Math 11000- 9.1 Notes

9.1 → Composite Functions and Inverse Functions

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9.2

Composite Functions

ESSENTIALS

Composition of Functions

The composite function $f \circ g$, the composition of f and g, is defined as

$(f \circ g)(x) = f(g(x))$.

$f \circ g$: $x \to \boxed{g} \xrightarrow{g(x)} \boxed{f} \to f(g(x))$

$-3 \to g(-3) \to f(g(-3))$

$g \circ f$: $x \to \boxed{f} \xrightarrow{f(x)} \boxed{g} \to g(f(x))$

Example

- Given $f(x) = 5x$ and $g(x) = 2 + x^2$,

  a) find $(f \circ g)(-3)$;

  b) find $(g \circ f)(x)$.

a) To find $(f \circ g)(-3)$, find $g(-3)$ and use that as the input for f.

$$\begin{aligned}(f \circ g)(-3) &= f(g(-3)) = f\left(2 + (-3)^2\right) && \text{Using } g(x) = 2 + x^2 \\ &= f(2+9) \\ &= f(11) \\ &= 5 \cdot 11 && \text{Using } f(x) = 5x \\ &= 55\end{aligned}$$

b) To find $(g \circ f)(x)$ substitute $f(x)$ for x in the equation for $g(x)$.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) = g(5x) && \text{Using } f(x) = 5x \\ &= 2 + (5x)^2 && \text{Using } g(x) = 2 + x^2 \\ &= 2 + 25x^2\end{aligned}$$

$f(x) = 5x$

$(f \circ f)(x) = f(f(x))$

$= f(5x) = 25x$

$g(x) = x^3$

$\Rightarrow (g \circ g)(x) = g(g(x)) = g(x^3) = (x^3)^3 = x^9$

Given $f(x) = 3x + 2$ and $g(x) = x^2 - 4$,
Find $(f \circ g)(2)$

$(f \circ g)(2) = f(g(2))$

$g(2) = 2^2 - 4 = 4 - 4 = 0$

$(f \circ g)(2) = f(g(2)) = f(0)$

$= 3 \times 0 + 2 = 0 + 2 = 2$

$(f \circ g)(2) = 2$

Given $f(x) = -2x + 1$ and $g(x) = x^3$,
Find $(f \circ g)(-1)$

$(f \circ g)(-1) = f(g(-1))$

$g(-1) = (-1)^3 = -1$

$(f \circ g)(-1) = f(g(-1)) = f(-1)$

$= -2(-1) + 1 = 2 + 1$

$\Rightarrow (f \circ g)(-1) = 3$

Given $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$,
Find $(f \circ g)(x)$

$(f \circ g)(x) = f(g(x)) = f(x^2 + 1)$

$= \sqrt{x^2 + 1}$

$\boxed{\sqrt{(x+1)^2} = x + 1 \quad \hookrightarrow \sqrt{x^2 + 1 + 2x}}$

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x})$

$= (\sqrt{x})^2 + 1 = x + 1$

Given $f(x) = \sqrt{x + 4}$ and $g(x) = \frac{5}{x}$,
Find $(f \circ g)(x)$

$(f \circ g)(x) = f(g(x))$

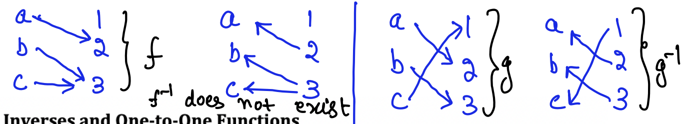
$= f\left(\frac{5}{x}\right)$

$= \sqrt{\frac{5}{x} + 4} = \sqrt{\frac{5 + 4x}{x}}$

$= \frac{\sqrt{5 + 4x}}{\sqrt{x}}$

$(g \circ f)(x) = g(f(x))$

$= g(\sqrt{x + 4}) = \frac{5}{\sqrt{x + 4}}$



Inverses and One-to-One Functions

ESSENTIALS

One-to-One Function

A function *f* is *one-to-one* if different inputs have different outputs. For every one-to-one function, an inverse function exists.

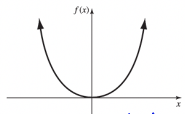
The Horizontal-Line Test

If it is impossible to draw a horizontal line that intersects a function's graph more than once, then the function is one-to-one.

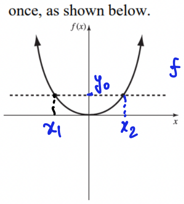
Example

- Determine if the function at right is one-to-one.

  The function is not one-to-one because there is a horizontal line that crosses the graph more than once, as shown below.

