Indiana University - Purdue University, Indianapolis

Math16600

Practice Test (Chapter 6)

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Name:____ [2 pts]

Instructions:

- No cell phones, calculators, watches, technology, hats - stow all in your bags.
- **Write your name** on this cover page. It carries 2 points.
- This test is closed book and closed notes.
- All work must be clearly shown for partial credit.
- If you wish for something not to be graded, please strike it out neatly.
- Box, circle, or otherwise clearly indicate your final answer.
- When you finish, return your test to the proctor, and leave the classroom.
- There are a total of **16 problems** including bonus problem.
  - Problems 1-10 are each worth 6 points.
  - Problems 11-15 are each worth 8 points.
  - The bonus problem is worth 8 points.
- You can score a **maximum of 110 points out of 100**.
- There are a total of **9 pages** including the cover page.

Problem 1: Given a one-to-one function $f(x) = 1 + 4x + \sin x$, $-\infty < x < \infty$. Find $f^{-1}(1)$ and $(f^{-1})'(1)$. [6 pts]

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

$$f^{-1}(1) = x$$
$$\Rightarrow 1 = f(x)$$
$$\Rightarrow 1 = 1 + 4x + \sin x$$
$$\Rightarrow 4x + \sin x = 0$$
$$\Rightarrow x = 0$$

$$\Rightarrow f^{-1}(1) = 0$$

$$f(x) = 1 + 4x + \sin x$$
$$f'(x) = 4 + \cos x$$

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{4+\cos 0}$$

$$\Rightarrow (f^{-1})'(1) = \frac{1}{5}$$

Problem 2: Simplify the expression $\cot(\sin^{-1} x)$. [6 pts]

Let $\theta = \sin^{-1} x \Rightarrow \sin\theta = \frac{x}{1} = \frac{P}{H}$

Let $P = x$, $H = 1$

$$P^2 + B^2 = H^2 \Rightarrow x^2 + B^2 = 1 \Rightarrow B^2 = 1 - x^2$$
$$\Rightarrow B = \sqrt{1-x^2}$$

$$\Rightarrow \cot(\sin^{-1}x) = \cot\theta = \frac{B}{P} = \frac{\sqrt{1-x^2}}{x}$$

$$\Rightarrow \cot(\sin^{-1}x) = \frac{\sqrt{1-x^2}}{x}$$

Problem 3: Compute the derivative

$$y = (\ln x)^{\tanh^3(x)}$$

$[f(x)]^{g(x)} \to$ use log. diff.

[6 pts]

Step 1

$$\ln y = \ln\left[(\ln x)^{\tanh^3(x)}\right] \Rightarrow \ln y = \tanh^3(x)\ln(\ln x)$$

Step 2 (diff.)

$$\frac{1}{y}\frac{dy}{dx} = \left[\tanh^3(x)\right]'\ln(\ln x) + \tanh^3(x)\left[\ln(\ln x)\right]'$$

$$= 3\tanh^2(x)\operatorname{sech}^2(x)\ln(\ln x) + \tanh^3(x)\frac{1}{\ln x}\frac{1}{x}$$

Step 3

$$\Rightarrow \frac{dy}{dx} = (\ln x)^{\tanh^3(x)}\left[3\tanh^2(x)\operatorname{sech}^2(x)\ln(\ln x) + \frac{\tanh^3(x)}{x\ln x}\right]$$

Problem 4: Compute the derivative

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$H(t) = \frac{\ln(1+t^2)}{1+e^{t^4}}$$

[6 pts]

$$H'(t) = \frac{\left[\ln(1+t^2)\right]'(1+e^{t^4}) - \ln(1+t^2)\left[1+e^{t^4}\right]'}{(1+e^{t^4})^2}$$

$$\left[\ln(1+t^2)\right]' = \left(\frac{1}{1+t^2}\right)2t = \frac{2t}{1+t^2}$$

$$\left[1+e^{t^4}\right]' = e^{t^4}(4t^3) = 4t^3e^{t^4}$$

$$\Rightarrow H'(t) = \frac{\left(\frac{2t}{1+t^2}\right)(1+e^{t^4}) - \ln(1+t^2)(4t^3e^{t^4})}{(1+e^{t^4})^2}$$

