Indiana University, Indianapolis

Spring 2025 Math-I 165

Practice Test 2b

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Name:____

Instructions:

- No cell phones, calculators, watches, technology, hats - stow all in your bags.
- **Write your name** on this cover page.
- This test is **closed book and closed notes**.
- All work must be clearly shown for partial credit.
- If you wish for something not to be graded, please strike it out neatly.
- Box, circle, or otherwise clearly indicate your final answer.
- When you finish, return your test to the proctor, and leave the classroom.
- There are a total of **12 problems including 2 bonus problems**.
  - Problems 1-10 are each worth 10 points.
  - The bonus problems are each worth 5 points.
- You can score a **maximum of 110 points out of 100**.
- There are a total of **7 pages** including the cover page.

Problem 1. Evaluate the limit: $\lim\limits_{x\to-\infty} \dfrac{2x^3+1}{\sqrt{x^6+1}-1}$. [10 pts]

Divide both numerator and denominator by x^3 :—

$$\Rightarrow \lim_{x\to-\infty} \frac{2x^3+1}{\sqrt{x^6+1}-1} = \lim_{x\to-\infty} \frac{\frac{1}{x^3}(2x^3+1)}{\frac{1}{x^3}\left(\sqrt{x^6+1}-1\right)}$$

$$= \lim_{x\to-\infty} \frac{\frac{2x^3}{x^3}+\frac{1}{x^3}}{\frac{1}{x^3}\sqrt{x^6+1}-\frac{1}{x^3}} = \lim_{x\to-\infty} \frac{2+\frac{1}{x^3}}{-\sqrt{\frac{1}{x^6}(x^6+1)}-\frac{1}{x^3}}$$

$\sqrt{\frac{1}{x^6}} = -\frac{1}{x^3}$ if $x<0$

$$= \lim_{x\to-\infty} \frac{2+\frac{1}{x^3}}{-\sqrt{1+\frac{1}{x^6}}-\frac{1}{x^3}} = \frac{2+\lim_{x\to-\infty}\frac{1}{x^3}}{-\sqrt{1+\lim_{x\to-\infty}\frac{1}{x^6}}-\lim_{x\to-\infty}\frac{1}{x^3}} = \frac{2+0}{-\sqrt{1+0}-0}$$

$$= -2$$

Problem 2. Evaluate the limit: $\lim\limits_{x\to\infty} \dfrac{\sin^4 x}{\sqrt{x}}$.

$$\lim_{x\to\infty} \frac{\sin^4 x}{\sqrt{x}} = \lim_{x\to\infty} \left(\frac{1}{\sqrt{x}}\right)\cdot\left(\sin^4 x\right)$$

$$= \left(\lim_{x\to\infty} \frac{1}{\sqrt{x}}\right)\cdot\left(\lim_{x\to\infty} \sin^4 x\right)$$

$\downarrow$ 0 $\qquad$ $\downarrow$ oscillates between 0 and 1

$$= 0 \cdot c \quad \text{where} \quad 0 \le c \le 1$$

$$= 0$$

(0 multiplied with any finite number gives 0)

Problem 3. Consider the function $f(x) = \dfrac{x}{1-x^2}$. Find the intervals where f increasing and the points of local maximum and minimum. [10 pts]

$$f'(x) = \frac{(1-x^2)[x]' - x[1-x^2]'}{(1-x^2)^2} \quad (\text{Quotient rule})$$

$$= \frac{1-x^2 - x(-2x)}{(1-x^2)^2} = \frac{1-x^2+2x^2}{(1-x^2)^2}$$

$$= \frac{1+x^2}{(1-x^2)^2}$$

always +ve

square is +ve

$\Rightarrow f'(x) > 0$ in the domain of f.

$\Rightarrow f$ is increasing in $(-\infty, \infty)$ and decreasing nowhere.

$\Rightarrow$ There are no points of local max or local min.

