Name:

Problem 1: Find the absolute maximum and absolute minimum values of the following functions on the given interval.

1. $f(x) = 2x^3 - 3x^{3} - 12x + 1$ on $[-2, 3]$. 2
2. $f(x) = 2\cos(x) + \sin(2x)$ on $[0, \pi/2]$.
3. $f(x) = x^{200}(1-x)^{100}$ on $[0, 1]$.
4. $f(x) = \dfrac{\sqrt{x}}{1+x^2}$ on $[0, 2]$.

1.1 $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$

$f'(x) = 0 \Rightarrow x = 2, -1$

$f(2) = 16 - 12 - 24 + 1 = -19$ <u>minimum value</u>

$f(-1) = -2 - 3 + 12 + 1 = 8$ <u>Maximum value</u>

$f(-2) = -16 - 12 + 24 + 1 = -3$

$f(3) = 54 - 27 - 36 + 1 = -8$

1.2 $f(x) = 2\cos x + \sin(2x) \Rightarrow f'(x) = -2\sin x + 2\cos 2x$

$= -2\sin x + 2(1 - 2\sin^2 x)$

$= 2 - 2\sin x - 4\sin^2 x$

$f'(x) = 0 \Rightarrow 1 - \sin x - 2\sin^2 x = 0$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0 \Rightarrow (2\sin x - 1)(\sin x + 1) = 0$

$\Rightarrow \sin x = \frac{1}{2}$ or $\sin x = -1 \longrightarrow$ not possible for $x \in [0, \frac{\pi}{2}]$

$\Downarrow$

$x = \frac{\pi}{6}$

$f(\frac{\pi}{6}) = 2\cos\frac{\pi}{6} + \sin\frac{\pi}{3} = 2\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$ <u>maximum value</u>

$f(0) = 2\cos 0 + \sin 0 = 2$

$f(\frac{\pi}{2}) = 2\cos\frac{\pi}{2} + \sin\pi = 0$ <u>minimum value</u>

(1.3) $f(x) = x^{200}(1-x)^{100}$

$$\Rightarrow f'(x) = 200\,x^{199}(1-x)^{100} + x^{200}(100)(-1)(1-x)^{99}$$

$$= 100\,x^{199}(1-x)^{99}\left[2(1-x) - x\right]$$

$$= 100\,x^{199}(1-x)^{99}(2-3x)$$

$f'(x) = 0 \Rightarrow x = 0, 1, \frac{2}{3}$

$\left.\begin{array}{l} f(0) = 0 \\ f(1) = 0 \end{array}\right\}$ minimum value

$$f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^{200}\left(1-\frac{2}{3}\right)^{100} = \frac{2^{200}}{3^{200}}\left(\frac{1}{3}\right)^{100} = \frac{2^{200}}{3^{300}}$$ maximum value

(1.4) $f(x) = \dfrac{\sqrt{x}}{1+x^2} \Rightarrow f'(x) = \dfrac{(1+x^2)\frac{1}{2\sqrt{x}} - \sqrt{x}\,(2x)}{(1+x^2)^2}$

$$\Rightarrow f'(x) = \frac{1+x^2 - 2x\sqrt{x}\,(2\sqrt{x})}{2\sqrt{x}\,(1+x^2)^2} = \frac{1+x^2-4x^2}{2\sqrt{x}\,(1+x^2)^2}$$

$\Rightarrow f'(x) = \dfrac{1-3x^2}{2\sqrt{x}(1+x^2)^2}$. Now, $f'(x) = 0 \Rightarrow 1-3x^2 = 0$

$\Rightarrow x = \pm\dfrac{1}{\sqrt{3}}$

Also, at $x = 0$, $f'(x)$ d.n.e.

But $x = \dfrac{-1}{\sqrt{3}}$ is not in domain.

Thus, critical points are $0, \dfrac{1}{\sqrt{3}}$.

