**M16600 Lecture Notes**
*Section 11.8: Power Series*

■ **Section 11.8** textbook exercises: # 3, 4, 6, 7, 9, 11, 12, 15 (these will take some time to do).

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**Definition of Power Series.** The ***power series centered at a*** is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where $x$ is the variable and $c_n$'s are constants called the **coefficients** of the series. Here, $a$ is a fixed number called the **center**.

*Example 1:* Here are some examples of power series

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

The center $a = 3$. The coefficients $c_n = \frac{1}{n}$

(b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$

The center $a = 0$. The coefficients $c_n = (-1)^n$

$$\sum_{n=0}^{\infty} (-1)^n x^{n+1}$$

$a=0$, $c_{n+1} = (-1)^n$

$\Rightarrow c_{n-1+1} = (-1)^{n-1}$

$\Rightarrow c_n = (-1)^{n-1}$

**Note:** A power series is a **function** in the variable $x$, where the domain is the set of all values of $x$ such that the series converges. The outputs are series.

For example, let $f(x) = \displaystyle\sum_{n=0}^{\infty} x^n$, i.e., $f(x) = x^0 + x^1 + x^2 + \cdots$
$= 1 + x + x^2 + x^3 + \cdots$

$f(3) = \displaystyle\sum_{n=0}^{\infty} 3^n$

↳ geometric series with $r = 3 \Rightarrow$ diverges

$f(\frac{1}{2}) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \quad \Rightarrow a = \left(\frac{1}{2}\right)^0 = 1$

↳ geometric series with $r = \frac{1}{2} < 1 \Rightarrow$ converges

$\Rightarrow f(\frac{1}{2}) = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2 \quad \Rightarrow f(\frac{1}{2}) = 2$

*Example 2:* For what values of $x$ is the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ convergent?

$$a_n = \frac{(x-3)^n}{n} \quad \Rightarrow r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

$$a_{n+1} = \frac{(x-3)^{n+1}}{n+1} \quad \Rightarrow \frac{a_{n+1}}{a_n} = \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} = (x-3)^{n+1-n} \frac{n}{n+1}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{(x-3)\,n}{n+1} \quad \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \frac{|(x-3)\,n|}{|n+1|} = \frac{|x-3|\,n}{n+1}$$

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x-3|\,n}{n+1} = \lim_{n\to\infty} \frac{|x-3|\,\cancel{n}}{\cancel{n}}$$

$$= |x-3|$$

By the ratio test, if $|x-3| < 1$ then the series converges.

$$\Rightarrow -1 < x-3 < 1 \Rightarrow 3-1 < x < 3+1$$

$$\Rightarrow 2 < x < 4$$

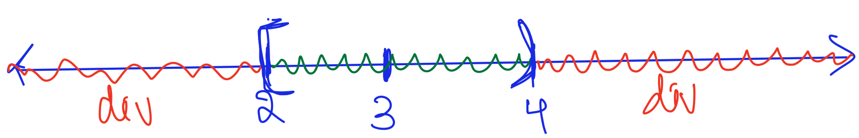
If $2 < x < 4$, the given power series converges ———— (1)

If $r > 1$, then the series diverges

$$|x-3| > 1 \Rightarrow x-3 > 1 \text{ or } x-3 < -1$$

$$\Rightarrow x > 4 \text{ or } x < 2$$

If $x > 4$ or if $x < 2$ then the given power series diverges ———— (2)



$x=2$, $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=1}^{\infty} \frac{(2-3)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

$\downarrow$ alternating series

$b_n = \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{n+1} < \frac{1}{n}$ $\Rightarrow$ By AST, we have convergence for $x=2$ — (3)

$x=4$, $\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$

$\hookrightarrow$ P-series with $P=1$

$\Rightarrow$ diverges

$\Rightarrow$ we have divergence for $x=4$ — (4)

The domain of $f(x) = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ is :

$2 \le x < 4$ or $x$ in the interval $[2,4)$

The point of focus for this section is to determine for what values of $x$ a power series is convergent. Hence, we have the following concepts.

