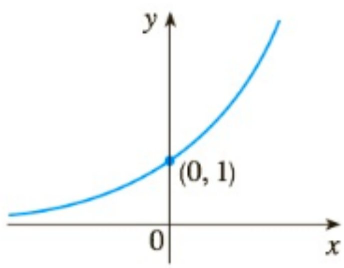
M16600 Lecture Notes

Section 6.2: Exponential Functions and Their Derivatives

SUMMARY:

- The general Exponential Functions $f(x) = b^x$ and their properties.
- The Natural Exponential Functions $f(x) = e^x$ and its calculus facts
- The derivative of e^x: $\dfrac{d}{dx}(e^x) = e^x$
- The integral (or antiderivative) of e^x: $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\lim_{x\to\infty} e^x = \infty$ and $\displaystyle\lim_{x\to-\infty} e^x = 0$
- The graph of $y = e^x$



I. Exponential Functions

Definition: An ***exponential function*** is a function of the form

$$f(x) = b^x$$

where b is a positive constant.

Warning: Exponential functions are not the same as ***power functions***

$x^2 \neq 2^x$

$x = 3 \Rightarrow x^2 = 9$ but $2^3 = 8$

$8^{-2/3} = \dfrac{1}{8^{2/3}} = \dfrac{1}{4}$

- If $x = n$, a positive number, then

$$b^n = \underbrace{b \cdot b \cdot b \cdots b \cdot b}_{n \text{ factors}}$$

$2^4 = 2\cdot2\cdot2\cdot2 = 16$

$2^2 = 2\cdot2 = 4$

$$b^{-n} = \frac{1}{b^n}$$

$2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{2\cdot2\cdot2} = \dfrac{1}{8}$, $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = \sqrt[3]{4\times4\times4} = 4$

$x = \frac{n}{d}$

$\sqrt[d]{b^n}$

- If $x = 0$, then $b^0 = 1$
- If x is a rational number then $b^x = b^{n/d} = \sqrt[d]{b^n}$

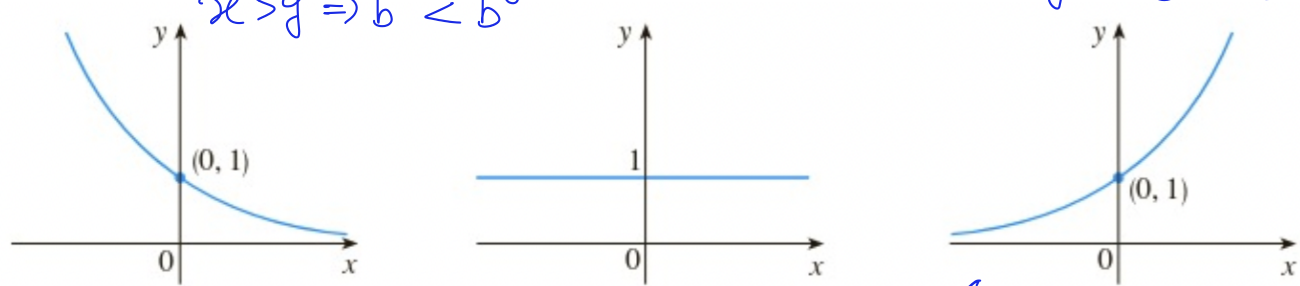
We can also define b^x for any irrational number x (see the discussion in the textbook, pages 408 and 409).

Properties of Exponential Functions: If $b > 0$ and $b \neq 1$, then $f(x) = b^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. If $a, b > 0$ and $x, y \in \mathbb{R}$, then we have the following

The base is either: $\underbrace{0 < b < 1}$ or $\underbrace{b > 1}$

- $b^x > 0$ for all x Range of $b^x = (0, \infty)$
- **Laws of Exponents:** $b^{x+y} = b^x b^y$, $b^{x-y} = \dfrac{b^x}{b^y}$, $(b^x)^y = b^{xy}$, $(ab)^x = a^x b^x$.
- Graphs and limits

$x > y \Rightarrow b^x < b^y$ $x > y \Rightarrow b^x > b^y$



(a) $y = b^x$, $0 < b < 1$ (b) $y = 1^x$ (c) $y = b^x$, $b > 1$

0.3^x, 0.75^x

$\boxed{\lim_{x \to \infty} b^x = 0, \quad \lim_{x \to -\infty} b^x = \infty}$

$\boxed{\begin{array}{l} \lim_{x \to \infty} b^x = \infty \\ \lim_{x \to -\infty} b^x = 0 \end{array}}$

