HW Exercises (Page 436) # 3, 5, 7, 9, 10, 11, 13, 15, 35, 43, 45, 47, 49, 51, 53, 71, 73, 75, 77, 78, 80.

$f(x) = \ln x = g^{-1}(x)$ where $g(x) = e^x \to g'(x) = e^x = g(x)$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Take $x = a$

$$f'(a) = (g^{-1})'(a) = \frac{1}{g'(g^{-1}(a))} = \frac{1}{g(g^{-1}(a))} = \frac{1}{a}$$

$$f'(a) = \frac{1}{a} \Rightarrow f'(x) = \frac{1}{x}$$

Chain Rule:

$$\boxed{\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}} \text{ and } \boxed{\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}}$$

$$\frac{d}{dx}(\ln u) = \frac{d}{du}(\ln u)\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$$

$u = g(x) \Rightarrow \frac{1}{g(x)}\frac{d}{dx}(g(x)) = \frac{g'(x)}{g(x)}$

Problem 1: Differentiate the following:

1. $y = \ln(x^2 + 1)$

$$\frac{dy}{dx} = \frac{1}{x^2+1} \times \frac{d}{dx}(x^2+1)$$

$$= \frac{1}{x^2+1} \times \left[\frac{d}{dx}(x^2) + \frac{d}{dx}(1)\right]$$

$$= \frac{1}{x^2+1} \times [2x + 0] = \frac{2x}{x^2+1}$$

2. $y = \ln(\sin x)$

$$\frac{dy}{dx} = \frac{1}{\sin x} \times \frac{d}{dx}(\sin x)$$

$$= \frac{1}{\sin x} \times \cos x = \frac{\cos x}{\sin x} = \cot x$$

3. $y = \sqrt{\ln x}$

$u = \ln x \Rightarrow y = \sqrt{u}$ $\qquad \frac{du}{dx} = \frac{d}{dx}(\ln x) = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{u}) = \frac{d}{du}(\sqrt{u})\frac{du}{dx} = \frac{1}{2}u^{\frac{1}{2}-1} \times \frac{du}{dx}$$

$$= \frac{1}{2}u^{-1/2} \times \frac{1}{x} = \frac{1}{2}(\ln x)^{-1/2} \times \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

$$\boxed{\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}}$$

$$\frac{du}{dx} = \frac{d}{dx}(u)$$

4.

$$\ln \frac{x+1}{\sqrt{x-2}}$$

$$y = \ln \frac{x+1}{\sqrt{x-2}} = \ln(x+1) - \ln\left(\sqrt{x-2}\right)$$

$$= \ln(x+1) - \ln(x-2)^{1/2} = \ln(x+1) - \frac{1}{2}\ln(x-2)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\ln(x+1) - \frac{1}{2}\ln(x-2)\right]$$

$$= \frac{d}{dx}\left[\ln(x+1)\right] - \frac{1}{2}\frac{d}{dx}\left[\ln(x-2)\right]$$

$$= \frac{1}{x+1}\,\cancel{\frac{d}{dx}(x+1)}^{\,1} - \frac{1}{2}\times\frac{1}{x-2}\times\cancel{\frac{d}{dx}(x-2)}^{\,1} = \frac{1}{x+1} - \frac{1}{2(x-2)}$$

$$= \frac{2x-4-x-1}{2(x+1)(x-2)}$$

$$= \frac{x-5}{2(x+1)(x-2)}$$

Problem 2: Find the absolute minimum value of $f(x) = x^2 \ln x$.

