The Weyl Group

Lemma (*) $\alpha_1, \dots, \alpha_t \in \Delta$ (simple roots), $\sigma_{\alpha_1}\sigma_{\alpha_2}\cdots\sigma_{\alpha_{t-1}}(\alpha_t) < 0$

then $\sigma_{\alpha_1}\cdots\sigma_{\alpha_t}$ is not minimal.

that is, $\exists$ s such that

$$\underbrace{\sigma_{\alpha_1}\cdots\sigma_{\alpha_t}}_{t} = \sigma_{\alpha_1}\cdots\sigma_{\alpha_{s-1}}\sigma_{\alpha_{s+1}}\cdots\sigma_{\alpha_{t-1}}.$$

Corollary

$\sigma_{\alpha_1}\cdots\sigma_{\alpha_t}$ is minimal $\Rightarrow \sigma_{\alpha_1}\cdots\sigma_{\alpha_{t-1}}(\alpha_t) > 0$

$\Rightarrow \sigma_{\alpha_1}\cdots\sigma_{t-1}\big(-\sigma_{\alpha_t}(\alpha_t)\big) > 0$

$\Rightarrow \underbrace{\sigma_{\alpha_1}\cdots\sigma_{\alpha_t}(\alpha_t) < 0}$

W = Subgrp generated by $\{\sigma_\alpha : \alpha \in \Phi\}$

W' = Subgrp. " " $\{\sigma_\alpha : \alpha \in \Delta\}$

To show $W = W'$

$W' \subseteq W$

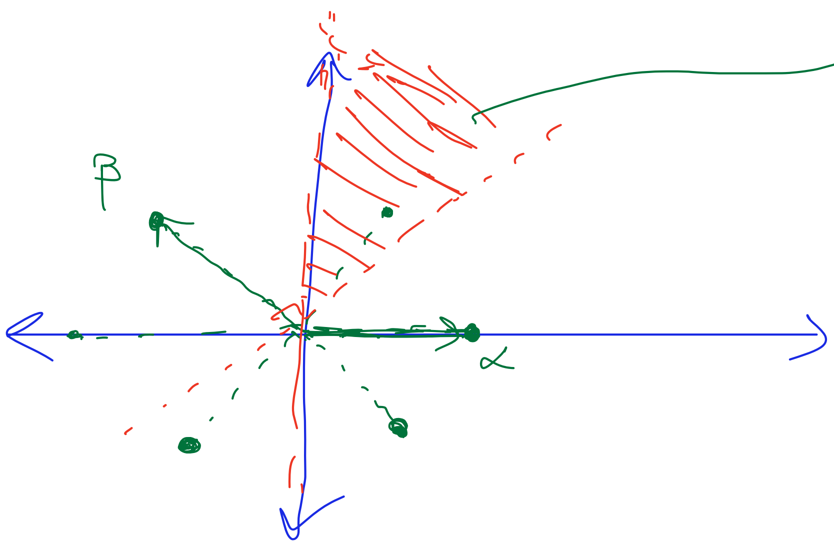
Lemma

Choose some $\overbrace{\text{regular}}$ $\gamma \in E$, then $\exists\ \sigma \in W'$

does not lie on reflecting hyperplanes

s.t. $\underbrace{(\sigma(\gamma), \alpha) > 0, \ \alpha \in \Delta}$

$\Rightarrow \sigma(\gamma) \in$ fundamental Weyl chamber w.r.t. Δ



Proof:

$$\delta = \frac{1}{2} \sum_{\alpha > 0} \alpha$$

$$\max \left\{ (\sigma(\gamma), \delta) : \sigma \in W' \right\} = (\sigma_0(\gamma), \delta)$$

$\boxed{\alpha \in \Delta}$

$$\Rightarrow (\sigma_\alpha \sigma_0(\gamma), \delta) \le (\sigma_0(\gamma), \delta)$$

$$\parallel$$

$$(\sigma_0(\gamma), \sigma_\alpha(\delta)) = (\sigma_0(\gamma), \delta - \alpha)$$