Problem 4. Let $f(x) = \dfrac{x}{1-x^2}$ be as in problem 2. Find all the asymptotes (vertical and horizontal) to the curve $y = f(x)$. [10 pts]

Vertical asymptotes

$$1-x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

$\Rightarrow$ There are two vertical asymptotes : $x=1$ and $x=-1$

Horizontal asymptotes

$$\lim_{x\to\infty} \frac{x}{x^2-1} = \lim_{x\to\infty} \frac{\frac{1}{x^2}\cdot x}{\frac{1}{x^2}(x^2-1)}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1-\frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\frac{1}{x}}{1-\lim_{x\to\infty}\frac{1}{x^2}} = \frac{0}{1-0} = 0$$

$$\lim_{x\to-\infty} \frac{x}{x^2-1} = \lim_{x\to-\infty} \frac{\frac{1}{x^2}\cdot x}{\frac{1}{x^2}(x^2-1)}$$

$$= \lim_{x\to-\infty} \frac{\frac{1}{x}}{1-\frac{1}{x^2}}$$

$$= \frac{\lim_{x\to-\infty}\frac{1}{x}}{1-\lim_{x\to-\infty}\frac{1}{x^2}}$$

$$= \frac{0}{1-0} = 0$$

$\Rightarrow$ There is one horizontal asymptote : $y=0$

Problem 5. Let $f(x) = \dfrac{x}{1-x^2}$ be as in problem 2. Find the intervals of concavity and the points of inflection of f. [10 pts]

$$f'(x) = \frac{1+x^2}{(1-x^2)^2} \Rightarrow f''(x) = \frac{(1-x^2)^2\,[1+x^2]' - (1+x^2)\,[(1-x^2)^2]'}{(1-x^2)^4}$$

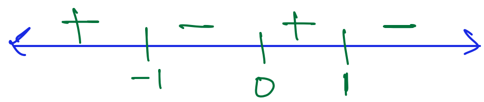
$$\Rightarrow f''(x) = \frac{(1-x^2)^2(2x) - (1+x^2)\,2(1-x^2)(-2x)}{(1-x^2)^4}$$

Factor $2x(1-x^2)$ from numerator.

$$= \frac{2x(1-x^2)\left[1-x^2 - (1+x^2)(-2)\right]}{(1-x^2)^4}$$

$$= \frac{2x(1-x^2)\left[1-x^2+2+2x^2\right]}{(1-x^2)^4} = \frac{2x(1-x^2)\overbrace{(x^2+3)}^{\text{always +ve}}}{\underbrace{(1-x^2)^4}_{\text{always +ve}}}$$

$\Rightarrow$ Sign of $f''(x)$ is same as sign of $2x(1-x^2)$



$\Rightarrow f$ is concave up in $(-\infty,-1)\cup(0,1)$ and concave down in $(-1,0)\cup(1,\infty)$

The points of inflection are $x = 0, -1, 1$.

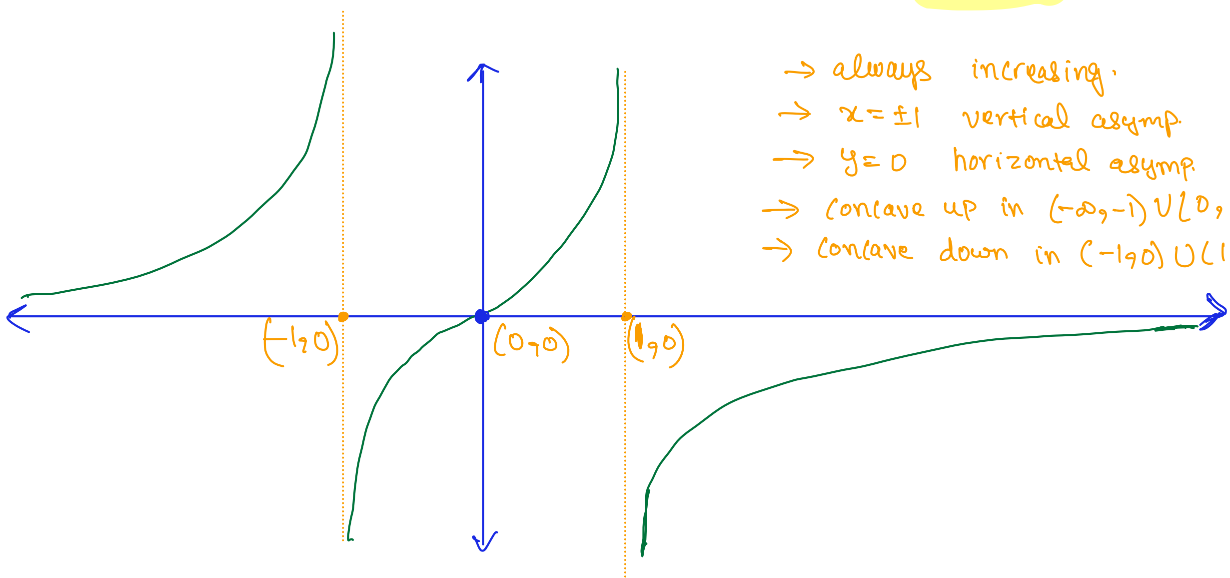
Problem 6. Find the x and y intercepts, and the domain of the function $f(x) = \dfrac{x}{1-x^2}$. Combine this with the information obtained from problems 2-5 to sketch the curve $y = f(x)$. [10 pts]