$$f'(x) = \frac{d}{dx}\left(\underline{x^2}\ \underline{\ln x}\right)$$

$$= x^2\frac{d}{dx}(\ln x) + \left[\frac{d}{dx}(x^2)\right]\ln x$$

$$= x^2 \times \frac{1}{x} + 2x\ln x = x + 2x\ln x$$

$$f'(x) = x(1+2\ln x) \cdot$$

Domain of $f = ??$

$(0, \infty)$

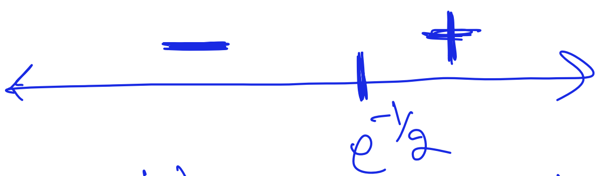
$x > 0$

$$1 + 2\ln x = 0 \Rightarrow 2\ln x = -1$$

$$\Rightarrow \ln x = \frac{-1}{2} \Rightarrow x = e^{-1/2}$$

$$\boxed{x > e^{-1/2}} \Rightarrow \ln x > \ln e^{-1/2}$$

$$\Rightarrow \ln x > \frac{-1}{2} \Rightarrow \boxed{2\ln x + 1 > 0}$$



$$f\left(e^{-1/2}\right) = \left(e^{-1/2}\right)^2 \ln e^{-1/2}$$

$$= e^{-1}\left(\frac{-1}{2}\right) = \frac{-1}{2e}$$

Problem 3: Sketch the curve $\ln(4-x^2)$.

1. Domain
2. Intercepts
3. Symmetry (even/odd)
4. Asymptotes
5. Intervals of Increasing/Decreasing
6. Local Maxima/Minima
7. Concavity/Convexity

$4-x^2>0$

$(-1)(4-x^2) < (-1)(0)$

$x^2-4<0$

$(x-2)(x+2)<0$

$x=2,\ x=-2$

$+$ -2 $-$ 2 $+$

$\boxed{-2<x<2}$ ← Domain of $\ln(4-x^2)$

Intercepts: $x=0 \Rightarrow y=\ln(4-0^2) = \ln 4$

$(0, \ln 4)$

$\ln(4-x^2)=0 \Rightarrow 4-x^2=e^0=1$

$\Rightarrow 4-x^2=1 \Rightarrow x^2=3 \Rightarrow x=\pm\sqrt{3}$ $(\sqrt{3},0),\ (-\sqrt{3},0)$

$f(-x)=\ln(4-(-x)^2)=\ln(4-x^2)=f(x) \Rightarrow f$ is even

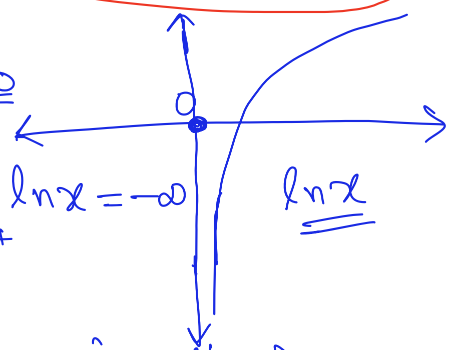
Asymptotes

Horizontal (None) $\to \lim_{x\to\pm\infty} f(x) = \ln(4-(\to\infty))$

Vertical $\to \lim_{x\to a} f(x)=\pm\infty$

$4-x^2=0 \Rightarrow x^2=4$

$\Rightarrow x=\pm 2$



$x=0$

$\lim_{x\to 0^+} \ln x = -\infty$

5. $f'(x)$: when is $\underbrace{f'(x)>0}_{\text{Increasing}}$, when is $\underbrace{f'(x)<0}_{\text{decreasing}}$

$$f'(x)=\frac{d}{dx}\left(\ln(4-x^2)\right)=\frac{1}{4-x^2}\cdot\frac{d}{dx}(4-x^2)=\frac{-2x}{\underbrace{4-x^2}_{\text{always +ve for } -2<x<2}}=\frac{2x}{x^2-4}$$

$-2<x<2 \Rightarrow 4-x^2>0$

$f'(x)>0 \Rightarrow \underbrace{x<0}_{\text{where}}$ and $f'(x)<0 \Rightarrow \underbrace{x>0}_{\text{where}}$

