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (R_1 \rightarrow R_1 + \alpha R_2), \quad E_{21}^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (R_1 \rightarrow R_2 + \alpha R_1) \\
 E_{23}^\alpha &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} (R_2 \rightarrow R_2 + \alpha R_3), \quad E_{32}^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \alpha & 1 \end{bmatrix} (R_3 \rightarrow R_3 + \alpha R_2) \\
 E_{13}^\alpha &= \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (R_1 \rightarrow R_1 + \alpha R_3), \quad E_{31}^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 0 & 1 \end{bmatrix} (R_3 \rightarrow R_3 + \alpha R_1)
 \end{aligned}$$

Note that  $E_{ij}^{-1} = E_{ij}$ ,  $(E_i^\alpha)^{-1} = E_i^{1/\alpha}$ ,  $(E_{ij}^\alpha)^{-1} = E_{ij}^{-\alpha}$  and that  $\det(E_{ij}) = -1$ ,  $\det(E_i^\alpha) = \alpha$ ,  $\det(E_{ij}^\alpha) = 1$ .

Also note that if  $E$  is any elementary matrix then  $\det(EB) = \det(E) \det(B)$ .

Now let  $E_A$  be the reduced row echelon form of  $A$  obtained by some  $s$  number of elementary row operations then  $E_A = E_s E_{s-1} \cdots E_2 E_1 A$  where  $E_i$ ,  $1 \leq i \leq s$  are elementary matrices.

So we have  $A = E_1^{-1}E_2^{-1} \cdots E_s^{-1}E_A$  where  $E_i^{-1}$ ,  $1 \leq i \leq s$  are again elementary matrices.

Thus,  $\det(AB) = \det(E_1^{-1} \cdots E_s^{-1}E_AB) = \det(E_1^{-1}) \cdots \det(E_s^{-1}) \det(E_AB)$

and  $\det(A) = \det(E_1^{-1}) \cdots \det(E_s^{-1}) \det(E_A)$ .

Now, if  $E_A$  has a zero row then so does  $E_AB$ , in which case  $\det(E_A) = 0 = \det(E_AB)$  and we have  $\det(AB) = 0 = \det(A) \Rightarrow \det(AB) = \det(A) \det(B) = 0$ .

Else, if  $E_A$  does not have a zero row, then  $E_A = I_3 \Rightarrow E_AB = I_3B = B$  and we have  $\det(AB) = \det(E_1^{-1}) \cdots \det(E_s^{-1}) \det(B)$ . Also,  $\det(A) = \det(E_1^{-1}) \cdots \det(E_s^{-1}) \det(I_3)$ .

Since  $\det(I_3) = 1$ , we have  $\det(AB) = \det(A) \det(B)$ .

(The same proof also works for  $n \times n$  matrices for any  $n$ )

**Problem 11:** Given a  $3 \times 3$  matrix  $A$  with a nonzero determinant, show that the determinant of its inverse matrix  $A^{-1}$  is  $1/\det A$ .

*Solution:*  $AA^{-1} = I_3 \Rightarrow \det(AA^{-1}) = \det(I_3) = 1$ . Using the result of previous problem we have  $\det(A) \det(A^{-1}) = 1$ . Since  $\det(A)$  is nonzero, we can divide both sides by  $\det(A)$ .

Therefore,  $\det(A^{-1}) = \frac{1}{\det(A)}$ .

**Problem 12:** Given a  $3 \times 3$  matrix  $A$  with a nonzero determinant, define the  $3 \times 3$  matrix  $C$  by  $C_{ij} = (-1)^{i+j} \det A(i|j)$  and show that  $AC^T = (\det A)I_3$ .

*Solution:* The diagonal entries  $(AC^T)_{ii}$  of  $AC^T$  for  $1 \leq i \leq 3$  are given by

$$(AC^T)_{ii} = A_{i1}(C^T)_{1i} + A_{i2}(C^T)_{2i} + A_{i3}(C^T)_{3i} = A_{i1}C_{i1} + A_{i2}C_{i2} + A_{i3}C_{i3}$$

$$\Rightarrow (AC^T)_{ii} = A_{i1}(-1)^{i+1} \det A(i|1) + A_{i2}(-1)^{i+2} \det A(i|2) + A_{i3}(-1)^{i+3} \det A(i|3)$$

which is the formula of  $\det(A)$  when expanding by the  $i^{\text{th}}$  row. Thus,  $(AC^T)_{ii} = \det(A)$ .

