Math 110 Notes

7.1-7.2

$(2)^2 = 4$

$(-2)^2 = (-2)\times(-2) = 4$

Exponents and Radicals
7.1- Radical Expressions and Functions

Square Roots and Square Root Functions

A function whose output is the square root of input

2 is a square root of 4.
−2 is also a square root of 4.

SQUARE ROOT- The number c is a square root of a if $c^2 = a$

For example,

1. −5 is a square root of 25 because $(-5)^2 = 25$
2. 7 is a square root of 49 because $7^2 = 49$
3. −3 is a square root of 9 because $(-3)^2 = 9$

EXAMPLE 1: Find the two square roots of 36

6 and −6

EXAMPLE 2: Find the two square roots of 49

7 and −7

PRINCIPAL SQUARE ROOT

- The positive number c for which $c^2 = a$ is called the principal square root of a.
- The principal square root of 0 is 0.

— The Pricipal square root of a is denoted by $\sqrt{a}$

EXAMPLE 3: Simplify each of the following

a) $\sqrt{25}$

$= 5$

b) $\sqrt{\frac{25}{64}}$

$= \frac{\sqrt{25}}{\sqrt{64}} = \frac{5}{8}$

c) $-\sqrt{64}$

$= -8$

d) $\sqrt{0.0049}$

$= \sqrt{\frac{49}{10000}}$

$= \frac{\sqrt{49}}{\sqrt{10000}} = \frac{7}{100}$

$= 0.07$

$\circledast$ $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

$\circledast$ $\sqrt{ab} = \sqrt{a}\sqrt{b}$

$(\underbrace{1000}_{3\ 0's})^2 = \underbrace{1000000}_{6\ 0's}$

There are three ways to read the principal square root of a , $\sqrt{a}$

1. Square root of a
2. root a
3. radical a

The following are radical expressions

$\sqrt{a}$, $\sqrt{a^2b}$, $\sqrt{a^3b^5}$, $\frac{\sqrt{a^4b^2}}{ab^2}$, $\sqrt{\frac{a^4b^2}{ab^2}}$

The expression under the radical sign is called the radicand, in the above expressions the radicands are

a, a^2b, a^3b^5, a^4b^2, $\frac{a^4b^2}{ab^2}$

On the calculator, the values for radial expressions are given as decimals

Example :

EXAMPLE 4: For each function, find the indicated function value

a) $f(x) = \sqrt{3x - 2}; f(1)$

$f(1) = \sqrt{3\times 1 - 2}$

$= \sqrt{3-2} = \sqrt{1} = 1$

b) $g(x) = \sqrt{6x + 4}\ ; g(3)$

$g(3) = \sqrt{6\times 3 + 4}$

$= \sqrt{18+4}$

$= \sqrt{22} = 4.69$

Expressions of the form $\sqrt{a^2}$

EXAMPLE 5: Evaluate $\sqrt{a^2}$ for each of the following values

a) 5

$\sqrt{5^2} = \sqrt{25}$

$= 5$

b) 0

$\sqrt{0^2} = \sqrt{0}$

$= 0$

c) -5

$\sqrt{(-5)^2} = \sqrt{25}$

$= 5 = -(-5)$

$$\boxed{\sqrt{a^2} = |a|}$$

If $a > 0$, $\sqrt{a^2} = a$

$\sqrt{0^2} = 0$

If $a < 0$, $\sqrt{a^2} = -a$

EXAMPLE 6: Simplify each expression. Assume that the variable can represent any real number

