Names of Team Members:

Problem 1: Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^2/s. How fast is the diameter of the balloon increasing when the radius is 25 cm?

$V = \frac{4\pi}{3} r^3$, $\frac{dV}{dt} = 100 \text{ cm}^2/\text{s}$

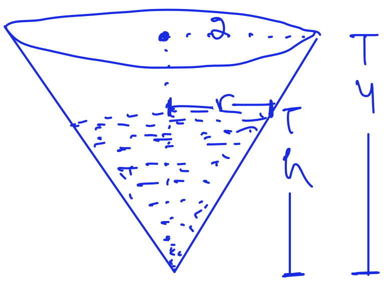
Diff. w.r.t. t $\Rightarrow \frac{dV}{dt} = \frac{4\pi}{3} \frac{d}{dt}(r^3) = \frac{4\pi}{3} \frac{d}{dr}(r^3) \frac{dr}{dt}$

$\Rightarrow 100 = \frac{4\pi}{3} \times 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$

when $r = 25$ cm, $100 = 4\pi(25)^2 \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{100}{4\pi \times 25 \times 25} \Rightarrow \boxed{\frac{dr}{dt} = \frac{1}{25\pi} \text{ cm/s}}$

Problem 2: A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.



$V = \frac{1}{3}\pi r^2 h$ (volume of water), $\frac{dV}{dt} = 2 \text{ m}^3/\text{min}$.

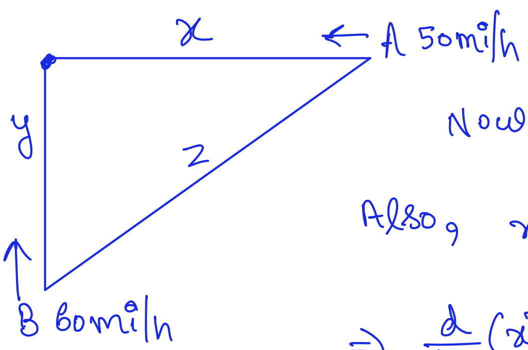
By similarity of triangles, $\frac{r}{2} = \frac{h}{4} \Rightarrow r = \frac{h}{2}$

$\Rightarrow V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{3}\frac{h^3}{4} = \frac{\pi h^3}{12}$

$\Rightarrow \frac{dV}{dt} = \frac{\pi}{12}\frac{d}{dt}(h^3) = \frac{\pi}{12} \times \frac{d}{dh}(h^3)\frac{dh}{dt} = \frac{\pi}{12} \times 3h^2 \frac{dh}{dt}$

$\Rightarrow 2 = \frac{\pi}{4} h^2 \frac{dh}{dt}$. when $h = 4$m, $2 = \frac{\pi}{4} \times 4^2 \frac{dh}{dt} \Rightarrow \boxed{\frac{dh}{dt} = \frac{1}{2\pi} \text{ m/min.}}$

Problem 3: Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 miles and car B is 0.4 miles from the intersection?



x, y, z are all decreasing $\Rightarrow \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are all $-$ve.

Now, $v_{\text{approach}} = \frac{dz}{dt}$, $\frac{dx}{dt} = -50$, $\frac{dy}{dt} = -60$

Also, $x^2 + y^2 = z^2$ $\Rightarrow \frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(z^2)$

$\Rightarrow \frac{d}{dx}(x^2)\frac{dx}{dt} + \frac{d}{dy}(y^2)\frac{dy}{dt} = \frac{d}{dz}(z^2)\frac{dz}{dt}$

$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$. when $x = 0.3$, $y = 0.4$

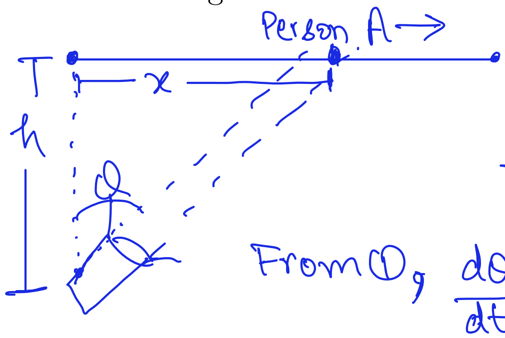
$z^2 = 0.3^2 + 0.4^2 = 0.25$

$\Rightarrow z = 0.5$

$\Rightarrow 2(0.3)(-50) + 2(0.4)(-60) = 2(0.5)\frac{dz}{dt}$

$\Rightarrow -30 - 48 = \frac{dz}{dt} \Rightarrow \boxed{\frac{dz}{dt} = -78 \text{ mi/h}}$