### Radius of Convergence and Interval of Convergence.

In example 2, we get $|x-3| < 1$. Geometrically, this implies the ***distance*** between $x$ and the center 3 is less than 1.

The ***Radius of Convergence*** of a power series is the greatest distance between $x$ and the center $a$ such that the series is convergent.

If $R$ is the radius of convergence, then the series is convergent for all $x$ such that $|x-a| < R$, where $a$ is the center of the power series.

In example 2, we find the interval $2 \leq x < 4$ for which the series is convergent. The interval $[2, 4)$ is called the *interval of convergence.*

**Note:** To find the interval of convergence, we had to test the endpoints $x = 2$ and $x = 4$ separately to determine whether the series is convergent of divergent. This will be the case in general.

The ***Interval of Convergence*** of a power series is the interval that consists of all values of $x$ for which the series converges.

$R = 1$

$I = [2, 4)$

*Example 3:* Find the radius of convergence and the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

$$a_n = \frac{(-3)^n x^n}{\sqrt{n+1}} \Rightarrow a_{n+1} = \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+1+1}} = \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \frac{\sqrt{n+1}}{(-3)^n x^n} = (-3)^{n+1-n} x^{n+1-n} \frac{\sqrt{n+1}}{\sqrt{n+2}}$$

$$= (-3)x \frac{\sqrt{n+1}}{\sqrt{n+2}} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{3|x|\sqrt{n+1}}{\sqrt{n+2}}$$

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3|x|\sqrt{n+1}}{\sqrt{n+2}} = 3|x| \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n+2}}$$

$$= 3|x| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3} \Rightarrow R = \frac{1}{3}$$

$$\Downarrow$$

For $-\frac{1}{3} < x < \frac{1}{3}$, we have convergence.

$$x = \frac{-1}{3} \Rightarrow \sum_{n=0}^{\infty} \frac{(-3)^n \left(\frac{-1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{\cancel{(-3)^n} \frac{1}{\cancel{(-3)^n}}}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

similar to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a P-series with $P=\frac{1}{2}$ $\Rightarrow$ diverges

By LCT, we have divergence at $x = \frac{-1}{3}$

$$x = \frac{1}{3} \Rightarrow \sum_{n=0}^{\infty} \frac{(-3)^n \frac{1}{3^n}}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}, \quad b_n = \frac{1}{\sqrt{n+1}}$$

By AST, this converges. $\Rightarrow I = \left(\frac{-1}{3}, \frac{1}{3}\right]$

*Example 4:* Find the radius of convergence and the interval of convergence of the series $\sum_{n=0}^{\infty} n!(x+1)^n$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|$$

$a_n = n!\,(x+1)^n$

$C_n = n!$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cancel{!}}{\cancel{n!}}\right| = \lim_{n\to\infty}(n+1) = \infty$$

$$\frac{\cancel{(n+1)\,n\,(n-1)\cdots 1}}{\cancel{n\,(n-1)\cdots 1}}$$

$$\frac{1}{R} = \infty \Rightarrow R = \frac{1}{\infty} = 0$$

$$\Rightarrow \boxed{R=0}$$

Alternatively $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(x+1)^{n+1}}{n!\,(x+1)^n}\right| = \lim_{n\to\infty}(n+1)|x+1| = \infty$ as long as $x \neq -1$

$r = \infty > 1 \Rightarrow$ divergence for $x \neq -1$

for $x \neq -1$

$$I = \{-1\}$$

*Example 5:* Find the radius of convergence and the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$a_n = \frac{x^n}{n!} \Rightarrow a_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!}\,\frac{n!}{x^n} = \frac{x^{n+1-n}\,\cancel{n!}}{(n+1)\,\cancel{n!}} = \frac{x}{n+1}$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$$

For every value of $x$, we have convergence (by Ratio test)

$\Rightarrow$ For $|x| < \infty$, we have convergence

$$\Rightarrow R = \infty, \quad I = (-\infty, \infty)$$