Problem 5: Compute the derivative

$$f(x) = \ln\left(x\, e^{-2x}\right)$$

$\boxed{\ln e = 1}$

[6 pts]

$\Rightarrow f(x) = \ln(x) + \ln(e^{-2x})$

$= \ln x + (-2x)\ln e$

$\Rightarrow f(x) = \ln x - 2x$

$\Rightarrow f'(x) = \frac{1}{x} - 2$

Chain Rule

$\frac{1}{x e^{-2x}} \frac{d}{dx}\left(x e^{-2x}\right)$

$= \frac{e^{-2x} + x(-2)e^{-2x}}{x e^{-2x}}$

$= \frac{\cancel{e^{-2x}}[1-2x]}{x\cancel{e^{-2x}}} = \frac{1-2x}{x}$

$= \frac{1}{x} - 2$

Problem 6: Compute the derivative

$$g(x) = \tan^{-1}\left(\frac{1}{x}\right)\ln(3x-1)$$

[6 pts]

$g'(x) = \left[\tan^{-1}\left(\frac{1}{x}\right)\right]' \ln(3x-1) + \tan^{-1}\left(\frac{1}{x}\right)\left[\ln(3x-1)\right]'$

$\left[\tan^{-1}\left(\frac{1}{x}\right)\right]' = \frac{1}{1+\left(\frac{1}{x}\right)^2}\left(\frac{-1}{x^2}\right)$

$= \frac{\cancel{x^2}}{x^2+1}\left(\frac{-1}{\cancel{x^2}}\right) = \frac{-1}{x^2+1}$

$\frac{d}{dx}\left(\tan^{-1} z\right) = \frac{d}{dz}\left(\tan^{-1} z\right)\frac{dz}{dx}$

$\qquad\qquad\qquad\qquad\;\; \frac{1}{1+z^2}\frac{dz}{dx}$

where $z = \frac{1}{x}$

$\Rightarrow \frac{dz}{dx} = \frac{-1}{x^2}$

$\left[\ln(3x-1)\right]' = \frac{3}{3x-1}$

$g'(x) = \frac{-\ln(3x-1)}{x^2+1} + \frac{3\tan^{-1}\left(\frac{1}{x}\right)}{3x-1}$

Problem 7: Evaluate the integral

$$\int \frac{x+1}{x^2+2x}\,dx \quad = \int \underbrace{\frac{1}{x^2+2x}}_{\frac{1}{u}} \underbrace{(x+1)\,dx}_{\frac{1}{2}du}$$

[6 pts]

Let $u = x^2 + 2x$

$\Rightarrow \frac{du}{dx} = 2x+2 = 2(x+1)$

$\Rightarrow du = 2(x+1)\,dx$

$\Rightarrow \frac{1}{2}du = (x+1)\,dx$

$= \int \frac{1}{u}\,\frac{1}{2}\,du$

$= \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|x^2+2x| + C$

Problem 8: A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420. Find the number of bacteria after 4 hours. [6 pts]

$N(0) = 100$, $N(1) = 420$. Find $N(4)$

$N(t) = N(0)\,e^{kt} \Rightarrow N(t) = 100e^{kt}$

Put $t=1 \Rightarrow N(1) = 100\,e^{k}$

$\Rightarrow 420 = 100\,e^{k}$

$\Rightarrow e^{k} = 4.2$

$\Rightarrow N(t) = 100\,(e^{k})^{t}$

$= 100\,(4.2)^{t}$

$N(4) = 100\,(4.2)^{4}$

Problem 9: Compute the limit

$$\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$$

[6 pts]