Problem 4: A person walks along straight (elevated) path at 4 ft/s. A searchlight is located on the ground 20 ft below the path and is kept focused on the person. At what rate is the searchlight rotating when the person is 15 ft from the point on the path closest to the searchlight?



$\tan\theta = \frac{x}{h} \Rightarrow x = h\tan\theta \Rightarrow x = 20\tan\theta$

$\frac{dx}{dt} = 4$ ft/s, $h = 20$ ft

$\frac{dx}{dt} = 20\frac{d}{dt}(\tan\theta)$

$\Rightarrow 4 = 20\frac{d}{d\theta}(\tan\theta)\frac{d\theta}{dt}$

$\Rightarrow 4 = 20\sec^2\theta\frac{d\theta}{dt}$ —— ①

From ①, $\frac{d\theta}{dt} = \frac{4}{20}\cos^2\theta$

when $x = 15$, $\cos\theta = \frac{h}{\sqrt{x^2+h^2}} = \frac{20}{\sqrt{20^2+15^2}} = \frac{20}{25} = \frac{4}{5}$

$\Rightarrow \boxed{\frac{d\theta}{dt} = \frac{1}{5}\times\frac{4^2}{5^2} = \frac{16}{125}}$

Problem 5: (Economics) Suppose $C(x)$ is the total cost that a company incurs in producing x units of a certain commodity. The function C it called the **cost function**. The instantaneous rate of change of cost with respect to the number of items produced, is called **marginal cost**, that is, marginal cost $= \frac{dC}{dx}$. If the cost function is given by $C(x) = 10,000 + 5x + 0.01x^2$, then what is the marginal cost at the production level of 500 items?

$C(x) = 10000 + 5x + 0.01x^2 \Rightarrow$ marginal cost $= \frac{dC}{dx} = 0 + 5 + 2(0.01)x$

At $x = 500$, marginal cost $= 5 + 0.02(500) = 5 + 10 = 15$

Thus, marginal cost (additional cost incurred to produce 500th item) = 15 dollars

Problem 6: (Biology) Considering the shape of a blood vessel to be a cylinder, the law of laminar flow states that the velocity of the blood is greatest at the central axis and decreases as the distance r from the central axis increases according to the rule $v(r) = k(R^2 - r^2)$, where k is some constant and R is the radius of the blood vessel. If the radius of a blood vessel is $R = 10\mu m$ and the velocity gradient ($\frac{dv}{dr}$) is $1\mu m/s$ at a distance of $r = 5\mu m$ from the central axis, find the velocity of blood at a distance of $9\mu m$ from the central axis.

$v(r) = k(R^2 - r^2)$ and $\frac{dv}{dr} = -1\,\mu m/s$ for $r = 5\,\mu m$

negative because $v(r)$ decreases as r increases

$\Rightarrow \frac{dv}{dr} = k(0 - 2r) = -2kr$. Thus, $-1 = -2k(5) \Rightarrow k = \frac{1}{10}$

$\Rightarrow v(r) = \frac{1}{10}(R^2 - r^2) = \frac{1}{10}(\underset{R=10}{10^2} - r^2) = 10 - \frac{r^2}{10}$

$\Rightarrow v(9) = 10 - \frac{9^2}{10} = 10 - \frac{81}{10} = 10 - 8.1 = 1.9$

$\Rightarrow$ At $9\,\mu m$ away from central axis, velocity of blood is $1.9\,\mu m/s$