Now if we look at the off-diagonal entry  $(AC^T)_{ij}$  for  $i \neq j$ . Then we have

$$(AC^T)_{ij} = A_{i1}(-1)^{j+1} \det A(j|1) + A_{i2}(-1)^{j+2} \det A(j|2) + A_{i3}(-1)^{j+3} \det A(j|3)$$

which is the formula (when expanding by  $j^{\text{th}}$  row) for determinant of a matrix whose  $i^{\text{th}}$  and  $j^{\text{th}}$  row are the same and we know that determinant of such a matrix is 0.

$$\text{Thus, } AC^T = \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix} = \det(A)I_3.$$

## 9. Matrices as Functions

**Problem 1:** Carry out the computations on both sides of the equation to see that they are equal.

$$\begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \left( \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$$

*Solution:*

$$\begin{aligned} \text{LHS} &= \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \left( \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \\ 19 \end{bmatrix} \\ \text{RHS} &= \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \\ -10 \end{bmatrix} + \begin{bmatrix} -14 \\ -18 \\ 29 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \\ 19 \end{bmatrix} \end{aligned}$$

**Problem 2:** Carry out the computations on both sides of the equation to see that they are equal.

$$\begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \left( 5 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right) = 5 \left( \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right)$$

*Solution:*

$$\begin{aligned} \text{LHS} &= \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \left( 5 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right) = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ -15 \end{bmatrix} = \begin{bmatrix} 55 \\ 50 \\ -50 \end{bmatrix} \\ \text{RHS} &= 5 \left( \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & -2 \\ -5 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right) = 5 \begin{bmatrix} 11 \\ 10 \\ -10 \end{bmatrix} = \begin{bmatrix} 55 \\ 50 \\ -50 \end{bmatrix} \end{aligned}$$

**Problem 3:** Given the matrix  $A = \begin{bmatrix} 3 & 1 \\ 4 & 0 \\ -5 & 2 \end{bmatrix}$  determine which of the following vectors are in the image of  $A$ :-

$$\begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} -5 \\ -8 \\ 12 \end{bmatrix}$$

*Solution:* Form the augmented matrix  $[A|v_1 \ v_2 \ v_3]$  and bring  $A$  in reduced row-echelon form.

$$\left[ \begin{array}{cc|cc} 3 & 1 & 4 & 5 & -5 \\ 4 & 0 & 4 & 3 & -8 \\ -5 & 2 & -3 & 2 & 12 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[ \begin{array}{cc|cc} 4 & 0 & 4 & 3 & -8 \\ 3 & 1 & 4 & 5 & -5 \\ -5 & 2 & -3 & 2 & 12 \end{array} \right] \xrightarrow{R_2 \rightarrow (1/4)R_2} \left[ \begin{array}{cc|cc} 1 & 0 & 1 & 3/4 & -2 \\ 3 & 1 & 4 & 5 & -5 \\ -5 & 2 & -3 & 2 & 12 \end{array} \right]$$



$$\xrightarrow[R_3 \rightarrow R_3 + 5R_1]{R_2 \rightarrow R_2 - 3R_1} \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 3/4 & -2 \\ 0 & 1 & 1 & 11/4 & 1 \\ 0 & 2 & 2 & 23/4 & 2 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 3/4 & -2 \\ 0 & 1 & 1 & 11/4 & 1 \\ 0 & 0 & 0 & 1/4 & 0 \end{array} \right]$$

We see that for the first and third vectors there are infinitely many solutions while the for the second vector there are no solutions. Therefore, the first and third vector lie in the image of  $A$  but the second vector does not.

**Problem 4:** Given the matrix  $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ -3 & 3 & 1 \end{bmatrix}$  determine which of the following vectors are in the image of  $A$ :-

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -5 \\ -4 \\ -3 \end{bmatrix}$$

*Solution:* Form the augmented matrix  $[A|v_1 \ v_2 \ v_3]$  and bring  $A$  in reduced row-echelon form.

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -4 & -5 \\ 2 & -1 & -2 & 1 & 2 & -4 \\ -3 & 3 & 1 & 0 & 3 & -3 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 + 3R_1]{R_2 \rightarrow R_2 - 2R_1} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & -4 & -5 \\ 0 & 1 & -2 & -1 & 10 & 6 \\ 0 & 0 & 1 & 3 & -9 & -18 \end{array} \right]$$

$$\xrightarrow{R_1 \rightarrow R_1 + R_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 6 & 1 \\ 0 & 1 & -2 & -1 & 10 & 6 \\ 0 & 0 & 1 & 3 & -9 & -18 \end{array} \right] \xrightarrow[R_2 \rightarrow R_2 + 2R_3]{R_1 \rightarrow R_1 + 2R_3} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -12 & -35 \\ 0 & 1 & 0 & 5 & -8 & -30 \\ 0 & 0 & 1 & 3 & -9 & -18 \end{array} \right]$$

We see that for all the three given vectors there is a unique solution. Therefore, all the three vectors lie in the image of  $A$ .

**Problem 5:** In the previous problem use the inverse of  $A$  to find the vectors in  $\mathbb{R}^3$  that are mapped by  $A$  to the three vectors listed.

