Indiana University, Indianapolis

Spring 2025 Math-I 165

Practice Test 1a

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Name:____

Instructions:

- No cell phones, calculators, watches, technology, hats - stow all in your bags.
- **Write your name** on this cover page.
- This test is **closed book and closed notes**.
- All work must be clearly shown for partial credit.
- If you wish for something not to be graded, please strike it out neatly.
- Box, circle, or otherwise clearly indicate your final answer.
- When you finish, return your test to the proctor, and leave the classroom.
- There are a total of **12 problems including 2 bonus problems**.
  - Problems 1-10 are each worth 10 points.
  - The bonus problems are each worth 5 points.
- You can score a **maximum of 110 points out of 100**.
- There are a total of **7 pages** including the cover page.

Problem 1. Evaluate the limit: $\lim_{x\to 1} \dfrac{x^3 - x^2 + x - 1}{x - 1}$. [10 pts]

$$\lim_{x\to 1} \frac{x^3-x^2+x-1}{x-1} \underset{=}{\text{D.S.}} \frac{1-1+1-1}{1-1} = \frac{0}{0}$$

$$\underbrace{x^3-x^2}+x-1 = \underbrace{x^2(x-1)+1(x-1)}$$
$$= (x-1)(x^2+1)$$

$$\lim_{x\to 1} \frac{x^3-x^2+x-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+1)}{\cancel{x-1}} = \lim_{x\to 1}(x^2+1)$$
$$= (1)^2+1 = 2$$

Problem 2. Find the points of discontinuity of the function $f(x) = \begin{cases} \sin x - 1 & x < 0\,, \\ \dfrac{|x-1|}{x-1} & 0 \le x < 1\,, \\ \cos(x-1) & x \ge 1\,. \end{cases}$

$\sin x$ is continuous everywhere

$\Rightarrow \sin x - 1$ is continuous for every $x < 0$

For $\underbrace{0 < x < 1}_{x-1<0}$, $f(x) = \dfrac{|x-1|}{x-1} = \dfrac{-(x-1)}{x-1} = -1$

$|x-1| = -(x-1)$ $\Rightarrow f(x)$ is continuous for $0<x<1$

For $x>1$, $f(x) = \cos(x-1)$ which is continuous everywhere

$x=0$ and $x=1$ can potentially be points of discontinuity.

$\underline{x=0}$

$$\underset{x=0}{\text{LHL}} = \lim_{x\to 0^-} f(x) = \sin(0)-1 = 0-1 = -1$$

$$f(0) = \underset{x=0}{\text{RHL}} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x-1|}{x-1} = \lim_{x\to 0^+} -1 = -1$$

$\Bigg\}$ Continuous

$\underline{x=1}$ $\quad \underset{x=1}{\text{LHL}} = \lim_{x\to 1^-} \dfrac{|x-1|}{x-1} = -1$, $\quad \underset{x=1}{\text{RHL}} = f(1) = \cos(1-1) = \cos 0 = 1$

$\underset{x=1}{\text{LHL}} \neq \underset{x=1}{\text{RHL}} \Rightarrow x=1$ is a pt. of discontinuity.

Problem 3. Use the limit definition of derivative to find $f'(0)$ if $f(x) = \dfrac{1}{\sqrt{1-x}}$. [10 pts]

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$\hookrightarrow f(0) = \frac{1}{\sqrt{1-0}} = 1$$

$$a=0 \quad \Rightarrow \quad f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{\sqrt{1-h}} - 1}{h} = \lim_{h\to 0} \frac{\frac{1-\sqrt{1-h}}{\sqrt{1-h}}}{h} = \lim_{h\to 0} \frac{1-\sqrt{1-h}}{h\sqrt{1-h}}$$

$$\stackrel{\text{D.S.}}{=} \frac{0}{0}$$

$$= \lim_{h\to 0} \frac{1-\sqrt{1-h}}{h\sqrt{1-h}} \times \frac{1+\sqrt{1-h}}{1+\sqrt{1-h}}$$

$$= \lim_{h\to 0} \frac{(1-\sqrt{1-h})(1+\sqrt{1-h})}{h\sqrt{1-h}\,(1+\sqrt{1-h})} = \lim_{h\to 0} \frac{1-(1-h)}{h\sqrt{1-h}\,(1+\sqrt{1-h})} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\sqrt{1-h}\,(1+\sqrt{1-h})}$$

$$1-1+h$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{1-h}\,(1+\sqrt{1-h})} \stackrel{\text{D.S.}}{=} \frac{1}{\sqrt{1}\,(1+\sqrt{1})} = \boxed{\frac{1}{2}}$$

