HW Exercises (Page 471) # 1, 3, 5(a)(b)(c), 8(a)(b)(c)(d), 9

Consider the following **differential equation** in y:

$$\frac{dy}{dt} = ky \text{ where } k \text{ is a constant.}$$

The only solutions of the above equation are the exponential functions

$$\boxed{y(t) = y(0)e^{kt}}$$

$$\frac{dy}{dt} = \frac{d}{dt}\left(y(0)e^{kt}\right) = y(0)\,\frac{d}{dt}\left(e^{kt}\right) = y(0)\,e^{kt}\,\cancel{\frac{d}{dt}(kt)} = k$$

$u = kt$

$$\frac{d}{dt}\left(e^{u}\right) = \frac{d}{du}\left(e^{u}\right)\frac{du}{dt} = e^{u}\frac{du}{dt}$$

$$= y(0)k\,e^{kt} = ky$$

- $k = 0$ implies $y(t)$ is a constant function.
- $k > 0$ implies $y(t)$ grows exponentially.
- $k < 0$ implies $y(t)$ decays exponentially.

Population Growth: Let $P(t)$ be the size of population at time t. Then

$$\frac{dP}{dt} = kP \quad \Rightarrow \quad P(t) = P(0)e^{kt} \qquad (k > 0)$$

Since we have

$$k = \frac{1}{P}\frac{dP}{dt}$$

the constant k is called the relative growth rate.

Problem 1: The cell of a bacterium divides into two cells every 20 minutes. The initial population of a culture is 50 cells.

1. Find the relative growth rate.
2. Find an expression for the number of cells after t hours.
3. Find the number of cells after 6 hours.
4. Find the rate of growth after 6 hours.
5. When will the population reach a million cells?

$20\text{ min} = \frac{20}{60}$ hours

$$P\left(\frac{20}{60}\right) = 2P(0)$$

1) $$P(t) = P(0)e^{kt} \xrightarrow[t=\frac{20}{60}]{} P\left(\frac{20}{60}\right) = P(0)e^{k \times \frac{20}{60}}$$

$$\Rightarrow 2P(0) = P(0)e^{k/3}$$

$$\Rightarrow 2 = e^{k/3} \Rightarrow \ln 2 = \ln e^{k/3} \Rightarrow \ln 2 = k/3$$

$$\Rightarrow k = 3\ln 2 \Rightarrow k = \ln 2^3 \Rightarrow k = \ln 8$$

2) $$P(t) = 50e^{(\ln 8)t} = 50e^{t\ln 8} = 50e^{\ln 8^t} = 50(8)^t$$

$P(t) = 50(8)^t$

$$e^{\ln f(x)} = f(x)$$

$$\frac{d}{dx}(6^x) = 6^x \ln 6$$

3) $P(6) = 50(8)^6$ cells

4) rate of growth $= P'(t)$. Find $P'(6)$

$P(t) = 50(8)^t \Rightarrow P'(t) = 50\frac{d}{dt}(8^t)$

$\Rightarrow P'(t) = 50(8)^t \ln 8 \Rightarrow P'(6) = 50(8)^6 \ln 8$

$P'(t) = k\,P(t) \Rightarrow P'(6) = k\,P(6) = (\ln 8)P(6)$

$= (\ln 8)\,50(8)^6$

5) t for which $P(t) = 10^6$ cells.

$\Rightarrow 50(8)^t = 10^6 \Rightarrow (8)^t = \frac{10^6}{50} = \frac{10^5}{5} = 2\times 10^4$

$= 20\,000$

$\Rightarrow 8^t = 20000$

$\Rightarrow \ln 8^t = \ln(20000) \Rightarrow t\ln 8 = \ln(20000)$

$$\boxed{t = \frac{\ln(20000)}{\ln 8}} \rightarrow \text{hours}$$

Radioactive Decay: Radioactive substances decay by spontaneously emitting radiation.

If $m(t)$ is the mass remaining after time t, from an initial mass of $m(0)$, then

$$\frac{dm}{dt} = km \text{ where } k \text{ is a negative constant}$$

Therefore,

$$m(t) = m(0)e^{kt}$$

The time required to decay to half of the initial mass is called half life ($t_{1/2}$) of a substance.

Problem 2: The half-life of radium-226 is 1590 years.

1. A sample of radium-226 has a mass of 100 mg, Find a formula for the mass of the sample that remains after t years. $m(0)$
2. Find the mass after 1000 years.
3. When will the mass be reduced to 30 mg?

$t_{1/2} = 1590$ years.

$$m(t_{1/2}) = \frac{m(0)}{2}$$

$$\Rightarrow \cancel{m(0)} e^{kt_{1/2}} = \frac{\cancel{m(0)}}{2}$$

$$\Rightarrow e^{kt_{1/2}} = \frac{1}{2} \Rightarrow \ln\left(e^{kt_{1/2}}\right) = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow kt_{1/2} = \ln\frac{1}{2} \Rightarrow t_{1/2} = \frac{1}{k}\ln\left(\frac{1}{2}\right) \Rightarrow \boxed{t_{1/2} = \frac{-\ln 2}{k}}$$

$$1590 = \frac{-\ln 2}{k}$$

$$\Rightarrow \boxed{k = \frac{-\ln 2}{1590}}$$

1.) $m(t) = m(0)e^{kt} = 100e^{kt}$

$$\Rightarrow m(t) = 100e^{\frac{-\ln 2}{1590}t}$$

$$\Rightarrow m(t) = 100e^{\ln(2)^{-t/1590}} = 100(2)^{\frac{-t}{1590}}$$

$$\Rightarrow m(t) = 100(2)^{-t/1590}$$

2) $m(1000) = 100(2)^{\frac{-1000}{1590}}$

3) t for which $m(t) = 30$ mg

$$\Rightarrow 100(2)^{-t/1590} = 30 \Rightarrow 2^{-t/1590} = \frac{30}{100}$$

$$\Rightarrow \ln 2^{-t/1590} = \ln \frac{30}{100}$$

$$\Rightarrow \frac{-t}{1590} \ln 2 = \ln 0.3$$

$$\Rightarrow t = \frac{(-\ln 0.3)1590}{\ln 2} \text{ years}$$