■ **Section 6.5** exercises, *page 471*, #1, 3, 5(a)(b)(c), 8(a)(b)(c)(d), 9

## SUMMARY

- Solve the Population Growth problems
- Solve the Radioactive Decay problems.

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For instance, if $y = f(t)$ is the number of individual in population of animals or bacteria at time $t$, then we can expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$, i.e., $f'(t) = kf(t)$ for some constant $k$.

In general, if
$y(t)$ is *the value of a quantity* $y$ *at time* $t$ and
***the rate of change*** *of* $y$ *with respect to* $t$ is proportional to its size $y(t)$ at any time, then

$$\frac{dy}{dt} = ky, \qquad \text{where } k \text{ is a constant.} \tag{1}$$

This equation is called a ***differential equation*** because it involves an unknown function $y$ and its derivative $\frac{dy}{dt}$.

***Solving a differential equation*** means finding the solution, or the original function $y(t)$, such that equation 1 is satisfied.

**Theorem**: The only solutions of the differential equation 1 are the exponential functions

$$y(t) = Ce^{kt}, \qquad \text{where } C = y(0), \text{ the } \textit{initial value} \text{ of the function } y.$$

Or we can simply write: $y(t) = y(0)e^{kt}$.

Check that $y(t)$ satisfies $\frac{dy}{dt} = ky$

◇ **Population Growth**: In the context of population growth, where $P(t)$ is the size of the population at time $t$, we can write

$$\frac{dP}{dt} = kP \quad \text{or} \quad k = \frac{1}{P}\frac{dP}{dt}$$ which is the growth rate divided by the population size.

Therefore, the constant $k$ is called ***relative growth rate***.

***The expression of the population function*** is

$$\boxed{P(t) = P(0)e^{kt}, \quad \text{where } P(0) \text{ is the initial population.}}$$

*Example 1*: The common inhabitant of human intestines is the bacterium *Escherichia coli*, named after the German pediatrician Theodor Escherich, who identified it in 1885. A cell of this bacterium in a nutrient-broth medium divideds into two cells every 20 minutes. The initial population of a culture is 50 cells.

(a) Find the relative growth rate. → Find $k$

(b) Find an expression for the number of cells after $t$ hours. → Find $N(t)$

(c) Find the number of cells after 6 hours. → Find $N(6)$

(d) Find the rate of growth after 6 hours. → Find $\left.\frac{dN}{dt}\right|_{t=6}$

(e) When will the population reach a ~~minion~~ million cells?

→ Find $t$ for which $N(t) = 10^6$.

(a) $N(0) = 50$

$N(t) = N(0)e^{kt} \Rightarrow N(t) = 50e^{kt}$

After 20 minutes = 1/3 hour, the population would be $2 \times 50 = 100$

$\Rightarrow N\left(\frac{1}{3}\right) = 100$

$\Rightarrow 50e^{k/3} = 100 \Rightarrow e^{k/3} = \frac{100}{50}$

$\Rightarrow e^{k/3} = 2 \Rightarrow \ln e^{k/3} = \ln 2 \Rightarrow \frac{k}{3}\ln e = \ln 2 \Rightarrow \frac{k}{3} = \ln 2$

$\Rightarrow \boxed{k = 3\ln 2}$

(b) $N(t) = 50\, e^{(3\ln 2)t} = 50\, e^{(\ln 8)t} = 50\left(e^{\ln 8}\right)^t$

- $3\ln 2 = \ln 2^3 = \ln 8$
- $e^{\ln 8} = 8$

$\Rightarrow N(t) = 50\,(8)^t$

(c) Put $t = 6$ in $N(t)$

$N(6) = 50\,(8)^6$

(d) $\left.\dfrac{dN}{dt}\right|_{t=6}$. We know that $N(t) = 50\,(8)^t$

$$\Rightarrow \frac{dN}{dt} = \frac{d}{dt}\left(50\,(8)^t\right) = 50\,\frac{d}{dt}\left(8^t\right)$$

