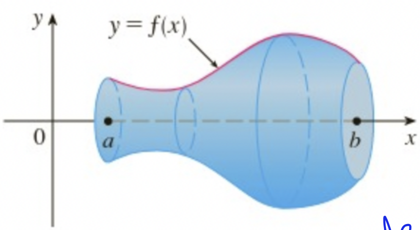
M16600 Lecture Notes

Section 8.2: Area of a Surface of Revolution

■ **Section 8.2** textbook exercises, *page 595*: # 1, 2, 3, 7.

A ***surface of revolution*** is formed when a curve is rotated about a line. How do we find the area of such a surface?

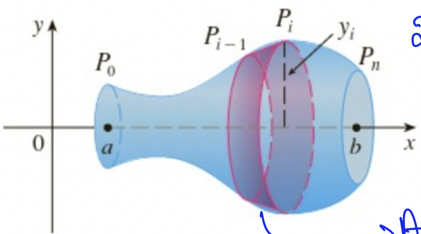


(a) Surface of revolution

$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

The area of the i band is $2\pi \underbrace{f(x_i^*)}_{y_i^*} \overbrace{\sqrt{1+\left[f'(x_i^*)\right]^2}\,\Delta x}^{\Delta s}$. See the discussion on page 591–592 of the textbook for more detail. Then an approximation of the surface area is

$$\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\,\Delta x$$



(b) Approximating band

$2\pi y\, ds$
$\shortparallel$
$2\pi r \times ds$
$\shortparallel$
Area = Circumference × width

Thus, the surface area is

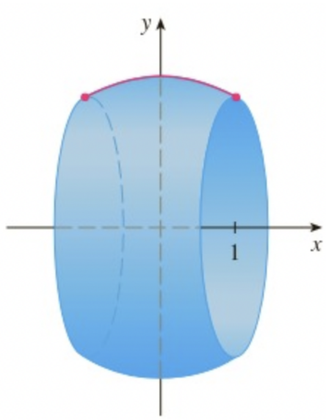
$$\lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\,\Delta x$$

$$= \int_a^b 2\pi \overset{y}{f(x)} \underbrace{\sqrt{1+\left[f'(x)\right]^2}\,dx}_{ds}.$$

Area of a Surface of Revolution about the x-axis. The surface area of a surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Example 1: The curve $y = \sqrt{4-x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2+y^2=4$. Find the area of the surface obtained by rotating this arc about the x-axis.



$a=-1,\ b=1,\ y=\sqrt{4-x^2}$

$\frac{dy}{dx} = \frac{1}{2\sqrt{4-x^2}} \times (4-x^2)' = \frac{-2x}{2\sqrt{4-x^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$

$S = \int_{-1}^{1} 2\pi\sqrt{4-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\,dx$

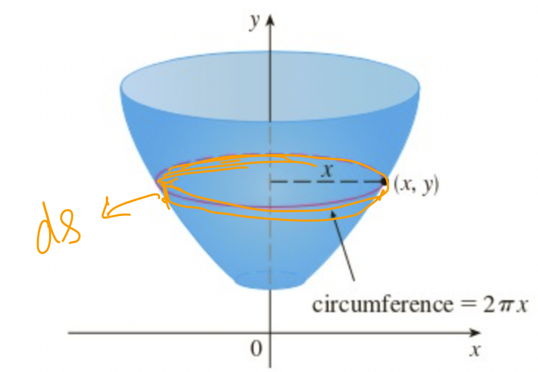
$$= 2\pi \int_{-1}^{1} \sqrt{4-x^2}\sqrt{1+\frac{x^2}{4-x^2}}\,dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4-x^2}\sqrt{\frac{4-\cancel{x^2}+\cancel{x^2}}{4-x^2}}\,dx$$

$$= 2\pi \int_{-1}^{1} \cancel{\sqrt{4-x^2}}\,\frac{\sqrt{4}}{\cancel{\sqrt{4-x^2}}}\,dx = 2\pi\int_{-1}^{1} 2\,dx = 2\pi\left[2x\right]_{-1}^{1}$$

$$= 2\pi\left[2-(-2)\right] = 8\pi$$

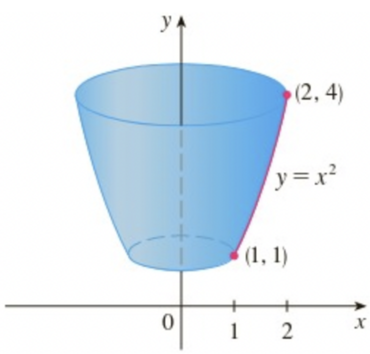
$dS = 2\pi x\, ds$



Area of a Surface of Revolution about the y-axis. The surface area of a surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the y-axis is

$$S = \int_a^b 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Example 2: The arc of the parabola $y = x^2$ from $(1,1)$ to $(2,4)$ is rotated about the y-axis. Find the area of the resulting surface.



