M16600 Lecture Notes

Section 11.4: The Comparison Tests

■ **Section 11.4** textbook exercises, *page 771*: #3, 5, 7, 10, 12, 13, 15, 19, 23, 25, 29.

In the comparison tests the idea is to compare the given series with a series that is known to be convergent or divergent.

For instance, say we would like to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is convergent or divergent. Note that the Test of Divergence fails for this series and this is not a geometric series or a p-series.

$\sum_{n=1}^{\infty} \frac{1}{2^n+1}$. TD $\lim_{n\to\infty} \frac{1}{2^n+1} = 0 \Rightarrow$ can converges or diverges

On the other hand, the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ reminds us of the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ with $r = \frac{1}{2}$; hence, the latter series is convergent. We can do the following comparison between these two series to determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.

$\int_1^{\infty} \frac{1}{2^x+1} \Rightarrow$ difficult.

$$2^n+1 > 2^n$$

$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} < \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$\hookrightarrow$ geometric series with $r = \frac{a_{n+1}}{a_n} = \frac{1}{2} < 1$ and $a = \frac{1}{2^1} = \frac{1}{2}$

has to be < 1

$\Rightarrow$ cannot be ∞

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges.

$a_n \le b_n$ for $n \geq n_0$ where n_0 is finite.

THE COMPARISON TEST. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for large enough n, then $\sum a_n$ is also Convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for large enough n, then $\sum a_n$ is also divergent.

$\hookrightarrow a_n \geq b_n$ for $n \geq n_0$

Remark: The Comparison Test is useful when testing series with sine or cosine functions.

Example 1: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{7^n}$ converges or diverges.

$7^n > 0$ for all $n \geq 1$ $\qquad -1 \le \sin(n) \le 1$

$1+\sin(n) \geq 0$ for all n. $\qquad 0 \le 1+\sin(n) \le 2$

$\dfrac{1+\sin(n)}{7^n} \geq 0$ for all n. $\Rightarrow$ Can apply comparison test.

$\dfrac{1+\sin(n)}{7^n} \le \dfrac{2}{7^n}$. Now, find $\displaystyle\sum_{n=1}^{\infty} \frac{2}{7^n}$ converges or not

$r = \dfrac{a_{n+1}}{a_n} = \dfrac{\cancel{2}}{7^{n+1}} \times \dfrac{7^n}{\cancel{2}} = \dfrac{1}{7} < 1$ $\qquad \dfrac{\frac{2}{7^{n+1}}}{\frac{2}{7^n}}$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{2}{7^n}$ converges

By comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin(n)}{7^n}$ also converges.

Question: Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 3}$ convergent or divergent?

$2^n - 3 \le 2^n \Rightarrow \dfrac{1}{2^n-3} \geq \dfrac{1}{2^n} \Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-3} \geq \sum_{n=1}^{\infty} \frac{1}{2^n}$

converges

$2^n - 3 \geq 2^{n-1}$ for $n \geq 3 \Rightarrow \dfrac{1}{2^n-3} \le \dfrac{1}{2^{n-1}}$ for $n \geq 3$

$2^n - 3 \geq \dfrac{2^n}{2}$

$\Rightarrow \displaystyle\sum_{n=3}^{\infty} \frac{1}{2^n-3} \le \sum_{n=3}^{\infty} \frac{1}{2^{n-1}}$ $\hookrightarrow$ converges

$2^n - \dfrac{2^n}{2} - 3 \geq 0 \Rightarrow \dfrac{2^n}{2} \geq 3 \Rightarrow 2^n \geq 6$

$\Rightarrow$ Converges by comparison test

If $n \geq 3$ then $2^n \geq 6$.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-3} = \frac{1}{2^1-3} + \frac{1}{2^2-3} + \sum_{n=3}^{\infty} \frac{1}{2^n-3} < \infty$

$\hookrightarrow$ converges

$n=3 \; \& \; -3$

The ***Limit Comparison Test*** helps us to determine the convergence or divergence of a series that is "similar" to a series which we're familiar with.

Definition of Similarity between Two Series. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \text{ a positive number },$$

then we say $\sum a_n$ and $\sum b_n$ are ***similar to*** each other.

The Limit Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are *similar series* with positive terms. Then **either** both series are convergent **or** both series are divergent.

In other words, similar series behave the same way regarding convergence or divergence.

Example 2: Show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+4}$ is similar to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$. Then use the Limit Comparison Test to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+4}$ is convergent or divergent.

$$a_n = \frac{1}{\sqrt{n}+4}, \quad b_n = \frac{1}{\sqrt{n}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+4}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt{n}+4} \times \sqrt{n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+4}$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} = 1$$

finite positive number

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+4} \text{ is similar to } \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

By limit-comparison test, either both converge / both diverge

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$: p-series with $p = \frac{1}{2} \Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

$\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+4}$ also diverges.

Remark: The Limit Comparison Test is very useful when working with series that remind us of geometric series or p-series.

Remark: To determine similar series, often we can drop the slower terms of the numerator and of the denominator then use algebra to simplify.

Example 3: Find the similar series of the given series then test for convergence and divergence.