Problem 7: (Chemistry) The rate of a chemical reaction of the form $aA + bB \to cC + dD$ is given by: Rate of reaction $= -\frac{1}{a}\frac{d[A]}{dt} = -\frac{1}{b}\frac{d[B]}{dt} = \frac{1}{c}\frac{d[C]}{dt} = \frac{1}{d}\frac{d[D]}{dt}$,
where [A] is the concentration of A in moles per liter and similarly of B, C, D.
Given the chemical reaction $2H_2 + O_2 \to 2H_2O$, suppose that the concentration of H_2O varies with time as $[H_2O](t) = 10 + 5t^{100}$ moles per liter. What is the rate of decrease of the concentrations of H_2 and O_2 in moles per liter per second at the time instant $t = 2s$.

$$-\frac{1}{2}\frac{d[H_2]}{dt} = -\frac{d[O_2]}{dt} = \frac{1}{2}\frac{d[H_2O]}{dt}. \qquad [H_2O](t) = 10+5t^{100}$$

$$\Rightarrow \frac{d[H_2O]}{dt} = \frac{d}{dt}(10+5t^{100}) = 0+5(100)t^{99} = 500t^{99}$$

$$\Rightarrow \frac{d[H_2]}{dt} = -(500)t^{99}, \quad \frac{d[O_2]}{dt} = -\frac{1}{2}(500)t^{99}.$$

Thus, at $t = 2s$,
$$\boxed{\frac{d[H_2]}{dt} = -500(2)^{99}, \quad \frac{d[O_2]}{dt} = -500(2)^{98}}$$

Problem 8: (Physics) A rod is placed along x-axis with its left end at the origin. Suppose that the mass of the rod varies along its length according to the rule $m(x) = \sqrt{kx+2} - 2$ for some constant $k > 0$. If the linear density $\rho = \frac{dm}{dx}$ is 0.5 kg/m at $x = 1$ m and the length of the rod is 17 m, then what is the mass in kilograms "at" the right end of the rod?

$$m(x) = \sqrt{kx+2} - 2 \quad (k>0)$$

$$\rho = \frac{dm}{dx} = \frac{d}{dx}\left(\sqrt{kx+2} - 2\right) = \frac{d}{dx}\left(\sqrt{kx+2}\right). \text{ Letting } z = kx+2 \Rightarrow \frac{dz}{dx} = k$$

$$= \frac{d}{dz}(\sqrt{z})\frac{dz}{dx} = \frac{1}{2\sqrt{z}} \times k = \frac{k}{2\sqrt{kx+2}}$$

It is given that $\rho = \frac{1}{2}$ at $x=1$

$$\Rightarrow \frac{k}{2\sqrt{k+2}} = \frac{1}{2} \Rightarrow k = \sqrt{k+2} \Rightarrow k^2 = k+2$$

$$\Rightarrow k^2 - k - 2 = 0 \Rightarrow (k-2)(k+1) = 0$$

$\Rightarrow k = 2$ or -1. But $k>0 \Rightarrow k = 2$

Thus, $m(17) = \sqrt{2(17)+2} - 2 = \sqrt{36} - 2 = \underline{\underline{4 \text{ kg}}}$.

Problem 9: (Thermodynamics) The isothermal compressibility of a fluid is defined to be $\beta = -\frac{1}{V}\frac{dV}{dP}$, where V is the volume of the fluid and P is the pressure of the fluid.
The volume (in cubic meters) of a sample of air at 25 degrees celsius was found to be related to pressure by the equation $V = \frac{5}{P}$. Find its isothermal compressibility at a pressure level of 10 kilo pascals.

$$\beta = \frac{-1}{V}\frac{dV}{dP}. \qquad \frac{dV}{dP} = 5\frac{d}{dP}\left(\frac{1}{P}\right) = 5\left(\frac{-1}{P^2}\right) = \frac{-5}{P^2}$$

$$\Rightarrow \beta = \frac{-1}{5/P} \times \frac{-5}{P^2} = \frac{-P}{5} \times \frac{-5}{P^2} = \frac{1}{P}$$

At $P = 10$ kilo Pa $= 10 \times 10^3$ Pa $= 10^4$ Pa, $\beta = \frac{1}{10^4}\ Pa^{-1}$

$$\Rightarrow \boxed{\beta = 10^{-4}\ Pa^{-1}}$$