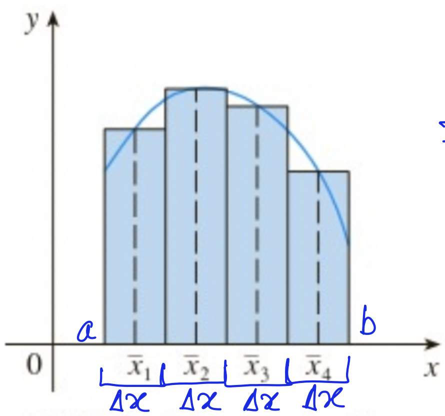
■ **Section 7.7** exercise: see the two bullets below

- Use (a) the Trapezoidal Rule, (b) The Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified value of n. (Round your answers to six decimal places.)
$$\int_1^3 \sqrt{x^3-1}\,dx, \qquad n=6$$
- *Optional exercises*: section 7.7, page 654, # 5, 16.

There are situations where it is difficult, or even impossible, to compute $\int_a^b f(x)\,dx$. Other times, when a function is determined from a scientific experiment through instrument readings or collect data, there may be no formula for the function. Therefore, it is needful to have methods for approximating definite integrals.

⋆ ***The Midpoint Rule*** *for approximating definite integrals*



(c) Midpoint approximation

$$f(\bar{x}_1)(\Delta x) + f(\bar{x}_2)\Delta x + f(\bar{x}_3)\Delta x + f(\bar{x}_4)\Delta x \simeq \int_a^b f(x)\,dx$$

$$4(\Delta x) = b-a$$

$$\Rightarrow \Delta x = \frac{b-a}{4}$$

In general for any n,

$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i(\Delta x) = a + i\frac{(b-a)}{n}$$

$$\bar{x}_i = a + \left(\frac{2i-1}{2}\right)\Delta x, \quad 1 \le i \le n$$

$$\int_a^b f(x)\,dx \simeq \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right]$$

⋆ ***The Trapezoidal Rule*** *for approximating definite integrals*

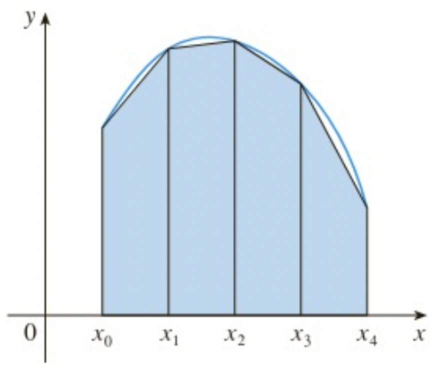


FIGURE 2
Trapezoidal approximation

n = the number of subintervals or the number of trapezoids

$\Delta x = \dfrac{b-a}{n}$ = the height of each trapezoid

T_n = the area of n trapezoids

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$$\int_a^b f(x)\,dx \approx T_n$$

$$x_i = a + i(\Delta x)$$

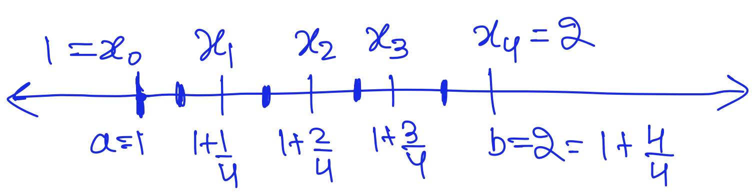
Example 1: Use (a) the Midpoint Rule and (b) the Trapezoidal Rule with $n = 4$ to approximate the integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

(a) $\displaystyle\int_1^2 \frac{1}{x}\,dx$

$a = 1, \quad b = 2$

$n = 4 \Rightarrow 4(\Delta x) = b - a = 2 - 1 = 1$

$\Rightarrow 4(\Delta x) = 1 \Rightarrow \Delta x = \frac{1}{4}$



$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

$x_1 - x_0 = \Delta x \Rightarrow x_1 = x_0 + \frac{1}{4} = 1 + \frac{1}{4}$

$x_2 - x_1 = \Delta x \Rightarrow x_2 = x_1 + \frac{1}{4} = 1 + 2\left(\frac{1}{4}\right)$

$x_3 - x_2 = \Delta x \Rightarrow x_3 = x_2 + \frac{1}{4} = 1 + 3\left(\frac{1}{4}\right)$