(a) $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$

$a_n = \frac{n^2+n+1}{n^4+n^2} \Rightarrow b_n = \frac{n^2}{n^4} \Rightarrow b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2+n+1}{n^4+n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \left(\frac{n^2+n+1}{n^4+n^2}\right) \times n^2$

$= \lim_{n\to\infty} \frac{n^4+n^3+n^2}{n^4+n^2} = \lim_{n\to\infty} \frac{n^4}{n^4} = 1 \rightarrow$ finite positive

$\sum_{n=1}^{\infty} \frac{1}{n^2}$: P-series with $P=2 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$ also converges.

(b) $\sum_{n=1}^{\infty} \frac{6^n+n}{5^n-2}$ $\qquad a_n = \frac{6^n+n}{5^n-2} \Rightarrow b_n = \frac{6^n}{5^n}$

$\Rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{6^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n \qquad \Rightarrow r = \frac{a_{n+1}}{a_n} = \frac{\left(\frac{6}{5}\right)^{n+1}}{\left(\frac{6}{5}\right)^n} = \frac{6}{5}$

$\hookrightarrow$ geometric series with $r = \frac{6}{5} > 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{6^n}{5^n}$ diverges

$\Rightarrow \sum_{n=1}^{\infty} \frac{6^n+n}{5^n-2}$ also diverges.

Example 4: Determine whether the series converges of diverges.

(a) $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n+9}}$

$a_n = \frac{5}{\sqrt{n+9}} \Rightarrow b_n = \frac{5}{\sqrt{n}}$

$\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}} = 5 \underbrace{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}}$

P-series with $P = \frac{1}{2}$

$\Rightarrow$ diverges.

$\Downarrow$

$\sum_{n=1}^{\infty} \frac{5}{\sqrt{n+9}}$ also diverges.

(b) $\sum_{n=1}^{\infty} \frac{2n(11+n)^{12}}{(8+3n)(1+n^2)^3}$

$b_n = \frac{2n(n)^{12}}{(3n)(n^2)^3} = \frac{2n^{13}}{3n(n^6)} = \frac{2n^{13}}{3n^7} = \frac{2}{3} n^{13-7} = \frac{2}{3} n^6$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{3} n^6 = \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{n^{-6}}$: P-series with $P = -6$

$\Rightarrow$ diverges.

By limit Comparison test, $\sum_{n=1}^{\infty} \frac{2n(11+n)^{12}}{(8+3n)(1+n^2)^3}$ diverges

(c) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{e^n + 3}$

$0 \le \cos^2 n \le 1 \Rightarrow \frac{0}{e^n+3} \le \frac{\cos^2 n}{e^n+3} \le \frac{1}{e^n+3} \Rightarrow 0 \le \frac{\cos^2 n}{e^n+3} \le \frac{1}{e^n+3}$

$\sum_{n=1}^{\infty} \frac{\cos^2 n}{e^n+3} \le \sum_{n=1}^{\infty} \frac{1}{e^n+3} \rightarrow$ similar to $\sum_{n=1}^{\infty} \frac{1}{e^n}$

$r = \frac{a_{n+1}}{a_n} = \frac{1/e^{n+1}}{1/e^n} = \frac{1}{e^{n+1}} \times e^n = \frac{1}{e}$

$\sum_{n=1}^{\infty} \frac{1}{e^n}$ is a geometric series

with $r = \frac{1}{e} \sim \frac{1}{2.71} < 1 \Rightarrow$ the series $\sum_{n=1}^{\infty} \frac{1}{e^n}$ Converges

By limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{e^n+3}$ also converges.

$\Downarrow$

By comparison test, $\sum_{n=1}^{\infty} \frac{\cos^2 n}{e^n+3}$ converges.

(d) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

$a_n = \frac{n}{e^n} \Rightarrow \lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = \frac{1}{\infty} = 0$

$f(x) = \frac{x}{e^x} \Rightarrow f(x) = xe^{-x} \rightarrow$ Positive on $[1, \infty)$ and Cont

$f'(x) = e^{-x} + x(-e^{-x})$
$= (1-x)\underbrace{e^{-x}}_{\text{always } +ve}$ $\Rightarrow f'(x) < 0$ for $x > 1 \Rightarrow f$ is decreasing.

when $x>1 \Rightarrow 1-x<0$

$\int_1^{\infty} xe^{-x}\,dx \Rightarrow \int \underset{u}{x}\, \underbrace{e^{-x}\,dx}_{dv} = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx$
$= -xe^{-x} - e^{-x}$
$\Rightarrow du = dx$ and $v = \int e^{-x}\,dx = -e^{-x}$
$= -e^{-x}(x+1)$

$\Rightarrow \lim_{t\to\infty} \int_1^t xe^{-x}\,dx = \lim_{t\to\infty} -e^{-x}(x+1)\Big|_1^t = \lim_{t\to\infty} -e^{-t}(t+1) - (-e^{-1}(1+1))$

$= \underbrace{\lim_{t\to\infty} -e^{-t}(t+1)}_{=0} + 2e^{-1}$

DS: $-e^{-\infty}(\infty+1) = 0\cdot\infty$ (indeterminate)

$\Rightarrow \lim_{t\to\infty} -e^{-t}(t+1) = \lim_{t\to\infty} \frac{-(t+1)}{e^t} = \lim_{t\to\infty} \frac{-1}{e^t} = \frac{-1}{e^{\infty}} = \frac{-1}{\infty}$
$= 0$

$\int_1^{\infty} xe^{-x}\,dx = 2e^{-1} = \frac{2}{e} < \infty \Rightarrow \int_1^{\infty} xe^{-x}\,dx$ converges

$\Rightarrow$ the series also converges.