Indiana University, Indianapolis

Spring 2025 Math-I 165

Test 3 (April 23, 2025)

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Name:____

Instructions:

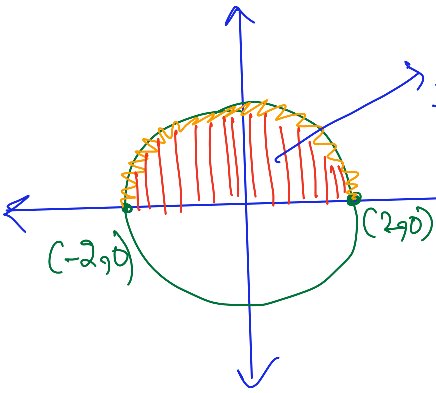
- No cell phones, calculators, watches, technology, hats - stow all in your bags.
- **Write your name** on this cover page.
- This test is **closed book and closed notes**.
- All work must be clearly shown for partial credit.
- If you wish for something not to be graded, please strike it out neatly.
- Box, circle, or otherwise clearly indicate your final answer.
- When you finish, return your test to the proctor, and leave the classroom.
- There are a total of **12 problems including 2 bonus problems**.
  - Problems 1-10 are each worth 10 points.
  - The bonus problems are each worth 5 points.
- You can score a **maximum of 110 points out of 100**.
- There are a total of **7 pages** including the cover page.

Problem 1. Evaluate the integral $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$ by interpreting it as area under a familiar curve. [10 pts]

$y = \sqrt{4-x^2}$

$y^2 = 4 - x^2$

$x^2 + y^2 = 4$



$\int_{-2}^{2} \sqrt{4-x^2}\, dx = A$

$A = \frac{1}{2}\left(\pi (2)^2\right) = 2\pi$

$\int_0^1 \sqrt{1-x^2}\, dx$

$\int_{-1}^{1} \left(x + \sqrt{1-x^2}\, dx\right) dx$

Problem 2. Use symmetry to compute the integral $\int_{-\pi/4}^{\pi/4} (\tan x + \sin x)\, dx$. [10 pts]

odd function

$\Rightarrow I = 0$

Problem 3. Compute the integral $\int_0^1 x\,dx$ by expressing it as limit of a sum.

Hint: Use $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. [10 pts]

$$\parallel$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{1}{n^2} \frac{n(n+1)}{2}$$

$$= \frac{1}{2}$$

Problem 4. A particle moves in a straight line with an acceleration of $a(t) = 1 + \sin t$. If the initial velocity of the particle at $t = 0$ was 0. Find velocity of the particle at $t = \pi$ seconds. [10 pts]

$$v(\pi) - \underbrace{v(0)}_{\parallel\ 0} = \int_0^{\pi} a(t)\,dt$$

Problem 5. Evaluate the indefinite integral $\int (\sin^2 x)(\cos x)\,dx$. [10 pts]

$u = \sin x$

$du = \cos x\, dx$

$$\int u^2\, du$$

$u = \cos x$

$du = -\sin x\, dx$

$$\int u \sin x\,(-du)$$

$$= -\int u\sqrt{1-u^2}\, du$$

Problem 6. Evaluate definite integral $\int_0^4 |t-2|\,dt$. [10 pts]

$$\| $$

$$\int_0^2 -(t-2)\,dt + \int_2^4 (t-2)\,dt$$

$$t-2 = \begin{cases} -\text{ve} & \text{if } t<2 \\ +\text{ve} & \text{if } t>2 \end{cases}$$

Alternatively

$$\int_0^4 |\underbrace{t-2}|\,dt = \int_{-2}^{2} |u|\,du = 2\int_0^2 |u|\,du$$

$u = t-2 \Rightarrow du = dt$

$$= 2\int_0^2 u\,du = 2\left.\frac{u^2}{2}\right|_0^2$$

$$= 4$$

Problem 7. Find area of the region bounded on three sides by the curves $y = \sin x$, $y = \cos x$ and the y-axis. [10 pts]

Problem 8. Find area of the region bounded by the parabolas $x = 1 - y^2$ and $x = y^2 - 1$. [10 pts]

Problem 9. Use the disk method to find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, the x-axis and $x = 1$ line; about the x-axis. [10 pts]

Problem 10. Use the shell method to find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and the x-axis; about the axis y-axis. [10 pts]

Bonus Problem 1. Evaluate the limit: $\lim_{n\to\infty} \frac{\pi}{n}\left[\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{3\pi}{n}\right) + \cdots + \sin\left(\frac{n\pi}{n}\right)\right]$ by expressing it as an integral and then evaluating the integral obtained. [5 pts]

$\sin\left(i\frac{\pi}{n}\right)$

$\Delta x = \frac{\pi}{n}$

$f(x) = \sin x$

$x_i = i\frac{\pi}{n}$

$x_i = a + i(\Delta x)$

$\Rightarrow \frac{b-a}{n} = \frac{\pi}{n} \Rightarrow b - a = \pi$

$b - 0 = \pi$

$\Rightarrow b = \pi$

$a + i\left(\frac{\pi}{n}\right) = i\frac{\pi}{n} \Rightarrow a = 0$

$$\lim_{n\to\infty} \frac{\pi}{n}\left[\sin\left(\frac{\pi}{n}\right) + \cdots + \sin\left(\frac{n\pi}{n}\right)\right] = \int_0^{\pi} \sin x \, dx$$

$$= -\cos x \Big|_0^{\pi} = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 1 + 1 = 2$$

Bonus Problem 2. Find the derivative of $f(x) = \int_{-x}^{\sin x} \sqrt{1-t^2}\, dt$. Assume $0 \le x \le \pi/2$. [5 pts]

$\int_{v(x)}^{u(x)} g(t)\, dt$

$$f'(x) = g(u(x))u'(x) - g(v(x))v'(x)$$

$$= \sqrt{1-\sin^2 x}\,(\sin x)' - \sqrt{1-(-x)^2}\,(-x)'$$

$$= \sqrt{1-\sin^2 x}\,\cos x - \sqrt{1-x^2}\,(-1)$$

$$= \sqrt{\cos^2 x}\,\cos x + \sqrt{1-x^2}$$

$$= |\cos x|\cos x + \sqrt{1-x^2} = \cos^2 x + \sqrt{1-x^2}$$