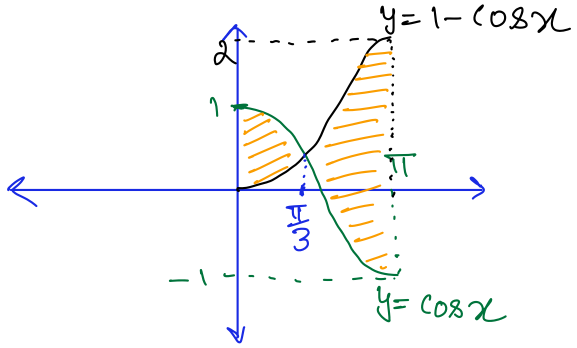
Name:

Problem 1: Sketch the region enclosed by the given curves and find its area.

1. $y = \cos x$, $y = 1 - \cos x$, $x = 0$, $x = \pi$.
2. $y = x^4$ and $y = 2 - |x|$.
3. $x = 2y^2$ and $x = y^2 + 4$.

(1)



Point of intersection :-

$$\cos x = 1 - \cos x$$

$$\Rightarrow 2\cos x = 1 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$$

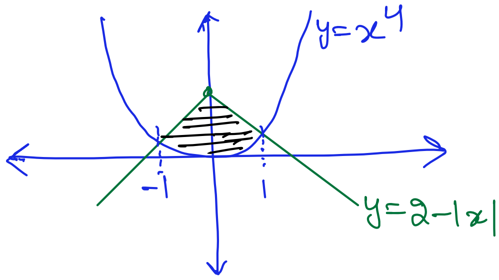
$$\Rightarrow A = \int_0^{\pi/3} [\cos x - (1 - \cos x)]\,dx + \int_{\pi/3}^{\pi} [1 - \cos x - \cos x]\,dx$$

$$= \int_0^{\pi/3} (2\cos x - 1)\,dx + \int_{\pi/3}^{\pi} (1 - 2\cos x)\,dx$$

$$= 2\sin x\Big|_0^{\pi/3} - x\Big|_0^{\pi/3} + x\Big|_{\pi/3}^{\pi} - 2\sin x\Big|_{\pi/3}^{\pi}$$

$$= 2\frac{\sqrt{3}}{2} - \frac{\pi}{3} + \left(\pi - \frac{\pi}{3}\right) - 2\left(0 - \frac{\sqrt{3}}{2}\right) = 2\sqrt{3} + \frac{\pi}{3}$$

(2) $y = x^4$ and $y = 2 - |x|$



Points of intersection :-

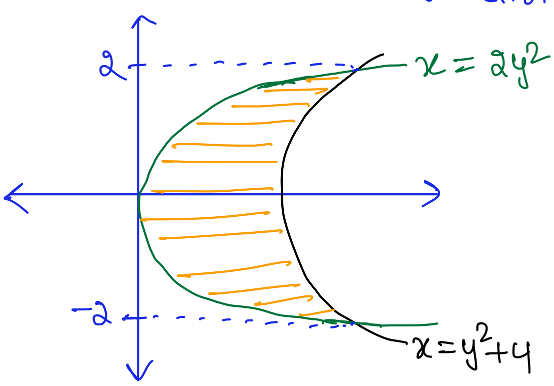
$$x^4 = 2 - x \Rightarrow x = 1$$

$$x^4 = 2 + x \Rightarrow x = -1$$

$$\Rightarrow A = \int_{-1}^{1} \underbrace{(2 - |x| - x^4)}_{\text{even function}}\,dx = 2\int_0^1 (2 - x - x^4)\,dx = 2\left[2x\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 - \frac{x^5}{5}\Big|_0^1\right]$$

$$= 2\left[2 - \frac{1}{2} - \frac{1}{5}\right] = \frac{13}{5}$$

(3)



Points of intersection :-

$$2y^2 = y^2 + 4 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

$$\Rightarrow A = \int_{-2}^{2} (y^2 + 4 - 2y^2)\,dy = \int_{-2}^{2} \underbrace{(4 - y^2)}_{\text{even function}}\,dy$$

