The Derivative at a Point

The derivative of a function $f(x)$ at $x = a$, denoted by $f'(a)$, is defined to be the limit

$$f'(a) := \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Problem 1 Evaluate the limit

$$\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$$

for the following functions:-

1. $f(x) = x^2$
2. $f(x) = x^3$
3. $f(x) = \frac{1}{x}$
4. $f(x) = \frac{1}{x^2}$
5. $f(x) = \sqrt{x}$

Your answers would be the value of derivative of the given function f at $x = 1$.

1. $$\lim_{x\to 1} \frac{x^2 - 1^2}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x\to 1} (x+1) = 1+1 = 2$$

2. $$\lim_{x\to 1} \frac{x^3 - 1^3}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x\to 1} (x^2+x+1) = 1^2+1+1 = 3$$

3. $$\lim_{x\to 1} \frac{\frac{1}{x} - \frac{1}{1}}{x-1} = \lim_{x\to 1} \frac{\frac{1-x}{x}}{x-1} = \lim_{x\to 1} \frac{-\cancel{(x-1)}}{x\cancel{(x-1)}} = \frac{-1}{1} = -1$$

4. $$\lim_{x\to 1} \frac{\frac{1}{x^2} - \frac{1}{1^2}}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x^2} - 1}{x-1} = \lim_{x\to 1} \frac{1-x^2}{x^2(x-1)}$$
$$= \lim_{x\to 1} \frac{-(x^2-1)}{x^2(x-1)} = \lim_{x\to 1} \frac{-\cancel{(x-1)}(x+1)}{x^2\cancel{(x-1)}} = \frac{-(1+1)}{1^2} = -2$$

5. $$\lim_{x\to 1} \frac{\sqrt{x} - \sqrt{1}}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \times \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to 1} \frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x}+1)}$$
$$= \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

The Derivative as a Function

The derivative of a function $f(x)$, denoted by $f'(x)$, is defined to be the function whose value at a given input x is the limit

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Problem 2 Evaluate the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

for the following functions:-

1. $f(x) = x^2$ $\longrightarrow$ $2x$
2. $f(x) = x^3$ $\longrightarrow$ $3x^2$
3. $f(x) = \frac{1}{x}$ $\longrightarrow$ $-1\ x^{-1-1}$
4. $f(x) = \frac{1}{x^2}$ $\longrightarrow$ $-2\ x^{-2-1}$
5. $f(x) = \sqrt{x}$ $\longrightarrow$ $\frac{1}{2}\ x^{\frac{1}{2}-1}$

nx^{n-1}

$\Uparrow$

Your answers would be the derivative of the given function f.

Try to see the pattern in your answers and find the derivative of $f(x) = x^n$.

① $\displaystyle \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h}$

$\displaystyle = \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = 2x+0 = \underline{\underline{2x}}$

② $\displaystyle \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$

$\displaystyle = \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}} = 3x^2 + 3x(0) + 0^2 = \underline{\underline{3x^2}}$

③ $\displaystyle \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{hx(x+h)} = \lim_{h\to 0} \frac{x-x-h}{hx(x+h)}$

$\displaystyle = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}x(x+h)} = \frac{-1}{x(x+0)} = \underline{\underline{\frac{-1}{x^2}}}$

(4)
$$\lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h \to 0} \frac{x^2 - (x+h)^2}{h x^2 (x+h)^2}$$

$$= \lim_{h \to 0} \frac{x^2 - (x^2 + 2xh + h^2)}{h x^2 (x+h)^2}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{h x^2 (x+h)^2} = \lim_{h \to 0} \frac{-2xh - h^2}{h x^2 (x+h)^2}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2x - h)}{\cancel{h} x^2 (x+h)^2} = \frac{-2x - 0}{x^2 (x+0)^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}$$

(assuming $x \neq 0$)

(5)
$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \times \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

$$= \lim_{h \to 0} \frac{(\cancel{x} + h) - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$