M16600 Lecture Notes
Section 7.8: Improper Integrals

■ **Section 7.8** textbook exercises, *page 574*: #2, 5, 7, 9, 11, 13, 19, 21, 27, 29, 31, 33.

GOALS

- Compute **improper integrals** of type I. E.g., $\displaystyle\int_1^\infty \frac{1}{x}\,dx$.
- Compute **improper integrals** of type II. E.g., $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

A definite integral $\displaystyle\int_a^b f(x)\,dx$ that we've encountered so far satisfies both of these conditions:

(i) The interval $[a, b]$ is finite and

(ii) The integrand $f(x)$ is continuous on $[a, b]$

If either one of the two conditions above fails, we say the definite integral to be ***improper***. Here are some examples of improper integrals

- ***Improper Integrals of Type I*** (condition (i) fails):

$$\int_1^\infty \frac{1}{x}\,dx, \qquad \int_{-\infty}^0 xe^x\,dx, \qquad \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx.$$

- ***Improper Integrals of Type II*** (condition (ii) fails):

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx, \qquad \int_0^1 \ln x\,dx, \qquad \int_{-1}^0 \frac{3}{x^3}\,dx, \qquad \int_0^3 \frac{1}{x-1}\,dx.$$

How to Compute Improper Integrals of Type I: Rewrite the integrals as follows:

- $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\left[\int_a^t f(x)\,dx\right]$
- $\displaystyle\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\left[\int_t^b f(x)\,dx\right]$
- $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$ where c is a constant

Definitions:

· The improper integral is ***convergent*** if the limit = a finite number (i.e., the limit exists)

· The improper integral is ***divergent*** if the limit = $\pm\infty$ or the limit does not exist.

Example 1: Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\int_1^\infty \frac{1}{x}\,dx$ $= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$

$$\int_1^t \frac{1}{x}\,dx = \ln|x| \Big|_1^t = \ln|t| - \ln|1| = \ln|t| - \cancel{\ln 1}^{\,0}$$
$$= \ln|t|$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|t| = \infty$$ $\Rightarrow$ the given improper integral is divergent.

(b) $\int_{-\infty}^0 xe^x\,dx$

$$= \lim_{t\to-\infty} \int_t^0 x e^x\,dx$$

$$\int \underbrace{x}_{u} \underbrace{e^x\,dx}_{dv} = x e^x - \int e^x\,dx = x e^x - e^x + C$$
$$= (x-1)e^x + C$$

$u = x \Rightarrow du = dx$

$dv = e^x\,dx \Rightarrow v = e^x$

$$\int_t^0 x e^x\,dx = (x-1)e^x \Big|_t^0 = (0-1)e^0 - (t-1)e^t$$
$$= -1 - (t-1)e^t$$

$$\int_{-\infty}^0 x e^x\,dx = \lim_{t\to-\infty} \left[-1 - (t-1)e^t\right]$$
$$= -1 - \boxed{\lim_{t\to-\infty} (t-1)e^t}$$

$\hookrightarrow (-\infty-1)e^{-\infty} = (-\infty)\,0$ (indeterminate)

$$= \lim_{t\to-\infty} \frac{(t-1)}{e^{-t}} = \frac{-\infty}{e^{\infty}} = \frac{\infty}{\infty}$$

$$= \lim_{t\to-\infty} \frac{1}{-e^{-t}} = \frac{1}{-e^{\infty}} = \frac{1}{-\infty} = 0$$

$$\int_{-\infty}^{0} x e^{x}\,dx = -1 - 0 = -1$$

$\Rightarrow$ The integral is convergent

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \lim_{s\to-\infty} \int_{s}^{t} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \lim_{s\to-\infty} \left(\tan^{-1} t - \tan^{-1} s\right)$$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \operatorname{Tan}^{-1}x \Big|_{t}^{0}$$

$$= \lim_{t\to-\infty} \left[\operatorname{Tan}^{-1}0 - \operatorname{Tan}^{-1}t\right]$$

$$= 0 - \lim_{t\to-\infty} \operatorname{Tan}^{-1}t$$

$$= 0 - (-\pi/2) = \pi/2$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to\infty} \operatorname{Tan}^{-1}x \Big|_{0}^{t} = \lim_{t\to\infty} \left[\operatorname{Tan}^{-1}t - \underbrace{\cancel{\operatorname{Tan}^{-1}0}}_{=0}\right]$$

$$= \lim_{t\to\infty} \operatorname{Tan}^{-1}t = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi \Rightarrow \text{the integral is convergent}$$

How to Compute Improper Integrals of Type II: Rewrite the integrals as follows:

- If f is only discontinuous at $x = b$, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \left[\int_a^t f(x)\,dx\right].$$

- If f is only discontinuous at $x = a$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \left[\int_t^b f(x)\,dx\right].$$

- If f is only discontinuous at $x = c$, where $a < c < b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 2: Determine whether the following integrals are convergent or divergent. Evaluate those that are convergent.

(a) $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

$\Rightarrow$ Pt. of discontinuity is $x=2$

f tends to ∞

$$= \lim_{t\to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big|_t^5$$

$$(x-2)^{-1/2} \xrightarrow{\text{Int.}} \frac{(x-2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{\sqrt{x-2}}{1/2}$$

$$= \lim_{t\to 2^+}\left[2\sqrt{5-2} - 2\sqrt{t-2}\right]$$

$$= 2\sqrt{3} - \lim_{t\to 2^+} 2\sqrt{t-2} = 2\sqrt{3} - \lim_{h\to 0} 2\sqrt{2+h-2}$$

$$= 2\sqrt{3} - \lim_{h\to 0} 2\sqrt{h} = 2\sqrt{3} - 0$$

$$= 2\sqrt{3}$$

$\Rightarrow$ The integral converges.

(b) $\displaystyle\int_0^3 \frac{1}{x-1}\,dx$

Pt. of discont. is at $x=1$

$$\Rightarrow \int_0^3 \frac{1}{x-1}\,dx = \underbrace{\int_0^1 \frac{1}{x-1}\,dx}_{\text{divergent}} + \underbrace{\int_1^3 \frac{1}{x-1}\,dx}_{\text{divergent.}}$$

$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \Big[\ln|x-1|\Big]_0^t$$

$$= \lim_{t\to 1^-}\left(\ln|t-1| - \ln|0-1|\right) = \lim_{t\to 1^-}\left(\ln|t-1| - \cancelto{0}{\ln 1}\right)$$

$$= \lim_{t\to 1^-} \ln|t-1| = \lim_{h\to 0} \ln|1-h-1| = \lim_{h\to 0} \ln|-h|$$

$$= \lim_{h \to 0} \ln(h) = -\infty$$

$$\Rightarrow \boxed{\int_0^3 \frac{1}{x-1}\,dx \quad \text{is divergent.}}$$

$$\int_1^3 \frac{1}{x-1}\,dx = \lim_{t \to 1^+} \int_t^3 \frac{1}{x-1}\,dx$$

$$= \lim_{t \to 1^+} \left(\ln|x-1|\right)_t^3$$

$$= \lim_{t \to 1^+} \ln|3-1| - \ln|t-1|$$

$$= \ln 2 - \lim_{t \to 1^+} \ln|t-1| = \ln 2 - \lim_{h \to 0} \ln|1+h-1|$$

$$= \ln 2 - \lim_{h \to 0} \ln h = \ln 2 - (-\infty)$$

$$= \infty$$