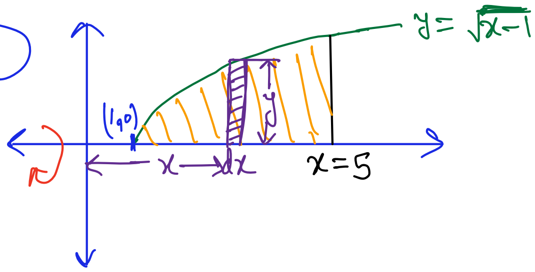
$$= 2\int_0^2 (4 - y^2)\,dy$$

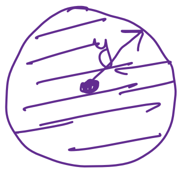
$$= 2\left[4y\Big|_0^2 - \frac{y^3}{3}\Big|_0^2\right] = 2\left[8 - \frac{8}{3}\right] = \frac{32}{3}$$

Problem 2: Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

1. $y = \sqrt{x-1}$, $y = 0$, $x = 5$ about x-axis.
2. $y^2 = x$, $x = 2y$ about the y-axis.
3. $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$ about $y = -1$.

①

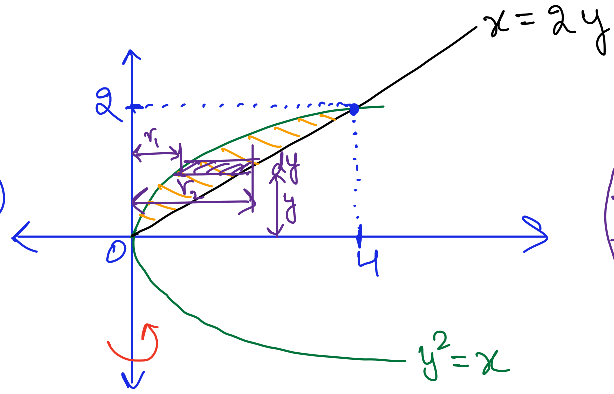


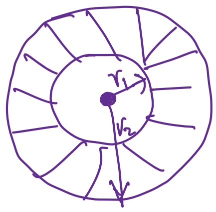


$$\Rightarrow dV = \pi y^2\, dx = \pi\left(\sqrt{x-1}\right)^2 dx$$

$$\Rightarrow V = \int_1^5 \pi(x-1)\,dx = \pi\left[\frac{x^2}{2}\Big|_1^5 - x\Big|_1^5\right] = \pi\left[\frac{24}{2} - 4\right] = 8\pi$$

②





Point of intersection :-

$$x = 2y = y^2 \Rightarrow y^2 - 2y = 0 \Rightarrow y = 0, 2$$

$$dV = \pi(r_2^2 - r_1^2)\,dy \Rightarrow V = \int_0^2 \pi(r_2^2 - r_1^2)\,dy$$

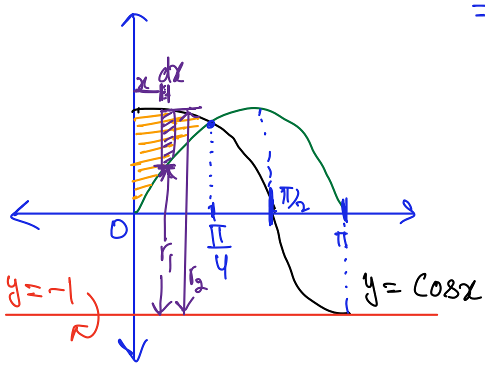
For r_2, the point to the right lies on $x = 2y \Rightarrow r_2 = x_2 = 2y$

For r_1, the point to the left lies on $y^2 = x \Rightarrow r_1 = x_1 = y^2$

$$\Rightarrow V = \int_0^2 \pi\left((2y)^2 - (y^2)^2\right)dy = \int_0^2 \pi(4y^2 - y^4)\,dy = \pi\left[\frac{4}{3}y^3\Big|_0^2 - \frac{y^5}{5}\Big|_0^2\right]$$

$$= \pi\left[\frac{32}{3} - \frac{32}{5}\right] = \frac{64\pi}{15}$$

③



$$V = \int_0^{\frac{\pi}{4}} \pi(r_2^2 - r_1^2)\,dx$$

For r_1, (the lower point lies on $y = \sin x$)