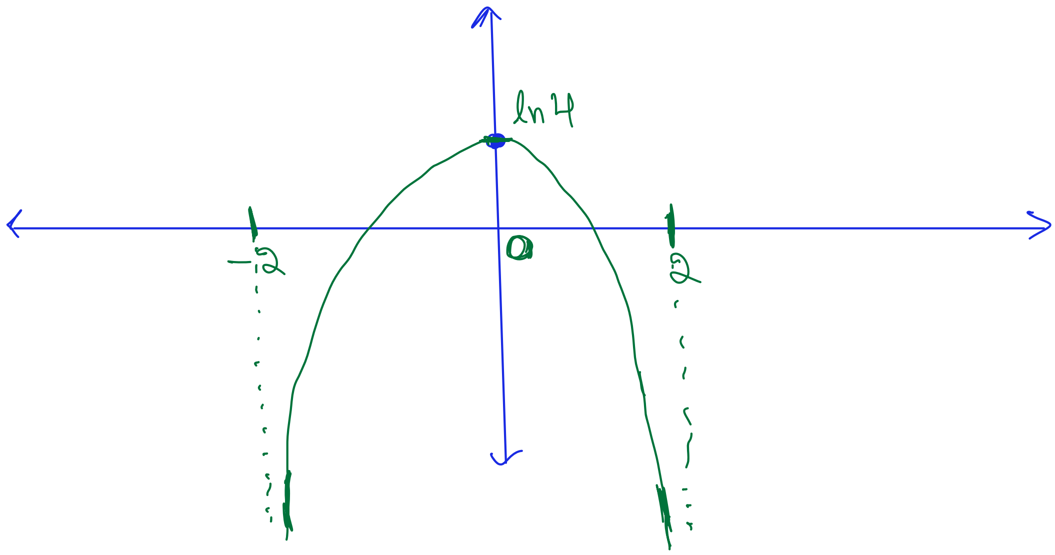
f is increasing f is decreasing.

6.

-2 $+$ 0 $-$ 2

$x=0$ is a Pt. of local maxima

$f(0)=\ln 4$



7.

$$f''(x) = \frac{d}{dx}\left(\frac{2x}{x^2-4}\right)$$

<u>H.W.</u> : Check that $f''(x) < 0$ for $-2 < x < 2$

$f(x) = \ln|x|$

Domain of f: All real numbers except $x=0$

$$\boxed{\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x}}$$

Domain of $\ln x = (0,\infty)$

$$|x| = \begin{cases} x & x>0 \\ -x & x<0 \end{cases}$$

$$\ln|x| = \begin{cases} \ln x & , x>0 \\ \ln(-x) & , x<0 \end{cases}$$

$$\frac{d}{dx}\left(\ln|x|\right) = \begin{cases} \frac{d}{dx}(\ln x) & , x>0 \\ \frac{d}{dx}(\ln(-x)) & , x<0 \end{cases}$$

$$= \begin{cases} \frac{1}{x} & , x>0 \\ \frac{1}{-x}\cdot\frac{d}{dx}(-x) & , x<0 \end{cases} = \begin{cases} \frac{1}{x} & , x>0 \\ \frac{-1}{x}\cdot(-1) & , x<0 \end{cases} = \frac{1}{x} \text{ for every nonzero real number}$$

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + c}$$

$n=-1$

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1}$$

Problem 4: Evaluate the following integrals:

1. $\displaystyle\int \frac{x}{x^2+1}\,dx$ $\longrightarrow$ Substitute $u = x^2+1$

$$\frac{du}{dx} = \frac{d}{dx}(x^2+1) = 2x \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$$

$$\Rightarrow \frac{du}{2} = x\,dx$$

$$I = \int \frac{x\,dx}{x^2+1} = \int \frac{1}{u}\,\frac{du}{2} = \frac{1}{2}\int\frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2+1| + c$$

2. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}\,dx$$

$$x=1 \Rightarrow u = \ln 1 = 0, \quad x = e \Rightarrow u = \ln e = 1$$

$$I = \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2} - 0 = \frac{1}{2}$$

Alternatively

$$\int u\,du = \frac{u^2}{2} + c = \frac{(\ln x)^2}{2} + c$$

$$I = \frac{(\ln x)^2}{2}\Big|_1^e = \frac{(\ln e)^2}{2} - \frac{(\ln 1)^2}{2} = \frac{1}{2} - 0 = \frac{1}{2}$$

3. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow du = \cos x\,dx$

$u = \cos x \Rightarrow \frac{du}{dx} = -\sin x \Rightarrow du = -\sin x\,dx$
$\Rightarrow -du = \sin x\,dx$

$I = \int \frac{1}{u}(-du) = -\int \frac{1}{u}\,du$
$= -\ln|u| + c = -\ln|\cos x| + c = \ln|\cos x|^{-1} + c$
$= \ln\left|\frac{1}{\cos x}\right| + c$
$= \ln|\sec x| + C$

$$\boxed{\frac{d}{dx}(\log_b x) = \frac{1}{x\ln b}}$$

$\log_b x = \frac{\ln x}{\ln b} \Rightarrow \frac{d}{dx}(\log_b x) = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b}\frac{d}{dx}(\ln x)$
$= \frac{1}{x\ln b}$

$$\boxed{\frac{d}{dx}(b^x) = b^x\,\ln b}$$

Proof. Use $e^{\ln b} = b$. $b^x = (e^{\ln b})^x = e^{(\ln b)x}$ □

$\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{(\ln b)x}\right)$ $\quad \frac{d}{dx}(e^x) = e^x$

$u = (\ln b)x$
$\frac{du}{dx} = \ln b$

$= \frac{d}{dx}(e^u) = \frac{d}{du}(e^u)\frac{du}{dx} = e^u\,\ln b$
$= e^{(\ln b)x}\,\ln b$
$= (e^{\ln b})^x\,\ln b$
$= b^x\,\ln b$

$$\boxed{\int b^x\,dx = \frac{b^x}{\ln b} + c,\ b \neq 1}$$

Problem 5: Evaluate the following:

1. $\frac{d}{dx}\Big(\log_{10}(2 + \sin x)\Big)$

$\frac{d}{dx}(\log_b x) = \frac{1}{x\ln b}$

$u = 2 + \sin x$

$\frac{d}{dx}(\log_{10} u) = \frac{d}{du}(\log_{10} u)\frac{du}{dx} = \frac{1}{u\ln 10}\cdot\frac{d}{dx}(2 + \sin x)$
$= \frac{1}{u\ln 10}\cdot\cos x = \frac{\cos x}{(2+\sin x)\ln 10}$

2. $\frac{d}{dx}(10^{x^2})$ $\quad u = x^2$

$$\boxed{\frac{d}{dx}(b^x) = b^x \ln b}$$

$$\frac{d}{dx}(10^u) = \frac{d}{du}(10^u)\frac{du}{dx} = (10^u \ln 10)\frac{d}{dx}(x^2) = (10^u \ln 10)2x$$

$$= 2x\, 10^{x^2} \ln 10$$

3. $\int_0^5 2^x \, dx$

$$\boxed{\int b^x \, dx = \frac{b^x}{\ln b} + C}$$

$$= \frac{2^x}{\ln 2}\Big|_0^5 = \frac{2^5}{\ln 2} - \frac{2^0}{\ln 2} = \frac{32-1}{\ln 2} = \frac{31}{\ln 2}$$

Logarithmic Differentiation:

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the properties of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

Problem 6: Differentiate

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}.$$

Step 1

$$\ln y = \ln\left[\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right] = \ln\left(x^{3/4}\sqrt{x^2+1}\right) - \ln(3x+2)^5$$

$$= \ln x^{3/4} + \ln\sqrt{x^2+1} - \ln(3x+2)^5$$

$$= \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Step 2

$$\frac{d}{dx}(\ln y) = \frac{3}{4}\frac{d}{dx}(\ln x) + \frac{1}{2}\frac{d}{dx}\left(\ln(x^2+1)\right) - 5\frac{d}{dx}\left[\ln(3x+2)\right]$$

$$\Rightarrow \frac{d}{dy}(\ln y)\frac{dy}{dx} = \frac{3}{4x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot\frac{d}{dx}(x^2+1) - 5\cdot\frac{1}{3x+2}\cdot\frac{d}{dx}(3x+2)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{3}{4x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - \frac{5}{3x+2}\cdot 3$$

