■ **Section 7.3** exercises, *page 531*: #1, 2, 5, 6, 8, 12, 14, 9, 22, 17, 11.

Trigonometric Substitution is a new method which oftentimes are useful in solving integrals that involves the following radicals. We will also give the appropriate trig substitution for each type of radical:

- $\sqrt{4-x^2} = \sqrt{2^2 - x^2}$ $\Rightarrow x = 2\sin\theta$
- $\sqrt{5-x^2} \Rightarrow x = \sqrt{5}\sin\theta$

$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

- $\sqrt{9+x^2} \Rightarrow x = 3\tan\theta$
- $\sqrt{7+x^2} \Rightarrow x = \sqrt{7}\tan\theta$

- $\sqrt{x^2-1} \Rightarrow x = \sec\theta$
- $\sqrt{x^2-3} \Rightarrow x = \sqrt{3}\sec\theta$

We might need these two formulas for integrals in this section:

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

Example 1: Evaluate $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

$x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta$

$\sin\theta = \dfrac{x}{3}$

$\dfrac{dx}{d\theta} = 3\cos\theta$

$$\Rightarrow I = \int \frac{(3\sin\theta)^2}{\sqrt{9-9\sin^2\theta}}\, 3\cos\theta\, d\theta$$

$$= \int \frac{9\sin^2\theta}{\sqrt{9(1-\sin^2\theta)}}\, 3\cos\theta\, d\theta = \int \frac{9\sin^2\theta}{\sqrt{9\cos^2\theta}}\, 3\cos\theta\, d\theta$$

$$= \int \frac{9\sin^2\theta}{\cancel{3\cos\theta}}\, \cancel{3\cos\theta}\, d\theta = \int 9\sin^2\theta\, d\theta$$

$$= 9\int \sin^2\theta\, d\theta$$

$$= 9\int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{9}{2}\int (1-\cos 2\theta)\, d\theta$$

$$= \frac{9}{2}\int 1\, d\theta - \frac{9}{2}\int \cos 2\theta\, d\theta$$

$$\boxed{\cos\theta = \frac{\sqrt{9-x^2}}{3}} \quad = \frac{9}{2}\theta - \frac{9}{2}\frac{\sin 2\theta}{2} + C = \frac{9}{2}\theta - \frac{9}{2}\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}} + C$$

$$\Uparrow$$

$$\sin\theta = \frac{x}{3} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{3}\right)$$

$$= \frac{9}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{\cancel{9}}{2}\left(\frac{x}{\cancel{3}}\right)\frac{\sqrt{9-x^2}}{\cancel{3}} + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C$$

Example 2: Compute $\displaystyle\int_2^3 \frac{1}{\sqrt{x^2-1}}\, dx$

$$\downarrow$$

$$\sqrt{x^2-1^2}$$

$$x = \sec\theta$$

$$\Rightarrow \frac{dx}{d\theta} = \sec\theta\tan\theta$$

$$\Rightarrow dx = \sec\theta\tan\theta\, d\theta$$

$$\boxed{\sec^2\theta - 1 = \tan^2\theta}$$

$$\int \frac{1}{\sqrt{x^2-1}}\, dx = \int \frac{1}{\sqrt{\sec^2\theta - 1}}\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{\tan^2\theta}}\sec\theta\tan\theta\, d\theta = \int \frac{1}{\cancel{\tan\theta}}\sec\theta\cancel{\tan\theta}\, d\theta$$

$$= \int \sec\theta\, d\theta = \int \frac{\sec\theta(\sec\theta+\tan\theta)}{(\sec\theta+\tan\theta)}\, d\theta$$

$$= \int \frac{(\sec^2\theta + \sec\theta\tan\theta)}{\sec\theta+\tan\theta}\, d\theta$$

$$u = \sec\theta + \tan\theta$$

$$\frac{du}{d\theta} = \sec\theta\tan\theta + \sec^2\theta$$

$$du = (\sec\theta\tan\theta + \sec^2\theta)\, d\theta$$

$$= \int \frac{1}{u}\, du = \ln|u| + C$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$\sec\theta = \frac{x}{1} = \frac{H}{B} \Rightarrow H = x, \quad B = 1$$

$$\Rightarrow P = \sqrt{H^2 - B^2} = \sqrt{x^2 - 1^2} = \sqrt{x^2-1}$$

$$\tan\theta = \frac{P}{B} = \frac{\sqrt{x^2-1}}{1} = \sqrt{x^2-1}$$

$$I = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|x + \sqrt{x^2-1}\right| + C$$

$$\int_2^3 \frac{1}{\sqrt{x^2-1}}\,dx = \ln\left|x + \sqrt{x^2-1}\right|\Big|_2^3$$

$$= \ln\left|3 + \sqrt{3^2-1}\right| - \ln\left|2 + \sqrt{2^2-1}\right|$$

$$= \ln\left|3 + \sqrt{8}\right| - \ln\left|2 + \sqrt{3}\right|$$

$$= \ln(3 + \sqrt{8}) - \ln(2 + \sqrt{3})$$

$$= \ln\left(\frac{3 + \sqrt{8}}{2 + \sqrt{3}}\right)$$

Example 3: Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

$$\sqrt{x^2+2^2} \Rightarrow x = 2\tan\theta$$

$$\Rightarrow dx = 2\sec^2\theta\, d\theta$$

$$\Rightarrow I = \int \frac{1}{(2\tan\theta)^2\sqrt{(2\tan\theta)^2+4}}\, 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}\, 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{4\tan^2\theta\sqrt{4(\tan^2\theta+1)}}\, 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{4\tan^2\theta\sqrt{4\sec^2\theta}}\, 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{4\tan^2\theta\,(\cancel{2}\,\cancel{\sec\theta})}\, \cancel{2}\sec^{\cancel{2}}\theta\, d\theta$$

$$= \int \frac{\sec\theta}{4\tan^2\theta}\, d\theta = \int \frac{\frac{1}{\cos\theta}}{4\,\frac{\sin^2\theta}{\cos^2\theta}}\, d\theta$$

$$= \int \frac{1}{4}\left(\frac{1}{\cancel{\cos\theta}}\right)\frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\, d\theta = \int \frac{\cos\theta}{4\sin^2\theta}\, d\theta$$

$$u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$$

$$= \int \frac{du}{4u^2} = \frac{1}{4}\int u^{-2}\,du = \frac{1}{4}\,\frac{u^{-2+1}}{-2+1} + C$$

$$= \frac{-1}{4}\,u^{-1} + C = \frac{-1}{4u} + C$$

$$= \frac{-1}{4\sin\theta} + C$$

$$x = 2\tan\theta \Rightarrow \tan\theta = \frac{x}{2} = \frac{P}{B}$$

$$P = x, \quad B = 2 \Rightarrow H = \sqrt{x^2 + 2^2} = \sqrt{x^2+4}$$

$$\Rightarrow \sin\theta = \frac{P}{H} = \frac{x}{\sqrt{x^2+4}}$$

$$\Rightarrow \int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \frac{-1}{4\,\frac{x}{\sqrt{x^2+4}}} + C = \frac{-\sqrt{x^2+4}}{4x} + C$$