Exercise (Pg 516): 1, 2, 5, 11, 9, 7, 10, 12, 23, 19, 26, 17.

The method of ***Integration by Parts*** corresponds to the Product Rule in differentiation.

There is one formula you need to remember

$$\boxed{\int u\,dv = uv - \int v\,du} \xrightarrow{\text{Diff.}} u\,dv = (uv)' - v\,du$$

$$d(uv) = u\,dv + v\,du$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

We will learn how this formula works in examples

Example 1: Find $\int x \sin x\, dx$

Note: u-substitution will not work for this problem.

$$\int \underbrace{x}_{u}\ \underbrace{\sin x\ dx}_{dv} = x(-\cos x) - \int(-\cos x)\,dx$$

$u = x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx \Rightarrow du = dx$

$dv = \sin x\ dx \Rightarrow v = \int \sin x\, dx = -\cos x \Rightarrow v = -\cos x$

$$\Rightarrow \int x \sin x\, dx = -x\cos x + \int \cos x\, dx$$

$$= -x\cos x + \sin x + C$$

ILATE or LIATE

I → Inverse Trigonometric fns.

L → Logarithmic functions

A → Algebraic functions

T → Trigonometric functions

E → exponential function

Preference for u ↑

Preference for dv ↓

Example 2: Evaluate $\int \underbrace{3x^3}_{u} \ln x\, dx$ $\quad dv$

- $u = \ln x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow du = \dfrac{dx}{x}$
- $dv = 3x^3\, dx$

$\Rightarrow v = \int 3x^3\, dx \Rightarrow v = \dfrac{3x^4}{4}$

$$\int (\ln x)(3x^3\, dx) = \frac{3x^4}{4}\ln x - \int \frac{3}{4}x^{\cancel{4}\,3}\frac{dx}{\cancel{x}}$$

$$= \frac{3}{4}x^4 \ln x - \frac{3}{4}\int x^3\, dx$$

$$= \frac{3}{4}x^4 \ln x - \frac{3}{4}\,\frac{x^4}{4} + C$$

$$= \frac{3}{4}x^4\left(\ln x - \frac{1}{4}\right) + C$$

Example 3: Find $\int \underbrace{t^2}_{u}\underbrace{e^t\, dt}_{dv}$

$u = t^2 \Rightarrow \dfrac{du}{dt} = 2t \Rightarrow du = 2t\, dt$

$dv = e^t\, dt \Rightarrow v = \int e^t\, dt \Rightarrow v = e^t$

$$\int t^2 e^t\, dt = t^2 e^t - \int e^t\, 2t\, dt = t^2 e^t - 2\boxed{\int t e^t\, dt}$$

Let $I' = \int \underbrace{t}_{u}\underbrace{e^t\, dt}_{dv}$ $\qquad u = t \Rightarrow \dfrac{du}{dt} = 1 \Rightarrow du = dt$

$\qquad\qquad ||$ $\qquad\qquad dv = e^t dt \Rightarrow v = \int e^t dt \Rightarrow v = e^t$

$$t e^t - \int e^t\, dt = t e^t - e^t$$

$$\int t^2 e^t\, dt = t^2 e^t - 2\left[t e^t - e^t\right] + C$$

$$= t^2 e^t - 2t e^t + 2e^t + C \quad = e^t\left(t^2 - 2t + 2\right) + C$$

Example 4: Calculate $\int_0^1 \underbrace{\tan^{-1} x}_{u}\, \underbrace{dx}_{dv}$

$u = \mathrm{Tan}^{-1} x \Rightarrow \frac{du}{dx} = \frac{1}{1+x^2} \Rightarrow du = \frac{dx}{1+x^2}$

$dv = dx \Rightarrow v = \int dx \Rightarrow v = x$

$$\int \mathrm{Tan}^{-1}x\,dx = x(\mathrm{Tan}^{-1}x) - \int x \,\frac{dx}{1+x^2}$$

$$= x(\mathrm{Tan}^{-1}x) - \int \frac{x}{1+x^2}\,dx$$

$u = 1+x^2 \Rightarrow \frac{du}{dx} = 2x$

$\Rightarrow du = 2x\,dx$

$\Rightarrow \frac{1}{2} du = x\,dx$

$$\int \frac{x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\, x\,dx$$

$$= \int \frac{1}{u}\,\frac{1}{2}\,du = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|$$

