M16600 Lecture Notes

Section 6.1: Inverse Functions

GOALS

– Given a function $f(x)$, understanding the **inverse of** f, denote by $f^{-1}(x)$.

– Find the derivative of f^{-1} at $x = a$. *Notation*: $(f^{-1})'(a)$.

RECALL:

– The *domain* of a function is the set of all *input values*.

– The *range* of a function is the set of all *output values*.

• $g(x) = \dfrac{1}{\sqrt{x-5}}$ ⇒ Find the domain of g. $(5, \infty)$

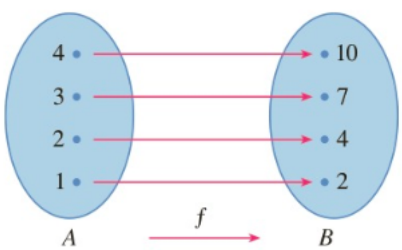
• $f(x) = \dfrac{1}{x-5}$ ⇒ All real range $y = \dfrac{1}{x-5}$

At $x=5$ $f(x)$ is not defined ⇒ $x=5$ is not in D_f

Domain of f = All real numbers except 5. $(-\infty, 5) \cup (5, \infty)$

I. Understanding Inverse Functions

Here is an example of a function $f(x)$



f is one-one

$4 \to 10$, $3 \to 7$, $2 \to 4$, $1 \to 4$, 2; g · g is not one-one

Range = $\{10, 4\}$

$f(3) =$

$f(1) =$

$f(4) =$

The domain of f is the set $A = \{4, 3, 2, 1\}$

The range of f is the set $B = \{10, 7, 4, 2\}$.

Definition: The ***inverse function of*** $f(x)$ is a new function, denoted by $f^{-1}(x)$. ⇒ exists only for one-one functions

The *domain* of f^{-1} = Range of f

The *range* of f^{-1} = Domain of f

$$f^{-1}(x) = y \iff x = f(y)$$ if and only if

Using the example of f above, evaluate:

$f^{-1}(2) =$ 1 because $2 = f(1)$

$f^{-1}(4) =$ 2 because $4 = f(2)$

Domain of f		Range of f
4	←	10
3	←	7
2	←	4
1	←	2
Range of f^{-1}		Domain of f^{-1}

Example 1: If $f(1) = 5$, $f(3) = 7$, and $f(8) = 3$, find $f^{-1}(7)$, $f^{-1}(5)$, $f^{-1}(3)$.

(1) $f^{-1}(7) = y \Rightarrow 7 = f(y) \Rightarrow y = 3$

$\Rightarrow f^{-1}(7) = 3$

(2) $f^{-1}(5) = 1$ (3) $f^{-1}(3) = 8$

Example 2: (a) Let $f(x) = x^3$, without an explicit formula of $f^{-1}(x)$, could you spot the answer for $f^{-1}(8)$?

$f^{-1}(8) = y \Rightarrow 8 = f(y) \Rightarrow 8 = y^3 \Rightarrow y = 2$

$\Rightarrow f^{-1}(8) = 2$

(b) Let $f(x) = x^3 + x + 1$ without an explicit formula of $f^{-1}(x)$, could you spot the answer for $f^{-1}(3)$?

$f^{-1}(3) = y \Rightarrow f(y) = 3 \Rightarrow y^3 + y + 1 = 3 \Rightarrow y^3 + y = 2 \Rightarrow y = 1$

$\Rightarrow f^{-1}(3) = 1$

(c) Find the inverse function of $f(x) = x^3$.

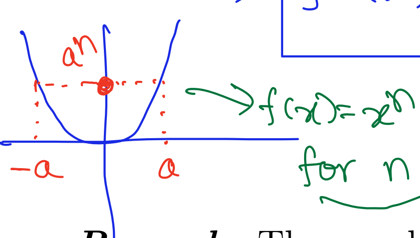
Let $f^{-1}(x) = y \Rightarrow x = f(y) \Rightarrow x = y^3$

Cube roots exist for all real numbers

$\Rightarrow y = \sqrt[3]{x}$

$\Rightarrow \boxed{f^{-1}(x) = \sqrt[3]{x}}$

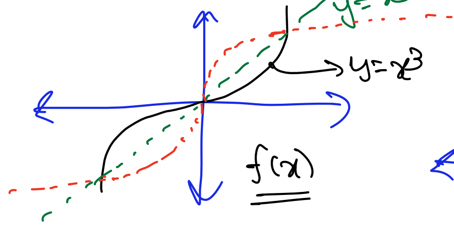
- For any odd positive integer n, if we have $f(x) = x^n$ then $f^{-1}(x) = \sqrt[n]{x}$

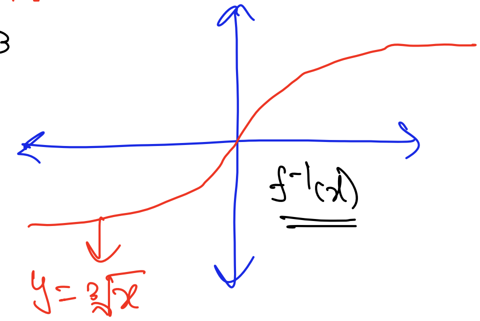


for n is even $\Rightarrow$ not one-one $\Rightarrow$ inverse does not exist.

