Learning objectives:

1. Rolle's theorem.
2. The Mean value theorem.
3. Applications.

Rolle's Theorem

Let f be a function that satisfies the following three conditions:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

Example 1. An object is moving in a straight line along the x-axis. Suppose the object was at position $x = 3$ at $t = 0$ and at $t = 2$. Show that at some instant between $t = 0$ and $t = 2$, the object was at rest.

Example 2. Prove that the equation $x^3 + x - 1$ has exactly one real root.

The Mean Value Theorem

Let f be a function that satisfies the following two conditions:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} ,$$

or equivalently

$$f(b) - f(a) = f'(c)(b - a) .$$

Example 3. Let $f(x) = x^3 - x$, $a = 0$, $b = 2$. Check/illustrate that the mean value theorem holds.

Example 4. Suppose $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

Example 5. If $f'(x) = 0$ for all x in an interval (a, b), then show that f is constant on (a, b).

Example 6. If $f'(x) = g'(x)$ for all x in an interval (a, b), then show that $f(x) = g(x) + c$ for some constant c whenever $a < x < b$.

Example 7. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for all x?

Example 8. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed.