Learning objectives:

1. Learn the chain rule analog for integration: called the substitution rule.
2. Apply the substitution rule to evaluate integrals.

If $F' = f$ then by chain rule $[F(g(x))]' = F'(g(x))\,g'(x) = f(g(x))\,g'(x)$.

Letting $u = g(x)$ we get that the antiderivative of $f(g(x))\,g'(x)$ is given by $F(u)$, which is the antiderivative of $f(u)$.

The substitution rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du\ .$$

Example 1. Evaluate the integral $\int 2x\,\sqrt{x^2+1}\,dx$.

Example 2. Evaluate $\int x^2 \cos(x^4 + 2)\,dx$.

Example 3. Evaluate $\int \sqrt{2x + 1}\,dx$.

Example 4. Evaluate $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Example 5. Evaluate $\int \cos 5x \; dx$.

Example 6. Evaluate $\int \sqrt{1+x^2}\, x^5\, dx$.

Example 7. Evaluate $\int \sqrt{\cot x}\, \csc^2 x\, dx$.

The substitution rule for definite integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du\ .$$

Example 8. Evaluate $\int_0^1 \cos(\pi t/2)\, dt$.

Example 9. Evaluate $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$.

Symmetry

Let f be continuous on $[-a, a]$.

1. If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.
2. If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Example 10. Evaluate the following integrals.

1. $\int_{-2}^{2} (x^6 + 1)\, dx$.

2. $\displaystyle\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\, dx$.