Section 1.6

(15) $$\lim_{t\to -3} \frac{t^2-9}{2t^2+7t+3} = L$$

At $t=-3$, denominator $= 2(-3)^2+7(-3)+3 = 2\times 9-21+3$
$= 18-21+3 = 0$

and numerator $= (-3)^2-9 = 9-9 = 0.$

$\Rightarrow$ we may be able to cancel out something from both numerator and denominator.

$$t^2-9 = t^2-3^2 = (t-3)(t+3)$$

$$2t^2+7t+3 = 2t^2+6t+t+3 = 2t(t+3)+1(t+3) = (t+3)(2t+1)$$

$$\Rightarrow L = \lim_{t\to -3} \frac{(t-3)\cancel{(t+3)}}{\cancel{(t+3)}(2t+1)} = \lim_{t\to -3} \frac{t-3}{2t+1} = \frac{-3-3}{2(-3)+1}$$

direct substitution rule.

$$\Rightarrow L = \frac{-6}{-6+1} = \frac{-6}{-5} \Rightarrow \boxed{L = \frac{6}{5}}$$

(18) $$\lim_{h\to 0} \frac{(2+h)^3-8}{h} = L \qquad (2+0)^3-8 = 8-8 = 0$$

At $h=0$, both numerator and denominator become 0.

$\Rightarrow$ we may be able to cancel out something from both.

$$(2+h)^3 = 2^3+h^3+3(2)(h)(2+h) = 8+h^3+6h(2+h)$$
$$= 8+h^3+12h+6h^2$$

$$\Rightarrow L = \lim_{h\to 0} \frac{\cancel{8}+h^3+12h+6h^2-\cancel{8}}{h} = \lim_{h\to 0} \frac{h^3+12h+6h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(h^2+12+6h)}{\cancel{h}} = \lim_{h\to 0} (h^2+12+6h)$$

$= 0^2 + 12 + 6\times 0$ (direct substitution rule).

$\Rightarrow \boxed{L = 12}$

(24) $\lim_{h\to 0} \frac{(3+h)^{-1} - 3^{-1}}{h} = L$

At $h=0$, both numerator and denominator become 0.

$\hookrightarrow (3+0)^{-1} - 3^{-1} = 3^{-1} - 3^{-1} = 0$

$\Rightarrow$ we may be able to cancel out something, and then use direct substitution.

$$\Rightarrow L = \lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h\to 0} \frac{\frac{3-(3+h)}{(3+h)3}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3} - \cancel{3} - h}{h(3+h)3} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(3+h)3}$$

$$= \lim_{h\to 0} \frac{-1}{(3+h)3} = \frac{-1}{(3+0)3} = \frac{-1}{9}$$

$$\Rightarrow \boxed{L = \frac{-1}{9}}$$

(31) $\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = L$

At $h=0$, both numerator and denominator become 0.

$\hookrightarrow (x+0)^3 - x^3 = x^3 - x^3 = 0.$

$\Rightarrow$ we try to simplify and cancel and then use direct substitution

$$(x+h)^3 = x^3 + h^3 + 3(x)(h)(x+h) = x^3 + h^3 + 3xh(x+h)$$
$$= x^3 + h^3 + 3x^2h + 3xh^2$$

$$\Rightarrow L = \lim_{h \to 0} \frac{\cancel{x^3} + h^3 + 3x^2h + 3xh^2 - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} \frac{h^3 + 3x^2h + 3xh^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(h^2 + 3x^2 + 3xh)}{\cancel{h}}$$

$$= \lim_{h \to 0} (h^2 + 3x^2 + 3xh) = 0^2 + 3x^2 + 3x \times 0 = 3x^2$$

$$\Rightarrow \boxed{L = 3x^2}$$

32) $$\lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = L$$

At $h=0$, both numerator and denominator become 0.

$$\hookrightarrow \frac{1}{(x+0)^2} - \frac{1}{x^2} = \frac{1}{x^2} - \frac{1}{x^2} = 0$$

$$\Rightarrow L = \lim_{h \to 0} \frac{x^2 - (x+h)^2}{h x^2 (x+h)^2} = \lim_{h \to 0} \frac{x^2 - (x^2 + h^2 + 2xh)}{h x^2 (x+h)^2}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} - \cancel{x^2} - h^2 - 2xh}{h x^2 (x+h)^2} = \lim_{h \to 0} \frac{-h^2 - 2xh}{h x^2 (x+h)^2}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-h - 2x)}{\cancel{h} x^2 (x+h)^2} = \lim_{h \to 0} \frac{-h - 2x}{x^2 (x+h)^2}$$

$$= \frac{-0 - 2x}{x^2 (x+0)^2} = \frac{-2x}{x^4}$$

$\hookrightarrow x \neq 0$ for it to be well-defined

$\Downarrow$ we can cancel x.

