M16600 Lecture Notes

Section 11.10: Taylor and Maclaurin Series

■ **Section 11.10** textbook exercises, *page 811*: #6, 8, 9, <u>19</u>, <u>21</u>, 23, 25, 35, 37, 54. For #54, use the series representation for $\sin x$ in Table 1, page 808.

Taylor Series is a power series with a formula for the coefficient c_n. How do we find the formula for the coefficients? We will start out with the general form for power series

$$\Rightarrow \quad f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1\underbrace{(x-a)} + c_2\underbrace{(x-a)}^2 + c_3\underbrace{(x-a)}^3 + c_4(x-a)^4 + \cdots,$$

then compute $f(a)$, $f'(a)$, $f''(a)$, $f'''(a)$, *etc.* and see if we can find a pattern for c_n:

$$f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots = c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f'(a) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots = c_1$$

$$f''(x) = 2c_2 + 3\cdot 2 c_3(x-a) + 4\cdot 3\, c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2$$

$$f'''(x) = 3\cdot 2\cdot c_3 + 4\cdot 3\cdot 2\, c_4(x-a) + 5\cdot 4\cdot 3\cdot c_5(x-a)^2 + \cdots$$

$$f'''(a) = 3\cdot 2\cdot c_3$$

$$f^{(iv)}(a) = 4\cdot 3\cdot 2\cdot c_4$$

$$f^{(n)}(a) = n(n-1)(n-2)\cdots\cdots 1\cdot c_n = n!\, c_n$$

$$\Rightarrow \boxed{c_n = \frac{f^{(n)}(a)}{n!}}$$

Taylor Series of $f(x)$ at a.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots\cdots \infty$$

$\leftarrow c_n$

A special case of Taylor series is when the center $a = 0$. This special is given a name called ***Maclaurin series***.

Maclaurin series (Taylor series centered at 0). $\quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$

$a = 0$

Example 1: Use the definition of Taylor series to find the first four nonzero terms of the series for $f(x) = \ln x$ centered at $a = 1$.

$$f(x) = \ln x = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

n	$f^{(n)}(1)$
0	0
1	1
2	$-1\cdot 1$
3	$2\cdot 1$
4	$-3\cdot 2\cdot 1$
5	$4\cdot 3\cdot 2\cdot 1$
6	$-5\cdot 4\cdot 3\cdot 2\cdot 1$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = \frac{-1}{x^2}$

$f'''(x) = \frac{2}{x^3} = 2x^{-3}$

$f^{(IV)}(x) = \frac{-6}{x^4} = -6x^{-4} = \frac{-3\cdot 2\cdot 1}{x^4}$

$f^{(V)}(x) = \frac{24}{x^5} = 24x^{-5} = \frac{4\cdot 3\cdot 2\cdot 1}{x^5}$

$f^{(VI)}(x) = \frac{24(-5)}{x^6} = \frac{-5\cdot 4\cdot 3\cdot 2\cdot 1}{x^6}$

$f^{(n)}(1) = (-1)^{n-1}(n-1)(n-2)(n-3)\cdots\cdots 1$

$\Rightarrow f^{(n)}(1) = (-1)^{n-1}(n-1)!$

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n$$

$$\frac{(n-1)!}{n!} = \frac{\cancel{(n-1)}\cancel{(n-2)}\cdots\cdot 1}{n\,\cancel{(n-1)}\cancel{(n-2)}\cdots 1} = \frac{1}{n}$$

$$\boxed{\ln x = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n}$$

Example 2: Find the Taylor series for $f(x) = \dfrac{1}{1+x}$ centered at $a = 2$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n$$

$f(x) = \frac{1}{1+x} = (1+x)^{-1}$ $\xrightarrow{x=2}$ $f(2) = \frac{1}{1+2} = \frac{1}{3}$