$\sigma_\alpha^2 = 1$

$$\Rightarrow (\sigma_0(\gamma), \alpha) \ge 0, \ \alpha \in \Delta$$

Suppose $(\sigma_0(\gamma), \alpha) = 0$ for some $\alpha \in \Delta$

$\Rightarrow \left(\gamma, \underbrace{\sigma_0^{-1}(\alpha)}_{\text{root}}\right) = 0$ $\Rightarrow$ γ lies on the reflecting hyperplane $P_{\sigma_0^{-1}(\alpha)}$

$\Rightarrow \Leftarrow$ γ is regular.

$\Rightarrow$ $\sigma_0(\gamma)$ lies in fund. weyl chamber w.r.t. Δ

Corollary

If Δ, Δ' are bases of Φ, then $\exists\ \sigma \in W$, $\sigma(\Delta') = \Delta$.

Proof :

regular pts $\gamma \in E$ $\longleftrightarrow$ Weyl chambers.

γ, γ'

For Δ', $\exists$ some regular $\gamma' \in E$ s.t. $\Delta' = \Delta(\gamma')$

$$\Delta(\gamma') = \left\{\beta \in \Phi : (\gamma', \beta) > 0 \text{ and } \beta \text{ is indecomposable}\right\}$$

$\exists\ \sigma \in W$ s.t $\sigma(\gamma')$ is in the fund. weyl chamber w.r.t Δ.

$$\Rightarrow \quad \Delta = \Delta(\sigma(\gamma')) = \sigma(\Delta(\gamma')) = \sigma(\Delta')$$

Theorem Every base Δ arises from ~~some~~ regular point

$$\Rightarrow \Delta = \Delta(\gamma)$$

⊛ Δ → define +ve
-ve

δ → gives $(\cdot , \cdot)$

(δ, β) ,

$$\left(\frac{1}{2}\sum_{\alpha>0}\alpha, \beta\right) = \frac{1}{2}\sum_{\alpha>0} \underbrace{(\alpha, \beta)}_{} \overset{?}{>} 0$$

$$\sigma_\beta(\delta) = \delta - \beta$$

$$\delta - 2\frac{(\delta, \beta)}{(\beta, \beta)}$$

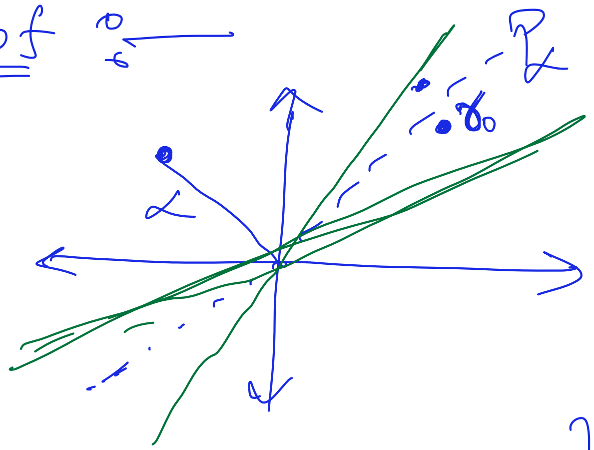
$$\underbrace{(\delta, \beta^\vee)}_{}$$

$$\underbrace{\Delta = \Delta(\delta)}_{}$$

Lemma

If $\alpha \in \Phi$, then $\exists\ \sigma \in W'$ s.t. $\sigma(\alpha) \in \Delta$.

Proof:



γ_0 regular and (γ_0, α) is positive

$\Rightarrow \alpha \in \Delta(\gamma_0)$

$\exists\ \sigma \in W'$ s.t.