a) $\sqrt{36t^2}$

$= \sqrt{36}\sqrt{t^2}$

$= 6|t|$

b) $\sqrt{\underbrace{(x+1)}_{a}{}^2}$

$= |x+1|$

$\sqrt{(-2+1)^2} = 1$

$x = -2 \Rightarrow x+1 = -2+1 = -1$

$|x+1| = |-2+1| = |-1| = 1$

c) $\sqrt{x^2 - 8x + 16}$

$= \sqrt{(x-4)^2}$

$= |x-4|$

d) $\sqrt{t^6}$

$\sqrt{t^{3\times 2}}$

$= \sqrt{(t^3)^2}$

$= |t^3|$

$6 = 3\times 2$

$(a^m)^n = a^{mn}$

$(t^2)^3$

e) $\sqrt{z^8}$

$= \sqrt{z^{4\times 2}}$

$= \sqrt{(z^4)^2}$

$= |z^4|$

$= z^4$

$x^2 - 8x + 16$

$(a-b)^2 = a^2 - 2ab + b^2$

$x^2 + 4^2 - 2\times x\times 4$

$2\times x\times 4 = 8x$

$\shortparallel$

$(x-4)^2$

EXAMPLE 7: Simplify each expression. Assume that the expressions being square are nonnegative. Thus absolute value notation is not necessary.

$2\times 3x\times 1$
$=6x$

a) $\sqrt{y^2}$

$= y$

b) $\sqrt{a^{10}}$

$=\sqrt{a^{5\times 2}}$

$=\sqrt{(a^5)^2}$

$=a^5$

c) $\sqrt{9x^2-6x+1}$

$=\sqrt{(3x)^2-2\times 3x\times 1+1^2}$

$=\sqrt{(3x-1)^2}$

$=3x-1$

Cube roots

We often need to know what number cubed produces a certain value. When such number is found, we say that we have found a cube root. For example

* c is a cube root of a if $c^3=a$

1. Cube root of $8=2$
2. Cube root of $-8=-2$

* $\sqrt[3]{8}=2$, $\sqrt[3]{-1}=-1$

EXAMPLE 8: For each function, find the indicated function value

a) $f(x)=\sqrt[3]{y}; f(125)$

$f(125)=\sqrt[3]{125}=5$

b) $g(x)=\sqrt[3]{x-1}\ ; g(-26)$

$g(-26)=\sqrt[3]{-26-1}=\sqrt[3]{-27}$

$=-3$

$\boxed{\sqrt[3]{a^3}=a}$, $\boxed{\sqrt[3]{ab}=\sqrt[3]{a}\ \sqrt[3]{b}}$, $\boxed{\sqrt[3]{\frac{a}{b}}=\frac{\sqrt[3]{a}}{\sqrt[3]{b}}}$

EXAMPLE 9: Simplify $\sqrt[3]{-8y^3}$

$=\sqrt[3]{-8}\times\sqrt[3]{y^3}$

$=-2y$

$\sqrt[n]{a} \Rightarrow n = \{2, 3, 4, 5, 6, \dots\}$ a can be +ve, 0, −ve $\sqrt[n]{a^n} = a$ (n odd)

ODD AND EVEN NTH ROOTS

$\sqrt[5]{a}$, $\sqrt[7]{a}$, $\sqrt[9]{a}$ → behave like $\sqrt[3]{a}$

$\sqrt[4]{a}$, $\sqrt[6]{a}$, $\sqrt[8]{a}$ → behave like $\sqrt{a}$ → $a \geq 0$

→ 1·1

$\sqrt[n]{a^n} = |a|$ (n even)

EXAMPLE 10: Simplify each expression

a) $\sqrt[5]{32}$

$= 2$

b) $\sqrt[5]{-32}$

$= -2$

c) $-\sqrt[5]{32}$

$= -2$

d) $-\sqrt[5]{-32}$

$= -(-2)$

$= 2$

e) $\sqrt[7]{x^7}$

$= x$

f) $\sqrt[9]{(t-1)^9}$

$= t - 1$

EXAMPLE 11: Simplify each expression, if possible. Assume that variables can represent any number.

a) $\sqrt[4]{81}$

$= 3$

b) $-\sqrt[4]{81}$

$= -3$

$$\sqrt[4]{2^4} = \sqrt[4]{16} = 2$$

$$\sqrt[4]{(-1)^4} = \sqrt[4]{1^4} = 1 = -(-1)$$

c) $\sqrt[4]{81x^4}$

$= \sqrt[4]{81} \times \sqrt[4]{x^4}$

$= 3|x|$

d) $\sqrt[6]{(y+7)^6}$

$= |y+7|$

EXAMPLE 12: Determine the domain of g if $g(x) = \sqrt[6]{\underbrace{7-3x}}$

$7 - 3x \geq 0$

$-3x \geq -7$

$\frac{-3x}{-3} \leq \frac{-7}{-3} \Rightarrow x \leq \frac{7}{3}$

Domain of $g = \boxed{\left(-\infty, \frac{7}{3}\right]}$

$\sqrt[7]{7-3x}$

$\longrightarrow$ $7-3x$ can be any real number

$\Rightarrow$ x can be any real number.

