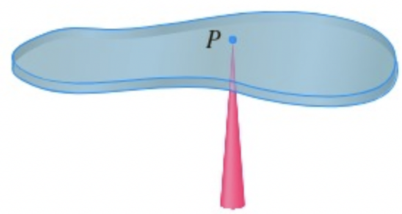
M16600 Lecture Notes

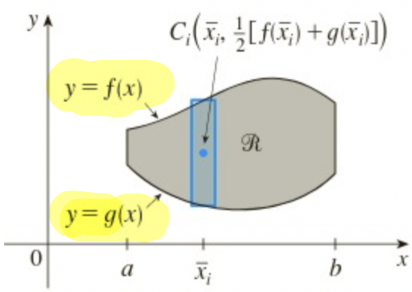
Section 8.3: Center of Mass of Centroid

■ **Section 8.3** textbook exercises, *page 595*: # 29, 30, 32.



Our main objective here is to find the point P on which a thin plate of any given shape balance horizontally as in the figure on the left. This point is called the ***center of mass*** of the plate (or the ***centroid*** of the plate).

We consider a flat plate (called *lamina*) that occupies a region $\mathcal{R}$ of the xy-plane.



The center of mass of the region $\mathcal{R}$ is located at the point $(\bar{x}, \bar{y})$, where

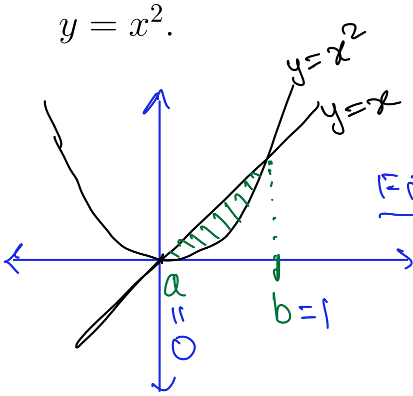
$$\bar{x} = \frac{1}{A}\int_a^b x\big[f(x) - g(x)\big]\,dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \tfrac{1}{2}\Big(\big[f(x)\big]^2 - \big[g(x)\big]^2\Big)\,dx$$

Here A is the area of the region $\mathcal{R}$.

See the discussion on page 600-603 of the textbook for more detail

Example: Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$.



$$A = \int_a^b |f(x) - g(x)|\,dx$$

Find Points of intersection of $y=x$, $y=x^2$

$$x = x^2 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$$

$$A = \int_0^1 |x - x^2|\,dx = \int_0^1 (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]\Bigg|_0^1 = \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{0}{2} - \frac{0}{3}\right)$$

$$= \frac{1}{6}$$

$$\Rightarrow A = \frac{1}{6}$$

$$\bar{x} = \frac{1}{1/6}\int_0^1 x(x - x^2)\,dx = 6\int_0^1 (x^2 - x^3)\,dx$$

$$\Rightarrow \bar{x} = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = 6\left[\left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{0}{3} - \frac{0}{4}\right)\right]$$

$$= 6 \times \frac{1}{12} = \frac{1}{2}$$

$$\bar{y} = \frac{1}{1/6}\int_0^1 \frac{1}{2}\left[x^2 - (x^2)^2\right] dx$$

$$= \frac{6}{2}\int_0^1 (x^2 - x^4)\, dx$$

$$= 3\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_0^1 = 3\left[\left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{0}{3} - \frac{0}{5}\right)\right]$$

$$= 3 \times \frac{2}{15} = \frac{2}{5}$$

The centroid is at $\left(\frac{1}{2}, \frac{2}{5}\right)$