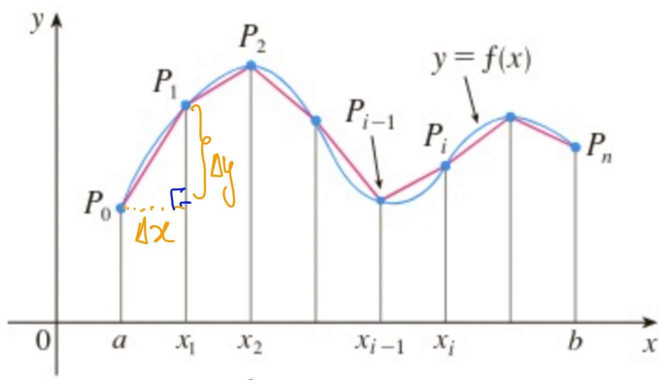
M16600 Lecture Notes

Section 8.1: Arc Length

■ **Section 8.1** textbook exercises, *page 589*: # 3, 5, 14, 11, 21.

How do we find the length of a curve $y = f(x)$, where $a \le x \le b$?



$\Delta x = x_{i+1} - x_i$

$\Delta y = f(x_{i+1}) - f(x_i)$

Length of the segment $P_i P_{i+1}$

$= \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x = \boxed{\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

length of $y = f(x)$ from $x = a$ to $x = b$

As $n \to \infty$, $\Delta x \to dx$, $\Delta y \to dy$

The Arc Length Formula. If $f'(x)$ is continuous on $[a, b]$, then the length of the curve $y = f(x)$, where $a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$\frac{dy}{dx} = f'(x)$

or we can use Leibniz notation for derivatives and write the arc length formula as

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Example 1: Find the length of the curve $y = \frac{2}{3}x^{3/2}$ from the point $(1, \frac{2}{3})$ to the point $(2, \frac{4}{3}\sqrt{2})$.

$f(1)$

$f(2) = \frac{2}{3} 2^{3/2} = \frac{4}{3}\sqrt{2}$

$a = 1$, $b = 2$

$y = \frac{2}{3} x^{3/2} \Rightarrow \frac{dy}{dx} = \frac{2}{3} \times \frac{3}{2} x^{\frac{3}{2}-1} = x^{1/2}$

$$L = \int_1^2 \sqrt{1 + (x^{1/2})^2}\, dx = \int_1^2 \sqrt{1+x}\, dx = \left.\frac{(1+x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^2$$

$$\Rightarrow L = \frac{2}{3}(1+x)^{3/2}\Big|_1^2 = \frac{2}{3}(1+2)^{3/2} - \frac{2}{3}(1+1)^{3/2}$$

$$= \frac{2}{3}\left[3^{3/2} - 2^{3/2}\right] = \frac{2}{3}\left(3\sqrt{3} - 2\sqrt{2}\right)$$

Example 2: Find the exact length of the curve $y = \ln(\sec x)$, where $0 \le x \le \pi/4$.

$a = 0, \quad b = \frac{\pi}{4}$

$$y = \ln(\sec x) \Rightarrow \frac{dy}{dx} = \frac{1}{\sec x} \times (\sec x)' = \frac{1}{\sec x} \times \sec x \operatorname{Tan} x$$

$$= \operatorname{Tan} x$$

$\boxed{\sec^2 x = 1 + \operatorname{Tan}^2 x}$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + (\operatorname{Tan} x)^2}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \operatorname{Tan}^2 x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\frac{\pi}{4}} \sec x\, dx$$

sec x is positive for $0 \le x \le \pi/4$

$$= \ln\left|\operatorname{Tan} x + \sec x\right|\Big|_0^{\pi/4}$$

$$= \ln\left|\operatorname{Tan}\frac{\pi}{4} + \sec\frac{\pi}{4}\right| - \ln\left|\operatorname{Tan} 0 + \sec 0\right|$$

$$= \ln\left|1 + \sqrt{2}\right| - \ln|0 + 1| = \ln(1+\sqrt{2}) - \ln 1$$

$$= \ln(1+\sqrt{2})$$