Indiana University, Indianapolis

Spring 2025 Math-I 165

Practice Test 1b

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Name:____

Instructions:

- No cell phones, calculators, watches, technology, hats - stow all in your bags.
- **Write your name** on this cover page.
- This test is **closed book and closed notes**.
- All work must be clearly shown for partial credit.
- If you wish for something not to be graded, please strike it out neatly.
- Box, circle, or otherwise clearly indicate your final answer.
- When you finish, return your test to the proctor, and leave the classroom.
- There are a total of **12 problems including 2 bonus problems**.
  - Problems 1-10 are each worth 10 points.
  - The bonus problems are each worth 5 points.
- You can score a **maximum of 110 points out of 100**.
- There are a total of **7 pages** including the cover page.

Problem 1. Evaluate the limit: $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{\sqrt{x-1}}$. [10 pts]

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt{x-1}} \overset{DS.}{=} \frac{\sqrt{1}-1}{\sqrt{1-1}} = \boxed{\frac{0}{0}}$$

Rationalize the numerator.

$$= \lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt{x-1}} \times \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x-1}\,(\sqrt{x}+1)}$$

$$= \lim_{x\to 1} \frac{(\sqrt{x})^2-(1)^2}{\sqrt{x-1}\,(\sqrt{x}+1)} = \lim_{x\to 1} \frac{(x-1)}{\sqrt{x-1}\,(\sqrt{x}+1)} = \lim_{x\to 1} \frac{(x-1)^1}{(x-1)^{1/2}(\sqrt{x}+1)}$$

$$= \lim_{x\to 1} \frac{(x-1)^{1-1/2}}{\sqrt{x}+1} = \lim_{x\to 1} \frac{\sqrt{x-1}}{\sqrt{x}+1} = \frac{\sqrt{1-1}}{\sqrt{1}+1} = \frac{0}{2} = 0$$

Problem 2. Evaluate the limit: $\lim_{x\to 1} \dfrac{\sin(2x-2)}{\tan(3x-3)}$. [10 pts]

$$\lim_{x\to 1} \frac{\sin(2x-2)}{\tan(3x-3)} \overset{DS.}{=} \frac{\sin(2-2)}{\tan(3-3)} = \frac{\sin 0}{\tan 0} = \frac{0}{0}$$

$$\boxed{\text{use } \lim_{x\to 0} \frac{\sin x}{x} = 1}$$

$$= \lim_{x\to 1} \frac{\sin(2x-2)}{\dfrac{\sin(3x-3)}{\cos(3x-3)}}$$

$$= \lim_{x\to 1} \frac{\sin(2x-2)}{\sin(3x-3)} \cos(3x-3)$$

$$= \lim_{x\to 1} \frac{\dfrac{\sin(2x-2)}{(2x-2)} \times (2x-2)}{\dfrac{\sin(3x-3)}{(3x-3)} \times (3x-3)} \cos(3x-3)$$

As $x\to 1$, $2x-2 \to 0$ and $3x-3 \to 0$

Letting $y = 2x-2$, $z = 3x-3$

$\cos(0) = 1$

$$= \frac{\lim_{y\to 0} \frac{\sin y}{y} =1}{\lim_{z\to 0} \frac{\sin z}{z} =1} \times \lim_{x\to 1} \frac{2\cancel{(x-1)}}{3\cancel{(x-1)}} \cos(3x-3) = \frac{1}{1} \times \frac{2}{3} \times 1 = \frac{2}{3}$$

Problem 3. Check the differentiability of the function $f(x) = |x-1| + |x+1|$ at $x = \pm 1$. [10 pts]

At $x=1$: $f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{|1+h-1| + |1+h+1| - 2}{h}$

$f(1) = |1-1| + |1+1| = 2$

$= \lim_{h\to 0} \frac{|h| + |2+h| - 2}{h}$

$2+h > 0$ for small h

$\Rightarrow |2+h| = 2+h$

$= \lim_{h\to 0} \frac{|h| + 2+h-2}{h}$

$= \lim_{h\to 0} \frac{|h|+h}{h} = \begin{cases} \lim_{h\to 0} \frac{-h+h}{h} & \text{if } h<0 \\ \lim_{h\to 0} \frac{h+h}{h} & \text{if } h>0 \end{cases}$

$= \begin{cases} 0 & \text{if } h<0 \\ 2 & \text{if } h>0 \end{cases} \Rightarrow \lim$ does not exist

$\Rightarrow f$ is not differentiable at $x=1$

At $x=-1$

$f(-1) = |-1-1| + |-1+1| = 2 \Rightarrow f'(-1) = \lim_{h\to 0} \frac{|-1+h-1| + |-1+h+1| - 2}{h} = \lim_{h\to 0} \frac{|2-h| + |h| - 2}{h}$

$2-h>0$ for small $h \Rightarrow |2-h| = 2-h$

$\Rightarrow f'(-1) = \lim_{h\to 0} \frac{2-h+|h|-2}{h} = \lim_{h\to 0} \frac{|h|-h}{h} = \begin{cases} 0 & \text{if } h>0 \\ -2 & \text{if } h<0 \end{cases} \Rightarrow \lim$ does not exist

