M16600 Lecture Notes

Section 7.4: Integration of Rational Functions by Partial Fractions

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■ **Section 7.4** exercises, *page 541*: #9, 12, 19, 23, 24, 10, 11, 20, 25.

Terminologies:

- **Rational Function:** a ratio of polynomials
- **Partial Fractions Decomposition:** is the technique of decomposing rational function into a combination of simpler fractions

  E.g., $\dfrac{x+5}{x^2+x-2} = \dfrac{2}{x-1} - \dfrac{1}{x+2}$
- **Integration by Partial Fractions:** is a method of integrating certain types of rational functions by first decomposing the rational function into simpler fractions then integrate.

  E.g., $\displaystyle\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx = 2\ln|x-1| - \ln|x+2| + C$

In order to perform the method of Integration by Partial Fractions, we need to be able to do these three processes:

1. Writing out the form of the partial fractions decomposition
2. Finding the values of the coefficients
3. Doing a u-substitution

Example 1 (Process 1): Write out the form of the partial fractions decomposition of the functions

Step 1: If the (highest degree of the numerator) is $\geq$ the (highest degree of the denominator), do long division
Step 2: Factor the denominator completely
Step 3: Treat ***Linear Factor*** (highest degree is 1) and ***Quadratic Factor*** (highest degree is 2) differently
Step 4: Take care of ***the multiplicity*** of each factor accordingly

(a) $\dfrac{x+5}{x^2+x-2} = \dfrac{x+5}{(x-1)(x+2)}$

Factor x^2+x-2

$= \underline{x^2-x}+2x-2$

$= x(x-1)+2(x-1)$

$= (x-1)(x+2)$

(a) $\left[\dfrac{x+5}{(x-1)(x+2)} = \dfrac{a}{x-1} + \dfrac{b}{x+2}\right]$

(b) $\dfrac{x^3-x+1}{x(x+4)^3(x^2+4)}$

$= \dfrac{a}{x} + \dfrac{b}{x+4} + \dfrac{c}{(x+4)^2} + \dfrac{d}{(x+4)^3} + \dfrac{e_1x+e_2}{x^2+4}$

(c) $\dfrac{x^3+x^2+1}{x^2(x-1)(x^2+x+1)(x^2+1)^2}$

$$= \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-1} + \frac{d_1 x + d_2}{x^2+x+1}$$

$$+ \frac{e_1 x + e_2}{x^2+1} + \frac{g_1 x + g_2}{(x^2+1)^2}$$

(a') $\dfrac{x+5}{(x-1)^2}$

$$= \frac{a}{x-1} + \frac{b}{(x-1)^2}$$

(a'') $\dfrac{x+5}{(x-1)^3}$

$$= \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{(x-1)^3}$$

Example 2 (Processes 1 and 2): Write out the form of the partial fraction decomposition of the functions then find the values of the coefficients

(a) $\left[\dfrac{x+5}{(x-1)(x+2)} = \dfrac{a}{x-1} + \dfrac{b}{x+2}\right] \times (x-1)(x+2)$

(a''') $\dfrac{x+5}{(x-1)^2(x+2)}$

$$= \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2}$$

$$\frac{\cancel{(x-1)(x+2)}\,(x+5)}{\cancel{(x-1)(x+2)}} = \frac{\cancel{(x-1)}(x+2)\,a}{\cancel{x-1}} + \frac{(x-1)\cancel{(x+2)}\,b}{\cancel{x+2}}$$

$$x+5 = a(x+2) + b(x-1) = ax + 2a + bx - b = (ax+bx) + 2a - b = (a+b)x + 2a - b$$

Compare coeff. of x^0 on both sides :—

$$5 = 2a - b$$

Compare coeff. of x^1 on both sides :—

$$1 = a + b$$

$[\, x = ax + bx \Rightarrow 1 = a + b \,]$

Solve for a and b :—

$$\begin{array}{r} 2a - \cancel{b} = 5 \\ a + \cancel{b} = 1 \\ \hline 3a = 6 \end{array}$$

