M16600 Lecture Notes

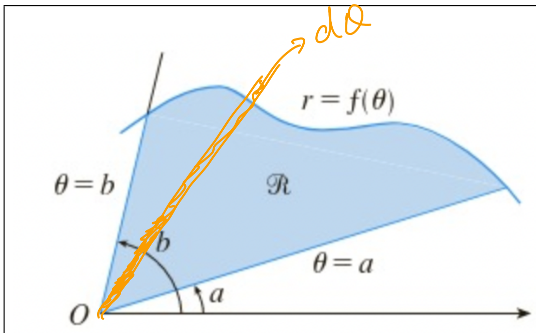
Section 10.4: Areas and Lengths in Polar Coordinates

■ **Section 10.4** textbook exercises, *page 712*: #1, 3, 5, 6, 7, 8

We discuss areas in polar coordinates in this lecture notes. The discussion on arc length in polar coordinates can be founded on pages 711–712 in the textbook.

Areas in Polar Coordinates. Let $\mathcal{R}$ be the region bounded by the polar curve $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leq 2\pi$.

dA = area of the sector having angle $d\theta = \frac{d\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \, d\theta$

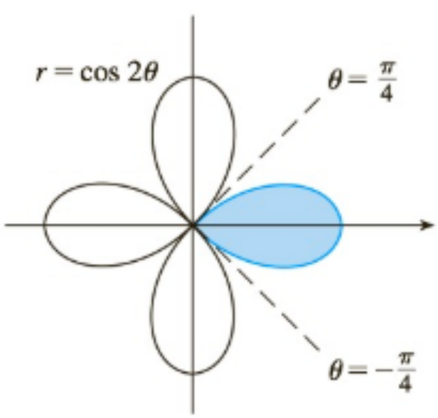


The ***area*** of the polar region $\mathcal{R}$ is

$$A = \int_{\theta=a}^{\theta=b} \tfrac{1}{2}\left[f(\theta)\right]^2 d\theta$$

or we can simply write $A = \displaystyle\int_{\theta=a}^{\theta=b} \tfrac{1}{2} r^2 \, d\theta.$

Example 1: Find the area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.



$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta \, d\theta$$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2 2\theta \, d\theta$$

$\cos^2 x = \frac{1 + \cos 2x}{2}$

$\cos^2 2\theta = \frac{1 + \cos 4\theta}{2}$

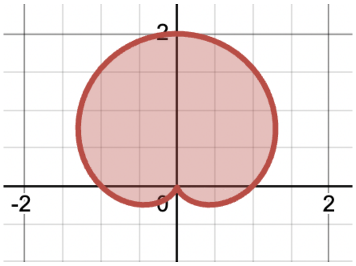
$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \frac{1 + \cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{4} \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta) \, d\theta$$

$$= \frac{1}{4} \left[\int_{-\pi/4}^{\pi/4} 1 \, d\theta + \int_{-\pi/4}^{\pi/4} \cos 4\theta \, d\theta \right] = \frac{1}{4} \left[\theta \Big|_{-\pi/4}^{\pi/4} + \frac{\sin 4\theta}{4} \Big|_{-\pi/4}^{\pi/4} \right]$$

$$= \frac{1}{4} \left[\left(\frac{\pi}{4} - \frac{-\pi}{4} \right) + \frac{1}{4} \left(\sin \frac{2\pi}{4} - \sin \frac{-2\pi}{4} \right) \right] = \frac{1}{4} \left[\frac{\pi}{2} + \frac{1}{4}(1 + 1) \right] = \frac{1}{4} \left(\frac{\pi}{2} + \frac{1}{2} \right) = \frac{\pi + 1}{8}$$

Example 2: Find the area of the region bounded by the cardioid $r = 1 + \sin\theta$.



$$A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} (1+\sin\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \left[1^2 + \sin^2\theta + 2\sin\theta \right] d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\sin^2\theta + \sin\theta \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \, d\theta + \int_0^{2\pi} \frac{1}{2} \sin^2\theta \, d\theta + \int_0^{2\pi} \sin\theta \, d\theta$$

$$= \frac{1}{2}\,\theta \Big|_0^{2\pi} + \frac{1}{2} \int_0^{2\pi} \frac{1-\cos 2\theta}{2} \, d\theta + -\cos\theta \Big|_0^{2\pi}$$

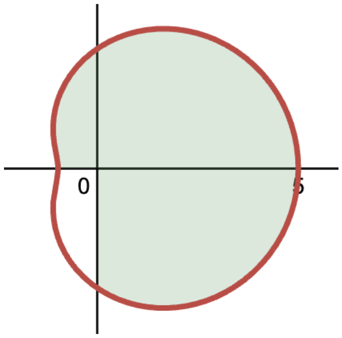
$$= \frac{1}{2}(2\pi) + \frac{1}{4} \int_0^{2\pi} 1 \, d\theta - \frac{1}{4} \int_0^{2\pi} \cos 2\theta \, d\theta + \left[-\cos 2\pi - (-\cos 0) \right]$$

$$= \pi + \frac{1}{4}(2\pi) - \frac{1}{4} \frac{\sin 2\theta}{2} \Big|_0^{2\pi} + \left[-1 + 1 \right]$$

$$= \pi + \frac{\pi}{2} - \frac{1}{8} \left[\sin 2(2\pi) - \sin 2(0) \right] + 0$$

$$= \frac{3\pi}{2} - \frac{1}{8} \left[0 - 0 \right] + 0 = \boxed{\frac{3\pi}{2}}$$

Example 3: Find the area of the shaded region



$r = 3 + 2\cos\theta$

$$A = \int_{-\pi/2}^{\pi} \frac{1}{2} r^2 \, d\theta = \int_{-\pi/2}^{\pi} \frac{1}{2} (3+2\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi} \left(9 + 4\cos^2\theta + 2\times 3\times 2\cos\theta\right) d\theta$$

$$\Rightarrow A = \frac{1}{2} \int_{-\pi/2}^{\pi} \left(9 + 4\cos^2\theta + 12\cos\theta\right) d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi} 9 \, d\theta + \frac{4}{2}\int_{-\pi/2}^{\pi} \cos^2\theta \, d\theta + \frac{12}{2}\int_{-\pi/2}^{\pi} \cos\theta \, d\theta$$

$$= \frac{9}{2}\,\theta\Big|_{-\pi/2}^{\pi} + 2\int_{-\pi/2}^{\pi} \frac{1+\cos 2\theta}{2} \, d\theta + 6\sin\theta\Big|_{-\pi/2}^{\pi}$$

$$= \frac{9}{2}\left(\pi - \left(-\pi/2\right)\right) + \int_{-\pi/2}^{\pi} 1 \, d\theta + \int_{-\pi/2}^{\pi} \cos 2\theta \, d\theta + 6\left[\sin\pi - \sin\frac{-\pi}{2}\right]$$

$$= \frac{9}{2}\left(\frac{3\pi}{2}\right) + \pi - \left(\frac{-\pi}{2}\right) + \frac{\sin 2\theta}{2}\Big|_{-\pi/2}^{\pi} + 6\left[0 - (-1)\right]$$

$$= \frac{27\pi}{4} + \frac{3\pi}{2} + \frac{1}{2}\left[\sin 2\pi - \sin(-\pi)\right] + 6\left[1\right]$$

$$= \frac{27\pi}{4} + \frac{3\pi}{2} + \frac{1}{2}\left[0 - 0\right] + 6$$

$$= \frac{27\pi}{4} + \frac{6\pi}{4} + 6 = \frac{33\pi}{4} + 6$$