Math17100 Section 22866 Quiz 11
Spring 2023, April 12

Name: [1 pt]

Problem 1: Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & -1 & 1 \end{bmatrix}$. Use the inverse matrix thus obtained to find x, y, z where $x + 2y + 2z = 1$, $x + 3y + 2z = 1$, $x - y + z = 1$. [12 pts]

Solution:

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & -3 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + 3R_2]{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -4 & 3 & 1 \end{array}\right] \xrightarrow{R_3 \to -R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 3 & -2 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 4 & -3 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 4 & 2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 4 & -3 & -1 \end{array}\right]$$

Therefore, we have

$$A^{-1} = \begin{bmatrix} -5 & 4 & 2 \\ -1 & 1 & 0 \\ 4 & -3 & -1 \end{bmatrix}$$

The linear system given is $A\vec{x} = b$ where $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$A\vec{x} = b \Rightarrow \vec{x} = A^{-1}b$. Thus we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 & 4 & 2 \\ -1 & 1 & 0 \\ 4 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the solution set of the given system is $x = 1$, $y = 0$, $z = 0$.

Problem 2: Given the reduced row-echelon matrix $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ find the solution set.

[7 pts]

Solution: From the given reduce row-echelon matrix we see that the variable z is free since the third column does not have any pivot position.

So, we let $z = t$ for some parameter $t \in \mathbb{R}$.

The second row then gives $y + 3z = 2 \Rightarrow y = 2 - 3t$.

The first row gives $x - z = 4 \Rightarrow x = 4 + t$.

Therefore, the solution set is

$$x = 4 + t \ , \quad y = 2 - 3t \ , \quad z = t \quad \text{where} \ \ t \in \mathbb{R}$$

Bonus Problem: Find the 3×4 augmented reduced row-echelon matrix whose solution set is the plane $x + y + z = 1$. [2 pts]

Solution: Since the solution is a plane which is two dimensional or a surface, we must have two free variables.

If x and y were free then we would have got $x = s$, $y = t$ and $z =$ some real number which gives the solution set to be the some plane parallel to xy-plane.

If x and z were free then we would have got $z = t$, $y = c_1 + c_2 t$, $x = s$, which is the plane $y - c_2 z = c_1$. Note that c_1, c_2 are some constant real numbers here.

Thus, to get the plane $x + y + z = 1$ which contains all the variables x, y and z, we must have y and z to be the free variables.

So we must have $y = s$, $z = t$ for some parameters t and s and $x = 1 - t - s$.

The reduced row-echelon matrix that gives this solution set is thus:-

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$