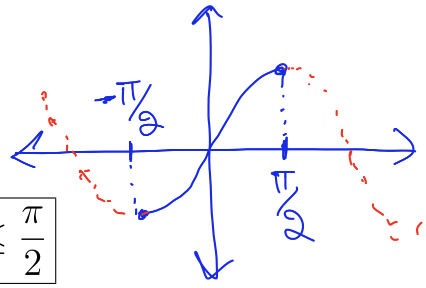

■ **Section 6.6** exercises, *page 481*: #1, 2, 3, 4, 5, 7, 12, 13, 22, 23, 25, 27, 31, 33, 59, 61, 65, 64, 67.

GOALS

- Compute the values of the **inverse trigonometric functions**, e.g., $\sin^{-1}(\frac{1}{2})$, $\cos^{-1}(0)$, $\tan^{-1}(\sqrt{3})$, etc.
- Compute or simplify expressions such as $\tan\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$, $\cos\left(\tan^{-1} x\right)$, etc.
- Compute derivatives and integrals involving inverse trigonometric functions.

In this section, we explore the inverse functions of trigonometric functions. The functions $\sin(x)$, $\cos(x)$, $\tan(x)$ are not one-to-one over their domains. However, if we restrict their domains, they will be one-to-one on the restricted domain. We then can find their inverse functions.

◇ **Inverse Sine Function**. *Notation*: $\sin^{-1}(x)$ or $\arcsin(x)$



$\sin\theta$ is one-to-one for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Thus, we have

$(\sin x)^{-1}$

$$\boxed{\sin^{-1} x = \theta \iff \sin\theta = x \quad \text{for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}}$$

Note: $\sin^{-1} x \neq \frac{1}{\sin x}$

$\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1] \Rightarrow \sin^{-1} : [-1,1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

Example 1: Evaluate (a) $\sin^{-1}(\frac{1}{2})$ (b) $\tan\left(\arcsin\frac{1}{3}\right)$

(a) $\sin^{-1}\left(\frac{1}{2}\right) = \theta \Rightarrow \sin\theta = \frac{1}{2}$, $-\frac{\pi}{2} \le \theta \le \pi/2$

$\Rightarrow \theta = \frac{\pi}{6} \Rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

(b) $\tan\left(\sin^{-1}\frac{1}{3}\right)$ Let $\sin^{-1}\frac{1}{3} = \theta$

$= \tan\theta$ $\Rightarrow \sin\theta = \frac{1}{3} = \frac{P}{H}$

$P = 1, H = 3$, $B^2 = H^2 - P^2$ $P^2 + B^2 = H^2$

$= 3^2 - 1^2 = 8$

$\Rightarrow B = \sqrt{8}$

$\Rightarrow \tan\theta = \frac{P}{B} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}} \Rightarrow \tan\left(\sin^{-1}\frac{1}{3}\right) = \frac{1}{2\sqrt{2}}$

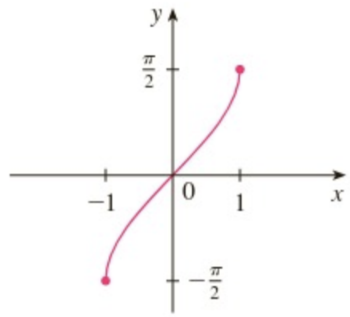


FIGURE 4
$y = \sin^{-1}x = \arcsin x$

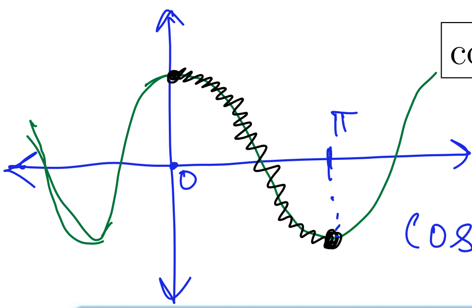
$$\boxed{\sin^{-1}(-x) = -\sin^{-1}x}$$

$$\sin^{-1}\left(\frac{-1}{2}\right) = -\sin^{-1}\frac{1}{2} = -\frac{\pi}{6}$$

$\sin^{-1}(\sin x) = x$	for $-\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$
$\sin(\sin^{-1}x) = x$	for $-1 \leqslant x \leqslant 1$

