Learning objectives:

1. Understand definition of continuity at a point.
2. Continuous functions on an interval.
3. Examples of continuous functions.
4. Continuity and composition of functions.
5. The intermediate value theorem and its applications.

Continuity at a point.

A function f is continuous at a number a if f is defined at a and

$$\lim_{x\to a} f(x) = f(a) \ .$$

If f is not continuous at a, then we say f is discontinuous at a.

Graphs of continuous functions.

If f is continuous at a then its graph cannot have a break at a.

Example 1.

Show that the following functions are discontinuous at the given point.

1. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ at $x = 2$.

2. $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0\,, \\ 1 & \text{if } x = 0\,, \end{cases}$ at $x = 0$.

3. $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2\,, \\ 1 & \text{if } x = 2\,, \end{cases}$ at $x = 2$.

4. $f(x) = \begin{cases} \dfrac{|x|}{x} & \text{if } x \neq 0 \,, \\ 0 & \text{if } x = 0 \,, \end{cases}$ at $x = 0$.

Types of discontinuities

A discontinuity of f at a is called:

1. removable discontinuity if it can be removed by redefining f at $x = a$,
2. infinite discontinuity if the function takes an infinite (left hand and/or right hand) limit at $x = a$,
3. jump discontinuity if both the left hand and right limits of the function at $x = a$ are finite but unequal.

Continuous from the right and from the left

A function f is said to be continuous from the right at the number a if

$$\lim_{x \to a^+} f(x) = f(a) \,,$$

and f is said to be continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a) \,.$$

Continuous on an interval

A function f is said to be continuous on an open interval (a, b) if it is continuous at every number in (a, b).

A function f is said to be continuous on a closed interval $[a, b]$ if it is continuous on (a, b), right continuous at a and left continuous at b.

Continuity on half-open intervals is defined similarly.

Example 2. Let $f(x) = \begin{cases} \dfrac{|x-1|}{x-1} & \text{if } x \neq 1 \,, \\ 1 & \text{if } x = 1 \,. \end{cases}$

Is f is continuous on the following intervals?

1. $[1, \infty)$.
2. $[0, 1]$.

Combinations of continuous functions

If f and g are continuous at a and suppose c is a constant real number then the following functions are also continuous at a:

1. $f + g$,
2. $f - g$,
3. cf ,
4. fg ,
5. $\dfrac{f}{g}$, if $g(a) \neq 0$.

Examples of continuous functions

1. Polynomials are continuous everywhere, that is at every real number.
2. Rational functions are continuous in their domains.
3. Root functions are continuous in their domains.
4. Trigionometric functions are continuous in their domains. In particular, the sine and cosine functions are continuous everywhere.

Example 3. On what intervals are the following functions continuous?

1. $f(x) = x^{1000} - 2x^{357} + 750$.
2. $g(x) = \dfrac{x^2 + x + 17}{x^2 - 1}$.
3. $h(x) = \sqrt{x} + \dfrac{x+1}{x-1} - \dfrac{x+1}{x-1}$.

Example 4. Evaluate $\lim\limits_{x \to \pi} \dfrac{\sin x}{\cos x + 2}$.

Composition of continuous functions

If f is continuous at b and $\lim\limits_{x \to a} g(x) = b$, then $\lim\limits_{x \to a} f(g(x)) = f(b)$.

In other words, if f is continuous at $\lim\limits_{x \to a} g(x)$, then

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) \ .$$

A consequence of the above statement is that if g is continuous at a and f is continuous at $g(a)$ then the composite function $f \circ g$ is continuous at a.

Example 5.

Where are the following functions continuous ?

1. $f(x) = \sin(x^2)$.
2. $g(x) = \dfrac{1}{\sqrt{x^2 + 7} - 4}$.

The intermediate value theorem.

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Example 6.

Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.