M16600 Lecture Notes

Section 6.8: Indeterminate Forms and L'Hospital's Rule

■ **Section 6.8** exercises, *page* : #9, 15, 19, 21, 27, 35, 37, 43, 47, 52, 53, 57, 59, 65.
Optional: Practice more problems from #8 to #68.

GOALS: Use L'Hospital's Rule to compute the limit of the following ***indeterminate form***

$0 \cdot \text{anything} = 0$

- **Indeterminate Quotient**: $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$
- **Indeterminate Product**: $0 \cdot \infty$ $(\to 0)(\to \infty) \to$ take one of the factors in denominator
- **Indeterminate Difference**: $\infty - \infty$ $\to$ common denominator
- **Indeterminate Power**: 0^0, ∞^0, 1^∞ $(\to 0)^{\to 0}$, $(\infty)^{\to 0}$, $(\to 1)^{\to \infty}$ $\Big\} \to$ ln on both sides

The Intuition of a Limit Statement: $\lim_{x\to 1}(x^2+2) = 3$. This equation states that as x approaches 1 (from the left and the right side of 1), the values of x^2+2 approaches ________.

Some Notation:

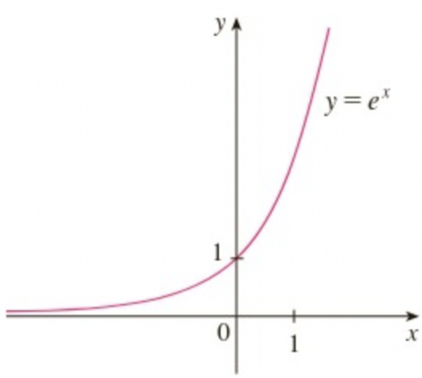
$x \to 1^+$ means x approaches 1 from the RIGHT, i.e., x is slightly BIGGER than 1 (e.g., $x = 1.01$, 1.000012, etc.)

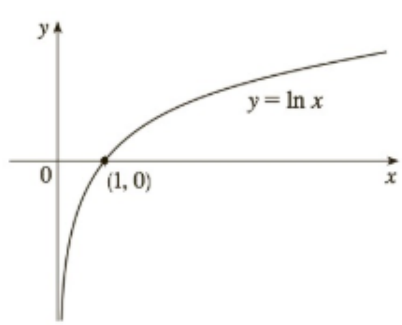
$x \to 1^-$ means x approaches 1 from the LEFT, i.e., x is a little SMALLER than 1 (e.g., $x = 0.99$, 0.999999, etc.)

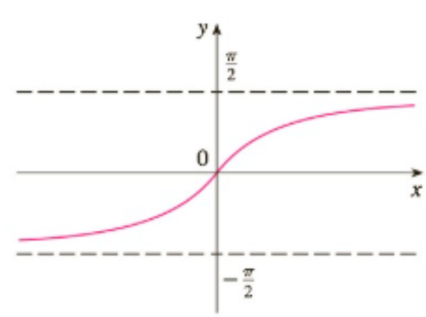
$x \to 1$ means x approaches 1 from both directions, left and right (i.e., x can take any values slightly less than or bigger than 1)

Warning: 1^- does NOT mean -1.

Limit Facts about e^x**,** $\ln x$**, and** $\arctan(x)$







$$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to\infty} \ln x = \infty \qquad \lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$$

$$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to 0^+} \ln x = -\infty \qquad \lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$$

Computing Limits: The FIRST step in computing limit is what I call ***"direct substitution"*** (D.S.) Keep in mind, $x \to 1$ means x is very close to 1 but never equal 1.

After we do "direct substitution", we either get a **determinate form** or an **indeterminate form**.

Determinate Forms

- A real number $\to$ the limit is this real number
- $\dfrac{\text{a number}}{\pm\infty}$ $= 0$
- $\dfrac{\text{a nonzero number}}{0}$ = Not Defined

$\dfrac{\text{a nonzero number}}{\to 0} = \pm\infty$

Indeterminate Forms

- $\dfrac{0}{0} \to$ in section 1.6, we learn some algebra techniques to find the limit. In this section, we can apply L'Hospital's rule.
- $\dfrac{\pm\infty}{\pm\infty} \to$ in section 3.4, we learn a technique to solve this case. In this section, we can apply *L'Hospital's Rule* for this indeterminate form.
- $0 \cdot \infty \to$ rewrite as indeterminate quotient form then apply *L'Hospital's Rule*.
- $\infty - \infty \to$ rewrite as indeterminate quotient form then apply *L'Hospital's Rule*.
- $0^0,\ \infty^0,\ 1^\infty \to$ apply the tool of natural log then rewrite into indeterminate quotient form then apply *L'Hospital's Rule*.

L'Hosptital's Rule: Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a).

Suppose that $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} \to \dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$. Then, by ***L'Hospital's Rule***, we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{1}$$

provide that the limit on the right side of the equation exists or is $\pm\infty$.

Note: L'Hospital's Rule also applies for $x \to a^+$, $x \to a^-$, or $x \to \pm\infty$.

