■ **Section 11.3** textbook exercises, *page 765*: #3, 5, 7, 21, 23, 22. **Note:** For # 21, 23, 22, show that the conditions of the Integral Test are true.

THE INTEGRAL TEST. Suppose f is a *continuous*, *positive*, *decreasing* function on $[1, \infty)$ and let $a_n = f(n)$. Then

(i) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^2} \qquad \text{we use} \qquad \int_4^\infty \frac{1}{(x-3)^2}\,dx$$

Also, it is not necessary that f be *always* decreasing. What is important is that f be *ultimately* decreasing, that is decreasing for x larger than some number N. f can be decreasing on $[N, \infty)$

Example 1: Use the Integral Test to test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence. Show that the conditions of the Integral Test are true for this problem.

$a_n = \frac{1}{n^2+1} \Rightarrow f(x) = \frac{1}{x^2+1}$

replace n with x

→ Is f continuous? Yes

→ Is f positive? Yes

→ Is f ultimately decreasing? Yes.

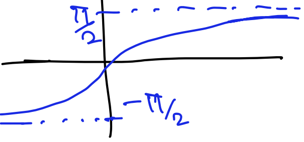
$x^2+1 > 0 \Rightarrow \frac{1}{x^2+1} > 0$

Check if the denominator becomes 0 or not → does not

→ As x increases, x^2+1 also increases ⇒ denominator is increasing

$\Rightarrow \frac{1}{x^2+1}$ decreases

$\Rightarrow$ Calculate $\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx$



$$= \lim_{t\to\infty}\left[\tan^{-1}(x)\Big|_1^t\right] = \lim_{t\to\infty}\left[\tan^{-1}t - \underbrace{\tan^{-1}1}_{\pi/4}\right]$$

$$= \lim_{t\to\infty}\tan^{-1}t - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \Rightarrow \text{the integral converges}$$

$\Rightarrow$ the series also converges.

Note that the series/sum is not $\pi/_4$.

Example 2: Use the Integral Test to test the series $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$ for convergence or divergence. Show that the conditions of the Integral Test are true for this problem.

$a_n = \dfrac{\ln n}{n} \Rightarrow f(x) = \dfrac{\ln x}{x}$. $\rightarrow$ Is f continuous on $[1,\infty)$? Yes.

replace n with x

$\rightarrow$ Is f Positive? Yes.

$\ln x \geq 0$ for $1 \leq x < \infty$

$x > 0$

$\rightarrow$ Is f decreasing? Yes

* x increases faster than $\ln x$

$$* \quad f'(x) = \frac{x(\ln x)' - \ln x\,(x)'}{x^2} = \frac{x \times \frac{1}{x} - \ln x}{x^2} = \frac{1-\ln x}{x^2}$$

when $\underbrace{x > e}_{\Downarrow\, f'(x)<0} \Rightarrow \ln x > \ln e = 1 \Rightarrow \ln x > 1 \Rightarrow 1 - \ln x < 0$

$f'(x) < 0 \Rightarrow f$ is decreasing on $[e,\infty)$

$$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}\,dx$$

$$\int \frac{\ln x}{x}\,dx \rightarrow \text{Substitute } u = \ln x \Rightarrow du = \frac{1}{x}\,dx$$

$$= \int u\,du = \frac{u^2}{2} = \frac{1}{2}(\ln x)^2$$

$$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}(\ln x)^2\right]_1^t = \lim_{t\to\infty}\left[\frac{1}{2}(\ln t)^2 - \frac{1}{2}(\ln 1)^2\right]$$

$$= \lim_{t\to\infty}\frac{1}{2}(\ln t)^2 = \frac{1}{2}\infty^2 = \infty$$

$\Rightarrow$ the integral diverges $\Rightarrow$ the series also diverges.