*Solution:* Find the inverse of  $A$ .

$$[A|I_3] = \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & -1 & -2 & 0 & 1 & 0 \\ -3 & 3 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow[R_3 \rightarrow R_3 + 3R_1]{R_2 \rightarrow R_2 - 2R_1} \left[ \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{R_1 \rightarrow R_1 + R_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array} \right] \xrightarrow[R_2 \rightarrow R_2 + 2R_3]{R_1 \rightarrow R_1 + 2R_3} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 1 & 2 \\ 0 & 1 & 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array} \right]$$

Thus,  $A^{-1} = \begin{bmatrix} 5 & 1 & 2 \\ 4 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$  and we have:-

$$A^{-1}v_1 = \begin{bmatrix} 5 & 1 & 2 \\ 4 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$$

$$A^{-1}v_2 = \begin{bmatrix} 5 & 1 & 2 \\ 4 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -12 \\ -8 \\ -9 \end{bmatrix}$$

$$A^{-1}v_3 = \begin{bmatrix} 5 & 1 & 2 \\ 4 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ -4 \\ -3 \end{bmatrix} = \begin{bmatrix} -35 \\ -30 \\ -18 \end{bmatrix}$$

Note that these three vector appeared as column vectors to the right of the vertical line in the final augmented matrix of the previous solution.

**Problem 6:** Given the matrix  $B = \begin{bmatrix} 5 & 3 & -2 \\ 2 & 0 & 4 \\ 3 & 1 & 2 \end{bmatrix}$  determine which of the following vectors are in the image of  $B$ :-

$$\begin{bmatrix} 9 \\ 6 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

*Solution:* Form the augmented matrix  $[A|v_1 \ v_2 \ v_3]$  and bring  $A$  in reduced row-echelon form.

$$\begin{aligned} & \left[ \begin{array}{ccc|ccc} 5 & 3 & -2 & 9 & 1 & 1 \\ 2 & 0 & 4 & 6 & 1 & -1 \\ 3 & 1 & 2 & 7 & 1 & 1 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[ \begin{array}{ccc|ccc} 2 & 0 & 4 & 6 & 1 & -1 \\ 5 & 3 & -2 & 9 & 1 & 1 \\ 3 & 1 & 2 & 7 & 1 & 1 \end{array} \right] \\ & \xrightarrow{R_1 \rightarrow (1/2)R_1} \left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 1/2 & -1/2 \\ 5 & 3 & -2 & 9 & 1 & 1 \\ 3 & 1 & 2 & 7 & 1 & 1 \end{array} \right] \xrightarrow{\substack{R_2 \rightarrow R_2 - 5R_1 \\ R_3 \rightarrow R_3 - 3R_1}} \left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 1/2 & -1/2 \\ 0 & 3 & -12 & -6 & -3/2 & 7/2 \\ 0 & 1 & -4 & -2 & -1/2 & 5/2 \end{array} \right] \\ & \xrightarrow{R_1 \leftrightarrow R_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 1/2 & -1/2 \\ 0 & 1 & -4 & -2 & -1/2 & 5/2 \\ 0 & 3 & -12 & -6 & -3/2 & 7/2 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 3R_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & 1/2 & -1/2 \\ 0 & 1 & -4 & -2 & -1/2 & 5/2 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{array} \right] \end{aligned}$$

So we have infinitely many solutions for first two vectors and no solution for the third vector.

Therefore, the first two vectors are in the image of  $B$  while the third vector is not.

**Problem 7:** Let  $A$  be a  $3 \times 3$  matrix. Show that the image under the function defined by  $A$  of a line through the origin is the point  $(0, 0, 0)$  or a line through the origin. Find an example for each case.

*Solution:* Any line passing through origin can be expressed by the equation  $\vec{r} = t\vec{v}$  where  $\vec{v} \neq \vec{0}$  is the direction vector of the line and  $t \in \mathbb{R}$ . When  $A$  is applied to  $\vec{v}$  we may have two cases: Either  $\vec{w} = A\vec{v} = \vec{0}$  or  $\vec{w} = A\vec{v} \neq \vec{0}$ .

Case 1:  $\vec{w} = \vec{0} \Rightarrow A\vec{r} = A(t\vec{v}) = tA\vec{v} = t\vec{w} = \vec{0}$ . Thus, the entire line is mapped to  $(0, 0, 0)$ .

Case 2:  $\vec{w} \neq \vec{0} \Rightarrow A\vec{r} = A(t\vec{v}) = tA\vec{v} = t\vec{w}$ . Thus, any vector on the line through  $\vec{0}$  and  $\vec{v}$  is mapped to a unique vector on the line through  $\vec{0}$  and  $\vec{w}$ . Further, suppose we have some vector on the line through  $\vec{0}$  and  $\vec{w}$ , then it must be  $s\vec{w}$  for some  $s \in \mathbb{R}$ . Then we have a vector  $s\vec{v}$  on the line through  $\vec{0}$  and  $\vec{v}$  which is mapped by  $A$  to  $s\vec{w}$ , since  $A(s\vec{v}) = sA\vec{v} = s\vec{w}$ . Thus, the image of  $A$  in this case is a line through the origin.