$\Rightarrow$ Domain $= \mathbb{R}$

7.2- Rational Numbers as Exponents
Rational Exponents

$a^{\frac{1}{2}} = \sqrt{a}, \quad a^{\frac{1}{3}} = \sqrt[3]{a}$

$a^{1/n} = \sqrt[n]{a}$

$a^{1/n}$ means $\sqrt[n]{a}$. When a is nonnegative, n can be any natural number greater than 1. When a is negative, n can be any odd natural number greater than 1.

THE DENOMINATOR OF THE EXPONENET IS THE INDEX OF THE RADICAL EXPRESSION

$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}}$

EXAMPLE 1: Write an equivalent expression using radical notation and, if possible simplify

a) $16^{\frac{1}{2}}$

$= 16^{\frac{1}{2}}$

$= 4$

$(4^2)^{\frac{1}{2}} = 4^{2 \times \frac{1}{2}} = 4$

b) $(-8)^{\frac{1}{3}}$

$= (-8)^{\frac{1}{3}}$

$= -2$

$\left((-2)^3\right)^{\frac{1}{3}} = (-2)^{3 \times \frac{1}{3}}$

$= -2$

c) $(abc)^{\frac{1}{5}}$

$(abc)^{\frac{1}{5}}$

$= a^{\frac{1}{5}} b^{\frac{1}{5}} c^{\frac{1}{5}}$

d) $(25x^{16})^{\frac{1}{2}}$)

$= 25^{\frac{1}{2}} (x^{16})^{\frac{1}{2}}$

$= 5 x^{16 \times \frac{1}{2}}$

$= 5x^8$

EXAMPLE 2: Write an equivalent expression using exponential notation

a) $\sqrt[5]{7ab}$

$= (7ab)^{\frac{1}{5}}$

$= 7^{\frac{1}{5}} a^{\frac{1}{5}} b^{\frac{1}{5}}$

b) $\sqrt[7]{\frac{x^3y}{4}}$

$= \left(\frac{x^3y}{4}\right)^{\frac{1}{7}}$

$= (x^3)^{\frac{1}{7}} y^{\frac{1}{7}} \left(\frac{1}{4}\right)^{\frac{1}{7}}$

$\left(\frac{1}{4}\right)^{\frac{1}{7}} \to \frac{1}{4^{1/7}}$

$= \frac{x^{\frac{3}{7}} y^{\frac{1}{7}}}{4^{1/7}}$

c) $\sqrt{5x}$

$\sqrt{5x} \to \sqrt[2]{5x}$

$= (5x)^{\frac{1}{2}}$

$= 5^{\frac{1}{2}} x^{\frac{1}{2}}$

POSITIVE RATIONAL EXPONENTS

For any natural numbers m and n $(n \neq 1)$ and any real number a for which $\sqrt[n]{a}$ exists,

$$a^{m/n} \quad \text{means} \quad (\sqrt[n]{a})^m, \quad \text{or} \quad \sqrt[n]{a^m}.$$

EXAMPLE 3: Write an equivalent expression using radical notation and simplify.