Problem 4 . Find derivative of the function $f(x) = \dfrac{x^9}{x+9}$. [10 pts]

use quotient rule

$$f'(x) = \frac{(x+9)[x^9]' - x^9[x+9]'}{(x+9)^2}$$

$$= \frac{(x+9)(9x^8) - x^9(1)}{(x+9)^2}$$

$$= \frac{9x^9 + 81x^8 - x^9}{(x+9)^2} = \frac{8x^9 + 81x^8}{(x+9)^2}$$

$$= \boxed{\frac{x^8(8x+81)}{(x+9)^2}}$$

Problem 5. A particle is moving along the x-axis so that its displacement varies with time as $s(t) = t^4 - t^3$. Find the time interval when the particle is speeding up. [10 pts]

speeding up: t for which $a(t) > 0$

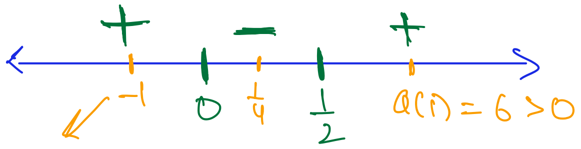
$$v(t) = s'(t) = \frac{d}{dt}(t^4 - t^3) = 4t^3 - 3t^2$$

$$a(t) = v'(t) = \frac{d}{dt}(4t^3 - 3t^2) = 12t^2 - 6t$$

$$a(t) = 6t(2t-1).$$

Put $a(t) = 0$

$\Rightarrow 6t(2t-1) \Rightarrow t = 0, t = \frac{1}{2}$



$a(-1) = -6(-3) = 18 > 0$ $\quad a(\frac{1}{4}) = \frac{3}{2}(\frac{-1}{2}) < 0$

$a(t) > 0$ when

either $\underbrace{t < 0}_{\text{reject}}$ or $t > \frac{1}{2}$

(time is taken to be +ve)

$\Rightarrow$ Particle is speeding for $t > \frac{1}{2}$ or $(\frac{1}{2}, \infty)$

Problem 6. The derivative of the function $f(x) = x^2(x^2+1)^{5/2}$ is $f'(x) = x(x^2+1)^{3/2}(ax^2+b)$. Find the numbers a and b. [10 pts]

$$f'(x) = \frac{d}{dx}\left[x^2 (x^2+1)^{5/2}\right]$$

use Product rule

$$= \frac{d}{dx}(x^2)\,(x^2+1)^{5/2} + x^2 \frac{d}{dx}\left[(x^2+1)^{5/2}\right]$$

$$= 2x(x^2+1)^{5/2} + x^2\left[\frac{5}{2}(x^2+1)^{3/2}(2x)\right]$$

from chain rule.

$$\Rightarrow f'(x) = \underbrace{2x(x^2+1)^{5/2} + 5x^3(x^2+1)^{3/2}}_{\text{can factor } x(x^2+1)^{3/2}}$$

$$= x(x^2+1)^{3/2}\left[2(x^2+1)^{\frac{5}{2}-\frac{3}{2}} + 5x^2\right] = x(x^2+1)^{3/2}\left[2(x^2+1) + 5x^2\right]$$

$$= x(x^2+1)^{3/2}(7x^2+2) \Rightarrow a = 7,\ b = 2$$

Problem 7. Differentiate the function $f(x) = \dfrac{x}{\sqrt{1+x+x^2}}$. [10 pts]

(use quotient rule)

$$\Rightarrow f'(x) = \frac{\sqrt{1+x+x^2}\,[x]' - x\left[\sqrt{1+x+x^2}\right]'}{(1+x+x^2)}$$

$$\frac{d}{dx}\left[(1+x+x^2)^{1/2}\right]$$

$$= \frac{1}{2}(1+x+x^2)^{-1/2} \cdot (1+x+x^2)'$$

$$= \frac{1}{2\sqrt{1+x+x^2}} \cdot (1+2x)$$

$$= \frac{\sqrt{1+x+x^2}\,(1) - x \cdot \dfrac{(1+2x)}{2\sqrt{1+x+x^2}}}{(1+x+x^2)}$$

cross multiply

$$= \frac{\dfrac{2(1+x+x^2) - x(1+2x)}{2\sqrt{1+x+x^2}}}{(1+x+x^2)} = \frac{2 + 2x + \cancel{2x^2} - x - \cancel{2x^2}}{2(1+x+x^2)\sqrt{1+x+x^2}}$$

$$= \frac{2+x}{2(1+x+x^2)\sqrt{1+x+x^2}}$$

Problem 8. A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$, the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate of the point changing at that instant? [10 pts]

