Name:

Problem 1: Find the following limits or show that it does not exist:-

1. $$\lim_{x\to\infty} \frac{\sqrt{x}+x^2}{2x-x^2}$$

2. $$\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} \quad \text{and} \quad \lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3}$$

3. $$\lim_{x\to\infty} \left(\sqrt{9x^2+x}-3x\right)$$

4. $$\lim_{x\to\infty} \sqrt{x}\sin\frac{1}{x}$$

1) $$\lim_{x\to\infty} \frac{\sqrt{x}+x^2}{2x-x^2} = \lim_{x\to\infty} \frac{\frac{\sqrt{x}}{x^2}+\frac{x^2}{x^2}}{\frac{2x}{x^2}-\frac{x^2}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x^{3/2}}+1}{\frac{2}{x}-1} = -1$$

2) $$\lim_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to\infty} \frac{\frac{1}{x^3}\sqrt{1+4x^6}}{\frac{2}{x^3}-\frac{x^3}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x^6}}\sqrt{1+4x^6}}{\frac{2}{x^3}-1}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} = \frac{\sqrt{0+4}}{0-1} = -2$$

$$\lim_{x\to-\infty} \frac{\sqrt{1+4x^6}}{2-x^3} = \lim_{x\to-\infty} \frac{\frac{1}{x^3}\sqrt{1+4x^6}}{\frac{2}{x^3}-1} = \lim_{x\to-\infty} \frac{\frac{-1}{\sqrt{x^6}}\sqrt{1+4x^6}}{\frac{2}{x^3}-1}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{\frac{1}{x^6}+4}}{\frac{2}{x^3}-1} = \frac{-\sqrt{4}}{-1} = 2$$

3) $$\lim_{x\to\infty} \left(\sqrt{9x^2+x}-3x\right) = \lim_{x\to\infty} \left(\sqrt{9x^2+x}-3x\right)\times\frac{\left(\sqrt{9x^2+x}+3x\right)}{\sqrt{9x^2+x}+3x}$$

$$= \lim_{x\to\infty} \frac{9x^2+x-(3x)^2}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x}$$

$$= \lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{1}{x}\sqrt{9x^2+x} + \frac{3x}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{\frac{9x^2+x}{x^2}} + 3}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{9+\frac{1}{x}} + 3} = \frac{1}{\sqrt{9+0} + 3} = \frac{1}{6}.$$

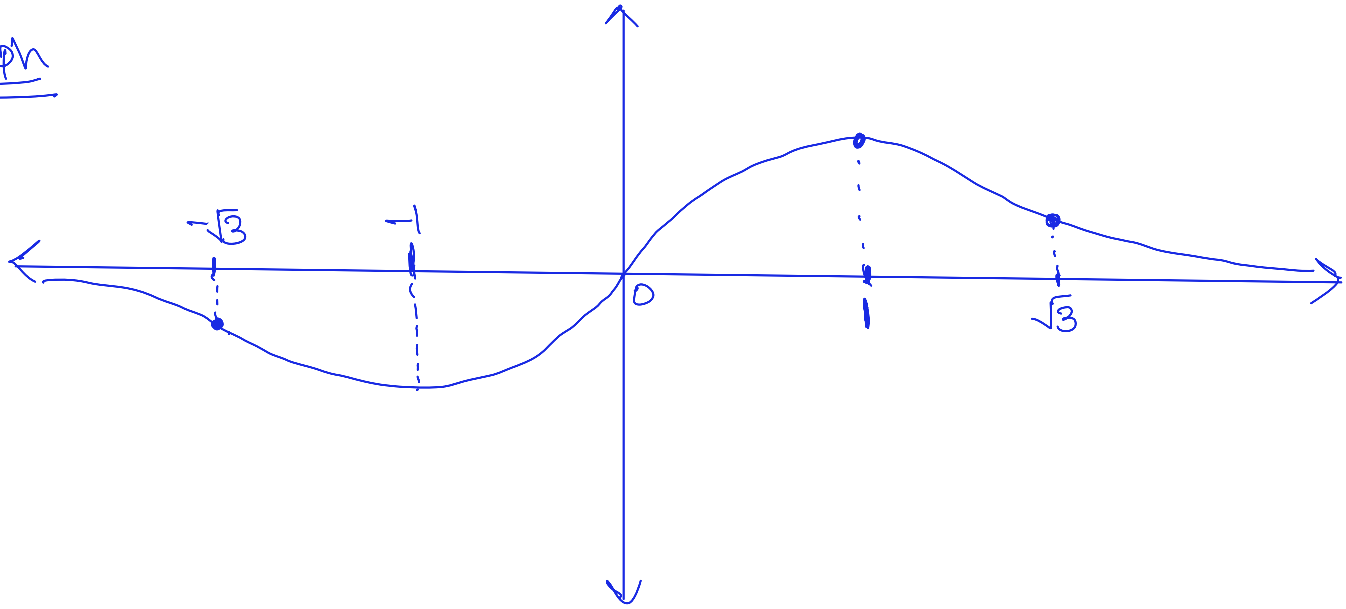
4) $\displaystyle \lim_{x\to\infty} \sqrt{x}\,\sin\frac{1}{x} = \lim_{x\to\infty} \sqrt{x}\,\frac{\sin\frac{1}{x}}{x\times\frac{1}{x}}$