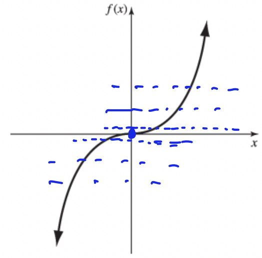


Not one-to-one



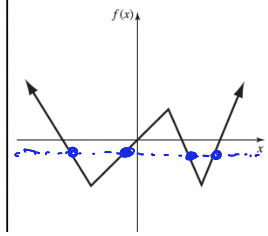
$f(x_1) = f(x_2) = y_0$

Determine whether the function is one-to-one.



Yes

Determine whether the function is one-to-one.



NO

ESSENTIALS

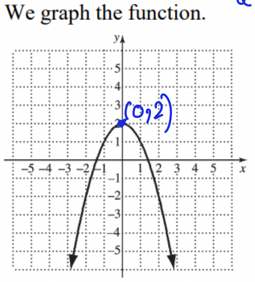
When the inverse of a function *f* is also a function, it is denoted f^{-1} (read "*f*-inverse").

To find a formula for f^{-1}, first make sure that *f* is one-to-one. Then,

1. Replace $f(x)$ with *y*.
2. Interchange *x* and *y*.
3. Solve for *y*.
4. Replace *y* with $f^{-1}(x)$.

Determine whether the function $f(x) = -x^2 + 2$ is one-to-one.

We graph the function.



$a = -1,\ b = 0,\ c = 2$

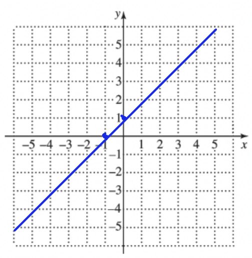
$\frac{-b}{2a} = \frac{-0}{2(-1)} = 0$

$f(0) = -0^2 + 2 = 2$

$(0, 2)$

NO

Determine whether the function $f(x) = x + 1$ is one-to-one.

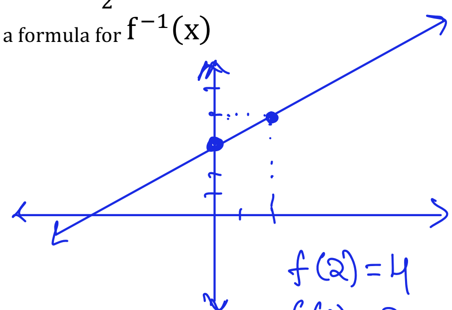


$f(0) = 0 + 1 = 1$

$f(-1) = -1 + 1 = 0$

YES

Determine whether the function $f(x) = \frac{1}{2}x + 3$ is one to one and if so, find a formula for $f^{-1}(x)$



$f(2) = 4$

$f(0) = 3$

Yes, f is one-one

$y = \frac{1}{2}x + 3$ Interchange x and y

$x = \frac{1}{2}y + 3$ Solve for y.

$\frac{1}{2}y = x - 3$

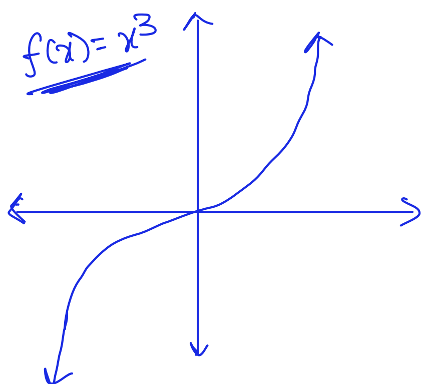
$y = 2(x - 3)$

$\Rightarrow y = 2x - 6$

$f^{-1}(x) = 2x - 6$

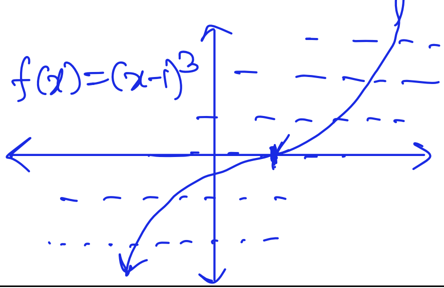
Determine whether the function $f(x) = (x-1)^3$ is one to one and if so, find a formula for $f^{-1}(x)$

$f(x) = x^3$



$x \to x - 1$

$f(x) = (x-1)^3$



Yes, f is one-one

$f(x) = a(x-h)^5 + k$

~~Determine whether the function $f(x) = (x-1)^3$ is one to one and if so, find a formula for $f^{-1}(x)$~~

$y = (x-1)^3$ Interchange x and y.

