Name: [1 pt]

Problem 1: Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$. [4 pts]

Let (x, y) be an arbitrary point on the parabola.

$\Rightarrow y^2 = 2x$

$\text{dist}((x,y),(1,4)) = \sqrt{(x-1)^2 + (y-4)^2}$

to be minimized $\rightarrow$ equivalent to minimizing $(x-1)^2 + (y-4)^2$ because $\sqrt{\cdot}$ is an increasing fn.

$f(x,y) = (x-1)^2 + (y-4)^2$. Putting $y^2 = 2x$ we have :—

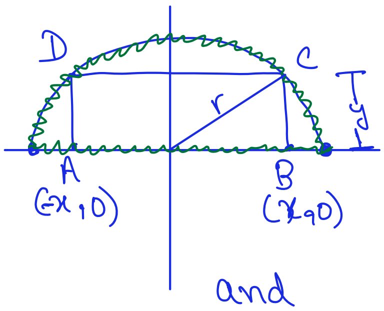
$$f(y) = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$$

$$f'(y) = 2\left(\frac{y^2}{2} - 1\right)\left(\frac{2y}{2}\right) + 2(y-4) = (y^2-2)y + 2y - 8 = y^3 - 8$$

$$f'(y) = 0 \Rightarrow y^3 - 8 = 0 \Rightarrow y = 2 \Rightarrow x = \frac{y^2}{2} = \frac{2^2}{2} = 2.$$

Thus, the point on $y^2 = 2x$ that is closest to $(1,4)$ is $(2,2)$.

Problem 2: Find the area of the largest rectangle that can be inscribed in a semicircle of radius r. [5 pts]



Let ABCD be a rectangle inscribed in the semicircle $y = \sqrt{r^2 - x^2}$ of radius r centred at $(0,0)$.

$\text{Area}(ABCD) = 2xy$

Then $y = \sqrt{r^2 - x^2}$

and we have $A(x) = 2x\sqrt{r^2 - x^2}$ where $0 \le x \le r$

$A(0) = 0 = A(r)$.

To find largest area, we find critical points of $A(x)$.

$$A'(x) = 2\sqrt{r^2-x^2} + 2x \times \frac{1}{2\sqrt{r^2-x^2}} \times (-2x)$$

$$= 2\sqrt{r^2-x^2} - \frac{2x^2}{\sqrt{r^2-x^2}} = \frac{2(r^2-x^2) - 2x^2}{\underbrace{\sqrt{r^2-x^2}}_{\text{always +ve}}}$$

Now, $A'(x) = 0 \Rightarrow 2(r^2-x^2) - 2x^2 = 0 \Rightarrow r^2 - x^2 - x^2 = 0 \Rightarrow 2x^2 = r^2 \Rightarrow x = \frac{\pm r}{\sqrt{2}}$

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = 2\frac{r}{\sqrt{2}}\sqrt{\frac{r^2}{2}} = 2\frac{r}{\sqrt{2}}\frac{r}{\sqrt{2}} = r^2$$

$$A\left(\frac{-r}{\sqrt{2}}\right) = -2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = -2\frac{r}{\sqrt{2}}\frac{r}{\sqrt{2}} = -r^2.$$

$\boxed{\text{Thus, largest area possible is } r^2}$