$\Rightarrow r_1 = 1 + \sin x$

For r_2 (the upper point lies on $y = \cos x$) $\Rightarrow r_2 = 1 + \cos x$

$$\Rightarrow V = \int_0^{\frac{\pi}{4}} \pi\left((1+\cos x)^2 - (1+\sin x)^2\right)dx = \pi\int_0^{\pi/4}\left(\cancel{1} + \cos^2 x + 2\cos x - \cancel{1} - \sin^2 x - 2\sin x\right)dx$$

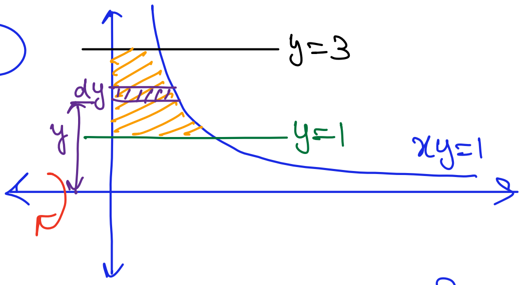
$$= \pi\int_0^{\pi/4}(\cos^2 x - \sin^2 x)\,dx + 2\pi\int_0^{\frac{\pi}{4}}\cos x\,dx - 2\pi\int_0^{\frac{\pi}{4}}\sin x\,dx$$

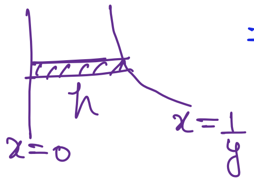
$$= \pi\int_0^{\frac{\pi}{4}}\cos 2x\,dx + 2\pi\sin x\Big|_0^{\pi/4} + 2\pi\cos x\Big|_0^{\pi/4} = \pi\,\frac{\sin 2x}{2}\Big|_0^{\frac{\pi}{4}} + 2\pi(\sqrt{2}-1) = \pi\left(2\sqrt{2} - \frac{3}{2}\right)$$

Problem 3: Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis.

1. $xy = 1$, $x = 0$, $y = 1$, $y = 3$ about x-axis.
2. $y = 4x - x^2$, $y = x$ about y-axis.
3. $x = 2y^2$, $x = y^2 + 1$ about $y = -2$.

①





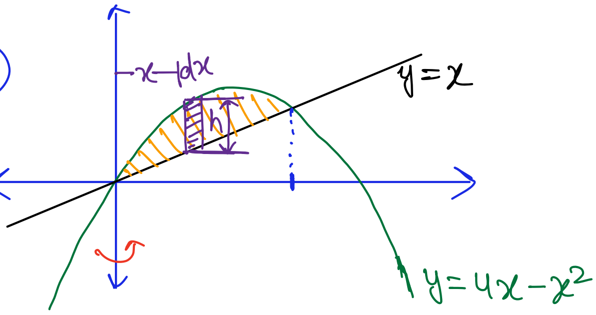
$\Rightarrow h = \frac{1}{y} - 0 = \frac{1}{y}$

$r = y$

$dV = 2\pi r h \, dy = 2\pi y \frac{1}{y} dy = 2\pi \, dy$

$$\Rightarrow V = \int_1^3 2\pi \, dy = 2\pi y \Big|_1^3 = 4\pi$$

②



$dV = 2\pi r h \, dx$

$r = x$

For h, the upper point lies on $y=4x-x^2$ and the lower point lies on $y=x$

we want h in terms of x.

For point of intersection :-

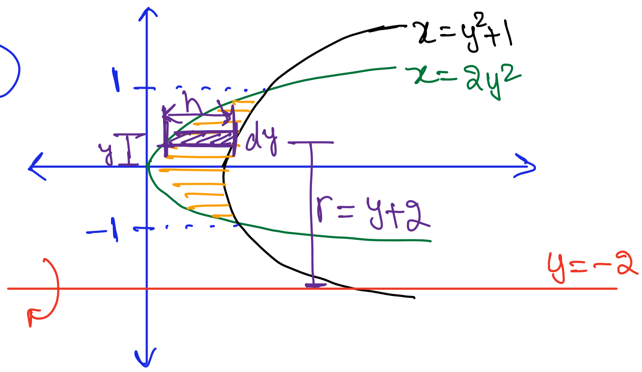
$4x - x^2 = x \Rightarrow x^2 - 3x = 0 \Rightarrow x = 0, 3$