$$= \frac{1}{2}\ln|1+x^2|$$

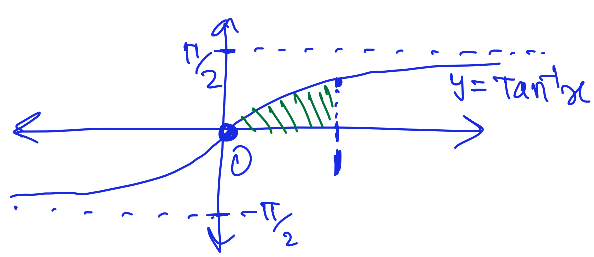
$$\Rightarrow \int \mathrm{Tan}^{-1}x\,dx = x(\mathrm{Tan}^{-1}x) - \frac{1}{2}\ln|1+x^2| + C$$

$$\int_0^1 \mathrm{Tan}^{-1}x\,dx = \left(1(\mathrm{Tan}^{-1}1) - \frac{1}{2}\ln|1+1^2|\right) - \left(0(\mathrm{Tan}^{-1}0) - \frac{1}{2}\ln|1+0^2|\right)$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) - \left(0 - \frac{1}{2}\cancelto{0}{\ln 1}\right)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

= Area of the shaded region.



Example 5: Find $\int e^x \sin x \, dx$

$$I = \int \underbrace{\sin x}_{u} \; \underbrace{e^x \, dx}_{dv}$$

$u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow du = \cos x \, dx$

$dv = e^x dx \Rightarrow v = \int e^x dx \Rightarrow v = e^x$

$$I = (\sin x) e^x - \int e^x \cos x \, dx \longrightarrow (1)$$

$$\int e^x \cos x \, dx = \int \underbrace{\cos x}_{u} \; \underbrace{e^x \, dx}_{dv}$$

$u = \cos x \Rightarrow du = -\sin x \, dx$

$dv = e^x dx \Rightarrow v = e^x$

$$= (\cos x) e^x - \int e^x (-\sin x) \, dx$$

$$= (\cos x) e^x + \int e^x \sin x \, dx$$

Put in (1)

$$I = (\sin x) e^x - \Big[(\cos x) e^x + \underbrace{\int e^x \sin x \, dx}_{I} \Big]$$

$$I = (\sin x) e^x - (\cos x) e^x - I$$

$$\Rightarrow 2I = (\sin x) e^x - (\cos x) e^x$$

$$\Rightarrow I = \frac{1}{2} \left[(\sin x) e^x - (\cos x) e^x \right]$$

$$\Rightarrow I = \frac{e^x}{2} (\sin x - \cos x) + C$$

$\int e^{2x} \sin 3x \, dx$

$\rightarrow \int e^{ax} \sin bx \, dx \rightarrow$ HW

$\rightarrow \int e^{ax} \cos bx \, dx \rightarrow$ HW.

Example : $\int e^{2x} \cos 3x \, dx$

$$I = \int \underbrace{\cos 3x}_{u} \; \underbrace{e^{2x} \, dx}_{dv}$$

$u = \cos 3x \Rightarrow \frac{du}{dx} = -3 \sin 3x$

$\int dv = \int e^{2x} dx \Rightarrow v = \frac{e^{2x}}{2}$

$$I = (\cos 3x) \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} (-3 \sin 3x \, dx)$$

$$= \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \underbrace{\int e^{2x} \sin 3x \, dx}_{} \longrightarrow (1)$$

$$\int e^{2x} \sin 3x \, dx = \int \underbrace{\sin 3x}_{u} \; \underbrace{e^{2x} \, dx}_{dv}$$

$u = \sin 3x$

$\Rightarrow du = 3 \cos 3x \, dx$

$v = \int e^{2x} dx = \frac{e^{2x}}{2}$

$$= (\sin 3x) \frac{e^{2x}}{2} - \int \frac{e^{2x}}{2} \, 3 \cos 3x \, dx$$

$$= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \underbrace{\int e^{2x} \cos 3x \, dx}_{I}$$

$$I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \left[\frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} I \right]$$

$$= \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} I$$

$$I + \frac{9}{4} I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x$$

$$\left(1 + \frac{9}{4}\right) I = \frac{1}{4} e^{2x} \left[2 \cos 3x + 3 \sin 3x \right]$$

$$\frac{13}{4} I = \frac{1}{4} e^{2x} (2\cos 3x + 3\sin 3x)$$

$$I = \frac{e^{2x}}{13} (2\cos 3x + 3\sin 3x) + C$$