$\Rightarrow f$ is not differentiable at $x=-1$

Problem 4 . Find derivative of the polynomial $f(x) = \frac{x^6}{6} - \frac{x^5}{5} + \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x$. [10 pts]

$f'(x) = \frac{1}{6}[x^6]' - \frac{1}{5}[x^5]' + \frac{1}{4}[x^4]' - \frac{1}{3}[x^3]' + \frac{1}{2}[x^2]' - [x]'$

$= \frac{1}{6}(6x^5) - \frac{1}{5}(5x^4) + \frac{1}{4}(4x^3) - \frac{1}{3}(3x^2) + \frac{1}{2}(2x) - 1$

$= x^5 - x^4 + x^3 - x^2 + x - 1$

Problem 5. The amount of charge flowing through a circuit varies with time as $q(t) = 0.5\sin(60t+\pi)$. Find the amount of current through the circuit at $t = 0$ seconds. [10 pts]

$$i(t) = q'(t) = \frac{d}{dt}\left[0.5\ \sin(60t+\pi)\right]$$

$$= 0.5\ \frac{d}{dt}\left[\sin(60t+\pi)\right]$$

$$= 0.5 \times \cos(60t+\pi) \times \frac{d}{dt}(60t+\pi)$$

$$= 0.5 \times \cos(60t+\pi) \times 60$$

$$= 0.5 \times 60 \times \cos(60t+\pi) = 30\cos(60t+\pi)$$

At $t=0$

$$i(0) = 30\cos(60\times 0+\pi) = 30\cos\pi = 30(-1) = -30 \text{ Amperes.}$$

Problem 6. Find the derivative of the function $y = \cot^2\sqrt{x}$. [10 pts]

use chain rule.

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cot^2\sqrt{x}\right]$$

Let $u = \cot\sqrt{x}$ $\Rightarrow \frac{dy}{dx} = \frac{d}{dx}[u^2] = \frac{d}{du}(u^2)\frac{du}{dx}$

$$= 2u\frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 2\cot\sqrt{x}\ \frac{d}{dx}(\cot\sqrt{x})$$

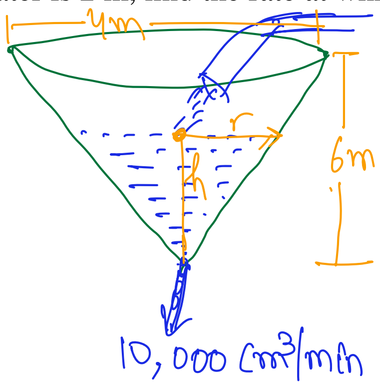
Let $z = \sqrt{x}$ $\Rightarrow \frac{dy}{dx} = 2\cot\sqrt{x}\cdot\frac{d}{dx}(\cot z)$

$$= 2\cot\sqrt{x}\cdot\frac{d}{dz}(\cot z)\frac{dz}{dx}$$

$$= 2\cot\sqrt{x}\cdot(-\csc^2 z)\cdot\frac{d}{dx}(\sqrt{x})$$

$$= -2(\cot\sqrt{x})(-\csc^2\sqrt{x})\cdot\frac{1}{2\sqrt{x}} = \frac{(\cot\sqrt{x})(\csc^2\sqrt{x})}{\sqrt{x}}$$

Problem 7. Water is leaking out of an inverted conical tank at a rate of 10,000 cm^3/min. At the same time water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter of the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank. [10 pts]



constant rate
C cm³/min.

Given $\frac{dh}{dt} = 20$ cm/min.

To find C when $h = 2$ m.

$V = \frac{1}{3}\pi r^2 h$ and $\frac{r}{h} = \frac{2}{6}$ (by similar triangles)

$\Rightarrow V = \frac{1}{3}\pi\left(\frac{2h}{6}\right)^2 h = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{1}{27}\pi h^3$ $\qquad \frac{d}{dt}(h^3)$

$\Rightarrow \frac{dV}{dt} = \frac{\pi}{27}\underbrace{\frac{d}{dh}(h^3)\frac{dh}{dt}} = \frac{\pi h^2}{9}\frac{dh}{dt} = \frac{\pi}{9}(200)^2 \times 20 = \frac{8\pi}{9}\times 10^5$ $\qquad$ (200 cm, 20)

$\parallel$

$C - 10{,}000 \Rightarrow C = \left(\frac{8\pi}{9}\times 10^5 + 10^4\right)$ cm³/min $= \left(\frac{80\pi}{9} + 1\right)\times 10^4$ cm³/min