$x_4 - x_3 = \Delta x \Rightarrow x_4 = x_3 + \frac{1}{4} = 1 + 4\left(\frac{1}{4}\right)$

$$\int_1^2 \frac{1}{x}\,dx = f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + f(\bar{x}_3)\Delta x + f(\bar{x}_4)\Delta x$$

$$= \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + f(\bar{x}_3) + f(\bar{x}_4)\right]$$

$$\overline{x_1} = \frac{x_0 + x_1}{2} = \frac{1 + 1 + \frac{1}{4}}{2} = 1 + \frac{1}{2}\left(\frac{1}{4}\right)$$

$$\overline{x_2} = \frac{x_1 + x_2}{2} = \frac{1 + \frac{1}{4} + 1 + 2\left(\frac{1}{4}\right)}{2} = 1 + \frac{3}{2}\left(\frac{1}{4}\right)$$

$$\overline{x_3} = \frac{x_2 + x_3}{2} = \frac{1 + 2\left(\frac{1}{4}\right) + 1 + 3\left(\frac{1}{4}\right)}{2} = 1 + \frac{5}{2}\left(\frac{1}{4}\right)$$

$$\overline{x_4} = \frac{x_3 + x_4}{2} = \frac{1 + 3\left(\frac{1}{4}\right) + 1 + 4\left(\frac{1}{4}\right)}{2} = 1 + \frac{7}{2}\left(\frac{1}{4}\right)$$

$$\overline{x_1} = a + \frac{1}{2}\Delta x, \quad \overline{x_2} = a + \frac{3}{2}\Delta x, \quad \overline{x_3} = a + \frac{7}{2}\Delta x, \ \ldots\ldots$$

$$\overline{x_i} = a + \left(\frac{2i-1}{2}\right)\Delta x, \quad 1 \le i \le n$$

$$\int_1^2 \frac{1}{x}\,dx = \frac{1}{4}\left[f(\overline{x_1}) + \ldots + f(\overline{x_4})\right]$$

$$= \frac{1}{4}\left[\frac{1}{\overline{x_1}} + \frac{1}{\overline{x_2}} + \frac{1}{\overline{x_3}} + \frac{1}{\overline{x_4}}\right]$$

$$= \frac{1}{4}\left[\frac{1}{1+\frac{1}{8}} + \frac{1}{1+\frac{3}{8}} + \frac{1}{1+\frac{5}{8}} + \frac{1}{1+\frac{7}{8}}\right]$$

$$= \frac{1}{4}\left[\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right] = 2\left[\frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15}\right]$$

$$= a_0 \cdot a_1 a_2$$

(b) $\displaystyle\int_1^2 \frac{1}{x}\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$

$\underline{\underline{n=4}}$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

$$x_4 = 1+4\left(\tfrac{1}{4}\right) = 2$$

$$x_0 = a \Rightarrow x_0 = 1, \quad x_1 = 1+\tfrac{1}{4} = \tfrac{5}{4}, \quad x_2 = 1+2\left(\tfrac{1}{4}\right) = \tfrac{3}{2}, \quad x_3 = 1+3\left(\tfrac{1}{4}\right) = \tfrac{7}{4},$$

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{1}{2}\left(\frac{1}{4}\right)\left[\frac{1}{1} + \frac{2}{5/4} + \frac{2}{3/2} + \frac{2}{7/4} + \frac{1}{2}\right]$$

$$= \frac{1}{8}\left[1 + \frac{8}{5} + \frac{4}{3} + \frac{8}{7} + \frac{1}{2}\right] = \frac{1}{8} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{16}$$

⋆ ***The Simpson's Rule*** *for approximating definite integrals*

Another rule for approximate integration results from using parabolas instead of straight line segments to approximate the curve.

