M16600 Lecture Notes

Section 11.6: Absolute Convergence and the Ratio and Root Tests

■ **Section 11.6** textbook exercises, *page 782*: #1, 2, 3, 4, 5, 7, 8, 9, 11, 14, 25, 26, 27, 31, 33, 35.

Definition of Absolute Convergence. A series $\sum a_n$ is called ***absolutely convergent*** if the series of absolute values $\sum |a_n|$ is convergent.

Example 1: Test for absolute convergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|(-1)^{n-1}|}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\downarrow$ convergent.

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \rightarrow$ diverges

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is not absolutely convergent.

Definition of Conditional Convergence. A series $\sum a_n$ is called ***conditionally convergent*** if

- it is **not** absolutely convergent, but
- it is convergent.

For example, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent because

$\hookrightarrow$ Alternating series.

$\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} b_n$

$\rightarrow \displaystyle\lim_{n\to\infty} b_n = 0$

$\rightarrow b_{n+1} \leq b_n$

$\Rightarrow$ convergent.

$b_n = \frac{1}{n}$

$\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0, \quad \frac{1}{n+1} \leq \frac{1}{n} \Rightarrow$ convergent.

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent.

Example 2: Determine whether the series is absolutely convergent or conditionally convergent.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}+1}$$

Absolute Convergence

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{\sqrt{n}+1}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+1}$$

similar

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+1} \text{ divergent } \xleftarrow{LCT} \text{ divergent } \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

$\Rightarrow$ Not absolutely convergent

Convergence

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}+1} \longrightarrow \text{Alternating series}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}+1} = \frac{\text{slower}}{\text{faster}} = 0$$

$$b_{n+1} = \frac{1}{\sqrt{n+1}+1} < \frac{1}{\sqrt{n}+1} = b_n$$

$\Rightarrow$ By AST given series is Convergent

$\Rightarrow$ Conditionally convergent

Theorem. If a series $\sum a_n$ is absolutely convergent then $\sum a_n$ is convergent.

For example, we know the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ is absolutely convergent from example 1. Therefore, by the theorem above, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ is automatically ___convergent___ without using Alternating Series Tests.

Example 3: Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent or divergent.

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\downarrow$ Convergent.

$0 \leq |\cos n| \leq 1$

$0 \leq \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$

By CT

$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ is convergent.

$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent.

$\Downarrow$ (by the theorem)

Convergent

The following test is very useful in determining whether a given series is absolutely convergent

The Ratio Test. Given $\sum a_n$. First, we compute $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

(i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| =$ a number < 1, then the series $\sum a_n$ is absolutely convergent (and therefore $\sum a_n$ is convergent).

(ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ (or $= \infty$), then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, then the Ratio Test is inconclusive; that is no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Example 4: Use the Ratio Test to determine whether the series is convergent or divergent

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{(n+1)^3\, 5^n}{3^n}$

$$a_n = (-1)^n \frac{(n+1)^3\, 5^n}{3^n}$$

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$\Rightarrow a_{n+1} = (-1)^{n+1}\frac{(n+1+1)^3\, 5^{n+1}}{3^{n+1}} = (-1)^{n+1}\frac{(n+2)^3\, 5^{n+1}}{3^{n+1}}$$

$$\frac{a_{n+1}}{a_n} = (-1)^{n+1}\frac{(n+2)^3\, 5^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(-1)^n (n+1)^3\, 5^n} = (-1)^{n+1-n}\frac{(n+2)^3\, 5^{n+1-n}}{3^{n+1-n}(n+1)^3}$$

$$= (-1)\frac{5}{3}\frac{(n+2)^3}{(n+1)^3}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{5}{3}\frac{(n+2)^3}{(n+1)^3} \Rightarrow \lim_{n\to\infty}\frac{5}{3}\frac{(n+2)^3}{(n+1)^3} = \lim_{n\to\infty}\frac{5\,\cancel{(n)^3}}{3\,\cancel{(n)^3}}$$

$$= \frac{5}{3} > 1$$

$\Rightarrow$ divergent.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n-1}}{n!}$

$$a_n = \frac{2^{n-1}}{n!}, \quad a_{n+1} = \frac{2^{n+1-1}}{(n+1)!} = \frac{2^n}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{2^n}{(n+1)!}\cdot\frac{n!}{2^{n-1}} = 2^{n-(n-1)}\frac{n!}{(n+1)!} = \frac{2\left(\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{1}\right)}{(n+1)\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cdots\cancel{1}}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{2}{n+1} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{n+1}$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2}{n+1} = \frac{\text{slower}}{\text{faster}} = 0 < 1$$

$\Rightarrow$ given series absolutely convergent $\Rightarrow$ convergent.

The following test is convenient to apply when nth powers occur.

The Root Test. Given $\sum a_n$.

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ = a number < 1, then the series $\sum a_n$ is absolutely convergent (and therefore $\sum a_n$ is convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ (or $= \infty$), then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.

Example 5: Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\left(\frac{2n+3}{3n+2}\right)^n\right|} = \sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \left(\left(\frac{2n+3}{3n+2}\right)^n\right)^{\frac{1}{n}}$$

$$= \left(\frac{2n+3}{3n+2}\right)^{\cancel{n}\cdot\frac{1}{\cancel{n}}}$$

$$= \frac{2n+3}{3n+2}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n+3}{3n+2}$$

$$= \lim_{n\to\infty} \frac{2\cancel{n}}{3\cancel{n}} = \frac{2}{3} < 1$$

$\Rightarrow$ the given series is abs Conv.

$\Rightarrow$ Convergent.