Problem 1: Determine whether the following sequence converges or diverges.
If it converges, find the limit.

$$a_n = \frac{2n^2 + \ln n}{n^2 + n + 1}$$

[6 pts]

Problem 2: Determine whether the series is convergent or divergent.
If it is convergent, find its sum.

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \frac{1}{48} + \cdots$$

[6 pts]

Problem 3: Determine whether the series is convergent or divergent:

$$\sum_{n=2}^{\infty} \frac{n}{\ln n}$$

Hint: Use Test for Divergence. [6 pts]

Problem 4: Determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

Hint: Use Limit Comparison Test. [6 pts]

Problem 5: Determine whether the series is convergent or divergent:

$$\frac{\ln 2}{\ln 3} - \frac{\ln 3}{\ln 4} + \frac{\ln 4}{\ln 5} - \frac{\ln 5}{\ln 6} + \frac{\ln 6}{\ln 7} \mp \cdots$$

Hint: Use Alternating Series Test. [6 pts]

Problem 6: Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

[6 pts]

Problem 7: Determine whether the series is convergent or divergent:

$$\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$$

Hint: If a series is absolutely convergent, then it is convergent. [6 pts]

Problem 8: Find the radius of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$$

[6 pts]

Problem 9: Find a power series representation for the function $f(x) = \dfrac{x}{1-x}$. [6 pts]

Problem 10: Find Maclaurin series for the function $f(x) = \cosh x$. [6pts]

Problem 11: Find the radius of convergence and interval of convergence of the power series:

$$\sum_{n=2}^{\infty} \frac{(x-1)^n}{2^n \ln n} \Rightarrow a_n = \frac{(x-1)^n}{2^n \ln(n)} \Rightarrow a_{n+1} = \frac{(x-1)^{n+1}}{2^{n+1} \ln(n+1)}$$

[8 pts]

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{2^{n+1} \ln(n+1)} \times \frac{2^n \ln(n)}{(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-1)}{2} \frac{\ln(n)}{\ln(n+1)} \right| = \frac{|x-1|}{2} \underbrace{\lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)}}_{=1} = \frac{|x-1|}{2}$$

$$\Rightarrow r = \frac{|x-1|}{2} < 1 \Rightarrow |x-1| < 2 \qquad \boxed{R=2}$$

$$-2 < x-1 < 2 \Rightarrow 1-2 < x < 1+2 \Rightarrow x \in (-1, 3)$$

$$x=-1 \Rightarrow \sum_{n=2}^{\infty} \frac{(-2)^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \longrightarrow \text{converges.}$$

$$\ln(n) < n$$

$$\frac{1}{\ln(n)} > \frac{1}{n}$$

$$x = 3 \Rightarrow \sum_{n=2}^{\infty} \frac{2^n}{2^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln(n)} > \sum_{n=2}^{\infty} \frac{1}{n} \rightarrow \text{diverges} \Rightarrow \boxed{I = [-1, 3)}$$

Problem 12: Find a power series representation of the function $f(x) = \ln(1-x)$ and determine its radius of convergence. [8 pts]

$$\ln(1-x) = -\int \frac{1}{1-x}\, dx$$

$$\int \frac{1}{1-x}\, dx = -\int \frac{1}{x-1}\, dx$$

$$= -\ln|x-1|$$

$$= -\ln|1-x|$$

$$= -\ln(1-x)$$

$$x<1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \infty$$

$$|x| < 1 \Rightarrow R = 1$$

$\downarrow$ integrate both sides

$$\int \frac{1}{1-x}\, dx = \int 1\, dx + \int x\, dx + \int x^2\, dx + \int x^3\, dx + \cdots \infty$$

$$\Rightarrow -\ln(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \infty$$

Put $x = 0 \Rightarrow -\ln 1 = C + 0 \Rightarrow 0 = C \Rightarrow C = 0$

$$\Rightarrow -\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \infty \Rightarrow \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \infty$$

$\Rightarrow \ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$

Problem 13: Find the Taylor series of $f(x) = \cos x$ about the point $x = \pi$. [8 pts]

$x = \pi$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(iv)}(x) = \cos x$

1st $f(\pi) = \cos\pi = -1$

2nd $f'(\pi)(x-\pi) = -\sin\pi\,(x-\pi) = 0$

3rd $\frac{f''(\pi)}{2}(x-\pi)^2 = -\frac{\cos\pi}{2}(x-\pi)^2 = \frac{1}{2}(x-\pi)^2$

4th $\frac{f'''(\pi)}{6}(x-\pi)^3 = \frac{\sin\pi}{6}(x-\pi)^3 = 0$

5th $\frac{f^{(iv)}(\pi)}{4!}(x-\pi)^4 = \frac{\cos\pi}{24}(x-\pi)^4 = -\frac{1}{4!}(x-\pi)^4$

$\cos x = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6 - \frac{1}{8!}(x-\pi)^8 \pm \cdots\cdots \infty$

$= \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(x-\pi)^{2n}}{(2n)!}$

Problem 14: Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

[8 pts]

Problem 15: Determine whether the series is convergent or divergent:

$$\sum_{n=2}^{\infty} n \tan(1/n)$$

[8 pts]

Bonus Problem: Find the radius of convergence of the Maclaurin series of the function $f(x) = 2^x$. [8 pts].

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f^{(n)}(0) \longrightarrow$
$n=0 \Rightarrow f(0)$
$n=1 \Rightarrow f'(0)$
$n=2 \Rightarrow f''(0)$
$n=3 \Rightarrow f'''(0)$
$\vdots$

$f(x) \quad 2^x$
$f'(x) \quad 2^x \ln 2$
$f''(x) \quad 2^x (\ln 2)^2$
$f'''(x) \quad 2^x (\ln 2)^3$
$\vdots$
$f^{(n)}(x) = 2^x (\ln 2)^n$

$$f^{(n)}(0) = 2^0 (\ln 2)^n = (\ln 2)^n$$

$$2^x = \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} x^n \Rightarrow r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$a_n = \frac{(\ln 2)^n}{n!} x^n \Rightarrow a_{n+1} = \frac{(\ln 2)^{n+1}}{(n+1)!} x^{n+1}$$

$$\Rightarrow \quad \frac{a_{n+1}}{a_n} = \frac{(\ln 2)^{n+1}}{(n+1)!} x^{n+1} \times \frac{n!}{(\ln 2)^n x^n} = \frac{(\ln 2)\, x}{(n+1)}$$

$$\Rightarrow r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(\ln 2)\, x}{n+1} \right| = \lim_{n\to\infty} \frac{\ln 2 \;\; |x|}{n+1} = 0 < 1$$

$\Rightarrow$ $r < 1$ for every value of x

$\Rightarrow$ the given series converges for every value of x.

$\Rightarrow$ $I = (-\infty, \infty)$ $\Rightarrow$ $R = \infty$