Remark: We can apply L'Hospital more than one times if needed.

Examples: Evaluate the following limits. **Warning:** Don't blindly use L'Hospital's rule for every problem, see if it applies.

(a) $\lim_{x\to 1} \dfrac{\ln x}{x-1}$ $= \dfrac{\ln 1}{1-1} = \dfrac{\to 0}{\to 0}$

$= \lim_{x\to 1} \dfrac{\frac{1}{x}}{1} = \lim_{x\to 1} \dfrac{1}{x} = \dfrac{1}{1} = 1$

(b) $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt[3]{x}}$ $= \dfrac{\infty}{\infty}$

$\dfrac{d}{dx}\left(x^{1/3}\right) = \dfrac{1}{3} x^{\frac{1}{3}-1}$

$= \dfrac{1}{3} x^{-2/3}$

$= \lim_{x\to\infty} \dfrac{\frac{1}{x}}{\frac{1}{3} x^{-2/3}}$

$= \lim_{x\to\infty} \dfrac{3x^{2/3}}{x} = \lim_{x\to\infty} \dfrac{3}{x^{1-2/3}} = \lim_{x\to\infty} \dfrac{3}{x^{1/3}}$

$\underline{\text{D.S.}} = \dfrac{3}{\to\infty} = 0$

(c) $\lim_{x\to\pi^-} \dfrac{\sin x}{1-\cos x}$ $\overset{\text{D.S.}}{=} \dfrac{\sin\pi}{1-\cos\pi} = \dfrac{0}{1-(-1)} = \dfrac{0}{2} = 0$

$\boxed{\parallel \quad \text{Incorrect} \quad \dfrac{\cos x}{\sin x} = \dfrac{\cos\pi}{\sin\pi} = \dfrac{-1}{\to 0} = \infty}$

(d) $\lim_{x\to\infty} \sqrt{x}e^{-x/2} \overset{D.S.}{=} \sqrt{\infty}\, e^{-\infty/2} = \infty \cdot 0$

$(\to\infty)\ (\to 0)$

$\lim_{x\to\infty} \sqrt{x} = \infty$

$\lim_{x\to\infty} e^{-x/2} = 0$

$\lim_{x\to\infty} e^{x/2} = \infty$

$\lim_{x\to\infty} x^{-1/2} = 0$

$\to \frac{1}{x^{1/2}} = \frac{1}{\infty} = 0$

$\lim_{x\to\infty} \frac{\sqrt{x}}{e^{x/2}} \overset{D.S.}{=} \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{\cancel{\frac{1}{2}} x^{\frac{1}{2}-1}}{\cancel{\frac{1}{2}} e^{x/2}} = \lim_{x\to\infty} \frac{x^{-1/2}}{e^{x/2}}$

$\overset{D.S.}{=} \frac{0}{\infty} = \boxed{0}$

Alternatively

$\lim_{x\to\infty} \frac{e^{-x/2}}{\frac{1}{\sqrt{x}}} \overset{D.S.}{=} \frac{e^{-\infty/2}}{\frac{1}{\infty}} = \frac{\to 0}{\to 0}$

$= \lim_{x\to\infty} \frac{\cancel{-\frac{1}{2}} e^{-x/2}}{\cancel{-\frac{1}{2}} x^{-\frac{1}{2}-1}} = \lim_{x\to\infty} \frac{e^{-x/2}}{x^{-3/2}}$

$= \frac{e^{-\infty/2}}{\frac{1}{\infty}} = \left(\frac{\to 0}{\to 0}\right)$

$\vdots$

(e) $\lim_{x\to 0^+} x \ln x$

$\overset{D.S.}{=} (\to 0)(-\infty)$

$\boxed{\lim_{x\to 0^+} \ln x = -\infty}$

$= \lim_{x\to 0^+} \frac{\ln x}{1/x}$ or $= \lim_{x\to 0^+} \frac{x}{\left(\frac{1}{\ln x}\right)}$

$= \frac{\ln 0^+}{\frac{1}{0}} = \frac{-\infty}{\infty} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \left(\frac{1}{\cancel{x}}\right)\left(\frac{-x^{\cancel{2}}}{1}\right)$

$= \lim_{x\to 0^+} -x$

$= -0 = \boxed{0}$

(f) $\lim_{x\to(\pi/2)^-} (\sec x - \tan x)$

$\overset{D.S.}{=} \sec \frac{\pi}{2}^- - \tan \frac{\pi}{2}^-$

$= \infty - \infty$

$= \lim_{x\to\frac{\pi}{2}^-} \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin x}{\cos x} \overset{D.S.}{=} \frac{1-\sin\frac{\pi}{2}^-}{\cos\frac{\pi}{2}^-}$

(Take common denominator)

$= \frac{1-1}{0} = \frac{0}{0}$

$\overset{\text{LH Rules}}{=}$

$\lim_{x\to\frac{\pi}{2}^-} \frac{\cancel{-}\cos x}{\cancel{-}\sin x} = \frac{\cos \pi/2}{\sin \pi/2} = \frac{0}{1} = \boxed{0}$