**Problem 8:** Let  $A$  and  $B$  be  $3 \times 3$  matrices. Each defines a function from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ . Show that the matrix  $BA$  defines the function obtained by composing the function defined by  $B$  with the function defined by  $A$  (first apply  $A$  then  $B$ ). What happens when the order of applying the functions is reversed?

*Solution:* We have  $T_A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $T_A(x) = Ax$  and  $T_B : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $T_B(x) = Bx$ . Now  $T_{BA}(x) = (BA)x = B(Ax) = BT_A(x) = T_B(T_A(x)) = (T_B \circ T_A)(x)$ .

Similarly,  $T_{AB}(x) = (AB)x = A(Bx) = A(T_B(x)) = T_A(T_B(x)) = (T_A \circ T_B)(x)$ .

**Problem 9:** Let  $A$  be a  $2 \times 2$  matrix. Show that the image of the unit square  $[0, 1] \times [0, 1]$  is a parallelogram with area  $|\det A|$ . Then show that the image of any square in the plane is a parallelogram with area equal to  $\det A$  times the area of the original square.

*Solution:* Let  $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  and  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .

The unit square  $[0, 1] \times [0, 1]$  is given by  $\{ue_1 + ve_2 : 0 \leq u \leq 1, 0 \leq v \leq 1\}$ .

The image of the unit square under  $A$  is then given by  $\{uAe_1 + vAe_2 : 0 \leq u, v \leq 1\}$ .

This is the parallelogram whose adjacent sides are  $Ae_1 = \begin{bmatrix} a \\ c \end{bmatrix}$  and  $Ae_2 = \begin{bmatrix} b \\ d \end{bmatrix}$ .

The area of this parallelogram is  $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = |ad - bc| = |\det A|$ .

Now suppose we have any square (not necessarily the unit square). Since rotation does not change area we can assume the square to be  $[0, l] \times [0, l]$  where  $l$  is the length of its side. The sides of this square are given by vectors  $le_1$  and  $le_2$ . By linearity of  $A$  the sides of the parallelogram which is the image of this square under  $A$  are given by vectors  $lAe_1, lAe_2$ . The area of this parallelogram is determinant of the matrix with column vectors  $lAe_1, lAe_2$ , that is,  $\det(lA) = l^2 \det A = \det A$  times the area of the original square.

**Problem 10:** Let  $A$  be a  $3 \times 3$  matrix. Show that the image of the unit cube  $[0, 1]^3$  is a parallelepiped with volume  $|\det A|$ . Then show the image of any cube is a parallelepiped with volume equal to  $|\det A|$  times the volume of the original cube. Use this to show that if  $A$  and  $B$  are two  $3 \times 3$  matrices then  $\det(AB) = \det A \det B$ .

*Solution:* Let  $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The unit cube  $[0, 1]^3$  is given by  $\{ue_1 + ve_2 + te_3 : 0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq t \leq 1\}$ .

The image of the unit cube under  $A$  is then given by  $\{uAe_1 + vAe_2 + tAe_3 : 0 \leq u, v, t \leq 1\}$ .

This is the parallelepiped whose adjacent sides are  $Ae_1, Ae_2$  and  $Ae_3$  which are respectively the first, second and third columns of  $A$ . The volume of such a parallelepiped is  $|\det A|$ .

Now suppose we have any cube (not necessarily the unit cube). Since rotation does not change volume we can assume the cube to be  $[0, l]^3$  where  $l$  is the length of its side. The sides of this cube are given by vectors  $le_1, le_2, le_3$ . By linearity of  $A$  the sides of the parallelepiped which is the image of this cube under  $A$  are given by vectors  $lAe_1, lAe_2, lAe_3$ .

The volume of this parallelepiped is determinant of the matrix with column vectors  $lAe_1, lAe_2, lAe_3$ , which is,  $|\det(lA)| = l^3 |\det A| = |\det A|$  times the volume of the original cube.

The oriented volume of the parallelepiped, say  $P$ , which is the image under  $AB$  of the unit cube is  $\det(AB)$ . This same volume can also be computed by first applying  $B$  to the unit cube which is a parallelepiped with oriented volume  $\det B$ , and then applying  $A$  to this parallelepiped to get the parallelepiped  $P$  which will have oriented volume  $\det A$  times  $\det B$ .