$\Rightarrow a = \frac{6}{3} = 2 \Rightarrow 1 = 2 + b \Rightarrow -1 = b$

$$\frac{x+5}{(x-1)(x+2)} = \frac{2}{x-1} - \frac{1}{x+2}$$

(b) $$\left[\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{a}{x} + \frac{b}{2x-1} + \frac{c}{x+2}\right] x(2x-1)(x+2)$$

$$x^2+2x-1 = \frac{a}{\cancel{x}}\cancel{x}(2x-1)(x+2) + \frac{b}{\cancel{2x-1}}x\cancel{(2x-1)}(x+2) + \frac{c}{\cancel{x+2}}x(2x-1)\cancel{(x+2)}$$

$$\left[x^2+2x-1 = a(2x-1)(x+2) + bx(x+2) + cx(2x-1)\right]$$

- Put $x=0$ on both sides.

$$-1 = a(-1)(2) + 0 + 0 \Rightarrow -1=-2a \Rightarrow a=\frac{1}{2}$$

- Put $x=-2$ on both sides.

$$(-2)^2+2(-2)-1 = 0 + 0 + c(-2)\left(2(-2)-1\right)$$

$$\Rightarrow 4-4-1 = -2c(-4-1) \Rightarrow -1=10c \Rightarrow c=\frac{-1}{10}$$

- Put $x=\frac{1}{2}$ on both sides.

$$\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 1 = 0 + b\,\frac{1}{2}\left(\frac{1}{2}+2\right) + 0$$

$$\Rightarrow \frac{1}{4}+1-1 = \frac{b}{2}\left(\frac{5}{2}\right) \Rightarrow \frac{1}{4} = \frac{5b}{4} \Rightarrow 1=5b \Rightarrow b=\frac{1}{5}$$

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{1}{2}\left(\frac{1}{x}\right) + \frac{1}{5}\left(\frac{1}{2x-1}\right) - \frac{1}{10}\left(\frac{1}{x+2}\right)$$

Example 3 (Process 3): Evaluate

$$\int f(x)\,dx = g(x) + C$$

$$\int f(ax+b)\,dx = \frac{g(ax+b)}{a} + C$$

$$\int \frac{1}{ax+b}\,dx = \frac{\ln|ax+b|}{a} + C$$

1. $\int \frac{1}{x+2}\,dx = \ln|x+2| + C$

2. $\int \frac{2}{x-1}\,dx = 2\int \frac{1}{x-1}\,dx = 2\ln|x-1| + C$

3. $\int \frac{1}{5}\frac{1}{2x-1}\,dx = \frac{1}{5}\int \frac{1}{2x-1}\,dx = \frac{1}{5}\,\frac{\ln|2x-1|}{2} + C$

$$= \frac{1}{10}\ln|2x-1| + C$$

4. $\int \frac{2}{(x-1)^2}\,dx = 2\int (x-1)^{-2}\,dx = 2\,\frac{(x-1)^{-2+1}}{-2+1} + C$

$$= \frac{2(x-1)^{-1}}{-1} + C = \frac{-2}{x-1} + C$$

Example 4: Evaluate $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx$

$$\left[\frac{5x+1}{(2x+1)(x-1)} = \frac{a}{2x+1} + \frac{b}{x-1}\right] \times (2x+1)(x-1)$$

$$5x+1 = a(x-1) + b(2x+1)$$

$2x+1=0$
$2x=-1$
$x=\frac{-1}{2}$

• Put $x=1$ on both sides.

$$5(1)+1 = 0 + b(2(1)+1)$$

$$\Rightarrow 6 = 3b \Rightarrow b = \frac{6}{3} = 2$$

• Put $x=\frac{-1}{2}$ on both sides.

$$5\left(\frac{-1}{2}\right)+1 = a\left(\frac{-1}{2}-1\right) + 0$$

$$\Rightarrow \frac{-5}{2}+1 = a\left(\frac{-3}{2}\right) \Rightarrow \cancel{\frac{-3}{2}} = a\left(\cancel{\frac{-3}{2}}\right) \Rightarrow a=1$$

