■ **Section 7.7** exercise: see the two bullets below

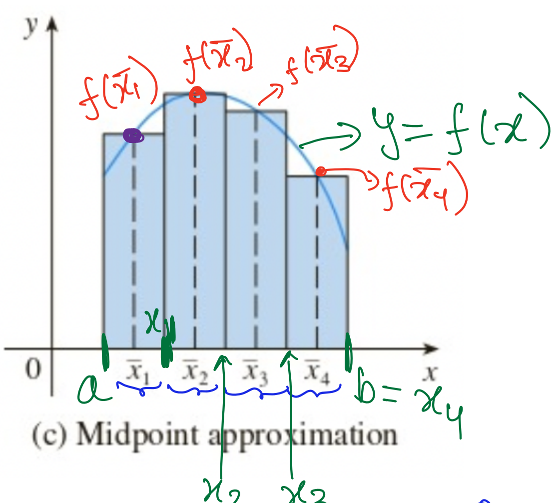
- Use (a) the Trapezoidal Rule, (b) The Midpoint Rule, and (c) Simpson's Rule to approximate the given integral with the specified value of $n$. (Round your answers to six decimal places.)

$$\int_1^3 \sqrt{x^3 - 1}\, dx, \qquad n = 6$$

- *Optional exercises*: section 7.7, page 654, # 5, 16.

There are situations where it is difficult, or even impossible, to compute $\int_a^b f(x)\,dx$. Other times, when a function is determined from a scientific experiment through instrument readings or collect data, there may be no formula for the function. Therefore, it is needful to have methods for approximating definite integrals.

⋆ ***The Midpoint Rule*** *for approximating definite integrals*



(c) Midpoint approximation

Recall: $\int_a^b f(x)\,dx$ = Area under the graph of $f$

$[a, b] \Rightarrow$ divide this interval into $n$ subintervals of $\underbrace{\text{equal width}}_{\Delta x}$

$$n(\Delta x) = b - a \Rightarrow \Delta x = \frac{b-a}{n}$$

Area of the $i_{th}$ rectangle $= f(\bar{x}_i)\,\Delta x$

$$\int_a^b f(x)\,dx \approx f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + \cdots + f(\bar{x}_n)\Delta x$$

$$= \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right]$$

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2} = \frac{a + (i-1)\Delta x + a + i(\Delta x)}{2},$$

$x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2(\Delta x)$, ..., $x_i = a + i(\Delta x)$

$A(\text{Trapezoid}) = \frac{1}{2}\,\text{width}\,(\text{Sum of lengths of Parallel sides})$

★ ***The Trapezoidal Rule*** *for approximating definite integrals*

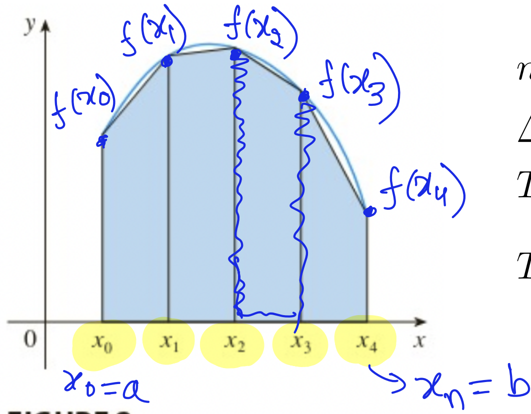



**FIGURE 2**
Trapezoidal approximation

$n =$ the number of subintervals or the number of trapezoids

$\Delta x = \dfrac{b-a}{n} =$ the height of each trapezoid

$T_n =$ the area of $n$ trapezoids

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$$\int_a^b f(x)\,dx \approx T_n$$

$\text{Area}(i_{th}\ \text{trapezoid}) = \frac{1}{2}\Delta x\left(f(x_{i-1}) + f(x_i)\right)$

$T_n = \frac{1}{2}\Delta x\left[f(x_0) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + \cdots + f(x_{n-1}) + f(x_n)\right]$

*Example 1*: Use (a) the Midpoint Rule and (b) the Trapezoidal Rule with $n = 4$ to approximate the integral $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

(a) $\displaystyle\int_1^2 \frac{1}{x}\,dx$, $n = 4$, $a = 1$, $b = 2$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

$x_0 = a = 1$, $x_1 = a + \Delta x = 1 + \frac{1}{4} = \frac{5}{4}$

$x_2 = a + 2(\Delta x) = 1 + 2\left(\frac{1}{4}\right) = 1 + \frac{1}{2} = \frac{3}{2}$

$x_3 = a + 3(\Delta x) = 1 + 3\left(\frac{1}{4}\right) = 1 + \frac{3}{4} = \frac{7}{4}$

$x_4 = a + 4(\Delta x) = 1 + 4\left(\frac{1}{4}\right) = 1 + 1 = 2$

$\bar{x}_1 = \dfrac{x_0 + x_1}{2} = \dfrac{1 + \frac{5}{4}}{2} = \dfrac{9/4}{2} = \dfrac{9}{8}$ ✓