$f'(x) = \frac{-1}{(1+x)^2}$ $\xrightarrow{x=2}$ $f'(2) = \frac{-1}{3^2}$

$f''(x) = \frac{2}{(1+x)^3}$ $\xrightarrow{x=2}$ $f''(2) = \frac{2}{3^3}$

$f'''(x) = \frac{-6}{(1+x)^4} = \frac{-3\cdot 2\cdot 1}{(1+x)^4}$ $\xrightarrow{x=2}$ $f'''(2) = \frac{-3\cdot 2\cdot 1}{3^4}$

$f^{(IV)}(x) = \frac{4\cdot 3\cdot 2\cdot 1}{(1+x)^5}$ $\xrightarrow{x=2}$ $f^{(IV)}(2) = \frac{4\cdot 3\cdot 2\cdot 1}{3^5}$

$$f^{(n)}(2) = (-1)^n \frac{n\cdot(n-1)(n-2)\cdots\cdots 1}{3^{n+1}} = \frac{(-1)^n\, n!}{3^{n+1}}$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n \cancel{n!}}{3^{n+1}}}_{f^{(n)}(2)} \frac{(x-2)^n}{\cancel{n!}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(x-2)^n$$

Example 3: Use the definition of Maclaurin series to find the Maclaurin series of $f(x) = e^x$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \longrightarrow \text{ want to find } f^{(n)}(0)$$

$$\left.\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \\ f''(x) &= e^x \\ f'''(x) &= e^x \\ &\vdots \end{aligned}\right\} \xrightarrow{x=0} \begin{aligned} f(0) &= e^0 = 1 \\ f'(0) &= 1 \\ f''(0) &= 1 \\ f'''(0) &= 1 \\ &\vdots \end{aligned} \qquad \Rightarrow f^{(n)}(0) = 1 \text{ for every } n$$

$$\left.\begin{aligned} e^{0.1} \\ e^{0.9} \end{aligned}\right\} \begin{aligned} &\rightarrow \sum_{n=0}^{\infty} \frac{0.1^n}{n!} \\ &\rightarrow \sum_{n=0}^{\infty} \frac{0.9^n}{n!} \end{aligned}$$

Choose up to what value of n we want to approximate

$$\longrightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example 4: Use the result in example 3 to find the Maclaurin series for

(a) $f(x) = e^{-x^2}$

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

$$y = -x^2$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

(b) $f(x) = xe^x$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow xe^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$$

Example 5: (a) Evaluate $\displaystyle\int e^{-x^2}\,dx$ as an infinite series. (Note, we cannot compute this indefinite integral using any of the integral techniques we've learned in chapter 7)

$$\int e^{-x^2}dx \quad , \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$$

$$\int e^{-x^2}dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n}{n!} x^{2n} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1}$$

(b) Evaluate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using the first four terms of the power series you found in part (a).

$$\int_0^1 e^{-x^2}dx \quad , \quad e^{-x^2} = \frac{(-1)^0}{0!} 0^{2n} + \frac{(-1)^1}{1!} x^{2(1)} + \frac{(-1)^2}{2!} x^{2(2)} + \frac{(-1)^3}{3!} x^{2(3)}$$

$$= 0 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6$$

$$e^{-x^2} = -x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6$$

$$\int_0^1 e^{-x^2}dx = \int_0^1 \left(-x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right)dx = -\int_0^1 x^2 dx + \frac{1}{2}\int_0^1 x^4 dx - \frac{1}{6}\int_0^1 x^6 dx$$

$$= -\left[\frac{x^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{x^5}{5}\right]_0^1 - \frac{1}{6}\left[\frac{x^7}{7}\right]_0^1$$

$$= \frac{-1}{3} + \frac{1}{2}\left(\frac{1}{5}\right) - \left(\frac{1}{6}\right)\left(\frac{1}{7}\right) = \frac{-1}{3} + \frac{1}{10} - \frac{1}{42}$$

$$= \frac{-140+42-10}{420} = \frac{-\cancel{108}^{\cancel{27}\;9}}{\cancel{420}_{\cancel{105}\;35}}$$

$$\int_0^1 e^{-x^2}dx \approx \frac{-9}{35}$$