Learning objectives:

1. Find areas of regions bounded between two or more curves.
2. We either divide a region in vertical strips and integrate with respect to x, or we divide a region in horizontal strips and integrate with respect to y.

Area using vertical strips

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and g are continuous and $f(x) \geq g(x)$, for $a \leq x \leq b$, is

$$A = \int_a^b [f(x) - g(x)]\, dx\,.$$

Example 1. Find the area of the region bounded by $y = x^2 + 1$, $y = x$, $x = 0$ and $x = 1$.

Example 2. Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.

Example 3. Find the area of the region enclosed by $y = x/\sqrt{x^2 + 1}$, $x = \sqrt{3}$ and the x-axis.

To find the area between the curves $y = f(x)$ and $y = g(x)$, when $f(x) \geq g(x)$ for some values of x while $g(x) \geq f(x)$ for some other values of x, we split the given region into several regions.

In general, the area between the curves $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$, $(a < b)$, is given by

$$A = \int_a^b |f(x) - g(x)|\, dx\,.$$

Here we keep in mind that

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{if } f(x) \geq g(x)\,, \\ g(x) - f(x) & \text{if } g(x) \geq f(x)\,. \end{cases}$$

Example 4. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$.

Area using horizontal strips.

Some regions are best treated by regarding x as a function of y.

If a region is bounded by the curves $x = f(y)$, $x = g(x)$, $y = c$ and $y = d$, $(c < d)$, then its area is given by

$$A = \int_c^d |f(y) - g(y)|\, dy\ .$$

Example 5. Find the area enclosed by the line $y = x-1$ and the parabola $y^2 = 2x+6$.