$$= \lim_{x\to\infty} \frac{\sqrt{x}}{x}\,\frac{\sin\frac{1}{x}}{\left(\frac{1}{x}\right)} = \left(\lim_{x\to\infty}\frac{1}{\sqrt{x}}\right)\left(\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}\right)$$

$$= 0 \times \lim_{y\to 0}\frac{\sin y}{y} \quad \text{where } y = \frac{1}{x}$$

$$= 0 \times 1 = 0.$$

Graph

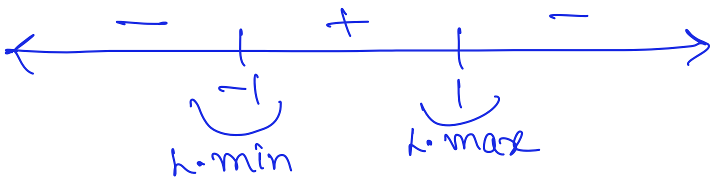


Problem 2: Find the horizontal asymptotes of the curve $y = \dfrac{x}{x^2+1}$ and use them, together with concavity and intervals of increase/decrease, to sketch the curve.

Horizontal asymptotes: $\displaystyle\lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{\frac{x}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 0$

$\displaystyle\lim_{x\to-\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{\frac{x}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 0$

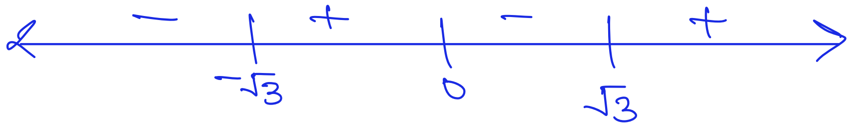
Intervals of Increase/Decrease: $\displaystyle f'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{-(x-1)(x+1)}{(x^2+1)^2}$



Concavity: $\displaystyle f''(x) = \frac{(x^2+1)^2(-2x)-(1-x^2)2(x^2+1)2x}{(x^2+1)^4}$

$\displaystyle = \frac{-2x(x^2+1)\left[x^2+1+2-2x^2\right]}{(x^2+1)^4} = \frac{-2x(3-x^2)}{(x^2+1)^3}$

$\displaystyle = \frac{2x(x-\sqrt{3})(x+\sqrt{3})}{(x^2+1)^3}$



Problem 3: Use the $\epsilon-\delta$ definition of a limit to prove that $\lim_{x\to3} x^2 = 9$ and $\lim_{x\to\infty} \frac{1}{x} = 0$.

To Prove: $\displaystyle\lim_{x\to3} x^2 = 9$.

Let $\epsilon > 0$ be arbitrary.

Need to find a $\delta > 0$ s.t. $0 < |x-3| < \delta \Rightarrow |x^2-9| < \epsilon$

$|x^2-9| = |x-3|\,|x+3|$.

we try to get an upper bound on $|x+3|$.

Since we are looking for small intervals around the Point $x=3$, it can be assumed that $|x-3|<1$.

$\Rightarrow -1 < x-3 < 1 \Rightarrow 3-1 < x < 3+1 \Rightarrow 2 < x < 4$

$\Rightarrow \quad 2+3 < x+3 < 4+3 \Rightarrow 5 < x+3 < 7 \Rightarrow |x+3| < 7$

as long as

$|x-3| < 1$

Now, if $|x-3| < \delta$ and $|x+3| < 7$, then

$$|x^2-9| = |x-3||x+3| < \delta \times 7 = 7\delta$$

So, 'may be' $7\delta = \varepsilon$ or $\delta = \frac{\varepsilon}{7}$ will work.

But to get $|x+3| < 7$ we also need $|x-3| < 1$

So, we also want $\delta \le 1$

Thus, we choose $\boxed{\delta = \min\left\{1, \frac{\varepsilon}{7}\right\}}$

Now, for this δ, $0 < |x-3| < \delta \Rightarrow |x-3| < \frac{\varepsilon}{7}$ and $|x-3| < 1$

— (I)

$|x-3| < 1 \Rightarrow |x+3| < 7$ — (II)

From (I) and (II) :—

$|x^2-9| = |x-3||x+3| < \frac{\varepsilon}{7} \times 7 = \varepsilon$, that is, $|x^2-9| < \varepsilon$

whenever $0 < |x-3| < \delta$

Hence Proved

To Prove $\lim_{x \to \infty} \frac{1}{x} = 0$

By defn., for every $\varepsilon > 0$, we want a number $M \in \mathbb{R}$

such that $x > M \Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon$

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we want $\left|\frac{1}{x}\right| < \varepsilon$

Since, lim is taken at $x \to \infty$, we can assume $x > 0$

$\Rightarrow \frac{1}{x} > 0 \quad \Rightarrow \quad \left|\frac{1}{x}\right| = \frac{1}{x}$.

So, we want that $\frac{1}{x} < \varepsilon \Rightarrow x > \frac{1}{\varepsilon}$

Thus, we can (may be) choose $\boxed{M = \frac{1}{\varepsilon}.}$

Now, suppose $M = \frac{1}{\varepsilon}$,

then $x > M \Rightarrow x > \frac{1}{\varepsilon} \Rightarrow \frac{1}{x} < \varepsilon \Rightarrow \left|\frac{1}{x}\right| < \varepsilon$

since $\varepsilon > 0$

$$\Rightarrow \left|\frac{1}{x} - 0\right| < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we have proved that

$$\lim_{x \to \infty} \frac{1}{x} = 0$$