2^x, 5^x, 3^x

7.5^x, π^x

$\pi \approx 3.14$

Example 1: (a) Find $\lim_{x \to \infty} (2^{-x} - 1)$.

$$\lim_{x \to \infty} (2^{-x} - 1)$$

$$= \lim_{x \to \infty} 2^{-x} - \lim_{x \to \infty} 1$$

$$= 0 - 1 = -1$$

$$\lim_{x \to \infty} \frac{1}{2^x} = \lim_{x \to \infty} \left(\frac{1}{2}\right)^x = 0$$

$\left(\frac{1}{2}\right)^1 = 0.5$

$\left(\frac{1}{2}\right)^2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = 0.25$

$\left(\frac{1}{2}\right)^3 = \frac{1}{8} = 0.125$

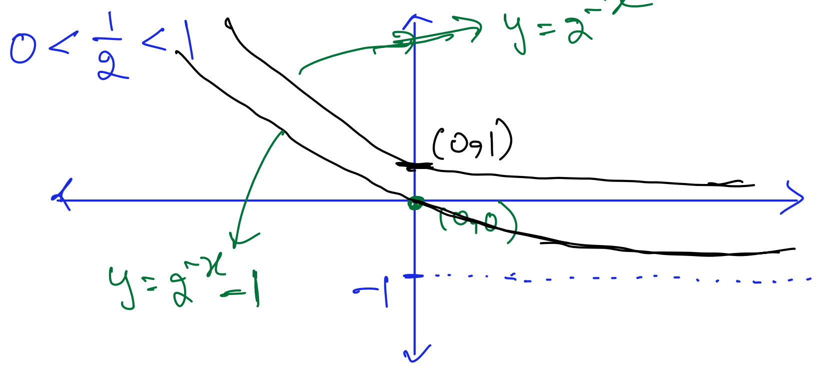
$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

$= \dfrac{1}{\lim_{x\to\infty} 2^x} = \dfrac{1}{\infty} = 0$

(b) Sketch the graph of the function $y = 2^{-x} - 1$.



- First sketch $y = 2^{-x} = \left(\frac{1}{2}\right)^x$
- Then shift the graph obtained down by 1 unit

Introducing the *Natural Exponential Function* $f(x) = e^x$, where e is an irrational number. Its approximate value to 20 decimal places is

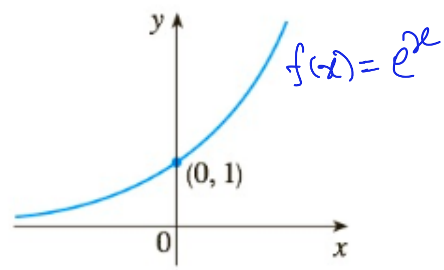
$f(x) = e^x$

$$e \approx 2.71828182845904523536 \quad , \quad e \simeq 2.72 \quad , \quad e \simeq 2.7$$

Read the discussion on *Derivatives of Exponential Functions*, page 412, for the motivation of defining the number e.

Some *Calculus facts* of the natural exponential function e^x.

- The derivative of e^x: $\dfrac{d}{dx}(e^x) = e^x$ $\Rightarrow f'(x) = f(x)$
- The integral of e^x: $\displaystyle\int e^x\,dx = e^x + C$
- $\lim\limits_{x\to\infty} e^x = \infty$ and $\lim\limits_{x\to-\infty} e^x = 0$
- The graph of $y = e^x$ *[Figure 14, section 6.2, textbook]*



Example 2: Rewrite the following expression into the form e^P, where P is some algebraic expression.

1. $e^x e^{x^2} = e^{x+x^2}$

2. $\dfrac{1}{e^x} = \dfrac{e^0}{e^x} = e^{0-x} = e^{-x}$

3. $\dfrac{e^{3x}}{e^2} = e^{3x-2}$

4. $(e^{x^2})^4 = e^{4x^2}$

Example 3: Differentiate

(a) $f(x) = e^{-3} + x^{-3} - e^x + e^{14}$

$$f'(x) = \frac{d}{dx}\left(e^{-3} + x^{-3} - e^x + e^{14}\right) = \frac{d}{dx}\left(e^{-3}\right) + \frac{d}{dx}\left(x^{-3}\right) - \frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(e^{14}\right)$$