Domain = All real numbers except when $1-x^2=0 \Rightarrow x=\pm1$

$= (-\infty,-1)\cup(-1,1)\cup(1,\infty)$

x-intercept $\Rightarrow y=0 \Rightarrow \dfrac{x}{1-x^2}=0 \Rightarrow x=0 \Rightarrow (0,0)$

y-intercept $\Rightarrow x=0 \Rightarrow f(0)=0 \Rightarrow (0,0)$



→ always increasing.
→ $x=\pm1$ vertical asymp.
→ $y=0$ horizontal asymp.
→ concave up in $(-\infty,-1)\cup(0,1)$
→ concave down in $(-1,0)\cup(1,\infty)$

Problem 7. Find all the points of local maximum and minimum of the function $f(x) = x\sqrt{2+x}$. [10 pts]

$$f'(x) = [x]'\sqrt{2+x} + x\left[\sqrt{2+x}\right]' \quad \text{[Product rule]}$$

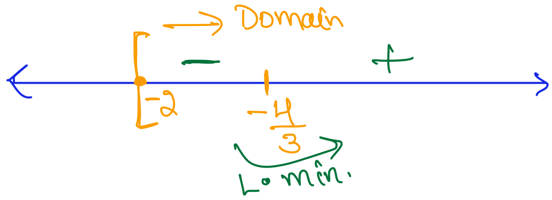
$$= \sqrt{2+x} + x\left(\frac{1}{2\sqrt{2+x}}\right)$$

cross multiply

$$= \frac{2(2+x)+x}{2\sqrt{2+x}} = \frac{4+2x+x}{2\sqrt{2+x}} = \frac{3x+4}{\underbrace{2\sqrt{2+x}}_{\text{always +ve}}}$$

Critical numbers

$3x+4=0 \Rightarrow x = -\frac{4}{3}$ and $\sqrt{2+x}=0 \Rightarrow x=-2$



Note that domain of f is given by $x+2 \geq 0 \Rightarrow x \geq -2$

$\Rightarrow x = -\frac{4}{3}$ is a point of local minimum and there are no points of local maximum.

Problem 8. Find the intervals of concavity of the function $f(x) = \sin^2 x - 2\cos x$, $0 \le x \le 2\pi$. [10 pts]

sign of $f''(x)$

$$f'(x) = 2\sin x(\cos x) - 2(-\sin x) = 2\sin x\cos x + 2\sin x$$

$$f''(x) = \underbrace{2[\sin x]'\cos x + 2\sin x[\cos x]'}_{\text{Product rule}} + 2[\sin x]'$$

$$= 2\cos^2 x - 2\sin^2 x + 2\cos x$$

because we have $\cos x$ here transform $f''(x)$ completely in terms of $\cos x$ | $\cos^2 x$

$$= 2\cos^2 x - 2(1-\cos^2 x) + 2\cos x$$

$$= 2\cos^2 x - 2 + 2\cos^2 x + 2\cos x$$

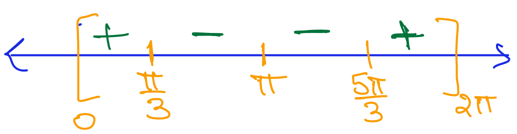
$$= 4\cos^2 x + 2\cos x - 2 = 2(\underbrace{2\cos^2 x + \cos x - 1}_{\text{Factorize}})$$

$$= 2\left[2\cos^2 x + 2\cos x - \cos x - 1\right]$$

$$= 2\left[2\cos x(\cos x + 1) - (\cos x + 1)\right] = 2(\cos x + 1)(2\cos x - 1)$$

$\Rightarrow \cos x = -1 \Rightarrow x = \pi$

or $\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$



$f''(0) > 0$

$f''\left(\frac{\pi}{2}\right) < 0$

$f''\left(\frac{3\pi}{2}\right) < 0$

$f''(2\pi) > 0$

$\Rightarrow f$ is concave up in $\left(0, \frac{\pi}{3}\right) \cup \left(\frac{5\pi}{3}, 2\pi\right)$ and concave down in $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$

Problem 9. Find two positive integers such that the sum of the first number and four times the second number is 1000 and the product of the numbers is as large as possible. [10 pts]

Let x be the first number and y be the second number.