Step 3

$$\frac{dy}{dx} = \underbrace{\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}}_{y}\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right]$$

The Power Rule (General Version): If n is any real number and $f(x) = x^n$, then

$$f'(x) = nx^{n-1} .$$

$y = x^n$

$|y| = |x^n| = |x|^n$

$\ln|y| = \ln|x|^n = n \ln|x|$

$\frac{d}{dx}(\ln|y|) = \frac{d}{dx}(n \ln|x|) \Rightarrow \frac{d}{dy}(\ln|y|)\frac{dy}{dx} = n\frac{d}{dx}(\ln|x|)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = n\frac{1}{x} = \frac{n}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{n}{x} \times y = \frac{n}{x} \times x^n = n x^{n-1}$

$\frac{d}{dx}(x^{2.45}) = 2.45\, x^{1.45}$

$\frac{d}{dx}(x^{\pi}) = \pi x^{\pi - 1}$

The following four cases may occur for exponents and bases:

1. $\frac{d}{dx}(b^n) = 0$ (b and n are constants)
2. $\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$
3. $\frac{d}{dx}[b^{g(x)}] = b^{g(x)} (\ln b)\, g'(x)$ $\quad \frac{d}{dx}(b^x) = b^x \ln b$
4. $\frac{d}{dx}[f(x)]^{g(x)}$: Use logarithmic differentiation.

Problem 7: Differentiate

$$y = x^{\sqrt{x}} .$$

Step 1 $\ln y = \ln x^{\sqrt{x}}$

$\Rightarrow \ln y = \sqrt{x} \ln x$

Step 2 $\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sqrt{x} \ln x)$

$\Rightarrow \frac{d}{dy}(\ln y)\frac{dy}{dx} = (\sqrt{x})' \ln x + \sqrt{x}(\ln x)'$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \ln x + \sqrt{x}\left(\frac{1}{x}\right) = \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$

Step 3 $\frac{1}{y}\frac{dy}{dx} = \frac{\ln x + 2}{2\sqrt{x}} \Rightarrow \frac{dy}{dx} = y\left[\frac{\ln x + 2}{2\sqrt{x}}\right]$

$\Rightarrow \frac{dy}{dx} = x^{\sqrt{x}}\left(\frac{\ln x + 2}{2\sqrt{x}}\right)$

$\boxed{f(x) = x^x} \Rightarrow y = x^x$, $\ln y = x \ln x \rightarrow \frac{1}{y}\frac{dy}{dx} = \ln x + 1$

$\hookrightarrow \frac{dy}{dx} = x^x(\ln x + 1)$

The number e as a limit:

$e \simeq 2.71$

$x = \frac{1}{n} \Rightarrow n = \frac{1}{x}$

$$\boxed{e = \lim_{x\to 0}(1+x)^{1/x} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}$$

Proof. Let $f(x) = \ln x$. Then $f'(1) = 1$. Now use definition of f'. □

$f'(x) = \frac{1}{x}$ $\xrightarrow{x=1}$

$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\ln(1+h) - \cancel{\ln(1)}^{\,0}}{h}$

$\Rightarrow \boxed{\lim_{h\to 0} \frac{\ln(1+h)}{h} = 1} \Rightarrow e^{\lim_{h\to 0}\frac{\ln(1+h)}{h}} = e^1$

$\Rightarrow \lim_{h\to 0} e^{\frac{\ln(1+h)}{h}} = e$

$\Rightarrow \lim_{h\to 0} e^{\ln(1+h)^{1/h}} = e \Rightarrow \boxed{\lim_{h\to 0}(1+h)^{\frac{1}{h}} = e}$

Example $\lim_{x\to 0}(1+4x)^{\frac{1}{x}}$

$4x = h$. As $x\to 0$, $4x \to 0 \Rightarrow h\to 0$

$L = \lim_{h\to 0}(1+h)^{1/(h/4)} = \lim_{h\to 0}(1+h)^{4/h} = \left[\lim_{h\to 0}(1+h)^{\frac{1}{h}}\right]^4$

$= e^4$