$\sigma(\Delta(\gamma_0)) = \Delta$

$\Rightarrow \sigma(\alpha) \in \Delta$

$W' = W$

Choose some $\alpha \in \Phi \setminus \Delta$,

want to show $\sigma_\alpha \in W'$

$\exists\ \sigma \in W'$ s.t. $\sigma(\alpha)$ is simple.

$\sigma_{\sigma(\alpha)} \in W'$

$\|$

$\sigma \sigma_\alpha \sigma^{-1} \in W' \Rightarrow \sigma_\alpha \in W'$

$\Rightarrow W' = W$

Lemma

$\sigma(\Delta) = \Delta$, for some $\sigma \in W$

$\Rightarrow \sigma = 1$

Proof :

$\sigma \neq 1 \Rightarrow$ write $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_t}$ such that t is minimal

$\Rightarrow \sigma(\alpha_t) < 0$ $\not\Leftarrow$

$\alpha_t \in \Delta \Rightarrow \sigma(\alpha_t) \in \Delta$

(*) For any $\sigma \in W$, $\ell(\sigma)$ is the minimal number of simple reflections

$\sigma_1, \ldots, \sigma_{\ell(\sigma)}$

s.t. $\sigma = \sigma_1 \cdots \sigma_{\ell(\sigma)}$

length of σ

$\to \ell(1) = 0$

(*) $n(\sigma)$ = number of +ve roots α for which $\sigma(\alpha) < 0$

Lemma

$\ell(\sigma) = n(\sigma)$, $\sigma \in W$.

Proof :

use induction $l(\sigma)$

$l(\sigma)=0 \Rightarrow \sigma=1 \Rightarrow n(\sigma)=0$

Hypothesis : If $\tau \in W$, $l(\tau) < l(\sigma)$ then $l(\tau)=n(\tau)$

Let $\sigma = \sigma_1 \cdots \cdots \sigma_{l(\sigma)}$

and let $\sigma_{l(\sigma)} = \sigma_\alpha \Rightarrow \sigma(\alpha) < 0$

Then $n(\sigma\sigma_\alpha) = n(\sigma) - 1$

$(\sigma\sigma_\alpha)(\alpha) > 0$

(*) σ_α permutes all +ve roots except α.

$\sigma_\alpha^2 = 1 \Rightarrow l(\sigma\sigma_\alpha) = l(\sigma) - 1$

$\Rightarrow l(\sigma\sigma_\alpha) = n(\sigma\sigma_\alpha)$

$\Rightarrow l(\sigma) - 1 = n(\sigma) - 1 \Rightarrow l(\sigma) = n(\sigma)$.

(*) Given Δ, let $C(\Delta)$ to be the fundamental weyl chamber w.r.t. Δ.

Lemma

Let $\mu, \nu \in C(\Delta)$. If $\sigma\mu = \nu$ for some $\sigma \in W$, then $\sigma = \sigma_1 \cdots \sigma_t$ s.t. $\underbrace{\sigma_i(\mu) = \mu, \ 1 \le i \le t}_{\Downarrow}$

$\mu = \nu$.

Proof :-

induction on $\ell(\sigma)$.

$\ell(\sigma) = 0 \Rightarrow \sigma = 1 \Rightarrow \mu = \nu$

$\ell(\sigma) > 0 \Rightarrow n(\sigma) > 0$

$\Rightarrow \exists \alpha \in \Delta$ s.t. $\sigma(\alpha) < 0$

$0 \ge (\nu, \sigma(\alpha)) = (\sigma^{-1}(\nu), \alpha) = (\mu, \alpha) \ge 0$

$\Rightarrow (\mu, \alpha) = 0 \Rightarrow \boxed{\sigma_\alpha(\mu) = \mu}$

$\ell(\sigma\sigma_\alpha) < \ell(\sigma)$

do the same with $\sigma\sigma_\alpha$

A_n $\quad i < j \Rightarrow \sigma(i) > \sigma(j)$