$$\Rightarrow \boxed{L = \frac{-2}{x^3}}$$

(37) $4x-9 \leq f(x) \leq x^2-4x+7$ for $x \geq 0$

Find $\lim_{x\to 4} f(x)$

By Squeeze Theorem,

$$\lim_{x\to 4}(4x-9) \leq \lim_{x\to 4} f(x) \leq \lim_{x\to 4}(x^2-4x+7)$$

direct substitution rule
(since these are polynomials)

$$\Rightarrow \quad 4(4)-9 \leq \lim_{x\to 4} f(x) \leq (4)^2-4(4)+7$$

$$\Rightarrow \quad 16-9 \leq \lim_{x\to 4} f(x) \leq 16-16+7$$

$$\Rightarrow \quad 7 \leq \lim_{x\to 4} f(x) \leq 7$$

$$\Rightarrow \quad \boxed{\lim_{x\to 4} f(x) = 7}$$

Section 1.8

(20) $f(x) = \begin{cases} \dfrac{x^2-x}{x^2-1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$. Check continuity at $a=1$

(Skip the graph of f for now)

$f(1) = 1$.

$\lim_{x\to 1} f(x) = \lim_{x\to 1} \dfrac{x^2-x}{x^2-1}$. At $x=1$, numerator $= 1^2-1=0$
denominator $= 1^2-1=0$
$\Rightarrow$ we may be able to cancel out.

$$= \lim_{x\to 1} \frac{x\cancel{(x-1)}}{\cancel{(x-1)}(x+1)} = \lim_{x\to 1}\frac{x}{x+1} = \frac{1}{1+1} = \frac{1}{2}$$

$\Rightarrow f(1) \neq \lim_{x\to 1} f(x) \Rightarrow$ <u>f is discontinuous at $x=1$</u>

(36) $\lim_{x\to\pi} \sin(x+\sin x) = L$

If f is continuous, then $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$

$\Rightarrow L = \lim_{x\to\pi} \sin(x+\sin x) = \sin\left(\lim_{x\to\pi}(x+\sin x)\right)$

x is continuous everywhere on $\mathbb{R}$

$\sin(x)$ is continuous everywhere on $\mathbb{R}$

$\Rightarrow$ $x+\sin(x)$ is continuous on $\mathbb{R}$.

If a function is continuous, we can use direct substitution rule.

follows from definition of continuity: $\lim_{x\to a} f(x) = f(a)$

$\Rightarrow L = \sin\left(\lim_{x\to\pi}(x+\sin x)\right) = \sin(\pi+\sin\pi)$

$= \sin(\pi+0) = \sin\pi = 0$

$\Rightarrow \boxed{L=0}$

(43) $f(x) = \begin{cases} x+2 & \text{if } x<0 \\ 2x^2 & \text{if } 0\le x\le 1 \\ 2-x & \text{if } x>1. \end{cases}$. Find Points of discontinuity.

$x+2$ (being a Polynomial) is continuous everywhere on $(-\infty,0)$

Similarly, $2x^2$ is continuous everywhere on $(0,1)$

and $2-x$ is continuous everywhere on $(1,\infty)$.

So, the only possible points of discontinuity are $x=0,1$.

<u>At $x=0$</u>

$LHL = \lim_{h\to0} f(0-h) = \lim_{h\to0} (0-h)+2 = \lim_{h\to0} -h+2 = 2$

$RHL = \lim_{h\to0} f(0+h) = \lim_{h\to0} 2(0+h)^2 = \lim_{h\to0} 2h^2 = 0.$

$f(0) = 2(0)^2 = 0.$

$LHL \neq RHL = f(0) \Rightarrow$ $\boxed{f \text{ is discontinuous at } x=0.}$

Since $RHL = f(0)$, $\boxed{f \text{ is continuous from right at } x=0.}$

At $x=1$

$$LHL = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} 2(1-h)^2 = 2(1)^2 = 2$$

$$RHL = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 2-(1+h) = 2-1 = 1$$

$$f(1) = 2(1)^2 = 2$$

$\Rightarrow f(1) = LHL \neq RHL \Rightarrow$ $\boxed{f \text{ is discontinuous at } x=1}$

Since, $LHL = f(1)$, $\boxed{f \text{ is continuous from left at } x=1}$

Graph :-