$$= 0 + (-3)x^{-3-1} - e^x + 0$$

$$= -3x^{-4} - e^x = \frac{-3}{x^4} - e^x$$

(b) $g(x) = e^{x^7 - 4x}$

$g(x) = f(h(x))$ where $h(x) = x^7 - 4x$, $f(x) = e^x$

$f(x^7 - 4x) = e^{x^7-4x}$

$f'(x) = e^x$

$h'(x) = 7x^6 - 4$

$$= f'(h(x))\, h'(x)$$

$$= e^{x^7-4x} \cdot (7x^6 - 4)$$

$$= (7x^6 - 4)\, e^{x^7-4x}$$

$$\frac{d}{dx}\left[e^{x^7-4x}\right] = \frac{d}{dx}\left[e^z\right] = \frac{d}{dz}\left(e^z\right)\frac{dz}{dx}$$

let $z = x^7 - 4x$

$$= e^z \cdot (7x^6 - 4)$$

$$= (7x^6 - 4)\, e^{x^7-4x}$$

$$\boxed{\frac{d}{dx}\left[e^{f(x)}\right] = f'(x)\, e^{f(x)}}$$

$$\frac{d}{dx}\left(e^{x^2}\right) = 2x\, e^{x^2}$$

$$\frac{d}{dx}\left(e^{x^3+x}\right) = (3x^2+1)\, e^{x^3+x}$$

(c) $y = \sqrt{x}\, e^{x/5} - \sin(5x)$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x}\, e^{x/5}\right) - \frac{d}{dx}\left(\sin(5x)\right)$$

$$= (\sqrt{x})' e^{x/5} + \sqrt{x}\left(e^{x/5}\right)' - 5\cos(5x)$$

$$= \frac{1}{2\sqrt{x}} e^{x/5} + \sqrt{x}\, \frac{1}{5} e^{x/5} - 5\cos(5x)$$

$$= \frac{e^{x/5}}{2\sqrt{x}} + \frac{\sqrt{x}\, e^{x/5}}{5} - 5\cos(5x)$$

$$\frac{d}{dx}\left(\sin(5x)\right) = \frac{d}{dx}\left(\sin z\right) = \frac{d}{dz}\left(\sin z\right)\frac{dz}{dx}$$

where $z = 5x$

$$= \cos z\, (5)$$

$$= 5\cos(5x)$$

(d) $h(x) = \dfrac{(e^x)^{23}}{1 - e^x}$ $\to e^{23x}$

$$h'(x) = \frac{(1-e^x)\left(e^{23x}\right)' - e^{23x}(1-e^x)'}{(1-e^x)^2}$$

$$= \frac{(1-e^x)\, 23\, e^{23x} - e^{23x}(0 - e^x)}{(1-e^x)^2}$$

(e) $f(t) = \tan(e^t)$

$$f'(t) = \frac{d}{dt}\left[\operatorname{Tan}(e^t)\right]$$

$$= \frac{d}{dt}\left(\operatorname{Tan}(z)\right) \text{ where } z = e^t$$

$$= \frac{d}{dz}\left(\operatorname{Tan} z\right)\frac{dz}{dt} = \sec^2(z)\frac{dz}{dt}$$

$$= \sec^2(e^t)\ e^t = e^t \sec^2(e^t)$$

(f) $y = e^{4\sin(x)}$

$$\frac{dy}{dx} = \frac{d}{dx}\left[e^{4\sin x}\right] = \frac{d}{dx}\left[e^z\right]$$

Let $z = 4\sin x$

$$\Rightarrow \frac{dz}{dx} = 4\cos x$$

$$= \frac{d}{dz}(e^z)\ \frac{dz}{dx}$$

$$= e^z\ (4\cos x)$$

$$= e^{4\sin x}\ (4\cos x)$$

$$= \frac{23e^{23x}(1-e^x) - e^{23x}\cdot(-e^x)}{(1-e^x)^2}$$

$$= \frac{23e^{23x}(1-e^x) + e^{23x+x}}{(1-e^x)^2}$$

$$= \frac{23e^{23x}(1-e^x) + e^{24x}}{(1-e^x)^2}$$

$$= \frac{23e^{23x} - 23e^{24x} + e^{24x}}{(1-e^x)^2}$$

$$= \frac{23e^{23x} - 22e^{24x}}{(1-e^x)^2}$$

Example 4: Evaluate the integral

(a) $\displaystyle\int (e^x - x^e + 1)\, dx$

$$= \int e^x\, dx - \int x^e\, dx + \int 1\, dx$$

$$= e^x - \frac{x^{e+1}}{e+1} + \frac{x^{0+1}}{0+1} + C$$

$$= e^x - \frac{x^{e+1}}{e+1} + x + C$$

$$\int e^x\, dx = e^x + C$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$$