$x = (y-1)^3$

Solve for y:

Take cube root on both sides:

$x^{1/3} = ((y-1)^3)^{1/3}$

$\sqrt[3]{x} = y - 1$

$\Rightarrow y - 1 = \sqrt[3]{x}$

$\Rightarrow y = \sqrt[3]{x} + 1$

$f^{-1}(x) = \sqrt[3]{x} + 1$

is also one-one any odd number.

Graphing Functions and Their Inverses

ESSENTIALS

Visualizing Inverses

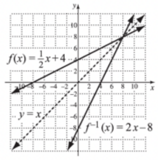
The graph of f^{-1} is a reflection of the graph of f across the line $y = x$.

Example

- Graph the function $f(x) = \frac{1}{2}x + 4$ and its inverse on the same set of axes.

First, find the inverse function.

1. Replace $f(x)$ with y: $y = \frac{1}{2}x + 4$
2. Interchange x and y: $x = \frac{1}{2}y + 4$
3. Solve for y: $x - 4 = \frac{1}{2}y$
   $2(x-4) = y$
   $2x - 8 = y$
4. Replace y with $f^{-1}(x)$: $f^{-1}(x) = 2x - 8$



Notice that the graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ across the line $y = x$.

Inverse Functions and Composition

ESSENTIALS

Composition and Inverses

If a function f is one-to-one, then f^{-1} is the unique function for which

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ and

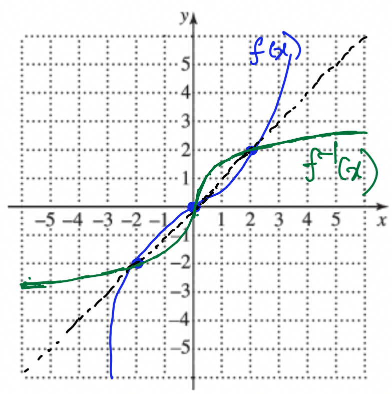
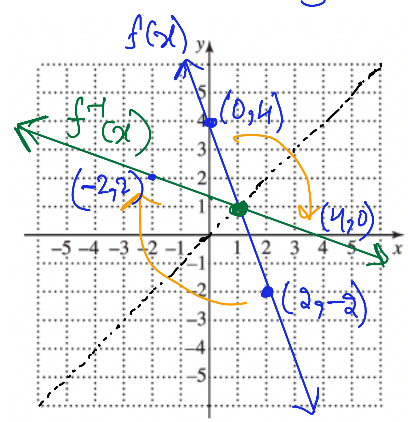
$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$.

$x \to \boxed{f} \to \boxed{f^{-1}} \to x$

$x \to \boxed{f^{-1}} \to \boxed{f} \to x$

Given f(x) show that that stated $f^{-1}(x)$ is the inverse of the function.

Graph the following functions and their inverses

$f(x) = \frac{1}{4}x^3$	$f(x) = -3x + 4$
$f^{-1}(x) = \sqrt[3]{4x}$ $y = \frac{1}{4}x^3 \Rightarrow x = \frac{1}{4}y^3$ $\Rightarrow 4x = y^3$ $\Rightarrow \sqrt[3]{4x} = y$ $\Rightarrow f^{-1}(x) = \sqrt[3]{4x}$  $f(0) = 0,\ f(2) = \frac{1}{4}(2)^3 = 2$ $f(-2) = \frac{1}{4}(-2)^3 = -2$	$f^{-1}(x) = \frac{-1}{3}x + \frac{4}{3}$ $y = -3x + 4$ $x = -3y + 4$ $x - 4 = -3y$ $\frac{x-4}{-3} = y$ $\Rightarrow y = \frac{-1}{3}x + \frac{4}{3}$  $f(0) = 4,\ f(2) = -2$

$f(x) = \sqrt[3]{x+2}, f^{-1}(x) = x^3 - 2$	$f(x) = x^3 + 5, f^{-1}(x) = \sqrt[3]{x-3}$
$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(\sqrt[3]{x+2})$ $= (\sqrt[3]{x+2})^3 - 2$ $= x + 2 - 2 = x.$ $(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(x^3 - 2)$ $= \sqrt[3]{x^3 - 2 + 2} = \sqrt[3]{x^3}$ $= (x^3)^{1/3} = x^{3 \times 1/3} = x$ Hence, it is the correct inverse	$(f^{-1} \circ f)(x) = f^{-1}(f(x))$ $= f^{-1}(x^3 + 5) = \sqrt[3]{x^3 + 5 - 3}$ $= \sqrt[3]{x^3 + 2} \neq x$ Hence, it is not correct inverse $(f \circ f^{-1})(x) = f(f^{-1}(x)) = f(\sqrt[3]{x-3})$ $= (\sqrt[3]{x-3})^3 + 5 = x - 3 + 5$ $= x + 2 \neq x$