$f(0) = 0 \rightarrow$ minimum value

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{3^{-\frac{1}{4}}}{1+\frac{1}{3}} = \frac{3^{3/4}}{4} \simeq 0.57 \rightarrow \text{maximum value}$$

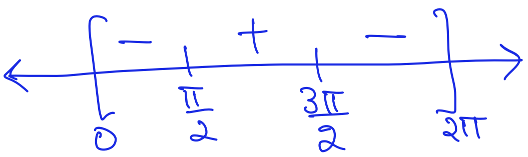
$$f(2) = \frac{\sqrt{2}}{1+2^2} = \frac{\sqrt{2}}{5} \simeq 0.283$$

Problem 2: For the following functions, find the intervals of increase/decrease, the points of local maxima/minima, intervals of concavity/convexity and the inflection points.

1. $f(x) = \cos^2 x - 2\sin x$, for $0 \le x \le 2\pi$.
2. $f(x) = \dfrac{x^2}{x-1}$.
3. $f(x) = x\sqrt{6-x}$.
4. $f(x) = 5x^{2/3} - 2x^{5/3}$.

2.1 $f(x) = \cos^2 x - 2\sin x \Rightarrow f'(x) = -2\cos x \sin x - 2\cos x$

$\Rightarrow f'(x) = -2\cos x \underbrace{(\sin x + 1)}_{\text{always +ve} \Rightarrow \text{no sign change}}$



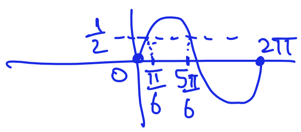
$\boxed{\begin{array}{l} \text{Inc. in } \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ \text{Dec. in } \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right) \\ \text{L. max at } x = \frac{3\pi}{2} \\ \text{L. min at } x = \frac{\pi}{2} \end{array}}$

$f'(x) = -\sin 2x - 2\cos x$

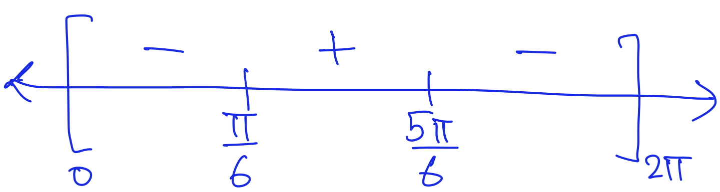
$\Rightarrow f''(x) = -2\cos 2x + 2\sin x$

$= 2\left[2\sin^2 x + \sin x - 1\right] = 2(2\sin x - 1)\underbrace{(\sin x + 1)}_{\text{always +ve}}$

$\Rightarrow \sin x = \frac{1}{2}$

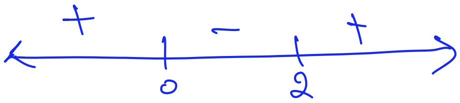


By graph we say $\sin x > \frac{1}{2}$ for $\frac{\pi}{6} < x < \frac{5\pi}{6}$ and $\sin x < \frac{1}{2}$ for $0 < x < \frac{\pi}{6}$, $\frac{5\pi}{6} < x < 2\pi$



$\boxed{\begin{array}{l} \text{Concave up in } \left(\frac{\pi}{6}, \frac{5\pi}{6}\right) \\ \text{Concave down in } \left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right) \\ \text{Points of Inflection} = \frac{\pi}{6}, \frac{5\pi}{6} \end{array}}$

(2.2) $f(x) = \dfrac{x^2}{x-1} \Rightarrow f'(x) = \dfrac{(x-1)2x - x^2}{(x-1)^2} = \dfrac{x^2-2x}{(x-1)^2} = \dfrac{x(x-2)}{\underset{\uparrow \text{ always +ve}}{(x-1)^2}}$



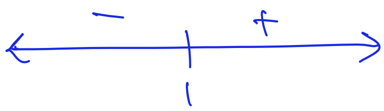
$\Rightarrow$

Inc.	in	$(-\infty, 0) \cup (2, \infty)$
Dec.	in	$(0, 2)$

L. max. at $x=0$, L. min at $x=2$

$$f''(x) = \frac{(x-1)^2(2x-2) - (x^2-2x)2(x-1)}{(x-1)^4} = \frac{2(x-1)\left[(x-1)^2 - (x^2-2x)\right]}{(x-1)^4}$$