## 10. Eigenvalues and Eigenvectors

**Problem 1:** Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 5 & 1 \\ 0 & -2 \end{bmatrix} \quad \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$$

*Solution:* (1)

$$A = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\text{Then } c_A(\lambda) = \begin{vmatrix} 5 - \lambda & 0 \\ 0 & -2 - \lambda \end{vmatrix} = (5 - \lambda)(-2 - \lambda).$$

$c_A(\lambda) = 0 \Rightarrow (5 - \lambda)(-2 - \lambda) = 0 \Rightarrow \lambda = 5$  or  $-2$ . Thus, the eigenvalues of  $A$  are 5 and  $-2$ .

*Notice that these are the diagonal entries of the given diagonal matrix.*

$$\text{Now let } \begin{bmatrix} x \\ y \end{bmatrix} \text{ be the eigenvector corresponding to } \lambda = 5. \text{ Then } \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$$

So we have  $5x = 5x$  and  $-2y = 5y \Rightarrow 7y = 0 \Rightarrow y = 0$  and  $x$  can be any real number.

Thus, the eigenvectors corresponding to  $\lambda = 5$  are  $\begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  where  $x \in \mathbb{R}$ .

In particular, an eigenvector corresponding to the eigenvalue of 5 is  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

$$\text{Now let } \begin{bmatrix} x \\ y \end{bmatrix} \text{ be the eigenvector corresponding to } \lambda = -2. \text{ Then } \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -2 \begin{bmatrix} x \\ y \end{bmatrix}$$

So we have  $5x = -2x$  and  $-2y = -2y \Rightarrow 7x = 0 \Rightarrow x = 0$  and  $y$  can be any real number.

Thus, the eigenvectors corresponding to  $\lambda = -2$  are  $\begin{bmatrix} 0 \\ y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  where  $y \in \mathbb{R}$ .

In particular, an eigenvector corresponding to the eigenvalue of  $-2$  is  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .

(2)

$$A = \begin{bmatrix} 5 & 1 \\ 0 & -2 \end{bmatrix}$$

$$\text{Then } c_A(\lambda) = \begin{vmatrix} 5 - \lambda & 1 \\ 0 & -2 - \lambda \end{vmatrix} = (5 - \lambda)(-2 - \lambda).$$

$c_A(\lambda) = 0 \Rightarrow (5 - \lambda)(-2 - \lambda) = 0 \Rightarrow \lambda = 5$  or  $-2$ . Thus, the eigenvalues of  $A$  are 5 and  $-2$ .

*Notice that these are the diagonal entries of the given upper triangular matrix.*

$$\text{Now let } \begin{bmatrix} x \\ y \end{bmatrix} \text{ be the eigenvector corresponding to } \lambda = 5. \text{ Then } \begin{bmatrix} 5 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$$

So we have  $5x + y = 5x$  and  $-2y = 5y \Rightarrow y = 0$  and  $x$  can be any real number.

Thus, the eigenvectors corresponding to  $\lambda = 5$  are  $\begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  where  $x \in \mathbb{R}$ .

In particular, an eigenvector corresponding to the eigenvalue of 5 is  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

Now let  $\begin{bmatrix} x \\ y \end{bmatrix}$  be the eigenvector corresponding to  $\lambda = -2$ . Then  $\begin{bmatrix} 5 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -2 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have  $5x + y = -2x$  and  $-2y = -2y \Rightarrow y = -7x$  and  $x$  can be any real number.

Thus, the eigenvectors corresponding to  $\lambda = -2$  are  $\begin{bmatrix} x \\ -7x \end{bmatrix} = x \begin{bmatrix} 1 \\ -7 \end{bmatrix}$  where  $x \in \mathbb{R}$ .

In particular, an eigenvector corresponding to the eigenvalue of  $-2$  is  $\begin{bmatrix} 1 \\ -7 \end{bmatrix}$ .

(3)

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$$

Then  $c_A(\lambda) = \begin{vmatrix} 5 - \lambda & 1 \\ 0 & 5 - \lambda \end{vmatrix} = (5 - \lambda)^2$ .

$c_A(\lambda) = 0 \Rightarrow (5 - \lambda)^2 = 0 \Rightarrow \lambda = 5$ . Thus, the only eigenvalue of  $A$  is 5.

*Notice that the diagonal entries of the given upper triangular matrix are also 5.*

Now let  $\begin{bmatrix} x \\ y \end{bmatrix}$  be the eigenvector corresponding to  $\lambda = 5$ . Then  $\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have  $5x + y = 5x$  and  $5y = 5y \Rightarrow y = 0$  and  $x$  can be any real number.

Thus, the eigenvectors corresponding to  $\lambda = 5$  are  $\begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  where  $x \in \mathbb{R}$ .

In particular, an eigenvector corresponding to the eigenvalue of 5 is  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

**Problem 2:** Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \quad \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}$$

*Solution:* (1)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix}$$

Then  $c_A(\lambda) = \begin{vmatrix} 1 - \lambda & 3 \\ 2 & -4 - \lambda \end{vmatrix} = (1 - \lambda)(-4 - \lambda) - 6 = \lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2)$ .