◇ **Inverse Cosine Function.** *Notation:* $\cos^{-1}(x)$ or $\arccos(x)$

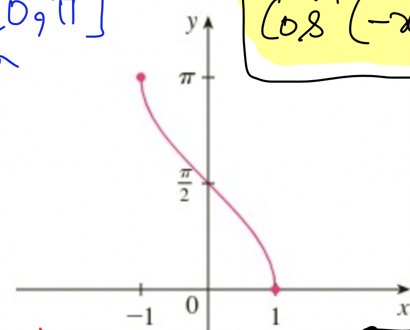
$$\boxed{\cos^{-1}x = \theta \iff \cos\theta = x \quad \text{for } 0 \le \theta \le \pi}$$



$\cos^{-1}: [-1,1] \to [0,\pi]$

$\cos: [0,\pi] \to [-1,1]$

$$\boxed{\cos^{-1}(-x) = \pi - \cos^{-1}x}$$



$\cos^{-1}(\cos x) = x$	for $0 \leqslant x \leqslant \pi$
$\cos(\cos^{-1}x) = x$	for $-1 \leqslant x \leqslant 1$

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Tan$^{-1}(x)$

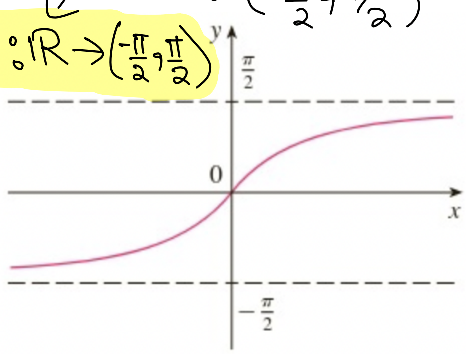
◇ **Inverse Tangent Function.** *Notation:* tan $-1(x)$ or $\arctan(x)$

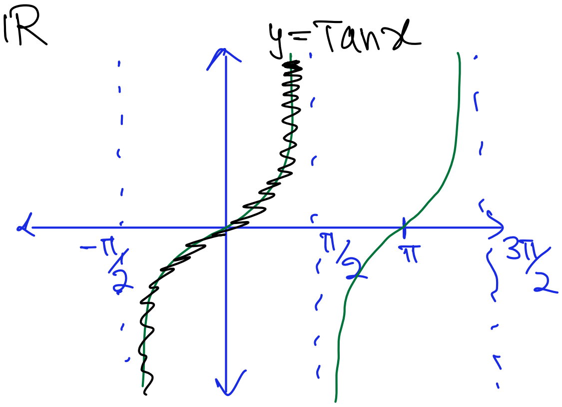
$\mathrm{Tan}^{-1}(-x) = -\mathrm{Tan}^{-1}x$

$$\boxed{\tan^{-1}x = \theta \iff \tan\theta = x \quad \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}}$$

$\mathrm{Tan}: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$

$\mathrm{Tan}^{-1}: \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$





$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\pi/3$$

Example 2: Evaluate (a) $\cos^{-1}(-1)$ and (b) $\arctan(\sqrt{3})$.

(a) $\cos^{-1}(-1)$

$= \pi - \cos^{-1}(1)$

$= \pi - 0 = \pi$

$= \mathrm{Tan}^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$

$\mathrm{Tan}^{-1}(-1) = -\mathrm{Tan}^{-1}(1) = -\dfrac{\pi}{4}$

Example 3: Simplify the expression $\cos\left(\tan^{-1}(x)\right)$

Let $\theta = \operatorname{Tan}^{-1} x \Rightarrow \operatorname{Tan}\theta = \dfrac{x}{1} = \dfrac{P}{B}$

$\cos(\operatorname{Tan}^{-1} x) = \cos\theta = \dfrac{B}{H}$

$P = x, \quad B = 1$

$P^2 + B^2 = H^2$

$x^2 + 1^2 = H^2 \Rightarrow H = \sqrt{x^2+1}$

$H = \pm\sqrt{x^2+1} \rightarrow$ Reject $-\sqrt{x^2+1}$

$$\cos\theta = \frac{1}{\sqrt{x^2+1}}$$

Derivative and Integral Formulas Involving Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}} \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2} \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}} \qquad \int -\frac{1}{\sqrt{1-x^2}}\,dx = \cos^{-1}(x) + C$$