(g) $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$ $\Rightarrow L = \lim_{x\to 0^+} \left(1+\sin 4x\right)^{\cot x}$

DS. $(1+0)^{\frac{1}{0}} = (\to 1)^{\to\infty}$ indeterminate

$$\Rightarrow \ln L = \ln\left(\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}\right) = \lim_{x\to 0^+} \ln\left(\left(1+\sin 4x\right)^{\cot x}\right)$$

$$= \lim_{x\to 0^+} \cot x \,\ln(1+\sin 4x) \underset{\text{DS.}}{=} \frac{\cos 0}{\sin 0}\ln(1+\sin 0)$$

$$= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\frac{1}{\cot x}} \qquad = \frac{1}{\to 0}\ln(1+0)$$

$$= (\to\infty)(\to 0)$$

$$= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} \underset{\text{DS.}}{=} \frac{\ln(1+0)}{\tan 0} = \frac{0}{0}$$

$$= \lim_{x\to 0^+} \frac{\left(\frac{1}{1+\sin 4x}\right)\frac{d}{dx}(1+\sin 4x)}{\sec^2 x} = \lim_{x\to 0^+} \frac{\left(\frac{1}{1+\sin 4x}\right)4(\cos 4x)}{\sec^2 x}$$

$$= \frac{\left(\frac{1}{1+\sin 0}\right)4\cos 0}{\sec^2 0}$$

$$= \frac{\left(\frac{1}{1+0}\right)(4)(1)}{(1)^2} = 4$$

$$\Rightarrow \ln L = 4$$

$$\Rightarrow L = e^4$$

(h) $\lim_{x\to 0^+} x^x \underset{\text{DS.}}{=} (\to 0)^{\to 0}$

$$L = \lim_{x\to 0^+} x^x$$

$$\Rightarrow \ln L = \lim_{x\to 0^+} \ln\left(x^x\right)$$

$$= \lim_{x\to 0^+} x\ln x = (\to 0)\ln 0^+ = (\to 0)(-\infty)$$

$$= \lim_{x\to 0^+} \frac{\ln x}{1/x} \underset{\text{DS.}}{=} \frac{-\infty}{\infty} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x \underset{\text{DS.}}{=} 0$$

$$\Rightarrow \ln L = 0 \Rightarrow L = e^0 \Rightarrow L = 1$$

(i) $\lim_{x \to \frac{\pi}{2}^+} \left(x - \frac{\pi}{2}\right)^{\tan x}$ D.S. $\left(\frac{\pi}{2} - \frac{\pi}{2}\right)^{\tan \pi/2} = (\to 0)^{\to \infty}$

$\Rightarrow L = \lim_{x \to \frac{\pi}{2}^+} \left(x - \frac{\pi}{2}\right)^{\tan x}$

$\Rightarrow \ln L = \lim_{x \to \frac{\pi}{2}^+} \ln\left(\left(x - \frac{\pi}{2}\right)^{\tan x}\right)$

$= \lim_{x \to \frac{\pi}{2}^+} \tan x \ \ln\left(x - \frac{\pi}{2}\right)$

D.S. $(-\infty) \ln\left(\frac{\pi}{2}^+ - \frac{\pi}{2}\right) = (-\infty) \ln 0^+$

$= (-\infty)(-\infty) = \infty$

$\Rightarrow \ln L = \infty$

$\Rightarrow L = e^{\infty} = \infty$

(j) $\lim_{x \to 0} (\sin x)^{\tan x} = (\to 0)^{\to 0}$

$L = \lim_{x \to 0} (\sin x)^{\tan x}$

$\Rightarrow \ln L = \lim_{x \to 0} \ln (\sin x)^{\tan x} = \lim_{x \to 0} (\tan x) \ln(\sin x)$

|| D.S.

$(\to 0) \ln 0^+ = 0(-\infty)$

$= \lim_{x \to 0} \frac{\ln(\sin x)}{\frac{1}{\tan x}}$

$$= \lim_{x\to 0} \frac{\ln(\sin x)}{\cot x} = \frac{\ln(\sin 0)}{\cot 0} = \frac{-\infty}{\infty}$$

$$\cot 0 \Rightarrow \frac{\cos 0}{\sin 0} = \frac{1}{\to 0} = \infty$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sin x}\cos x}{-\csc^2 x}$$

$$\frac{d}{dx}\left(\frac{1}{\tan x}\right) = \frac{-1}{\tan^2 x}\sec^2 x$$

$$= -\frac{\cos^2 x}{\sin^2 x}\sec^2 x$$

$$= \frac{-1}{\sin^2 x} = -\csc^2 x$$

$$= \lim_{x\to 0}\left(\frac{\cos x}{\cancel{\sin x}}\right)(-1)\left(\sin^{\cancel{2}} x\right)$$

$$= \lim_{x\to 0} (-1)(\cos x)(\sin x)$$

D.S.

$$(-1)(\cos 0)(\sin 0)$$

$$= (-1)(1)(0) = 0$$

$$\Rightarrow \ln L = 0 \Rightarrow L = e^0 = 1$$