Name:

Critical Points: Given a function $y = f(x)$, the critical points of f are those points in the domain of f, for which either $f'(x) = 0$ or $f'(x)$ does not exist.

Example: Find the critical points of the function $f(x) = x^{3/5}(4 - x)$.

Solution: $f'(x) = x^{3/5}\dfrac{d}{dx}(4 - x) + (4 - x)\dfrac{d}{dx}(x^{3/5}) = x^{3/5}(-1) + (4 - x)\frac{3}{5}x^{-2/5}$

$\Rightarrow f'(x) = -x^{3/5} + \dfrac{12 - 3x}{5x^{2/5}} = \dfrac{-5x + 12 - 3x}{5x^{2/5}} = \dfrac{12 - 8x}{5x^{2/5}}$.

Thus, $f'(x) = 0 \Rightarrow \dfrac{12 - 8x}{5x^{2/5}} = 0 \Rightarrow 12 - 8x = 0 \Rightarrow x = \dfrac{12}{8} = \dfrac{3}{2}$.

We also see that at $x = 0$, the denominator goes to 0 and $f'(x)$ does not exist.

Therefore, the critical points of the given function f are 0 and $\dfrac{3}{2}$.

Problem 1: Find the critical points of the function $f(x) = 2x^3 + 3x^2 - 12x + 7$.

$f'(x) = 6x^2 + 6x - 12$

$f'(x) = 0 \Rightarrow 6x^2 + 6x - 12 = 0 \Rightarrow 6(x^2 + x - 2) = 0$

$\Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0$

$\Rightarrow \boxed{x = -2 \text{ or } x = 1 \text{ are the critical points}}$

Problem 2: Find the critical points of the function $f(x) = \dfrac{x^2}{x + 2}$.

Note that -2 is not in the domain of f and hence cannot be a critical point.

$f'(x) = \dfrac{(x+2)(2x) - x^2(1)}{(x+2)^2} = \dfrac{2x^2 + 4x - x^2}{(x+2)^2}$

$\Rightarrow f'(x) = \dfrac{x^2 + 4x}{(x+2)^2} = \dfrac{x(x+4)}{(x+2)^2}$

$f'(x) = 0 \Rightarrow x(x+4) = 0 \Rightarrow x = 0$ or $x = -4$

$f'(x)$ d.n.e. $\Rightarrow x = -2$. But $x = -2$ is not in domain.

Thus, $\underline{0 \text{ and } -4}$ are the critical points of $f(x)$

Problem 3: Find the critical points of the following two functions:-

1. $f(x) = x + \sqrt{x}$
2. $g(x) = x - \sqrt{x}$

$f'(x) = 1 + \frac{1}{2\sqrt{x}} \Rightarrow f'(x) = \frac{2\sqrt{x}+1}{2\sqrt{x}}$

$f'(x)$ d.n.e at $x=0$ and $f'(x)=0 \Rightarrow 2\sqrt{x}+1=0 \Rightarrow \sqrt{x} = \frac{-1}{2}$ $\downarrow$ not possible

$\Rightarrow$ $\boxed{\text{0 is the only critical point of } f.}$

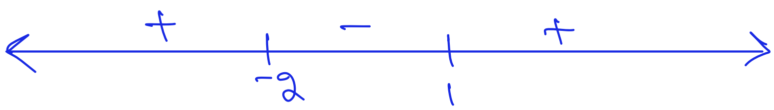
$g'(x) = 1 - \frac{1}{2\sqrt{x}} \Rightarrow g'(x) = \frac{2\sqrt{x}-1}{2\sqrt{x}}$

$g'(x)$ d.n.e. at $x=0$ and $g'(x)=0 \Rightarrow 2\sqrt{x}-1=0 \Rightarrow \sqrt{x} = \frac{1}{2}$
$\Rightarrow x = \frac{1}{4}$

$\Rightarrow$ $\boxed{0 \text{ and } \tfrac{1}{4} \text{ are the critical points of } g.}$

Problem 4: For $f(x) = 2x^3 + 3x^2 - 12x + 7$ (as in problem 1), find the intervals where $f'(x) > 0$ and the intervals where $f'(x) < 0$.

$f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x+2)(x-1)$



Thus, $f'(x) > 0 \Rightarrow \boxed{x \in (-\infty, -2) \cup (1, \infty)}$

and $f'(x) < 0 \Rightarrow \boxed{x \in (-2, 1)}$

Inflection Point: A point a in the domain of a function $y = f(x)$ is called an inflection point if f is continuous at a and $f''(a) = 0$ or does not exist.

Problem 5: Find inflection points for $f(x) = x^2 + \frac{1}{x}$.

$$f'(x) = 2x - \frac{1}{x^2} \Rightarrow f''(x) = 2 - (-2x^{-2-1})$$
$$= 2 + \frac{2}{x^3}$$

$$\Rightarrow f''(x) = \frac{2x^3+2}{x^3} = \frac{2(x^3+1)}{x^3}$$

$f''(x)$ d.n.e. at $x=0$. But 0 is not in domain of f.

And $f''(x) = 0 \Rightarrow 2(x^3+1) = 0 \Rightarrow x^3+1=0$

$\Rightarrow (x+1)(x^2-x+1) = 0 \Rightarrow x+1=0 \Rightarrow x=-1$

or $x^2-x+1=0$

no solutions.

$\boxed{\text{Thus, } -1 \text{ is the only inflection point}}$

Problem 6: Find inflection points for $f(x) = x^3$.

$f'(x) = 3x^2$, $f''(x) = 6x$

$f''(x) = 0 \Rightarrow 6x = 0 \Rightarrow x = 0$

$\boxed{\text{Thus, } 0 \text{ is the only inflection point of } f.}$