$= -\sin^{-1}x + \frac{\pi}{2} + C$

$\int -\dfrac{1}{\sqrt{1-x^2}}\,dx = -\int \dfrac{1}{\sqrt{1-x^2}}\,dx$

$= -\left[\sin^{-1}x + C\right]$

$= -\sin^{-1}x - C$

$$\boxed{\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}}$$

Example 4: Differentiate

(a) $H(x) = 2\tan^{-1}(x) + \arcsin(2x^2) + \cos^{-1}\left(\tan x\right)$

$H'(x) = 2\dfrac{d}{dx}\left[\operatorname{Tan}^{-1}(x)\right] + \dfrac{d}{dx}\left[\sin^{-1}(2x^2)\right] + \dfrac{d}{dx}\left[\cos^{-1}(\underbrace{\operatorname{Tan} x}_{z})\right]$

$= \dfrac{2}{1+x^2} + \dfrac{4x}{\sqrt{1-4x^4}} + \dfrac{-\sec^2 x}{\sqrt{1-\operatorname{Tan}^2 x}}$

$u = 2x^2 \Rightarrow du/dx = 4x$

$\dfrac{d}{dx}\left[\sin^{-1}(2x^2)\right] = \dfrac{d}{du}\left[\sin^{-1}u\right]\dfrac{du}{dx} = \dfrac{1}{\sqrt{1-u^2}} \times 4x = \dfrac{4x}{\sqrt{1-4x^4}}$

$z = \operatorname{Tan} x \Rightarrow \dfrac{dz}{dx} = \sec^2 x$

$\dfrac{d}{dx}\left[\cos^{-1}(z)\right]$

$= \dfrac{d}{dz}\left[\cos^{-1}(z)\right]\dfrac{dz}{dx}$

$= \dfrac{-1}{\sqrt{1-z^2}} \times \sec^2 x$

(b) $f(x) = x\arctan(\sqrt{x})$

$f'(x) = \dfrac{d}{dx}(x)\operatorname{Tan}^{-1}\sqrt{x} + x\dfrac{d}{dx}(\operatorname{Tan}^{-1}\underbrace{\sqrt{x}}_{u=\sqrt{x}})$

$\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$

$= \operatorname{Tan}^{-1}\sqrt{x} + x\dfrac{d}{du}(\operatorname{Tan}^{-1}u)\dfrac{du}{dx}$

$= \operatorname{Tan}^{-1}\sqrt{x} + x\left(\dfrac{1}{1+(\sqrt{x})^2}\right)\dfrac{1}{2\sqrt{x}} = \boxed{\operatorname{Tan}^{-1}\sqrt{x} + \dfrac{\sqrt{x}}{2(1+x)}}$

Example 5: Evaluate $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$, $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$

(a) $\displaystyle\int \frac{1}{15\sqrt{1-x^2}}\,dx$

$$= \frac{1}{15}\int \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{15}\sin^{-1}x + C$$

(b) $\displaystyle\int \frac{3}{1+x^2}\,dx$

$$= 3\int \frac{1}{1+x^2}\,dx = 3\tan^{-1}x + C$$

(c) $\displaystyle\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$

$u = \tan x \Rightarrow \frac{du}{dx} = \sec^2 x$

$\Rightarrow du = \sec^2 x \, dx$

$$I = \int \frac{\sec^2 x \, dx}{\sqrt{1-\tan^2 x}}$$

$$= \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}(u) + C$$

$$= \sin^{-1}(\tan x) + C$$

(d) $\int_0^1 \frac{x}{1+x^4}\,dx$. **Note**: Evaluate all expressions into real numbers for your final answer.

$u = x^2$ $\qquad\qquad$ $\frac{d}{dx}(x^2) = 2x$

$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$I = \int_0^1 \frac{x\,dx}{1+x^4} = \int_{0^2}^{1^2} \frac{1}{2}\frac{du}{1+u^2}$

Don't forget to change the limits!

$= \int_0^1 \frac{1}{2}\frac{du}{1+u^2} = \frac{1}{2}\int_0^1 \frac{du}{1+u^2} = \frac{1}{2}\operatorname{Tan}^{-1}(u)\Big|_0^1$

$= \frac{1}{2}\left[\operatorname{Tan}^{-1}(1) - \operatorname{Tan}^{-1}(0)\right]$

$= \frac{1}{2}\left[\pi/4 - 0\right] = \frac{1}{2}\times\frac{\pi}{4} = \boxed{\frac{\pi}{8}}$