a) $27^{\frac{2}{3}}$

$= (3^3)^{2/3}$

$= 3^{3 \times \frac{2}{3}}$

$= 3^2$

$= 9$

b) $25^{\frac{3}{2}}$

$= (5^2)^{3/2}$

$= 5^{2 \times \frac{3}{2}}$

$= 5^3$

$= 125$

EXAMPLE 4: Write an equivalent expression using exponential notation

a) $\sqrt[3]{9^4}$

$(9^4)^{\frac{1}{3}}$

$= 9^{\frac{4}{3}}$

b) $\left(\sqrt[4]{7xy}\right)^5$

$\left((7xy)^{\frac{1}{4}}\right)^5$

$= (7xy)^{5/4}$

$= 7^{5/4}\, x^{5/4}\, y^{5/4}$

NEGATIVE RATIONAL EXPONENTS

$a^{-2} = \left(\frac{1}{a}\right)^2 = \frac{1}{a^2}$

NEGATIVE RATIONAL EXPONENTS

For any rational number m/n and any nonzero real number a for which $a^{m/n}$ exists,

$$a^{-m/n} \quad \text{means} \quad \frac{1}{a^{m/n}}.$$

CAUTION! A negative exponent does not indicate that the expression in which it appears is negative: $a^{-1} \neq -a$.

$= \frac{1}{a}$

EXAMPLE 5: Write an equivalent expression with positive exponents and, if possible, simplify

a) $9^{-\frac{1}{2}}$

$= \frac{1}{9^{1/2}}$

$= \frac{1}{3}$

b) $(5xy)^{-\frac{4}{5}}$

$= \frac{1}{(5xy)^{4/5}}$

$= \frac{1}{5^{4/5} x^{4/5} y^{4/5}}$

c) $64^{-\frac{2}{3}}$

$= \frac{1}{64^{2/3}} = \frac{1}{(4^3)^{2/3}}$

$= \frac{1}{4^{3\times\frac{2}{3}}} = \frac{1}{4^2} = \frac{1}{16}$

d) $4x^{-\frac{2}{3}}y^{\frac{1}{5}}$

$= 4 \times \frac{1}{x^{2/3}} \times y^{1/5}$

$= \frac{4y^{1/5}}{x^{2/3}}$

e) $\left(\frac{3r}{7s}\right)^{-\frac{5}{2}}$

$= \frac{1}{\left(\frac{3r}{7s}\right)^{5/2}} = \frac{1}{\frac{(3r)^{5/2}}{(7s)^{5/2}}} = \frac{(7s)^{5/2}}{(3r)^{5/2}}$

$= \frac{7^{5/2} s^{5/2}}{3^{5/2} r^{5/2}}$

(*) $\frac{1}{a/b} = \frac{b}{a}$

(*) $\frac{a/b}{c/d} = \frac{a}{b} \times \frac{d}{c}$

$\frac{1}{1/2} = 2$, $\frac{1/2}{1/3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2}$

LAW OF EXPONENTS

LAWS OF EXPONENTS

For any real numbers a and b and any rational exponents m and n for which a^m, a^n, and b^m are defined:

1. $a^m \cdot a^n = a^{m+n}$ When multiplying, add exponents if the bases are the same.
2. $\dfrac{a^m}{a^n} = a^{m-n}$ When dividing, subtract exponents if the bases are the same. (Assume $a \neq 0$.)
3. $(a^m)^n = a^{m \cdot n}$ To raise a power to a power, multiply the exponents.
4. $(ab)^m = a^m b^m$ To raise a product to a power, raise each factor to the power and multiply.

$\frac{1}{4} - \frac{1}{2} = \frac{1}{4} - \frac{2}{4} = \frac{1-2}{4}$

$= \frac{-1}{4}$

EXAMPLE 6: Use the laws of exponents to simplify

a) $3^{\frac{1}{5}} \cdot 3^{\frac{3}{5}}$

$= 3^{\frac{1}{5}+\frac{3}{5}}$

$= 3^{4/5}$

b) $\dfrac{a^{\frac{1}{4}}}{a^{\frac{1}{2}}}$

$= a^{\frac{1}{4}-\frac{1}{2}} = a^{-\frac{1}{4}} = \dfrac{1}{a^{1/4}}$

c) $\left(7.2^{\frac{2}{3}}\right)^{\frac{3}{4}}$

$= 7^{3/4} \times \left(2^{2/3}\right)^{3/4}$

$= 7^{3/4} \times 2^{\cancel{2}/\cancel{3} \times \cancel{3}/\cancel{4}\ 2}$

$= 7^{3/4}\ 2^{1/2}$

$(7 \cdot 2)^{\cancel{2}/\cancel{3} \times \frac{\cancel{3}}{\cancel{4}\ 2}} = 7 \cdot 2^{1/2}$

d) $\left(a^{-\frac{1}{3}} b^{\frac{2}{5}}\right)^{\frac{1}{2}}$

$= \left(\frac{1}{a^{1/3}}\, b^{2/5}\right)^{\frac{1}{2}} = \left(\frac{b^{2/5}}{a^{1/3}}\right)^{1/2}$