$c_A(\lambda) = 0 \Rightarrow (\lambda + 5)(\lambda - 2) = 0 \Rightarrow \lambda = -5$  or  $2$ . Thus, the eigenvalues of  $A$  are  $-5$  and  $2$ .

Now let  $\begin{bmatrix} x \\ y \end{bmatrix}$  be the eigenvector corresponding to  $\lambda = -5$ . Then  $\begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -5 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have  $x + 3y = -5x$  and  $2x - 4y = -5y \Rightarrow y = -2x$  and  $x$  can be any real number.

Thus, the eigenvectors corresponding to  $\lambda = -5$  are  $\begin{bmatrix} x \\ -2x \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \end{bmatrix}$  where  $x \in \mathbb{R}$ .

In particular, an eigenvector corresponding to the eigenvalue of  $-5$  is  $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ .

Now let  $\begin{bmatrix} x \\ y \end{bmatrix}$  be the eigenvector corresponding to  $\lambda = 2$ . Then  $\begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have  $x + 3y = 2x$  and  $2x - 4y = 2y \Rightarrow x = 3y$  and  $y$  can be any real number.

Thus, the eigenvectors corresponding to  $\lambda = 2$  are  $\begin{bmatrix} 3y \\ y \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  where  $y \in \mathbb{R}$ .

In particular, an eigenvector corresponding to the eigenvalue of  $2$  is  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ .

(2)

$$A = \begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix}$$

Then  $c_A(\lambda) = \begin{vmatrix} -1 - \lambda & 3 \\ -3 & 5 - \lambda \end{vmatrix} = (-1 - \lambda)(5 - \lambda) + 9 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$ .

$c_A(\lambda) = 0 \Rightarrow (\lambda - 2)^2 = 0 \Rightarrow \lambda = 2$ . Thus, the only eigenvalue of  $A$  is  $2$ .

Now let  $\begin{bmatrix} x \\ y \end{bmatrix}$  be the eigenvector corresponding to  $\lambda = 2$ . Then  $\begin{bmatrix} -1 & 3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$

So we have  $-x + 3y = 2x$  and  $-3x + 5y = 2y \Rightarrow y = x$  and  $x$  can be any real number.

Thus, the eigenvectors corresponding to  $\lambda = 2$  are  $\begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  where  $x \in \mathbb{R}$ .

In particular, an eigenvector corresponding to the eigenvalue of  $2$  is  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

(3)

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}$$

Then  $c_A(\lambda) = \begin{vmatrix} 1 - \lambda & -5 \\ 1 & -3 - \lambda \end{vmatrix} = (1 - \lambda)(-3 - \lambda) + 5 = \lambda^2 + 2\lambda + 2$ .

$c_A(\lambda) = 0$  has no real roots. Thus, the given matrix has no real eigenvalues.

**Problem 3:** Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \quad \begin{bmatrix} -4 & 5 \\ -1 & -2 \end{bmatrix} \quad \begin{bmatrix} -1 & 5 \\ -1 & 3 \end{bmatrix}$$

*Solution:* (1)

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$$

$$\text{Then } c_A(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 3 & -2-\lambda \end{vmatrix} = (-\lambda)(-2-\lambda) - 3 = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1).$$

$$c_A(\lambda) = 0 \Rightarrow (\lambda + 3)(\lambda - 1) = 0 \Rightarrow \lambda = -3 \text{ or } 1. \text{ Thus, the } \boxed{\text{eigenvalues of } A \text{ are } -3 \text{ and } 1}.$$

$$\text{Now let } \begin{bmatrix} x \\ y \end{bmatrix} \text{ be the eigenvector corresponding to } \lambda = -3. \text{ Then } \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -3 \begin{bmatrix} x \\ y \end{bmatrix}$$

So we have  $y = -3x$  and  $3x - 2y = -3y \Rightarrow y = -3x$  and  $x$  can be any real number.

$$\text{Thus, the eigenvectors corresponding to } \lambda = -3 \text{ are } \begin{bmatrix} x \\ -3x \end{bmatrix} = x \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ where } x \in \mathbb{R}.$$

$$\text{In particular, } \boxed{\text{an eigenvector corresponding to the eigenvalue of } -3 \text{ is } \begin{bmatrix} 1 \\ -3 \end{bmatrix}}.$$

$$\text{Now let } \begin{bmatrix} x \\ y \end{bmatrix} \text{ be the eigenvector corresponding to } \lambda = 1. \text{ Then } \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

So we have  $y = x$  and  $3x - 2y = y \Rightarrow y = x$  and  $x$  can be any real number.

$$\text{Thus, the eigenvectors corresponding to } \lambda = 1 \text{ are } \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } x \in \mathbb{R}.$$

$$\text{In particular, } \boxed{\text{an eigenvector corresponding to the eigenvalue of } 1 \text{ is } \begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$