$$\int e^u\, du = e^u + C$$

$$\frac{1}{e^x} = \frac{e^0}{e^x} = e^{0-x} = e^{-x}$$

(b) $\displaystyle\int_0^1 \frac{3}{e^x}\,dx$

$$I = \int \frac{3}{e^x}\,dx = \int 3e^{-x}\,dx = 3\int e^{-x}\,dx$$

$$I = 3\int e^u(-du)$$

let $u = -x \Rightarrow \frac{du}{dx} = -1 \Rightarrow du = -dx$

$\Rightarrow -du = dx$

Diff w.r.t. x

$$= -3\int e^u\,du = -3e^u + C$$

$$= -3e^{-x} + C$$

$$\Rightarrow \int_0^1 \frac{3}{e^x}\,dx = -3e^{-x}\Big|_0^1$$

$$= -3e^{-1} - (-3e^{-0})$$

$$= -3e^{-1} - (-3) = -3e^{-1} + 3$$

(c) $\displaystyle\int xe^{x^2}\,dx$

let $u = x^2$

$\Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \rightarrow dx = \frac{1}{2x}du$

$$I = \int x e^u\,dx$$

$$= \int \cancel{x} e^u \frac{1}{2\cancel{x}}du = \int e^u \frac{1}{2}du = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C$$

$$= \frac{1}{2}e^{x^2} + C$$

(d) $\displaystyle\int e^x \sqrt[4]{e + e^x}\,dx$

let $u = e + e^x$

$\Rightarrow \frac{du}{dx} = e^x \Rightarrow du = e^x\,dx$

$\Rightarrow dx = \frac{du}{e^x}$

* If we had $\int x^2 e^{x^2}\,dx$ then the substitution $u = x^2$ would not have worked.

$$I = \int \cancel{e^x}\sqrt[4]{u}\,\frac{du}{\cancel{e^x}} = \int \sqrt[4]{u}\,du = \int u^{\frac{1}{4}}\,du = \frac{u^{\frac{1}{4}+1}}{\frac{1}{4}+1} + C$$

$$= \frac{u^{5/4}}{5/4} + C = \frac{4}{5} u^{5/4} + C$$

$$= \frac{4}{5}\left(e + e^x\right)^{5/4} + C$$

(e) $\displaystyle\int_{\pi/2}^{\pi} \sin x \, e^{\cos x}\, dx$

Let $u = \cos x$

$\Rightarrow \dfrac{du}{dx} = -\sin x \Rightarrow du = -\sin x \, dx \rightarrow dx = \dfrac{du}{-\sin x}$

$$I = \int_{\cos\frac{\pi}{2}}^{\cos\pi} \cancel{\sin x}\, e^u \frac{du}{-\cancel{\sin x}} = \int_{\cos\pi/2}^{\cos\pi} -e^u \, du = -\int_0^{-1} e^u \, du$$

$$= -e^u\Big|_0^{-1} = -e^{-1} - \left(-e^0\right) = -e^{-1} + e^0 = 1 - e^{-1}$$

(f) $\displaystyle\int \frac{2e^x}{(3+e^x)^3}\, dx$

Let $u = 3 + e^x \Rightarrow \dfrac{du}{dx} = e^x \Rightarrow du = e^x \, dx \Rightarrow dx = \dfrac{du}{e^x}$

$$I = \int \frac{2\cancel{e^x}}{u^3} \frac{du}{\cancel{e^x}} = \int \frac{2}{u^3} du = 2\int \frac{1}{u^3} du = 2\int u^{-3} du$$

$$= 2\,\frac{u^{-3+1}}{-3+1} + C = 2\,\frac{u^{-2}}{-2} + C = \frac{-1}{u^2} + C$$

$$= \frac{-1}{(3+e^x)^2} + C$$

Section 6.2 exercises, *page 418*, #7, <u>9</u>, 23, 24, 26, 31, 33, 37, 39, 42, 83, 85, 86, 87, 90, 91, 94. If computing the derivative, you don't need to simplify the answers. Underline problems are optional.