$a=1,\ b=2$

$y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

$$S = \int_1^2 2\pi x \sqrt{1+(2x)^2}\,dx$$

$u = 1+4x^2$
or

$$= 2\pi\int_1^2 x\sqrt{1+4x^2}\,dx$$

$u = 4x^2 \Rightarrow du = 8x\,dx \Rightarrow \frac{du}{8} = x\,dx$

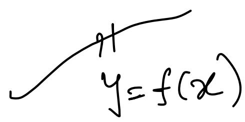
$$S = 2\pi\int_{4(1)^2}^{4(2)^2} \sqrt{1+u}\,\frac{du}{8} = \frac{2\pi}{8}\int_4^{16}\sqrt{1+u}\,du$$

$$= \frac{2\pi}{8} \left. \frac{(u+1)^{3/2}}{3/2} \right|_{4}^{16} = \frac{2\pi}{8} \times \frac{2}{3}\left[17^{3/2} - 5^{3/2}\right]$$

$$= \frac{4\pi}{24}\left[17\sqrt{17} - 5\sqrt{5}\right]$$

$$= \frac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right]$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



$$= \sqrt{(dx)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right]} = \sqrt{(dx)^2 + \frac{(dy)^2}{\cancel{(dx)^2}}\cancel{(dx)^2}}$$

$$= \sqrt{(dx)^2 + (dy)^2} = \sqrt{(dy)^2\left[1 + \left(\frac{dx}{dy}\right)^2\right]}$$

$$= \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Useful when $x = g(y)$

$$\boxed{S = \int_{f(a)}^{f(b)} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy}$$

$x = g(y)$

$\boxed{y = f(x)}$

$a \le x \le b$

$\boxed{y = \ln x \Rightarrow x = e^{y}}$

Example 2

$$\underbrace{y = x^2}_{\downarrow}, \quad 1 \le x \le 2$$

$$\boxed{x = \sqrt{y}} \Rightarrow \frac{dx}{dy} = \frac{d}{dy}\left(\sqrt{y}\right)$$

$$= \frac{d}{dy}\left(y^{1/2}\right) = \frac{1}{2} y^{1/2 - 1} = \frac{1}{2} y^{-1/2}$$

$$= \frac{1}{2\sqrt{y}}$$

$$S = \int_{f(a)}^{f(b)} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$f(1) = 1^2 = 1$

$f(2) = 2^2 = 4$

$$= \int_1^4 2\pi x \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2}\, dy$$

$$= \int_1^4 2\pi \sqrt{y} \sqrt{1 + \frac{1}{4y}}\, dy = 2\pi \int_1^4 \sqrt{y} \sqrt{\frac{4y+1}{4y}}\, dy$$

$$= 2\pi \int_1^4 \sqrt{y}\, \frac{\sqrt{4y+1}}{\sqrt{4y}}\, dy = 2\pi \int_1^4 \cancel{\sqrt{y}}\, \frac{\sqrt{4y+1}}{\sqrt{4}\,\cancel{\sqrt{y}}}\, dy$$

$$= 2\pi \int_1^4 \frac{\sqrt{4y+1}}{2}\, dy = \frac{2\pi}{2} \int_1^4 (4y+1)^{1/2}\, dy$$

$$= \pi \left. \frac{(4y+1)^{1/2 + 1}}{\frac{1}{2} + 1} \times \frac{1}{4} \right]_1^4 = \pi \left. \frac{(4y+1)^{3/2}}{3/2} \frac{1}{4} \right]_1^4$$

$$= \pi \frac{2}{3} \times \frac{1}{4} (4y+1)^{3/2} \Big|_1^4 = \frac{\pi}{6}\left[(4(4)+1)^{3/2} - (4(1)+1)^{3/2}\right]$$

$$= \frac{\pi}{6}\left[17^{3/2} - 5^{3/2}\right]$$

Example 3 $\quad y = \frac{x^2}{4} - \frac{1}{2}\ln x, \quad 1 \le x \le 2$

Rotate $y = f(x)$ about y-axis.

Find the area of the resulting surface.

$$\frac{dy}{dx} = \frac{1}{4}(2x) - \frac{1}{2}\left(\frac{1}{x}\right) = \frac{x}{2} - \frac{1}{2x}$$

$$S = \int_1^2 2\pi x \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\, dx$$

$$= \int_1^2 2\pi x \sqrt{1 + \left(\frac{x}{2}\right)^2 + \left(\frac{1}{2x}\right)^2 - 2 \times \frac{x}{2} \times \frac{1}{2x}}\, dx$$

$$= \int_1^2 2\pi x \sqrt{1 + \left(\frac{x}{2}\right)^2 + \left(\frac{1}{2x}\right)^2 - \frac{1}{2}}\, dx$$

$$= \int_1^2 2\pi x \sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{1}{2x}\right)^2 + \frac{1}{2}}\, dx$$

$$= \int_1^2 2\pi x \sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{1}{2x}\right)^2 + 2\left(\frac{x}{2}\right)\left(\frac{1}{2x}\right)}\, dx$$

$$= \int_1^2 2\pi x \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\, dx$$

$$= \int_1^2 2\pi x \left(\frac{x}{2} + \frac{1}{2x}\right) dx = 2\pi \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2}\right) dx$$

$$= 2\pi \left[\int_1^2 \frac{x^2}{2}\, dx + \int_1^2 \frac{1}{2}\, dx\right]$$

$$= 2\pi \left[\frac{1}{6}\left(2^3 - 1^3\right) + \frac{1}{2}\right] \doteq 2\pi \left[\frac{7}{6} + \frac{1}{2}\right] = 2\pi \times \frac{5}{3}$$

$$= \frac{10\pi}{3}$$