(2)

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

$$\text{Then } c_A(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 3 & -\lambda \end{vmatrix} = (2-\lambda)(-\lambda) - 3 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1).$$

$$c_A(\lambda) = 0 \Rightarrow (\lambda - 3)(\lambda + 1) = 0 \Rightarrow \lambda = 3 \text{ or } -1. \text{ Thus, the } \boxed{\text{eigenvalues of } A \text{ are } 3 \text{ and } -1}.$$

$$\text{Now let } \begin{bmatrix} x \\ y \end{bmatrix} \text{ be the eigenvector corresponding to } \lambda = 3. \text{ Then } \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

So we have  $2x + y = 3x$  and  $3x = 3y \Rightarrow y = x$  and  $x$  can be any real number.

$$\text{Thus, the eigenvectors corresponding to } \lambda = 3 \text{ are } \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } x \in \mathbb{R}.$$

$$\text{In particular, } \boxed{\text{an eigenvector corresponding to the eigenvalue of } 3 \text{ is } \begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$

$$\text{Now let } \begin{bmatrix} x \\ y \end{bmatrix} \text{ be the eigenvector corresponding to } \lambda = -1. \text{ Then } \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} x \\ y \end{bmatrix}$$

So we have  $2x + y = -x$  and  $3x = -y \Rightarrow y = -3x$  and  $x$  can be any real number.

Thus, the eigenvectors corresponding to  $\lambda = -1$  are  $\begin{bmatrix} x \\ -3x \end{bmatrix} = x \begin{bmatrix} 1 \\ -3 \end{bmatrix}$  where  $x \in \mathbb{R}$ .

In particular, an eigenvector corresponding to the eigenvalue of  $-1$  is  $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ .

(3)

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

Then  $c_A(\lambda) = \begin{vmatrix} 3-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda) + 2 = \lambda^2 - 4\lambda + 5$ .

$c_A(\lambda) = 0$  has no real roots. Thus, the given matrix has no real eigenvalues.

(4)

$$A = \begin{bmatrix} -4 & 5 \\ -1 & -2 \end{bmatrix}$$

Then  $c_A(\lambda) = \begin{vmatrix} -4-\lambda & 5 \\ -1 & -2-\lambda \end{vmatrix} = (-4-\lambda)(-2-\lambda) + 5 = \lambda^2 + 6\lambda + 13$ .

$c_A(\lambda) = 0$  has no real roots. Thus, the given matrix has no real eigenvalues.

(5)

$$A = \begin{bmatrix} -1 & 5 \\ -1 & 3 \end{bmatrix}$$

Then  $c_A(\lambda) = \begin{vmatrix} -1-\lambda & 5 \\ -1 & 3-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda) + 5 = \lambda^2 - 2\lambda + 2$ .

$c_A(\lambda) = 0$  has no real roots. Thus, the given matrix has no real eigenvalues.

**Problem 4:** Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 2 & 1 & -6 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & -1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -2 \end{bmatrix}$$

*Solution:* (1)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 2 & 1 & -6 \end{bmatrix}$$

Then  $c_A(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -2-\lambda & 0 \\ 2 & 1 & -6-\lambda \end{vmatrix} = \underbrace{(-6-\lambda)(-\lambda(-2-\lambda))}_{\text{Expanding by first column}} = -\lambda(\lambda+2)(\lambda+6)$ . The

real roots of  $c_A(\lambda) = 0$  are  $\lambda = 0, -2, -6$ . Thus, the eigenvalues of  $A$  are  $\lambda = 0, -2, -6$ .



(2)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\text{Then } c_A(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 1 & -1-\lambda \end{vmatrix} = \underbrace{(1-\lambda)(\lambda^2 + \lambda - 2)}_{\text{(Expanding by first column)}} = -(\lambda - 1)^2(\lambda + 2)$$

The real roots of  $c_A(\lambda) = 0$  are  $\lambda = 1, -2$ . Thus, the eigenvalues of  $A$  are  $\lambda = 1$  and  $\lambda = -2$ .

(3)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\text{Then } c_A(\lambda) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = \underbrace{-\lambda(\lambda^2 - \lambda + 1) + 1(1)}_{\text{(Expanding by first column)}} = -(\lambda^3 - \lambda^2 + \lambda - 1)$$

$$= -(\lambda - 1)(\lambda^2 + 1)$$

The real roots of  $c_A(\lambda) = 0$  are  $\lambda = 1$ . Thus, the only (real) eigenvalue of  $A$  is  $\lambda = 1$ .