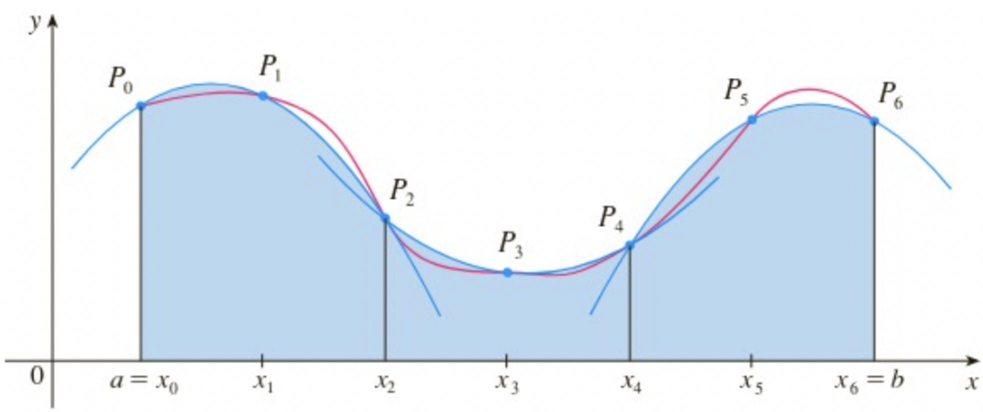


FIGURE 7

n = the number of subintervals. n must be **even** for Simpson's Rule.

$\Delta x = \dfrac{b-a}{n}$ = the length of each subinterval

S_n = the area under the parabolas by using Simpson's Rule

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-3}) + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

Note the pattern of coefficients: 1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 4, 1.

$$\int_a^b f(x)\,dx \approx S_n$$

You can read the discussion on page 559–560 of the textbook to see how the formula for S_n is derived.

$$x_i = a + i(\Delta x), \quad 0 \le i \le n$$

Example 2: Use Simpson's rule with $n = 8$ to approximate $\int_0^1 e^{x^2}\, dx$.

$$n = 8 \Rightarrow \Delta x = \frac{1-0}{8} = \frac{1}{8}$$

$$\int_0^1 e^{x^2}\,dx \approx \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8)\Big]$$

$x_0 = 0$

$x_1 = 0 + \frac{1}{8} = \frac{1}{8}$

$x_2 = \frac{2}{8}$

$x_3 = \frac{3}{8}$

$x_4 = \frac{4}{8}$

$x_5 = \frac{5}{8}$

$x_6 = \frac{6}{8}$

$x_7 = \frac{7}{8}$

$x_8 = \frac{8}{8} = 1$

$$x_i = 0 + i(\Delta x) \Rightarrow x_i = \frac{i}{8}$$

$$\int_0^1 e^{x^2}\,dx \approx \frac{1}{3}\left(\frac{1}{8}\right)\Big[e^{0^2} + 4e^{\left(\frac{1}{8}\right)^2} + 2e^{\left(\frac{2}{8}\right)^2} + 4e^{\left(\frac{3}{8}\right)^2} + 2e^{\left(\frac{4}{8}\right)^2} + 4e^{\left(\frac{5}{8}\right)^2} + 2e^{\left(\frac{6}{8}\right)^2} + 4e^{\left(\frac{7}{8}\right)^2} + e^{1^2}\Big]$$

$$\approx \frac{1}{24}\Big[1 + 4e^{1/64} + 2e^{1/16} + 4e^{9/64} + 2e^{1/4} + 4e^{25/64} + 2e^{9/16} + 4e^{49/64} + e\Big]$$

$$\int_1^2 x^2\,dx \qquad (\text{Simpson's rule}, n=4)$$

$$\approx \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\Big]$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

$$x_0 = 1, \quad x_1 = 1 + \frac{1}{4} = \frac{5}{4}, \quad x_2 = 1 + \frac{2}{4} = \frac{3}{2}, \quad x_3 = 1 + \frac{3}{4} = \frac{7}{4}, \quad x_4 = 2$$

$$\int_1^2 x^2\,dx \simeq \frac{1}{3}\left(\frac{1}{4}\right)\left[1^2 + 4\left(\frac{5}{4}\right)^2 + 2\left(\frac{3}{2}\right)^2 + 4\left(\frac{7}{4}\right)^2 + 2^2\right]$$

$$= \frac{1}{12}\left[1 + \frac{25}{4} + \frac{9}{2} + \frac{49}{4} + 4\right]$$

$$= \frac{1}{12}\left[1 + 6.25 + 4.5 + 12.25 + 4\right]$$

$$= \frac{1}{12}\left[5 + 4.5 + 18.5\right] = \frac{28}{12} = \frac{7}{3} \simeq 2.33$$

Error

$$\int_1^2 x^2\,dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{2^3 - 1^3}{3} = \frac{7}{3}$$