$\Rightarrow h = y_{upper} - y_{lower} = 4x - x^2 - x = 3x - x^2$

$$\Rightarrow V = \int_0^3 2\pi x (3x - x^2) \, dx = 2\pi \int_0^3 (3x^2 - x^3) \, dx = 2\pi \left[x^3 \Big|_0^3 - \frac{x^4}{4} \Big|_0^3 \right]$$

$$= 2\pi \left[27 - \frac{81}{4} \right] = \frac{27\pi}{2}$$

③



Points of intersection :-

$y^2 + 1 = 2y^2 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

$dV = 2\pi r h \, dy \Rightarrow V = \int_{-1}^{1} 2\pi r h \, dy$

$r = y + 2$

$h = x_2 - x_1$ ← lies on left curve

x_2: lies on right curve

$\Rightarrow h = y^2 + 1 - 2y^2 = 1 - y^2$

$$\Rightarrow V = \int_{-1}^{1} 2\pi (y+2)(1-y^2) \, dy$$

$$\Rightarrow V = \underbrace{\int_{-1}^{1} 2\pi y (1-y^2)}_{\text{odd}} + \underbrace{\int_{-1}^{1} 4\pi (1-y^2) \, dy}_{\text{even}} = 0 + 8\pi \int_0^1 (1-y^2) \, dy$$

$$= 8\pi \left(y \Big|_0^1 - \frac{y^3}{3} \Big|_0^1 \right) = \frac{16\pi}{3}$$

Problem 4: Find the average value of the following functions on the given interval.

1. $f(x) = \cos^4 x \sin x$ on $[0, \pi]$.

2. $g(t) = \dfrac{t}{\sqrt{3+t^2}}$ on $[1, 3]$.

$$f_{av} = \frac{1}{\pi}\int_0^{\pi} \cos^4 x \sin x \, dx \Rightarrow f_{av} = \frac{1}{\pi}\int_{\cos 0}^{\cos\pi} y^4 (-dy)$$

Put $y = \cos x \Rightarrow dy = -\sin x \, dx$

$$\Rightarrow f_{av} = \frac{-1}{\pi}\int_1^{-1} y^4 \, dy = \frac{1}{\pi}\int_{-1}^{1} y^4 \, dy = \frac{2}{\pi}\int_0^1 y^4 \, dy = \frac{2}{\pi}\frac{y^5}{5}\Big|_0^1 = \frac{2}{5\pi}$$

even

$$g_{av} = \frac{1}{3-1}\int_1^3 \frac{t}{\sqrt{3+t^2}} \, dt .$$ Substitute $y = 3+t^2 \Rightarrow dy = 2t\,dt \Rightarrow t\,dt = \frac{dy}{2}$

$$g_{av} = \frac{1}{2}\int_{3+1^2}^{3+3^2} \frac{1}{\sqrt{y}} \frac{dy}{2} = \frac{1}{4}\int_4^{12} y^{-1/2} \, dy = \frac{1}{4} \frac{y^{-1/2+1}}{-1/2+1}\Big|_4^{12}$$

$$= \frac{1}{4} \, 2\sqrt{y}\Big|_4^{12} = \frac{1}{2}\left(\sqrt{12} - \sqrt{4}\right) = \frac{1}{2}\left(2\sqrt{3} - 2\right) = \sqrt{3} - 1$$

Problem 5: When a particle is located at a distance x meters from the origin, a force of $\cos(\pi x/3)$ newtons acts on it. How much work is done in moving the particle from $x = 1$ to $x = 2$?

$$W = \int_1^2 \cos\left(\frac{\pi x}{3}\right) dx = \int_{\pi/3}^{2\pi/3} \cos y \, \frac{3}{\pi} dy = \frac{3}{\pi}\int_{\pi/3}^{2\pi/3} \cos y \, dy$$

$y = \frac{\pi x}{3} \Rightarrow x = \frac{3}{\pi} y \Rightarrow dx = \frac{3}{\pi} dy$

$$= \frac{3}{\pi} \sin y \Big|_{\pi/3}^{2\pi/3}$$

$$= \frac{3}{\pi}\left[\sin\frac{2\pi}{3} - \sin\frac{\pi}{3}\right]$$

$$= \frac{3}{\pi}\left[\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right] = 0$$