(4)

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -2 \end{bmatrix}$$

Then

$$c_A(\lambda) = \begin{vmatrix} -1-\lambda & -1 & 1 \\ 1 & -2-\lambda & 3 \\ 1 & 3 & -2-\lambda \end{vmatrix} = \underbrace{(-1-\lambda) \begin{vmatrix} -2-\lambda & 3 \\ 3 & -2-\lambda \end{vmatrix} - 1 \begin{vmatrix} -1 & 1 \\ 3 & -2-\lambda \end{vmatrix} + 1 \begin{vmatrix} -1 & 1 \\ -2-\lambda & 3 \end{vmatrix}}_{\text{Expanding by first column}}$$

$$= (-1-\lambda)(\lambda^2 + 4\lambda - 5) - (\lambda - 1) + (\lambda - 1) = -(\lambda + 1)(\lambda^2 + 4\lambda - 5) = -(\lambda + 1)(\lambda - 1)(\lambda + 5)$$

The real roots of  $c_A(\lambda) = 0$  are  $\lambda = 1, -1, -5$ . Thus, the eigenvalues of  $A$  are  $\lambda = 1, -1, -5$ .

**Problem 5:** Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} -1 & 2 & 0 \\ 3 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & 2 \\ 1 & -3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ -2 & 2 & -2 \\ 0 & -3 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

*Solution:* (1)

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 0 & -2 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\text{Then } c_A(\lambda) = \begin{vmatrix} -1-\lambda & 2 & 0 \\ 3 & -\lambda & -2 \\ 0 & 3 & -1-\lambda \end{vmatrix} = \underbrace{(-1-\lambda)(\lambda^2 + \lambda + 6) - 3(-2-\lambda)}_{\text{(Expanding by first column)}}$$

$$= -\lambda^3 - 2\lambda^2 - 7\lambda - 6 + 6 + 3\lambda = -\lambda(\lambda^2 + 2\lambda + 4)$$

The real roots of  $c_A(\lambda) = 0$  are  $\lambda = 0$ . Thus, the (real) eigenvalues of  $A$  are  $\lambda = 0$ .

(2)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -2 & 2 \\ 1 & -3 & 0 \end{bmatrix}$$

Then

$$c_A(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & -2-\lambda & 2 \\ 1 & -3 & -\lambda \end{vmatrix} = \underbrace{(1-\lambda)(\lambda^2 + 2\lambda + 6) + 1(-3 + 2 + \lambda)}_{\text{(Expanding along first row)}} = -(\lambda-1)(\lambda^2 + 2\lambda + 5)$$

The real roots of  $c_A(\lambda) = 0$  are  $\lambda = 1$ . Thus, the (real) eigenvalues of  $A$  are  $\lambda = 1$ .

(3)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 2 & -2 \\ 0 & -3 & 0 \end{bmatrix}$$

$$\begin{aligned} \text{Then } c_A(\lambda) &= \begin{vmatrix} 1-\lambda & 2 & 1 \\ -2 & 2-\lambda & -2 \\ 0 & -3 & -\lambda \end{vmatrix} = \underbrace{(1-\lambda)(\lambda^2 - 2\lambda - 6) + 2(-2\lambda + 3)}_{\text{(Expanding along first column)}} \\ &= -\lambda^3 + 3\lambda^2 + 4\lambda - 6 - 4\lambda + 6 = -\lambda^2(\lambda - 3) \end{aligned}$$

The real roots of  $c_A(\lambda) = 0$  are  $\lambda = 0, 3$ . Thus, the eigenvalues of  $A$  are  $\lambda = 0$  and  $\lambda = 3$ .

(4)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned} \text{Then } c_A(\lambda) &= \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = \underbrace{-\lambda(2 - 3\lambda + \lambda^2 + 1) + 1(1)}_{\text{Expanding along first column}} \\ &= -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3 \end{aligned}$$

The real roots of  $c_A(\lambda) = 0$  are  $\lambda = 1$ . Thus, the only eigenvalue of  $A$  is  $\lambda = 1$ .

**Problem 6:** Show geometrically by looking at the image of the function defined by a  $2 \times 2$  matrix that a  $2 \times 2$  matrix is not invertible if and only if it has 0 for an eigenvalue.

*Solution:*

**Problem 7:** Show that for each  $c \in \mathbb{R}$  the function defined by the matrix  $A = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$  maps the graphs of the hyperbola  $xy = c$  to itself.

*Solution:*

**Problem 8:** Show that the function defined by the matrix  $D = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$  maps every circle centered at the origin to itself. Then prove it rotates the circle by  $30^\circ$ .

*Solution:*

## 11. Complex Eigenvalues and Eigenvectors

**Problem 1:** Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} -4 & -3 \\ 6 & 2 \end{bmatrix} \quad \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ 5 & 4 \end{bmatrix}$$

*Solution:*

**Problem 2:** Find the eigenvalues and eigenvectors for the matrices

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \quad \begin{bmatrix} 5 & 7 \\ -2 & -1 \end{bmatrix}$$

*Solution:*