■ **Section 11.4** textbook exercises, *page 771*: #3, 5, 7, 10, 12, 13, 15, 19, 23, 25, 29.

In the comparison tests the idea is to compare the given series with a series that is known to be convergent or divergent.

For instance, say we would like to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is convergent or divergent. Note that the Test of Divergence fails for this series and this is not a geometric series or a p-series.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n+1} = \frac{\text{slower}}{\text{faster}} = 0$$

On the other hand, the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ reminds us of the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ with $r = \frac{1}{2}$; hence, the latter series is convergent. We can do the following comparison between these two series to determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.

Compare $\frac{1}{2^n+1}$ and $\frac{1}{2^n}$

$$\frac{2^n+1}{1} > \frac{2^n}{1} \Rightarrow \frac{1}{2^n+1} < \frac{1}{2^n}$$

$$\left(x > y \Rightarrow \frac{1}{x} < \frac{1}{y} \right)$$

$$\Rightarrow 0 \le \sum_{n=1}^{\infty} \frac{1}{2^n+1} \le \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$r = \frac{a_{n+1}}{a_n} = \frac{1}{2^{n+1}} 2^n = \frac{1}{2}$$

$\Rightarrow |r| = \frac{1}{2} < 1 \Rightarrow$ series on the right is convergent.

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n+1}$ is finite

$\Rightarrow$ the given series is convergent

The Comparison Test. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for large enough n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for large enough n, then $\sum a_n$ is also divergent.

Remark: The Comparison Test is useful when testing series with sine or cosine functions.

Example 1: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{7^n}$ converges of diverges.

$-1 \le \sin(n) \le 1$

$1-1 \le 1+\sin(n) \le 1+1$

$0 \le 1+\sin(n) \le 2$

for every natural number n.

$$\Rightarrow \frac{0}{7^n} \le \frac{1+\sin(n)}{7^n} \le \frac{2}{7^n}$$

$$0 \le \frac{1+\sin(n)}{7^n} \le \frac{2}{7^n}$$

$$\Rightarrow 0 \le \sum_{n=1}^{\infty} \frac{1+\sin(n)}{7^n} \le \sum_{n=1}^{\infty} \frac{2}{7^n}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{7^n} \Longrightarrow$ converges.

$\Rightarrow |r| = \frac{1}{7} < 1$

$$r = \frac{a_{n+1}}{a_n} = \frac{2}{7^{n+1}} \frac{7^n}{2} = \frac{1}{7}$$

By the CT, $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin(n)}{7^n}$ is also convergent.

Question: Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 3}$ convergent or divergent?

$2^n - 3 < 2^n$

$$\frac{1}{2^n-3} > \frac{1}{2^n}$$

$$\Rightarrow \underbrace{\sum_{n=1}^{\infty} \frac{1}{2^n-3}}_{\text{Cannot say.}} \ge \sum_{n=1}^{\infty} \frac{1}{2^n}$$

finite

geometric series with $r = \frac{1}{2}$

	2^n-3		2^{n-1}
$n=1$	$2^1-3=-1$	$>$	$2^0 = 1$
$n=2$	$2^2-3=1$	$>$	$2^1=2$
$n=3$	$2^3-3=5$	$>$	$2^2=4$
$n=4$	$2^4-3=13$	$>$	$2^3=8$
	$\vdots$		

$\Rightarrow 2^n - 3 > 2^{n-1}$ for $n \ge 3$

$\Rightarrow \dfrac{1}{2^n-3} < \dfrac{1}{2^{n-1}}$ for $n \ge 3$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-3}$ converges because $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges

The ***Limit Comparison Test*** helps us to determine the convergence or divergence of a series that is "similar" to a series which we're familiar with.

DEFINITION OF SIMILARITY BETWEEN TWO SERIES. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \text{ a positive number },$$

then we say $\sum a_n$ and $\sum b_n$ are ***similar to*** each other.

THE LIMIT COMPARISON TEST: Suppose $\sum a_n$ and $\sum b_n$ are *similar series* with positive terms. Then **either** both series are convergent **or** both series are divergent.

In other words, similar series behave the same way regarding convergence or divergence.

Example 2: Show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+4}$ is similar to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$. Then use the Limit Comparison Test to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+4}$ is convergent or divergent.

$a_n = \dfrac{1}{\sqrt{n}+4}$, $b_n = \dfrac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}+4} \cdot \sqrt{n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+4}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} = 1 \quad (\text{Positive number} \neq 0, \neq \infty)$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+4} \underset{(\text{similar})}{\sim} \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$$

divergent because its a p-series, $p = \frac{1}{2}$

$\Rightarrow$ By the LCT, $\displaystyle\sum_{n=2} \frac{1}{\sqrt{n}+4}$ is also divergent.

Remark: The Limit Comparison Test is very useful when working with series that remind us of geometric series or p-series.

Remark: To determine similar series, often we can drop the slower terms of the numerator and of the denominator then use algebra to simplify.

Example 3: Find the similar series of the given series then test for convergence and divergence.

(a) $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$ $\backsim$ $\sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2} \Longrightarrow$ converges.