$\bar{x}_2 = \dfrac{x_1 + x_2}{2} = \dfrac{5/4 + 3/2}{2} = \dfrac{11/4}{2} = \dfrac{11}{8}$ ✓

$$\overline{x}_3 = \frac{x_2+x_3}{2} = \frac{\frac{3}{2}+\frac{7}{4}}{2} = \frac{13/4}{2} = \frac{13}{8} \checkmark$$

$$\overline{x}_4 = \frac{x_3+x_4}{2} = \frac{\frac{7}{4}+2}{2} = \frac{15/4}{2} = \frac{15}{8} \checkmark$$

$$\int_a^b f(x)\,dx = \Delta x\left[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_n)\right]$$

$$\int_1^2 \frac{1}{x}\,dx = \frac{1}{4}\left[f\left(\frac{9}{8}\right) + f\left(\frac{11}{8}\right) + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right)\right]$$

$$= \frac{1}{4}\left[\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right]$$

$$= \checkmark$$

(b) $T_4 = \frac{1}{2}\Delta x\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$

$\Delta x = \frac{1}{4}$, $x_0 = 1$, $x_1 = \frac{5}{4}$, $x_2 = \frac{3}{2}$, $x_3 = \frac{7}{4}$, $x_4 = 2$

$f(x) = \frac{1}{x}$ $\Rightarrow T_4 = \frac{1}{2}\,\frac{1}{4}\left[\frac{1}{1} + 2\,\frac{4}{5} + 2\,\frac{2}{3} + 2\,\frac{4}{7} + \frac{1}{2}\right]$

$= \checkmark \qquad \simeq 0.697$

$$\int_1^2 \frac{1}{x}\,dx = \ln|x|\Big|_1^2 = \ln 2 - \ln 1 = \ln 2$$

⋆ ***The Simpson's Rule*** *for approximating definite integrals*

Another rule for approximate integration results from using parabolas instead of straight line segments to approximate the curve.

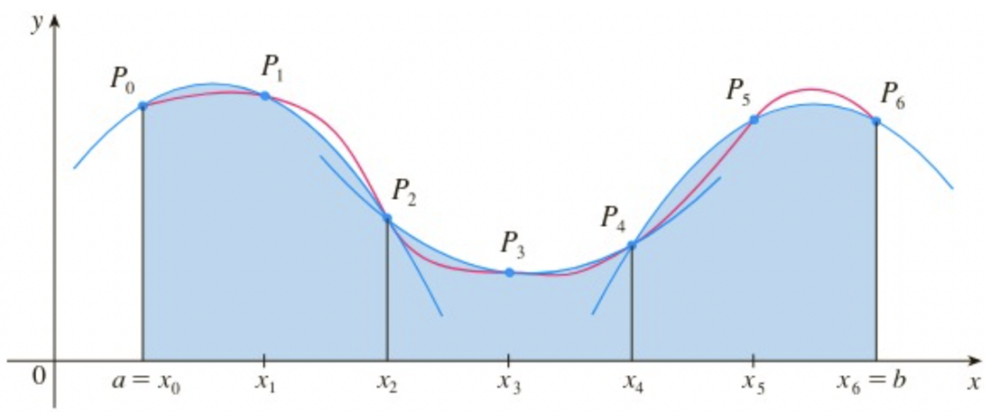



**FIGURE 7**

$n =$ the number of subintervals. $n$ must be **even** for Simpson's Rule.

$\Delta x = \dfrac{b-a}{n} =$ the length of each subinterval

$S_n =$ the area under the parabolas by using Simpson's Rule

$S_n = \dfrac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-3}) + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$

Note the pattern of coefficients: 1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 4, 1.

$$\int_a^b f(x)\, dx \approx S_n$$

You can read the discussion on page 559–560 of the textbook to see how the formula for $S_n$ is derived.

*Example 2*: Use Simpson's rule with $n = 8$ to approximate $\int_0^1 e^{x^2}\,dx$.

$$S_8 = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8)\right]$$

$b = 1$, $a = 0$, $\Delta x = \frac{b-a}{n} = \frac{1-0}{8} = \frac{1}{8}$

$x_0 = a = 0$, $x_1 = \frac{1}{8}$, $x_2 = \frac{2}{8}$, $x_3 = \frac{3}{8}$, $x_4 = \frac{4}{8}$, $x_5 = \frac{5}{8}$,

$x_6 = \frac{6}{8}$, $x_7 = \frac{7}{8}$, $x_8 = \frac{8}{8} = 1 = b$

$$S_8 = \frac{1}{24}\left[e^{0^2} + 4e^{1/64} + 2e^{4/64} + 4e^{9/64} + 2e^{16/64} + 4e^{25/64} + 2e^{36/64} + 4e^{\frac{49}{64}} + e^{1^2}\right]$$

$$\uparrow$$

$1$