D.S.: $\infty \sin\left(\frac{\pi}{\infty}\right) = \infty \sin(0) = \infty \cdot 0$

$$L = \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}} \overset{D.S.}{=} \frac{\sin\left(\frac{\pi}{\infty}\right)}{\frac{1}{\infty}} = \frac{\sin 0}{0} = \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right) \pi \cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}}$$

$$\left[\sin\left(\frac{\pi}{x}\right)\right]' = \cos\left(\frac{\pi}{x}\right) \frac{d}{dx}\left(\frac{\pi}{x}\right)$$

$$= \lim_{x\to\infty} \pi \cos\left(\frac{\pi}{x}\right) \overset{D.S.}{=} \pi \cos\left(\frac{\pi}{\infty}\right) = \pi \cos 0 = \pi$$

Problem 10: Compute the limit

$$\lim_{x\to 0} \frac{x^2}{1-\cos x} \overset{D.S.}{=} \frac{0^2}{1-\cos 0} = \frac{0}{1-1} = \frac{0}{0}$$

[6pts]

$$= \lim_{x\to 0} \frac{(x^2)'}{(1-\cos x)'} = \lim_{x\to 0} \frac{2x}{\sin x} \overset{D.S.}{=} \frac{2(0)}{\sin 0} = \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{2}{\cos x} \overset{D.S.}{=} \frac{2}{\cos 0} = \frac{2}{1} = 2$$

Problem 11: Use logarithmic differentiation to compute $\dfrac{dy}{dx}$ where

Step1: Take ln

$$y = \frac{x\sqrt[4]{x^4+4}}{x^2-2x}$$

$$\ln y = \ln\left[\frac{x\sqrt[4]{x^4+4}}{x^2-2x}\right]$$

[8 pts]

Step2: Simplify: $\ln y = \ln\left(x\sqrt[4]{x^4+4}\right) - \ln\left(x^2-2x\right)$

$= \ln x + \ln\left(\sqrt[4]{x^4+4}\right) - \ln\left(x(x-2)\right)$

$\Rightarrow \ln y = \cancel{\ln x} + \frac{1}{4}\ln(x^4+4) - \cancel{\ln x} - \ln(x-2)$

$\Rightarrow \ln y = \frac{1}{4}\ln(x^4+4) - \ln(x-2)$

Step3: Diff.: $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{\cancel{4}}\dfrac{\cancel{4}x^3}{x^4+4} - \dfrac{1}{x-2} \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{x^3}{x^4+4} - \dfrac{1}{x-2}$

$\Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{-2(x^3+2)}{(x^4+4)(x-2)} \Rightarrow \boxed{\dfrac{dy}{dx} = \dfrac{x\sqrt[4]{x^4+4}}{(x^2-2x)}\left[\dfrac{-2(x^3+2)}{(x^4+4)(x-2)}\right]}$

$= \dfrac{\cancel{x^4} - 2x^3 - \cancel{x^4} - 4}{(x^4+4)(x-2)} = \dfrac{-2(x^3+2)}{(x^4+4)(x-2)}$

Problem 12: Compute the limit

$$L = \lim_{x\to\infty}(x^2+2x)^{1/x^3}$$

[8 pts]

D.S. $\left[\infty^2 + 2(\infty)\right]^{\frac{1}{\infty^3}} = (\infty)^{\frac{1}{\infty}} = (\infty)^0$ (Indeterminate)

$\ln L = \lim\limits_{x\to\infty} \ln(x^2+2x)^{1/x^3} = \lim\limits_{x\to\infty} \dfrac{1}{x^3}\ln(x^2+2x)$

$= \lim\limits_{x\to\infty} \dfrac{\ln(x^2+2x)}{x^3} \overset{\text{D.S.}}{=} \dfrac{\ln(\infty^2+2(\infty))}{\infty^3} = \dfrac{\ln\infty}{\infty} = \dfrac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \dfrac{\left(\frac{1}{x^2+2x}\right)(2x+2)}{3x^2} = \lim\limits_{x\to\infty} \dfrac{2(x+1)}{3x^2(x^2+2x)} \overset{\text{D.S.}}{=} \dfrac{\infty}{\infty}$

