M16600 Lecture Notes

Section 10.2: Calculus with Parametric Curve

■ **Section 10.2** textbook exercises, *page 695*: #3, 4, 5, 7(a), 17, 11, 13. For #11, 13, only compute $\frac{d^2y}{dx^2}$, don't need to do concavity.

GOALS: Given a parametric curve $x = x(t)$ and $y = y(t)$

- Compute $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$
- Find the **slope of the tangent line** to the given parametric curve at a point.
- Write an **equation of the tangent line** to the given parametric curve at a point.
- Find points on parametric curves such that the tangent line is *horizontal* or *vertical*

Recall:

- Let $y = y(x)$ be a curve in the xy-plane (e.g $y = x^2 + 1$). Then

  the **SLOPE** of the TANGENT LINE to $y = y(x)$ at the point $x = a$ is $y'(a)$.

- The point-slope formula for **an equation of a line** is $y - y_1 = m(x - x_1)$ where (x_1, y_1) is one point on the line and m is the slope of the line.

Given a parametric curve: $x = x(t), y = y(t)$. We can compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$. How do we find $\frac{dy}{dx}$ so that we can compute the slope of a tangent line to this parametric curve?

Note that we can write $y(t)$ as the composite function $y(t) = y\big(x(t)\big)$, where $x(t)$ is the inner function. Then by the Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Therefore,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

Geometrically, $\frac{dy}{dx}$ represents the ***slope formula*** of tangent lines to the parametric curve $x = x(t), y = y(t)$ at any point. To find the ***slope of the tangent line*** at one specific when $t = a$, we evaluate $\frac{dy}{dx}$ at $t = a$. Notation: $\left.\dfrac{dy}{dx}\right|_{t=a}$.

Given parametric equations $x = x(t), y = y(t)$, the second derivative of y with respect to x is

$$\left[\ \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}\ \right]$$

Example 1: Let $x = t^2 - 3$ and $y = t^3 - 3t$. Find

(a) $\frac{dx}{dt}$ and $\frac{dy}{dt}$

$$x = t^2 - 3 \Rightarrow \frac{dx}{dt} = 2t$$

$$y = t^3 - 3t \Rightarrow \frac{dy}{dt} = 3t^2 - 3$$

(b) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t} \Rightarrow \boxed{\frac{dy}{dx} = \frac{3(t^2-1)}{2t}}$

(c) the slope of the tangent line to the given parametric curve when $t = -2$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=-2} = \frac{3((-2)^2-1)}{2(-2)} = \frac{3(4-1)}{-4} = \frac{-9}{4}$$

(d) an equation of the tangent line to the given parametric curve when $t = -2$

$$y - y_1 = m(x - x_1) \Rightarrow y - y_1 = \frac{-9}{4}(x - x_1)$$

$$y_1 = y(-2) = (-2)^3 - 3(-2) = -8 + 6 = -2$$

$$x_1 = x(-2) = (-2)^2 - 3 = 4 - 3 = 1$$

$$y - (-2) = \frac{-9}{4}(x-1) \Rightarrow y + 2 = \frac{-9}{4}(x-1) \Rightarrow 4y + 8 = -9x + 9 \Rightarrow \boxed{9x + 4y = 1}$$

(e) an equation of the tangent line to the given parametric curve at the point $(-2, 2)$

$$x = t^2 - 3, \quad y = t^3 - 3t, \quad \frac{dy}{dx} = \frac{3(t^2-1)}{2t}$$

$$(x, y) = (-2, 2) \Rightarrow t^2 - 3 = -2, \quad t^3 - 3t = 2$$

$$\Downarrow$$

$$t^2 = 3 - 2 = 1$$

$$\Rightarrow t^2 = 1 \Rightarrow t = \pm 1$$

not a solution

$t = 1$: $1^3 - 3(1) = -2 \neq 2$

$t = -1$: $(-1)^3 - 3(-1) = -1 + 3 = 2$

$$\Rightarrow (-2, 2) \text{ corresponds to } t = -1 \Rightarrow \left.\frac{dy}{dx}\right|_{t=-1} = \frac{3((-1)^2-1)}{2(-1)}$$

$x_1 = -2, \ y_1 = 2$

$$= \frac{3(1-1)}{-2} = 0$$

$\Rightarrow m = 0 \Rightarrow (y-2) = 0\,(x-(-2))$

$\Rightarrow y-2 = 0 \Rightarrow y = 2$

$\uparrow$ tangent line at $(-2, 2)$

$$\boxed{\begin{array}{l} \text{when } m=\infty,\ \frac{1}{m}=0 \\ x - x_1 = \frac{1}{m}(y-y_1) \\ \Rightarrow x - x_1 = 0 \Rightarrow x = x_1 \end{array}}$$

Example 2: Find an equation of the tangent line to the parametric curve

$$x = t - \sin t, \qquad y = 1 - \cos t$$

at $t = \pi/3$.

