MATH 16600 Practice Final Exam, *Version 2*

1 Let $f(x) = \ln\left[\frac{(3x+1)^2}{\sqrt{x-1}}\right]$. Use the properties of logarithmic functions to decompose $f(x)$ completely then find $f'(x)$.

2 Strontium-90 has a half-life of 28 days. A sample has a mass of 50 mg initially. Find the mass remaining after 40 days. **Note:** You don't need to simplify your final answer.

$m(0) = 50$ mg

$t_{1/2} = 28$ days.

$\underbrace{m(40) = ??}$

$m(t) = m(0)\, e^{-kt}$

where $k > 0$

$m(t) = 50\, e^{-kt}$

$k = ??$

$\Rightarrow$ after 28 days $m(28) = \frac{1}{2} m(0)$

$= \frac{1}{2} \times 50$

$= 25$

$25 = 50\, e^{-k(28)}$

$\frac{25}{50} = e^{-k(28)} \Rightarrow \frac{1}{2} = e^{-k(28)}$

$\Rightarrow \ln\left(\frac{1}{2}\right) = \ln e^{-28k} \Rightarrow \ln 2^{-1} = -28k \ln e$

$\Rightarrow (-1)\ln 2 = -28k$

3 Find the limit. $\lim_{x \to 0} \frac{e^{-x} - 1 + x}{x^2}$.

$$\Rightarrow k = \frac{-\ln 2}{-28} = \frac{\ln 2}{28}$$

$$m(t) = 50\, e^{-\frac{\ln 2}{28} t}$$

$$m(40) = 50\, e^{-\frac{\ln 2}{28} \cdot 40}$$

$$= 50\, e^{\ln 2^{-\frac{40}{28}}}$$

$$= 50\, (2)^{-\frac{40}{28}}$$

$$= \frac{50}{2^{10/7}}$$

4 Evaluate the integral $\int \frac{1}{x^2 + 4x - 12} dx$

1) Factorize the denominator

$$x^2 + 4x - 12 = x^2 - 2x + 6x - 12$$

$$= x(x-2) + 6(x-2)$$

$$= (x-2)(x+6)$$

2) $$\frac{1}{(x-2)(x+6)} = \frac{a}{x-2} + \frac{b}{x+6}$$

3) Find a and b

$$\Rightarrow 1 = a(x+6) + b(x-2)$$

Put $x = -6 \Rightarrow 1 = b(-6-2) \Rightarrow b = \frac{-1}{8}$

Put $x = 2 \Rightarrow 1 = a(2+6) \Rightarrow a = \frac{1}{8}$

Alternatively

$x \to a + b = 0$

$x^0 \to 6a - 2b = 1$

4) $$\frac{1}{(x-2)(x+6)} = \frac{1}{8}\,\frac{1}{x-2} - \frac{1}{8}\,\frac{1}{x+6}$$

$$\int \frac{1}{(x-2)(x+6)}\,dx = \frac{1}{8}\int \frac{1}{x-2}\,dx - \frac{1}{8}\int \frac{1}{x+6}\,dx$$

$$= \frac{1}{8}\ln|x-2| - \frac{1}{8}\ln|x+6| + C$$

- $$\frac{1}{(x^2-2)(x+6)} = \frac{1}{(x-\sqrt{2})(x+\sqrt{2})(x+6)}$$

$$= \frac{a}{x-\sqrt{2}} + \frac{b}{x+\sqrt{2}} + \frac{c}{x+6}$$

- $$\frac{1}{(x^2+2)(x+6)} = \frac{ax+b}{x^2+2} + \frac{c}{x+6}$$

5 Evaluate the integral. $\int \sin(2x)\cos^2 x \, dx$.

6 Evaluate the integral. $\int x^{2021} \ln x \, dx$.

By Part

$$\int x^{2021} \ln x \, dx = (\ln x)\frac{x^{2022}}{2022} - \int \frac{x^{2022}}{2022}\frac{1}{x}\,dx$$

$u \rightarrow du = \frac{1}{x}\,dx$

$dv = x^{2021}\,dx$

$v = \frac{x^{2022}}{2022}$

$$= (\ln x)\frac{x^{2022}}{2022} - \frac{1}{2022}\int x^{2021}\,dx$$

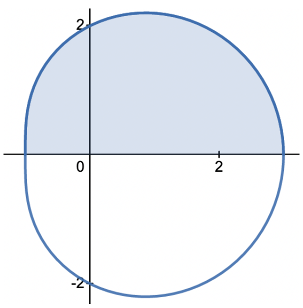
$$= (\ln x)\frac{x^{2022}}{2022} - \frac{x^{2022}}{(2022)^2}$$

7 Evaluate the integral. $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx.$

8 Set up an integral that represents the length of the curve $y = \frac{1}{8}x^2 - \ln x$, $1 \leq x \leq 2$.

9 Set up an integral that represents the area of the surface obtained by rotating the curve $y = \sin x$, $0 \le x \le \pi/2$, about the y-axis.

10 Find the area of the shaded region



$r = 2 + \cos\theta$

11 Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter. $x = t^3 + t,\ y = \ln t;\ \ t = 1.$

12 Determine whether $\int_1^\infty \frac{1}{x^2+1}\,dx$ is convergent or divergent. Evaluate the integral if it is convergent.

13 $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$. Determine whether the series is convergent or divergent.

14 Determine whether the given geometric series is convergent or divergent. Find its sum if it is convergent. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{10^n}$

15 Test the series for convergence or divergence. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$.

16 Determine whether the series is absolutely convergent, conditionally convergent, or divergent. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.

17 Find the radius of convergence and interval of convergence of the series. $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$.

18 Given $f(x) = 1 + x + x^3 + \sin x$, find $f^{-1}(1)$ and $(f^{-1})'(1)$.

19 Use the definition of Taylor series to find the first **four** nonzero terms of the series for $f(x) = \dfrac{1}{3 - 2x}$ centered at $a = 1$.