$= \dfrac{(b^{2/5})^{1/2}}{(a^{1/3})^{1/2}} = \dfrac{b^{\frac{2}{5} \times \frac{1}{2}}}{a^{\frac{1}{3} \times \frac{1}{2}}} = \dfrac{b^{\frac{1}{5}}}{a^{1/6}}$

SIMPLIFYING RADICAL EXRPESSIONS

TO SIMPLIFY RADICAL EXPRESSIONS

1. Convert radical expressions to exponential expressions.
2. Use arithmetic and the laws of exponents to simplify.
3. Convert back to radical notation as needed.

EXAMPLE 7: Use rational exponents to simplify. DO not use exponents that are fraction in the final answer

a) $\sqrt[6]{(5x)^3}$

$= \left((5x)^3\right)^{\frac{1}{6}}$

$= (5x)^{3 \times \frac{1}{6}} = (5x)^{1/2}$

$= \boxed{\sqrt{5x}}$

b) $\sqrt[5]{t^{20}}$

$= (t^{20})^{\frac{1}{5}} = t^{20 \times \frac{1}{5}}$

$= \boxed{t^4}$

c) $\left(\sqrt[3]{ab^2c}\right)^{12}$

$= \left[(ab^2c)^{1/3}\right]^{12}$

$= (ab^2c)^{\frac{1}{3} \times \cancel{12}^{4}}$

$= (ab^2c)^4$

$= a^4 (b^2)^4 c^4$

$= \boxed{a^4 b^8 c^4}$

d) $\sqrt{\sqrt[3]{x}}$

$= \sqrt{x^{1/3}}$

$= (x^{1/3})^{\frac{1}{2}}$

$= x^{1/3 \times \frac{1}{2}}$

$= \boxed{x^{1/6}}$

Quiz 10

① multiply $(x^2+4x+2)(x^2-x-1)$

$= x^2(x^2-x-1)+4x(x^2-x-1)+2(x^2-x-1)$

x^4	$-x^3$	$-x^2$
$4x^3$	$-4x^2$	$-4x$
$2x^2$	$-2x$	-2

$x^2 \times x^2$

$4x \times (-x)$

$x^2 \times -1$

$x^2 \times -x$

$= x^4+3x^3-3x^2-6x-2$

② (a) $(4x+5y)^2 = (4x)^2+(5y)^2+2\times 4x \times 5y$

$(a+b)^2 = a^2+b^2+2ab$

$= 4^2x^2+5^2y^2+2\times4\times5\,xy$

$= 16x^2+25y^2+40xy.$

(b) $\left(\frac{1}{3}p-5q\right)\left(\frac{1}{3}p+5q\right) = \left(\frac{1}{3}p\right)^2-(5q)^2$

$(a-b)(a+b) = a^2-b^2$

$= \left(\frac{1}{3}\right)^2p^2-5^2q^2$

$= \frac{1^2}{3^2}p^2-5^2q^2$

$= \frac{1}{9}p^2-25q^2$

Factorization

(*) Factorize $\overset{a}{2x^2} + 3x - \overset{c}{2}$

$2x \cdot 2 = -4$

$= 1x - 4 = -1 \times 4$

$= 2x - 2 = -2 \times 2$

$\underbrace{2x^2 - x}\,\underbrace{+4x - 2}$

$= x(2x-1) + 2(2x-1)$

$= (x+2)(2x-1)$

(*) $a^2 - b^2 = (a-b)(a+b)$

Factorize : $16x^2 - 25y^2$

$= (4x)^2 - (5y)^2 = (4x-5y)(4x+5y)$