$$= \frac{2(x-1)\left[x^2-2x+1-x^2+2x\right]}{(x-1)^4} = \frac{2}{(x-1)^3}$$



$\Rightarrow$

Concave up in $(1, \infty)$

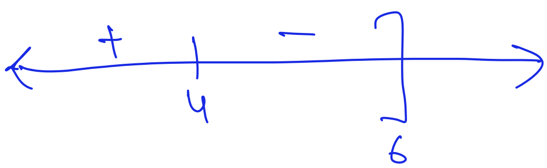
Concave down in $(-\infty, 1)$

Inflection point at $x=1$

(2.3) $f(x) = x\sqrt{6-x} \Rightarrow f'(x) = \sqrt{6-x} + x\left(\dfrac{1}{2\sqrt{6-x}} \times -1\right)$

$$= \frac{2(6-x) - x}{2\sqrt{6-x}} = \frac{12-3x}{2\sqrt{6-x}}$$

$\Rightarrow f'(x) = \dfrac{-3(x-4)}{2\sqrt{6-x}}$. Domain of $f = (-\infty, 6]$



$\Rightarrow$

Inc.	in	$(-\infty, 4)$
Dec.	in	$(4, 6]$
L. max.	at	$x=4$
No	L. min	

$$f''(x) = \frac{2\sqrt{6-x}\,(-3) - (12-3x) \times \cancel{2} \times \frac{1}{\cancel{2}\sqrt{6-x}} \times -1}{4(6-x)}$$

$$\Rightarrow f''(x) = \frac{-6(6-x)+12-3x}{4(6-x)\sqrt{6-x}} = \frac{-36+6x+12-3x}{4(6-x)\sqrt{6-x}}$$

$$\Rightarrow f''(x) = \frac{3x-24}{4(6-x)\sqrt{6-x}} = \frac{3(x-8)}{4(6-x)\sqrt{6-x}}$$

always $-$ve in the domain $(-\infty, 6]$

always $+$ve

always $+$ve in the domain $(-\infty, 6]$

$\Rightarrow f''(x) < 0$ everywhere in the domain of f.

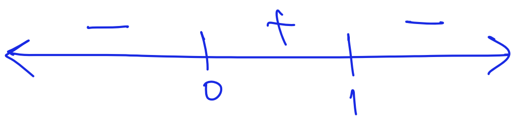
$\Rightarrow$

f is always concave down.
f is never concave up.
No inflection point.

(2.4) $f(x) = 5x^{2/3} - 2x^{5/3}$

$$\Rightarrow f'(x) = \frac{10}{3}x^{\frac{2}{3}-1} - \frac{10}{3}x^{\frac{5}{3}-1} = \frac{10}{3}x^{\frac{-1}{3}} - \frac{10}{3}x^{2/3}$$

$$= \frac{10}{3}\left[\frac{1}{x^{1/3}} - x^{2/3}\right] = \frac{10}{3}\left(\frac{1-x}{x^{1/3}}\right) = \frac{-10}{3}\,\frac{(x-1)}{x^{1/3}}$$



$\Rightarrow$

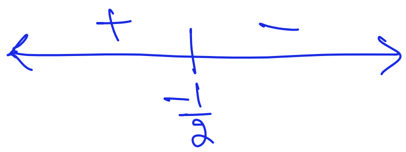
Inc. in $(0, 1)$
Dec. in $(-\infty, 0) \cup (1, \infty)$
L. max at $x = 1$
L. min at $x = 0$

$$f''(x) = \frac{10}{3}\left[\frac{d}{dx}\left(x^{-\frac{1}{3}}\right) - \frac{d}{dx}\left(x^{2/3}\right)\right]$$

$$= \frac{10}{3}\left[\frac{-1}{3}x^{\frac{-4}{3}} - \frac{2}{3}x^{\frac{-1}{3}}\right] = \frac{-10}{9}\left[\frac{1}{x^{4/3}} + 2x^{\frac{-1}{3}}\right]$$

$$= \frac{-10}{9}\left[\frac{1+2x}{x^{4/3}}\right] = \frac{-10}{9}\,\frac{(1+2x)}{x^{4/3}}$$

$= (x^{1/3})^4$ is always $+$ve



$\Rightarrow$

Concave up in $\left(-\infty, -\frac{1}{2}\right)$

Concave down in $\left(-\frac{1}{2}, \infty\right)$

Points of inflection at $x = -\frac{1}{2}$