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2+n+1}{n^4+n^2} \cdot n^2 = \lim_{n\to\infty} \frac{n^4+n^3+n^2}{n^4+n^2}$

$= \lim_{n\to\infty} \frac{n^4}{n^4} = 1$ $\Rightarrow$ the two series are similar

$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$ converges.

(b) $\sum_{n=1}^{\infty} \frac{6^n+n}{5^n-2}$ $\backsim$ $\sum_{n=1}^{\infty} \frac{6^n}{5^n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6^n+n}{5^n-2} \cdot \frac{5^n}{6^n} = \lim_{n\to\infty} \frac{\cancel{6^n}}{\cancel{5^n}} \cdot \frac{\cancel{5^n}}{\cancel{6^n}} = 1$

$\Rightarrow$ the two series are similar.

$\sum_{n=1}^{\infty} \frac{6^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ $\Rightarrow r = \frac{b_{n+1}}{b_n} = \frac{\left(\frac{6}{5}\right)^{n+1}}{\left(\frac{6}{5}\right)^n} = \frac{6}{5}$

$|r| = \frac{6}{5} > 1$ $\Rightarrow$ geometric series with common ration greater than 1

$\Rightarrow$ diverges.

$\Rightarrow \sum_{n=1}^{\infty} \frac{6^n+n}{5^n-2}$ also diverges.

Example 4: Determine whether the series converges of diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\sqrt{n+9}}$ $\backsim$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $\Rightarrow \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{5}{\sqrt{n+9}}\sqrt{n} = 5$

$\hookrightarrow$ P- series with $P = \frac{1}{2}$

$\Rightarrow$ diverges.

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{5}{\sqrt{n+9}}$ also diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n(11+n)^{12}}{(8+3n)(1+n^2)^3}$ $\backsim$ $\displaystyle\sum_{n=1}^{\infty} \frac{2n\,(n)^{12}}{(3n)\,(n^2)^3} = \sum_{n=1}^{\infty} \frac{2n^{13}}{3\,n^{1+6}} = \frac{2}{3}\sum_{n=1}^{\infty} n^6$

$\frac{2}{3}\displaystyle\sum_{n=1}^{\infty} n^6 = \frac{2}{3}\sum_{n=1}^{\infty} \frac{1}{n^{-6}}$ $\rightarrow$ P-series with $P = -6 < 1$

$\Rightarrow$ diverges.

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{2n(11+n)^{12}}{(8+3n)(1+n^2)^3}$ also diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{e^n+3}$ $\quad -1 \le \cos(n) \le 1 \Rightarrow 0 \le \cos^2(n) \le 1$

$$\Rightarrow \underbrace{\frac{0}{e^n+3}}_{=\,0} \le \frac{\cos^2(n)}{e^n+3} \le \frac{1}{e^n+3}.$$

$$\Rightarrow 0 \le \sum_{n=1}^{\infty} \frac{\cos^2(n)}{e^n+3} \le \sum_{n=1}^{\infty} \frac{1}{e^n+3} \underset{\text{similar}}{\backsim} \sum_{n=1}^{\infty} \frac{1}{e^n}$$

$r = \frac{a_{n+1}}{a_n} = \frac{1}{e^{n+1}} \cdot e^n = \frac{1}{e}$

geometric series with $|r| = \frac{1}{e} < 1$ $\Rightarrow$ converges.

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n+3}$ converges (LCT) $\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{e^n+3}$ converges (CT)

(d) $\sum_{n=1}^{\infty} \frac{n}{e^n}$ $\Rightarrow \lim_{n\to\infty} \frac{n}{e^n} = \frac{\text{slower}}{\text{faster}} = 0$

replace n with x

$$f(x) = \frac{x}{e^x} = x e^{-x}$$

For $[1, \infty)$, f is continuous and positive.

Is f ultimately decreasing?

$$f'(x) = e^{-x} + x(-e^{-x}) = e^{-x}(1-x)$$

$$f'(x) < 0 \Rightarrow \underbrace{e^{-x}}_{\text{Positive}}(1-x) < 0 \Rightarrow (1-x) < 0 \Rightarrow x > 1$$

$\Rightarrow$ Yes, f is decreasing on $(1, \infty)$

$$\int_1^{\infty} x e^{-x}\, dx = \lim_{t\to\infty} \int_1^t x e^{-x}\, dx$$

$$\int x e^{-x} = -x e^{-x} - \int (-e^{-x})\, dx = -x e^{-x} + \int e^{-x}\, dx$$
$$= -x e^{-x} - e^{-x} + C$$

$$\Rightarrow \lim_{t\to\infty} \left[-t e^{-t} - e^{-t} - (-e^{-1} - e^{-1}) \right]$$

$$= \lim_{t\to\infty} -t e^{-t} - \lim_{t\to\infty} e^{-t} + 2e^{-1}$$

$$= -\lim_{t\to\infty} \frac{t}{e^t} - 0 + 2e^{-1} = -\lim_{t\to\infty} \frac{1}{e^t} + 2e^{-1} = 2e^{-1}$$

$\Rightarrow$ The integral converges $\Rightarrow$ the series also converges.