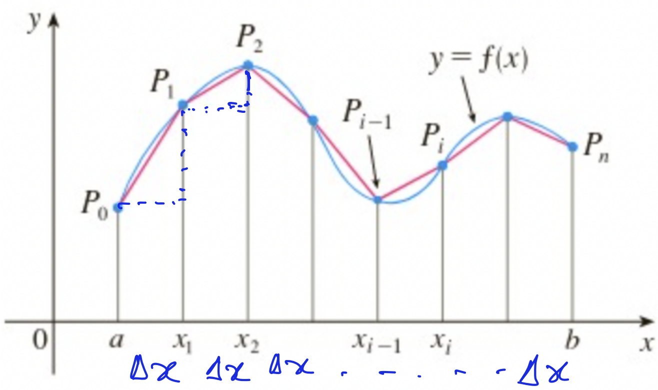
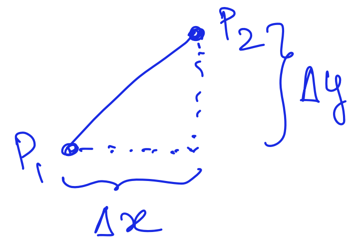
■ **Section 8.1** textbook exercises, *page 589*: # 3, 5, 14, <u>11</u>, <u>21</u>.

How do we find the length of a curve $y = f(x)$, where $a \le x \le b$?





$$P_1P_2 = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

$$\lim_{\substack{\Delta x \to 0 \\ n \to \infty}} \sum_{i=1}^{n} \Delta x \sqrt{1 + \left(\left(\frac{\Delta y}{\Delta x}\right)_i\right)^2}$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

The Arc Length Formula. If $f'(x)$ is continuous on $[a, b]$, then the length of the curve $y = f(x)$, where $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

or we can use Leibniz notation for derivatives and write the arc length formula as

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 1: Find the length of the curve $y = \frac{2}{3}x^{3/2}$ from the point $(1, \frac{2}{3})$ to the point $(2, \frac{4}{3}\sqrt{2})$.

$\underbrace{(2, \tfrac{4}{3}\sqrt{2})}_{(2,\, f(2))}$ $\qquad$ $\underbrace{(1, \tfrac{2}{3})}_{(1,\, f(1))}$

$$f(2) = \frac{2}{3}\, 2^{3/2} = \frac{2}{3}\, 2\sqrt{2} = \frac{4}{3}\sqrt{2}$$

$$f(1) = \frac{2}{3}(1)^{3/2} = \frac{2}{3}$$

from $a = 1$ to $b = 2$

$$y = \frac{2}{3}x^{3/2}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\frac{dy}{dx} = \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}}\, x^{3/2 - 1}$$

$$= x^{1/2}$$

$$L = \int_1^2 \sqrt{1+\left(x^{1/2}\right)^2}\, dx = \int_1^2 \sqrt{1+x}\, dx$$

$$= \frac{(x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_1^2 = \frac{(x+1)^{3/2}}{3/2}\Bigg|_1^2 = \frac{2}{3}(x+1)^{3/2}\Bigg|_1^2$$

$$= \frac{2}{3}\left[(2+1)^{3/2} - (1+1)^{3/2}\right] = \frac{2}{3}\left[3^{3/2} - 2^{3/2}\right] = \frac{2}{3}\left(3\sqrt{3} - 2\sqrt{2}\right)$$

Example 2: Find the exact length of the curve $y = \ln(\sec x)$, where $0 \leq x \leq \pi/4$.

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \ln(\sec x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sec x}(\sec x \operatorname{Tan} x) = \operatorname{Tan} x$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1+(\operatorname{Tan} x)^2}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{1+\operatorname{Tan}^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\, dx = \int_0^{\frac{\pi}{4}} \sec x\, dx$$

$$= \ln|\sec x + \operatorname{Tan} x|\Big]_0^{\pi/4}$$

$$\boxed{\int \sec x\, dx = \ln|\sec x + \operatorname{Tan} x| + C}$$

$$= \ln\left|\sec\frac{\pi}{4} + \operatorname{Tan}\frac{\pi}{4}\right| - \ln|\sec 0 + \operatorname{Tan} 0|$$

$$= \ln|\sqrt{2}+1| - \ln|1+0| = \ln(1+\sqrt{2}) - \cancel{\ln 1} \to 0$$

$$\Rightarrow L = \ln(\sqrt{2}+1)$$

$$\sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

$$\tan\frac{\pi}{4} = 1$$

$$\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

$$\tan 0 = 0$$