$$= 50\left(8^t\right)\ln 8$$

$$= (50\ln 8)\, 8^t$$

$$\left.\frac{dN}{dt}\right|_{t=6} = (50\ln 8)\, 8^6$$

(e) $N(t) = 10^6 \Rightarrow 50\,(8)^t = 10^6$

$$\Rightarrow (8)^t = \frac{10^6}{50}$$

$$\Rightarrow \ln 8^t = \ln \frac{10^6}{50} \Rightarrow t(\ln 8) = \ln\frac{10^6}{50}$$

$$\Rightarrow t(\ln 8) = \ln 10^6 - \ln 50 \Rightarrow t = \frac{\ln 10^6 - \ln 50}{\ln 8} \text{ hours.}$$

◇ **Radioactive Decay**: Radioactive substances decay by spontaneously emitting radiation. If $m(t)$ is the mass remaining from an initial mass $m(0)$ of the substance after time $t$, then

$$\frac{dm}{dt} = km \qquad \text{where } k \text{ is a } \textit{negative} \text{ constant.}$$

In other words, radioactive substances decay at a rate proportional to the remaining mass. This means ***the expression of the remaining mass*** $m$ ***after time*** $t$ is given by

$$\boxed{m(t) = m(0)e^{kt}}$$

Physicists express the rate of decay in terms of ***half-life***, the time required for ~~half of~~ any given quantity to decay. to half of its values.

*Example 2*: The half-life of radium-226 is 1590 years.

(a) A sample of radium-226 has a mass of 100mg. Find a formula for the mass of the sample that remains after $t$ year.

(b) Find the mass after 1000 years.

(c) When will the mass be reduced to 30 mg?

$$100\text{ mg} \xrightarrow{t_{1/2}} 50\text{ mg}$$

(a) $t_0 = 1590$ years

$m(0) = 100$ mg.

$$m(t) = m(0)\, e^{kt}$$
$$= 100\, e^{kt}$$

$$1590 = -\frac{\ln 2}{k}$$

$$\Rightarrow (1590)k = -\ln 2$$

$$\Rightarrow k = -\frac{\ln 2}{1590}$$

$$\Rightarrow m(t) = 100\, e^{-\frac{\ln 2}{1590} t}$$

$$m(t_0) = \frac{m(0)}{2}$$

$$\Rightarrow \cancel{m(0)}\, e^{kt_0} = \frac{\cancel{m(0)}}{2}$$

$$\Rightarrow e^{kt_0} = \frac{1}{2}$$

$$\Rightarrow \ln e^{kt_0} = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow kt_0 = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$\Rightarrow \boxed{t_0 = -\frac{\ln 2}{k}}$$

$t_0$ is the half-life.

$$\Rightarrow m(t) = 100\left(e^{\ln 2}\right)^{-t/1590} \qquad \cdot\ e^{\ln 2} = 2$$

$$m(t) = 100\,(2)^{-t/1590}$$

(b) Put $t = 1000 \Rightarrow m(1000) = 100\,(2)^{-\frac{1000}{1590}}$ mg.

(c) when is $m(t) = 30$

$$\Rightarrow 100\,(2)^{-t/1590} = 30 \Rightarrow 2^{-\frac{t}{1590}} = \frac{30}{100}$$

$$\Rightarrow \ln 2^{-t/1590} = \ln\left(\frac{30}{100}\right) = \ln\left(\frac{3}{10}\right) = \ln 3 - \ln 10$$

$$\Rightarrow \frac{-t}{1590}\ln 2 = \ln 3 - \ln 10 \Rightarrow t = \frac{(\ln 3 - \ln 10)1590}{-\ln 2}$$

$$\Rightarrow t = \frac{(\ln 10 - \ln 3)\,1590}{\ln 2} \text{ years.}$$