■ **Section 7.2** exercises, *page 524*: #1, 3, 7, 21, 23, 25, 13, 27, 17, 11, 29.

In this section, there are no new methods of integration. We mainly concern about **integrals that involve only trigonometric functions**, which we will call ***Trigonometric Integrals***.

Then main tools we are going to use to solve trigonometric integrals are

- The method of u-substitution
- Trigonometric identities

$$\sin^2 x + \cos^2 x = 1 \qquad 1+\tan^2 x = \sec^2 x \qquad \sin(2x) = 2\sin x\,\cos x$$
$$\cos^2 x = \tfrac{1}{2}\big[1+\cos(2x)\big] \qquad \sin^2 x = \tfrac{1}{2}\big[1-\cos(2x)\big]$$

- Sometimes, we will need to do integration by parts

$$\boxed{\int \sin^m x \cos^n x\, dx}$$

→ If m is odd, then substitute $u = \cos x$

→ If n is odd, then substitute $u = \sin x$

Example 1: Evaluate $\displaystyle\int \sin^5 x \cos^2 x\, dx$

$u = \cos x$

$\Rightarrow du = -\sin x\, dx \Rightarrow \sin x\, dx = -du$

$$I = \int \sin^5 x \cos^2 x\, dx$$
$$= \int \sin^4 x \cos^2 x\,(\sin x\, dx)$$
$$= \int (\sin^4 x \cos^2 x)(-du)$$

$[\sin^2 x]^2 \cos^2 x$

$\sin^4 x[1-\sin^2 x]$ needed when substitution is $u = \sin x$

$(1-\cos^2 x)^2 \cos^2 x \qquad \Rightarrow I = \int (1-u^2)^2 u^2 (-du)$

$$\Rightarrow I = \int (1-2u^2+u^4)u^2(-du) = -\int (u^2-2u^4+u^6)\,du$$

Example 2: Find $\int \cos^3 x\,dx$

$$= -\left[\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right] + C$$

$$= -\left[\frac{\cos^3 x}{3} - 2\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7}\right] + C$$

$u = \sin x$

$\Rightarrow du = \cos x\,dx$

$$I = \int \cos^3 x\,dx = \int \cos^2 x\,(\cos x\,dx)$$

$\sin^2 x + \cos^2 x = 1$

$$= \int [\cos^2 x]\,du = \int [1 - \sin^2 x]\,du$$

$$= \int (1-u^2)\,du = \int du - \int u^2\,du = u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

Example 3: Evaluate $\int_0^\pi \sin^2 x\,dx$

$$= \int_0^\pi \frac{1-\cos(2x)}{2}\,dx = \int_0^\pi \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]dx$$

$$= \int_0^\pi \frac{1}{2}\,dx - \int_0^\pi \frac{1}{2}\cos(2x)\,dx$$

$$= \frac{1}{2}\int_0^\pi dx - \frac{1}{2}\int_0^\pi \cos(2x)\,dx$$

$$= \frac{1}{2}\,x\Big|_0^\pi - \frac{1}{2}\left[\frac{\sin(2x)}{2}\right]_0^\pi = \frac{1}{2}[\pi - 0] - \frac{1}{4}[\sin 2\pi - \sin 0] = \pi/2$$

$$I' = \int \cos(\underbrace{2x}_{u})\,dx$$

HW.

$$\boxed{\begin{array}{l}\text{If } \int f(x)\,dx = g(x) + C \\ \text{then } \int f(ax+b)\,dx = \frac{1}{a}g(ax+b) + C\end{array}}$$

$\int \cos(3x+8) = \frac{\sin(3x+8)}{3} + C$

$u = 2x \Rightarrow du = 2\,dx$

$I' = \int \cos u \,\frac{du}{2}$

$= \frac{1}{2}\int \cos u \, du$

$= \frac{\sin u}{2} = \frac{\sin(2x)}{2}$

Example 4: Find $\int \tan^6 x \sec^4 x \, dx$

$u = \tan x \Rightarrow du = \sec^2 x \, dx$

$I = \int u^6 \sec^4 x \frac{du}{\sec^2 x}$

$= \int u^6 \sec^2 x \, du$

$= \int u^6 (1 + \tan^2 x) \, du$

$= \int u^6 (1+u^2) \, du = \int u^6 du + \int u^8 du$

$= \frac{u^7}{7} + \frac{u^9}{9} + C = \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$

$\boxed{\begin{array}{l} \int \tan^m x \sec^n x \, dx \\ \to n \text{ is even}: u = \tan x \\ \to m \text{ is odd}: u = \sec x \end{array}}$

Example 5: Find $\int \tan^5 \theta \sec^7 \theta \, d\theta$

$u = \sec\theta \Rightarrow \frac{du}{d\theta} = \sec\theta\tan\theta \Rightarrow du = \sec\theta\tan\theta\, d\theta$

$I = \int \tan^5\theta \sec^7\theta \frac{du}{\sec\theta\tan\theta} = \int \underbrace{\tan^4\theta \sec^6\theta}_{} \, du$

want to write this entirely in terms of u

$I = \int \tan^4\theta \, u^6 \, du$

use $\tan^2\theta = \sec^2\theta - 1$

$I = \int [\sec^2\theta - 1]^2 u^6 \, du = \int (u^2-1)^2 u^6 \, du$

$= \int (u^4 - 2u^2 + 1) u^6 \, du = \int [u^{10} - 2u^8 + u^6] \, du$

$$= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + C$$

Extra Examples:

- $\int \tan^3 x\,dx$ (Example 7, textbook, page 523).
- $\int \sec^3 x\,dx$ (Example 8, textbook, page 523).
- $\int \sin(4x)\cos(5x)\,dx$ (Example 9, textbook, page 524)

$$= \frac{\sec^{11}\theta}{11} - 2\frac{\sec^9\theta}{9} + \frac{\sec^7\theta}{7} + C$$

Example 6: Compute $\int \sin(2x)\cos^2 x\,dx$.

$$I = \int 2\sin x\cos x\cos^2 x\,dx$$

$$= 2\int \sin x\cos^3 x\,dx$$

Both work

$\boxed{u = \sin x}$ $\quad u = \cos x \Rightarrow du = -\sin x\,dx$

or $\sin x\,dx = -du$

$$I = 2\int \cos^3 x\,(\sin x\,dx)$$

$$= 2\int u^3(-du) = -2\int u^3\,du$$

$$= -2\frac{u^4}{4} + C = -\frac{u^4}{2} + C$$

$$= -\frac{\cos^4 x}{2} + C$$

$u = \sin x \Rightarrow du = \cos x\,dx$

$$I = 2\int \sin x\cos^2 x\,(\cos x\,dx) = 2\int u(1-\sin^2 x)\,du$$

$$= 2\int u(1-u^2)\,du$$

$$= 2\int (u-u^3)\,du = 2\left[\frac{u^2}{2} - \frac{u^4}{4}\right] + C$$

$$= \sin^2 x - \frac{1}{2}\sin^4 x + C$$

HW:

If $\int f(x)\,dx = g(x) + C$

then $\int f(ax+b)\,dx = \frac{1}{a}\,g(ax+b) + C$

$u = ax+b \Rightarrow du = a\,dx$

$$I = \int f(ax+b)\,dx = \int f(u)\,\frac{du}{a} = \frac{1}{a}\int f(u)\,du$$

$$= \frac{1}{a}\,g(u) + C = \frac{1}{a}\,g(ax+b) + C$$

$$\int \cos(10x)\,dx = \frac{\sin(10x)}{10}$$