Learning objectives:

1. Compute limits using the limit laws.
2. Compute limits using the direct substitution property.
3. To be able to apply the squeeze theorem.

Limit Laws

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then we have

1. $\lim_{x\to a}[f(x) \pm g(x)] = L \pm M$.
2. $\lim_{x\to a} f(x)g(x) = LM$.
3. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ provided that $M \neq 0$.
4. $\lim_{x\to a} cf(x) = cL$.
5. $\lim_{x\to a} c = c$ where c is a constant.
6. $\lim_{x\to a} x = a$.
7. $\lim_{x\to a}[f(x)]^n = L^n$.
8. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$, given that $L \geq 0$ if n is even.

Example 1.

Evaluate the limit

$$\lim_{x\to 0} \frac{3x^2 + \sqrt{x} + 1}{2x^2 - x + 2} .$$

Direct substitution property

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a) .$$

Example 2.

Evaluate the limit $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$.

Example 3.

Evaluate $\lim_{h \to 0} \dfrac{(3 + h)^2 - 9}{h}$.

Example 4.

Find $\lim\limits_{t\to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$.

Example 5.

Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Example 6.

If $f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4\,, \\ 8-2x & \text{if } x < 4\,, \end{cases}$ then determine whether $\lim_{x\to 4} f(x)$ exists.

The Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a, except possibly at a itself, and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L\,,$$

then

$$\lim_{x\to a} g(x) = L\,.$$

Example 7.

Show that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

Example 8.

Evaluate $\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$.

Example 9.

Evaluate $\lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$.

Example 10.

Evaluate $\lim_{t \to 0} \dfrac{\sqrt{1+t} - \sqrt{1-t}}{t}$.