$= \lim\limits_{x\to\infty} \dfrac{2}{[3x^4+6x^3]'} = \lim\limits_{x\to\infty} \dfrac{2}{12x^3+18x^2} \overset{\text{D.S.}}{=} \dfrac{2}{\infty} = 0$

$\Rightarrow \ln L = 0 \Rightarrow L = e^0 \Rightarrow L = 1$

Problem 13: Evaluate the integral

$$\int_0^1 \frac{x^5}{\sqrt{1-x^6}}\,dx = \int_0^1 \frac{1}{\sqrt{1-x^6}}\,\underbrace{x^5\,dx}$$

[8 pts]

$u = 1 - x^6$

$\Rightarrow \frac{du}{dx} = -6x^5$

$\Rightarrow du = -6x^5\,dx$

$\Rightarrow \underbrace{\frac{-1}{6}\,du} = x^5\,dx$

$$= \int_{1-0^6}^{1-1^6} \frac{1}{\sqrt{u}}\,\frac{-1}{6}\,du = \frac{-1}{6}\int_1^0 \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{6}\int_0^1 u^{-1/2}\,du = \frac{1}{6}\left.\frac{u^{-1/2+1}}{-1/2+1}\right|_0^1$$

$$= \frac{2}{6}\,u^{1/2}\Big|_0^1 = \frac{1}{3}\left[1^{1/2} - 0^{1/2}\right] = \frac{1}{3}$$

Problem 14: Evaluate the integral

$$\int \left(\frac{1-x}{x}\right)^2 dx$$

[8 pts]

$$\int \frac{(1-x)^2}{x^2}\,dx = \int \frac{1+x^2-2x}{x^2}\,dx$$

$$= \int \left(\frac{1}{x^2} + \frac{x^2}{x^2} - \frac{2x}{x^2}\right)dx = \int\left(\frac{1}{x^2} + 1 - \frac{2}{x}\right)dx$$

$$= \int \frac{1}{x^2}\,dx + \int 1\,dx - 2\int \frac{1}{x}\,dx$$

$$= \frac{x^{-2+1}}{-2+1} + x - 2\ln|x| + C$$

$$= \frac{-1}{x} + x - 2\ln|x| + C$$

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u)$$

Problem 15: Evaluate the integral

$$\int \frac{x^2}{\sqrt{1-x^6}}\,dx$$

[8 pts]

Substitute $u = x^3$

$\Rightarrow du = 3x^2\,dx$

$\Rightarrow \frac{1}{3}\,du = x^2\,dx$

$$= \int \frac{1}{\sqrt{1-x^6}}\,x^2\,dx$$

$$= \int \frac{1}{\sqrt{1-u^2}}\,\frac{1}{3}\,du = \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{3}\sin^{-1}(u) + C$$

$$= \frac{1}{3}\sin^{-1}(x^3) + C$$

Bonus Problem: Evaluate the integral

$$\int e^{7x}\,(4-e^{7x})^7\,dx$$

[8 pts].

Substitute $u = 4 - e^{7x} \Rightarrow \frac{du}{dx} = -7e^{7x}$

$\Rightarrow du = -7e^{7x}\,dx$

$\Rightarrow \frac{-1}{7}\,du = e^{7x}\,dx$

$$\int (4-e^{7x})^7\,\underbrace{e^{7x}\,dx}_{\frac{-1}{7}du} = \int u^7\left(\frac{-1}{7}\,du\right) = \frac{-1}{7}\int u^7\,du$$

$$= \frac{-1}{7}\,\frac{u^8}{8} + C$$

$$= \frac{-1}{56}\,(4-e^{7x})^8 + C$$