Remark: The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Cancellation Equations:





If x is in the domain of f, then

$$f^{-1}(f(x)) = x$$

If x is in the domain of f^{-1}, then

$$f(f^{-1}(x)) = x$$

For $f(x) = x^3$, $\sqrt[3]{x^3} \overset{?}{=} x$, $(\sqrt[3]{x})^3 = x$

Fact: Only ***one-to-one*** functions have inverses.

One-to-One Functions: A function f is ***one-to-one*** if

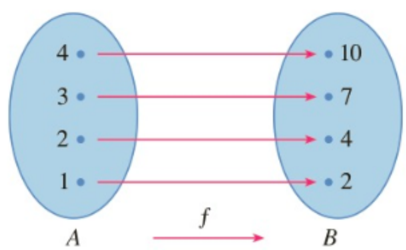
$x_1 = x_2$ whenever $f(x_1) = f(x_2)$

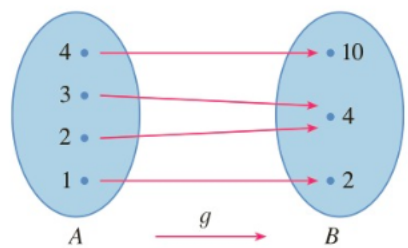
$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

In other words, a function is one-to-one if every output comes from ONLY ONE input.





Check: $f(x) = x^2$ is not one-one. $-1 \neq 1$ but $f(-1) = (-1)^2 = 1^2 = f(1)$

Horizontal Line Test: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

II. The derivative of $f^{-1}(x)$ at $x = a$. Notation: $(f^{-1})'(a)$

Derivative Notation

Functions	Derivatives (a new function)
$f(x)$	$f'(x)$
$y = f(x)$	$\frac{dy}{dx}$ or $\frac{d}{dx}[f(x)]$

Example 3: Let $f(x) = x^3$, then inverse function of f is $f^{-1}(x) = \sqrt[3]{x}$.

(a) Evaluate $f'(1)$

$f'(x) \xrightarrow{x=1} f'(1)$

$f'(x) = \frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$

$\Rightarrow f'(1) = 3(1)^2 = 3$

(b) Evaluate $(f^{-1})'(1)$

$f^{-1}(x) = \sqrt[3]{x}$

$(f^{-1})'(x) = \frac{d}{dx}(\sqrt[3]{x})$

$= \frac{d}{dx}(x^{1/3})$

$= \frac{1}{3}x^{1/3 - 1} = \frac{1}{3}x^{-2/3}$

Power rule

$$\boxed{\frac{d}{dx}(x^n) = n x^{n-1} \quad n \text{ is a rational number}}$$

$(f^{-1})'(1) = \frac{1}{3}(1)^{-2/3} = \frac{1}{3}$

There is another way of finding $(f^{-1})'(1)$ in example 3 without knowing the explicit formula of $f^{-1}(x)$.

Theorem: If f is one-to-one differentiable function with inverse function f^{-1} and $f'\big(f^{-1}(a)\big) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'\big(f^{-1}(a)\big)} \quad \text{where } a \text{ is a number} \tag{1}$$

Example 4: Let $f(x) = x^5 + x^3 + x$, use the formula (1) to find $(f^{-1})'(3)$

$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$

$y = 1 \to y^5 + y^3 + y = 3$

$y = 0 \to 0^5 + 0^3 + 0 = 0$

$y = \cancel{-1}$

Suppose $f^{-1}(3) = y \Rightarrow 3 = f(y) \Rightarrow 3 = y^5 + y^3 + y \Rightarrow y = 1$

$\Rightarrow f^{-1}(3) = 1$

$\Rightarrow (f^{-1})'(3) = \frac{1}{f'(1)} = \frac{1}{9}$

$f(x) = x^5 + x^3 + x$

$f'(x) = 5x^4 + 3x^2 + 1$

$\Rightarrow f'(1) = 5(1)^4 + 3(1)^2 + 1$

$= 5 + 3 + 1 = 9$

$\Rightarrow (f^{-1})'(3) = \frac{1}{9}$

Example 5: Let $f(x) = 2x + \cos x$. f is a one-to-one function. Find $(f^{-1})'(1)$.

$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$

$y = \pm 1 \to y = 1 \to 2(1) + \cos 1$

$y = 0 \to 2(0) + \cos 0 = 1$

Suppose $f^{-1}(1) = y \Rightarrow 1 = f(y) \Rightarrow 1 = 2y + \cos y \Rightarrow y = 0$

$\Rightarrow f^{-1}(1) = 0$

$\Rightarrow (f^{-1})'(1) = \frac{1}{f'(0)}$

$f(x) = 2x + \cos x$

$f'(x) = 2 + (-\sin x)$

$= 2 - \sin x$

$\Rightarrow (f^{-1})'(1) = \frac{1}{2}$

$f'(0) = 2 - \sin 0 = 2$

Section 6.1 Exercises, *page 406*: # 3, 5, 7, 17, 23, 39, 40, 41, 42, 43