$\Rightarrow x + 4y = 1000 \quad \Rightarrow x = 1000 - 4y$

$P = xy \quad \Rightarrow P(y) = y(1000 - 4y)$

want to maximize as a function of y.

$\Rightarrow P'(y) = 1000 - 8y$

$P'(y) = 0 \Rightarrow 1000 - 8y = 0 \Rightarrow y = \frac{1000}{8} = 125$

$+$ $\quad$ $-$ (125)

$\Rightarrow y = 125$ gives absolute maximum.

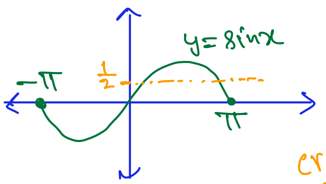
$\Rightarrow x = 1000 - 4(125) = 500.$

$\Rightarrow$ 500 and 125 are the two numbers.

Problem 10. Use the closed interval method to find the absolute maximum and minimum values of the function $f(x) = x + 2\cos x$ on the interval $[-\pi, \pi]$. [10 pts]

$f'(x) = 1 - 2\sin x$

$f'(x) = 0 \Rightarrow 1 - 2\sin x = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$



critical numbers

$f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + 2\cos\frac{\pi}{6} = \frac{\pi}{6} + \sqrt{3} \simeq 2.25$

$f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} + 2\cos\frac{5\pi}{6} = \frac{5\pi}{6} - \sqrt{3} \simeq 1.12$

endpoints

$f(-\pi) = -\pi + 2\cos(-\pi) = -\pi - 2 \simeq -5.14$

$f(\pi) = \pi + 2\cos\pi = \pi - 2 \simeq 1.14$

$\Rightarrow$ The absolute maximum value is $\frac{\pi}{6} + \sqrt{3} \simeq 2.25$

and the absolute minimum value is $-\pi - 2 \simeq -5.14$

Bonus Problem 1. Show that the equation $3x + 2\cos x + 5 = 0$ has exactly one real root. [5 pts]

Let $f(x) = 3x + 2\cos x + 5$. $\Rightarrow f(0) = 0 + 2\cos 0 + 5 = 7 > 0$

and $f(-\pi) = -3\pi + 2\cos(-\pi) + 5 = -3\pi - 2 + 5$
$= -3\pi + 3 = -6.42 < 0$

Note that f is continuous everywhere

and $f(0) > 0$ while $f(-\pi) < 0$

$\Rightarrow$ By intermediate value theorem we must have some number c between $-\pi$ and 0 for which $f(c) = 0$

$\Rightarrow$ Given equation has at least one real root.

Now suppose there is some number $d \neq c$ such that $f(d) = 0$

$\Rightarrow$ By Rolle's theorem (f is continuous and differentiable everywhere so we can apply this theorem) there must be some number a between c and d for which $f'(a) = 0$. But we have $f'(x) = 3 - 2\sin x > 0$ since $\sin x < 1$

$\Rightarrow$ Such an a is not possible. $\Rightarrow$ we cannot have the second root d.

$\Downarrow$

given equation has exactly one real root.

Bonus Problem 2. For what values of the constants a and b is $(1, 3)$ a point of inflection of the curve $y = ax^3 + bx^2$? [5 pts]

Note that $(1,3)$ must lie on the curve

$\Rightarrow$ when $x = 1$, we should have $y = 3$

$\Rightarrow 3 = a(1)^3 + b(1)^2 = a + b$

or $a + b = 3$ —— (I)

If $(1,3)$ is inflection point then $y'' = 0$ at $x = 1$

$y' = 3ax^2 + 2bx$

$y'' = 6ax + 2b \Rightarrow 6a(1) + 2b = 0 \Rightarrow 6a + 2b = 0$

or $3a + b = 0$ —— (II)

Solving (I) and (II) we have :-

$$\begin{array}{r} a + b = 3 \\ -\ 3a + b = 0 \\ \hline -2a + 0 = 3 \end{array}$$

subtract

$\Rightarrow a = \frac{-3}{2} \Rightarrow \frac{-3}{2} + b = 3 \Rightarrow b = 3 + \frac{3}{2} = \frac{9}{2}$

$\Rightarrow$ required values of a and b are $a = \frac{-3}{2}$, $b = \frac{9}{2}$