Problem 8. Let $z = \frac{\pi}{\sqrt[5]{r - r^2}}$. Then find $\frac{dz}{dr}$. [10 pts]

$$\frac{dz}{dr} = \frac{d}{dr}\left(\frac{\pi}{(r-r^2)^{1/5}}\right)$$

$$= \pi \frac{d}{dr}\left[(r-r^2)^{-\frac{1}{5}}\right] \quad \text{chain rule}$$

$$= \pi\left[\frac{-1}{5}(r-r^2)^{\frac{-1}{5}-1} \times \frac{d}{dr}(r-r^2)\right]$$

$$= -\frac{\pi}{5}(r-r^2)^{\frac{-6}{5}}(1-2r)$$

$$= \frac{-\pi(1-2r)}{5(r-r^2)^{6/5}}$$

Problem 9. Find the differential dy for the function $y = \tan\sqrt{t}$. [10 pts]

$$dy = \frac{d}{dt}\left(\tan\sqrt{t}\right)\ dt$$

Let $u = \sqrt{t}$

$$\Rightarrow dy = \frac{d}{du}(\tan u)\frac{du}{dt}\ dt$$

$$= \left(\sec^2 u\right)\frac{d}{dt}(\sqrt{t})\ dt$$

$$= \left(\sec^2\sqrt{t}\right)\frac{1}{2\sqrt{t}}\ dt$$

$$\Rightarrow dy = \left(\frac{\sec^2\sqrt{t}}{2\sqrt{t}}\right) dt$$

Problem 10. Find the slope of tangent to the curve $x^3 + y^3 = xy^2 + x^2y$ at the point $(1, 1)$. [10 pts]

Diff both sides w.r.t. x :—

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(xy^2) + \frac{d}{dx}(x^2y)$$

↓ Product rule ↓

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = \overbrace{y^2 + x\frac{d}{dx}(y^2)} + \overbrace{2xy + x^2\frac{dy}{dx}}$$

$$\Rightarrow 3x^2 + \underbrace{3y^2\frac{dy}{dx}} = y^2 + \underbrace{2xy\frac{dy}{dx}} + 2xy + \underbrace{x^2\frac{dy}{dx}}$$

$$\Rightarrow \left(3y^2 - 2xy - x^2\right)\frac{dy}{dx} = y^2 + 2xy - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 + 2xy - 3x^2}{3y^2 - 2xy - x^2}$$

$$\Rightarrow m_T = \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=1}} = \frac{(1)^2 + 2 - 3(1)^2}{3(1)^2 - 2 - 1} = \frac{1 + 2 - 3}{3 - 2 - 1} = \boxed{\frac{0}{0}}$$

$$= \frac{\frac{y^2}{x^2} + 2\frac{xy}{x^2} - \frac{3x^2}{x^2}}{3\,{}^{y^2}\!/_{x^2} - \frac{2xy}{x^2} - {}^{x^2}\!/_{x^2}} = \frac{h^2 + 2h - 3}{3h^2 - 2h - 1} \quad \text{where } h = {}^{y}\!/_{x}$$

$$\Rightarrow m_T = \lim_{h\to 1}\frac{h^2 + 2h - 3}{3h^2 - 2h - 1} = \lim_{h\to 1}\frac{(h+3)\cancel{(h-1)}}{(3h+1)\cancel{(h-1)}} = \frac{4}{4} = 1$$

$$\boxed{\text{As } \begin{matrix} x\to 1 \\ y\to 1\end{matrix},\ h\to 1}$$

Bonus Problem 1. If $xy + y^3 = 1$, find the value of d^2y/dx^2 at the point where $x = 0$. [5 pts]

diff w.r.t. x

$x=0 \Rightarrow (0)y + y^3 = 1$

$\Rightarrow y^3 = 1 \Rightarrow y = 1$

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^3) = \frac{d}{dx}(1)$$

Product rule Chain rule

$$\Rightarrow x\frac{dy}{dx} + y + 3y^2\frac{dy}{dx} = 0 \Rightarrow (x+3y^2)\frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x+3y^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{-(x+3y^2)\frac{dy}{dx} - (-y)[x+3y^2]'}{(x+3y^2)^2}$$

diff.

(Quotient rule)

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-(x+3y^2)\frac{dy}{dx} + y\left[1 + 6y\frac{dy}{dx}\right]}{(x+3y^2)^2}.$$

At $x=0$, $y=1$, $\frac{dy}{dx} = \frac{-1}{3}$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=0} = \frac{(-3)\left(\frac{-1}{3}\right) + 1\left(1 + 6\left(\frac{-1}{3}\right)\right)}{(3)^2} = \frac{1 + (1-2)}{9} = 0$$

Bonus Problem 2. Find equation of the line normal to the hyperbola $x^2 - y^2 = 1$ at the point $(1, 0)$. [5 pts]

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$\Rightarrow 2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}$$

$$\Rightarrow m_T = \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=0}} = \frac{1}{0}$$

$$\Rightarrow m_N m_T = -1 \Rightarrow m_N = \frac{-1}{m_T} = \frac{-1}{1/0} = 0$$

$\Rightarrow$ Tangent is vertical and normal is horizontal.

In other words, $m_T = \infty$, $m_N = 0$

Eqn. of normal :—

$$\frac{y-0}{x-1} = 0 \Rightarrow y = 0$$