Given $\dfrac{dy}{dt} = -3$ cm/s, To find $\dfrac{dx}{dt} = ?$ when $x=4$, $y=2$

$$xy = 8$$

Differentiate both sides w.r.t. t

product rule

$$\frac{d}{dt}(xy) = \frac{d}{dt}(8)$$

$$\left[\frac{d}{dt}(x)\right] y + x\left[\frac{d}{dt}(y)\right] = 0$$

$$y\frac{dx}{dt} + x\frac{dy}{dt} = 0 \Rightarrow 2\frac{dx}{dt} + 4(-3) = 0$$

$$\Rightarrow 2\frac{dx}{dt} - 12 = 0 \Rightarrow 2\frac{dx}{dt} = 12 \Rightarrow \frac{dx}{dt} = 6 \text{ cm/s}$$

Problem 9. Let $x^2y + y^3 - \csc x = 1$. Differentiate implicitly to find dy/dx. [10 pts]

Diff both sides w.r.t. x :—

$$\frac{d}{dx}(x^2y) + \frac{d}{dx}(y^3) - \frac{d}{dx}(\csc x) = \frac{d}{dx}(1)$$

Product rule   Chain rule

$$\Rightarrow \left[\frac{d}{dx}(x^2)\right]y + x^2\frac{dy}{dx} + \frac{d}{dy}(y^3)\frac{dy}{dx} - (-\csc x\cot x) = 0$$

$$\Rightarrow 2xy + x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} + \csc x\cot x = 0$$

$$\Rightarrow x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} + 2xy + \csc x\cot x = 0$$

$$\Rightarrow (x^2 + 3y^2)\frac{dy}{dx} = -(2xy + \csc x\cot x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(2xy + \csc x\cot x)}{(x^2 + 3y^2)}$$

Problem 10. The error in measuring the volume of a cylinder was 2%. If there was no error in measuring the height of the cylinder, then find the error in measuring the radius. [10 pts]

Given $\frac{dV}{V} \times 100 = 2$ To find $\frac{dr}{r} \times 100$

no error in height $\Rightarrow$ h can be treated like a constant.

$$V = \pi r^2 h$$

$$\Rightarrow \frac{d}{dr}(V) = \frac{d}{dr}(\pi r^2 h) = \pi h\frac{d}{dr}(r^2)$$

$$= \pi h(2r)$$

$$\Rightarrow \frac{dV}{dr} = 2\pi r h \Rightarrow dV = (2\pi r h)\,dr$$

$$\Rightarrow \frac{dV}{V} = \frac{2\cancel{\pi}\cancel{r}\cancel{h}}{\cancel{\pi}r^{\cancel{2}}\cancel{h}}dr = 2\frac{dr}{r} \Rightarrow \frac{dr}{r} = \frac{1}{2}\left(\frac{dV}{V}\right)$$

$$\Rightarrow \frac{dr}{r} = \frac{1}{2}\left(\frac{2}{100}\right) = \frac{1}{100} \Rightarrow \frac{dr}{r}\times 100 = 1$$ or error in radius is 1%

Bonus Problem 1. Find equation of the line tangent to the parabola $y^2 = -2x$ at the point $(-2, -2)$.
[5 pts]

$$y^2 = -2x \Rightarrow \frac{d}{dx}(y^2) = -2\frac{d}{dx}(x)$$

$$\Rightarrow \frac{d}{dy}(y^2)\frac{dy}{dx} = -2$$

$$\Rightarrow 2y\frac{dy}{dx} = -2 \Rightarrow \frac{dy}{dx} = \frac{-2}{2y} \Rightarrow \frac{dy}{dx} = \frac{-1}{y}$$

$$\Rightarrow m_T = \left.\frac{dy}{dx}\right|_{\substack{x=-2\\y=-2}} = \frac{-1}{-2} = \frac{1}{2}$$

$$\Rightarrow \frac{y-(-2)}{x-(-2)} = \frac{1}{2} \Rightarrow \frac{y+2}{x+2} = \frac{1}{2}$$

$$\Rightarrow 2y + 4 = x + 2 \Rightarrow x - 2y - 2 = 0$$

Bonus Problem 2. Find equation of the line normal to the hyperbola $xy = 5$ at the point $(5, 1)$.
[5 pts]

$$xy = 5 \Rightarrow y = \frac{5}{x} \Rightarrow \frac{dy}{dx} = 5\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = 5\left(\frac{-1}{x^2}\right) = \frac{-5}{x^2}$$

$$\Rightarrow m_T = \left.\frac{dy}{dx}\right|_{\substack{x=5\\y=1}} = \frac{-5}{5^2} = \frac{-1}{5}$$

$$\Rightarrow m_N\, m_T = -1 \Rightarrow m_N\left(\frac{-1}{5}\right) = -1$$

$$\Rightarrow m_N = 5 \Rightarrow \frac{y-1}{x-5} = 5$$

$$\Rightarrow y - 1 = 5x - 25 \Rightarrow y = 5x - 24$$

$$\Rightarrow 5x - y - 24 = 0$$