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \frac{1}{2x+1}\,dx + \int \frac{2}{x-1}\,dx$$

$$= \frac{\ln|2x+1|}{2} + 2\int \frac{1}{x-1}\,dx$$

$$= \frac{1}{2}\ln|2x+1| + 2\ln|x-1| + C$$

Example 5: Evaluate $\displaystyle\int \frac{4x}{(x-1)^2(x+1)}\,dx$

$$\left[\frac{4x}{(x-1)^2(x+1)} = \frac{a}{(x-1)} + \frac{b}{(x-1)^2} + \frac{c}{x+1}\right](x-1)^2(x+1)$$

$$4x = a(x-1)(x+1) + b(x+1) + c(x-1)^2$$

Put $x=1$ on both sides :—

$$4(1) = a(0) + b(1+1) + c(0) \Rightarrow 4 = 2b \Rightarrow b = 2$$

Put $x=-1$ on both sides :—

$$4(-1) = a(0) + b(0) + c(-1-1)^2 \Rightarrow -4 = 4c \Rightarrow c = -1$$

Put $x=0$ on both sides :—

$$4(0) = a(0-1)(0+1) + b(0+1) + c(0-1)^2$$

$$\Rightarrow 0 = -a + b + c \Rightarrow a = b + c \Rightarrow a = 2 + (-1) = 1$$

$$\int \frac{4x}{(x-1)^2(x+1)}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{2}{(x-1)^2}\,dx - \int \frac{1}{x+1}\,dx$$

$$= \ln|x-1| + 2\,\frac{(x-1)^{-2+1}}{-2+1} - \ln|x+1| + C$$

$$= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

$x = au$

It is useful to remember this integral formula

$$\boxed{\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

When $a = 1$, the above formula becomes one we already know $\displaystyle\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + C.$

Example 6: Evaluate $\displaystyle\int \frac{2x^2-x+1}{x^3+x}\,dx$

$$\frac{2x^2-x+1}{x^3+x} = \frac{2x^2-x+1}{x(x^2+1)}$$

$$\left[\frac{2x^2-x+1}{x(x^2+1)} = \frac{a}{x} + \frac{bx+c}{x^2+1}\right] x(x^2+1)$$

$$\Rightarrow 2x^2-x+1 = a(x^2+1) + (bx+c)x$$

$$\Rightarrow 2x^2-x+1 = ax^2+a+bx^2+cx$$

$$\Rightarrow 2x^2-x+1 = (a+b)x^2 + cx + a$$

compare coeff. of x^2, x, x^0 :-

$2 = a+b \longrightarrow b = 2-a = 2-1 = 1$

$-1 = c$ $\qquad \Rightarrow b = 1$

$1 = a$

$$\int \frac{2x^2-x+1}{x(x^2+1)}\,dx = \int \frac{1}{x}\,dx + \int \frac{x-1}{x^2+1}\,dx$$

$$= \ln|x| + \int \frac{x-1}{x^2+1}\,dx$$

$$\int \frac{x-1}{x^2+1}\,dx = \int \left(\frac{x}{x^2+1} - \frac{1}{x^2+1} \right) dx$$

$$= \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x$$

$u = x^2+1$

$\Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$\Rightarrow \frac{1}{2}\,du = x\,dx$

$$\int \frac{x}{x^2+1}\,dx = \int \frac{1}{x^2+1}\,x\,dx$$

$$= \int \frac{1}{u}\,\frac{1}{2}\,du = \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2+1|$$

$$\int \frac{x-1}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + C$$

$$\int \frac{2x^2-x+1}{x(x^2+1)}\,dx = \ln|x| + \int \frac{x-1}{x^2+1}\,dx$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\,\mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{1+x^2}\,dx = \mathrm{Tan}^{-1}x + C$$

$\downarrow$

$x = au \Rightarrow dx = a\,du$

$$\int \frac{1}{a^2u^2+a^2}\,a\,du = \int \frac{1}{a^2(u^2+1)}\,a\,du = \frac{a}{a^2}\int \frac{1}{u^2+1}\,du$$

$$= \frac{1}{a}\,\mathrm{Tan}^{-1}u + C$$

$x = au \Rightarrow u = \frac{x}{a}$

$$= \frac{1}{a}\,\mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + C$$