$x = t - \sin t \Rightarrow \frac{dx}{dt} = 1 - \cos t$

$y = 1 - \cos t \Rightarrow \frac{dy}{dt} = \sin t$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1-\cos t}}$$

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3}}{1-\cos\pi/3} = \frac{\frac{\sqrt{3}}{2}}{1-\frac{1}{2}} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

$\Rightarrow \boxed{m = \sqrt{3}}$

$(x_1, y_1) = (x(\pi/3), y(\pi/3))$

$x_1 = x(\pi/3) = \frac{\pi}{3} - \sin(\pi/3) = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$

$y_1 = y(\pi/3) = 1 - \cos(\pi/3) = 1 - \frac{1}{2} = 1/2$

$$y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)$$

$$y - \frac{1}{2} = \sqrt{3}\,x - \frac{\pi}{\sqrt{3}} + \frac{3}{2} \Rightarrow y = \sqrt{3}\,x - \frac{\pi}{\sqrt{3}} + 2$$

Facts:

- The tangent line is ***horizontal*** at the values of t where $\dfrac{dy}{dx} = 0$.
- The tangent line is ***vertical*** at the values of t where $\dfrac{dy}{dx}$ is undefined.

Example 3: Let $\mathcal{C}$ be the parametric curve given by $x = t^3 - 3t$ and $y = t^3 - 3t^2$. Find

(a) Find the points on the curve $\mathcal{C}$ where the tangent line is horizontal.

diff. diff.

– Find $\dfrac{dy}{dx}$ first. $\dfrac{dx}{dt} = 3t^2 - 3$, $\dfrac{dy}{dt} = 3t^2 - 6t$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 6t}{3t^2 - 3}$$

Tangent line is horizontal when $\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{3t^2 - 6t}{3t^2 - 3} = 0$

$$\Rightarrow 3t^2 - 6t = 0 \Rightarrow 3t(t-2) = 0 \Rightarrow t = 0 \text{ or } t - 2 = 0$$

$$\Rightarrow t = 0 \text{ or } t = 2$$

$t = 0 \Rightarrow (0^3 - 3(0), 0^3 - 3(0)^2) = (0, 0)$, $t = 2 \Rightarrow (2^3 - 3(2), 2^3 - 3(2)^2) = (2, -4)$

(b) Find the points on the curve $\mathcal{C}$ where the tangent line is vertical.

$\dfrac{dy}{dx} = \dfrac{3t^2 - 6t}{3t^2 - 3}$, $x(t) = t^3 - 3t$, $y(t) = t^3 - 3t^2$

Tangent line is vertical when $\dfrac{dy}{dx}$ is undefined, i.e. the denominator of $\dfrac{dy}{dx}$ becomes 0.

$$\Rightarrow 3t^2 - 3 = 0$$

$$\Rightarrow 3t^2 = 3 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$$

$t = 1 \Rightarrow (1^3 - 3(1), 1^3 - 3(1)^2) = (-2, -2)$

$t = -1 \Rightarrow ((-1)^3 - 3(-1), (-1)^3 - 3(-1)^2) = (-1+3, -1-3) = (2, -4)$

Example 4: Let $x = 2t^3$ and $y = 2 + t^2$, find $\dfrac{d^2y}{dx^2}$.

$$\frac{dx}{dt} = 6t^2 \quad , \quad \frac{dy}{dt} = 2t$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{6t^2} = \frac{1}{3t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{1}{3t}\right)}{6t^2}$$

$$= \frac{\frac{1}{3}\frac{d}{dt}\left(\frac{1}{t}\right)}{6t^2} = \frac{\frac{1}{3}\left(\frac{-1}{t^2}\right)}{6t^2} = \frac{\frac{-1}{3t^2}}{6t^2}$$

$$= \frac{-1}{18t^4}$$