M16600 Lecture Notes

Section 11.2: Series

■ **Section 11.2** textbook exercises, *page 755*: #6, 15, 22, 23, 24, 26, 29, 31, 33, 37, 46, 47.

Definition of Series. An ***infinite series*** (or just ***series***) is an infinite SUM of the terms of the sequence $\{a_n\}$

Series Notation:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$$

Note: n does not have to start from 1.

E.g., $\sum_{n=1}^{\infty} 2^n =$ $2^1 + 2^2 + 2^3 + 2^4 + \cdots \to \infty$

Here, $a_n =$ 2^n

Partial Sums of a Series. If we have a series $\sum_{n=1}^{\infty} a_n$ then

- the first partial sum $s_1 =$ a_1
- the second partial sum $s_2 =$ $a_1 + a_2$
- the 3^{rd} partial sum $s_3 =$ $a_1 + a_2 + a_3$
- the n^{th} partial sum $s_n =$ $a_1 + a_2 + a_3 + \cdots\cdots + a_n$

Example 1: Find the 4^{th} partial sum of $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$s_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

Definition of Convergent and Divergent Series. Given a series $\sum_{n=1}^{\infty} a_n$, we can establish a sequence of its *partial sums* $\{s_n\} = \{\ s_1,\ s_2,\ s_3, \ldots, s_n, \ldots\ \}$

We can compute $\lim_{n\to\infty} s_n$. If

$$\begin{cases} \lim_{n\to\infty} s_n = \pm\infty, & \text{then } \sum_{n=1}^{\infty} a_n \text{ is } \textbf{divergent} \\ \lim_{n\to\infty} s_n = S, \text{ a finite number}, & \text{then } \sum_{n=1}^{\infty} a_n \text{ is } \textbf{convergent} \text{ and } \sum_{n=1}^{\infty} a_n = S \end{cases}$$

Remark: By writitng $\sum_{n=1}^{\infty} a_n = S$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number S.

Example 2: Given the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Calculate the first eight terms of the sequence of partial sums correct to the four decimal places. Does it appear that the series is convergent or divergent?

$$s_1 = a_1 = \frac{1}{2}$$

$$s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{2^2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$s_3 = \underbrace{a_1 + a_2}_{} + a_3 = s_2 + a_3 = \frac{3}{4} + \frac{1}{2^3} = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$

$$s_4 = s_3 + a_4 = \frac{7}{8} + \frac{1}{2^4} = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$$

$$s_5 = s_4 + a_5 = \frac{15}{16} + \frac{1}{2^5} = \frac{15}{16} + \frac{1}{32} = \frac{31}{32}$$

$$\vdots$$

$$s_n = \frac{2^n - 1}{2^n} \Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n}}{\cancel{2^n}} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

series is convergent

Series with Names. There are three special series which come up fairly often in Chapter 11.

$\dfrac{a_{n+1}}{a_n} = r$ (indep. of n) $\underline{\text{for every } n}$

- ***Geometric Series:***

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

r is called the **common ratio** of the geometric series.

Remark: For a GEOMETRIC series, the first term is always a and the second term is always ar.

E.g., $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n}$ is a geometric series. Find a and r for this geometric series.

$a_n = \dfrac{2}{3^n} \Rightarrow a_{n+1} = \dfrac{2}{3^{n+1}}$

$r = \dfrac{a_{n+1}}{a_n} = \dfrac{\cancel{2}}{3^{n+1}} \cdot \dfrac{3^n}{\cancel{2}} = 3^{n-n-1} = 3^{-1} = \dfrac{1}{3}$

$r = \dfrac{1}{3}$, $a = \dfrac{2}{3^1} = \dfrac{2}{3}$

Convergence/Divergence Test for a Geometric Series.

$$\begin{cases} \text{The geometric series } \displaystyle\sum_{n=1}^{\infty} ar^{n-1} \text{ is } \textbf{divergent} \text{ if} \quad |r| \geq 1 \\ \text{The geometric series } \displaystyle\sum_{n=1}^{\infty} ar^{n-1} \text{ is } \textbf{convergent} \text{ if} \quad |r| < 1 \quad \text{and} \quad \displaystyle\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \end{cases}$$

Example 3: Is the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ convergent or divergent? If it converges, find its sum

$\longrightarrow a_n = \dfrac{1}{2^n} \Rightarrow a_{n+1} = \dfrac{1}{2^{n+1}}$

$r = \dfrac{a_{n+1}}{a_n} = \dfrac{1}{2^{n+1}} \cdot \dfrac{2^n}{1} = \dfrac{1}{2^{n+1-n}} = \dfrac{1}{2}$ $\Rightarrow a = \dfrac{1}{2^1} = \dfrac{1}{2}$

$|r| = \frac{1}{2} < 1 \Rightarrow$ the given series is convergent.

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$

also works for $\sum_{n=a}^{\infty} \frac{1}{n^p}$ where a can be any natural number.

- ***The*** p***-Series:*** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a real number. (section 11.3)

Convergence/Divergence Test for a p**-Series.**

$$\begin{cases} \text{The } p\text{-series } \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \textbf{divergent} \text{ if } & p \leq 1 \\ \text{The } p\text{-series } \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is } \textbf{convergent} \text{ if } & p > 1. \end{cases}$$

Here are examples of p-series.

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

$\to$ $p = 3$ $\Rightarrow$ Converges

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$\to$ $p = \frac{1}{2}$ ≤ 1 $\Rightarrow$ diverges

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$\to$ $p = 1$ $\Rightarrow$ diverges

- ***Telescoping Series:***

$$\sum_{n=1}^{\infty}(a_n - a_{n+1})$$

An example of a telescoping series is $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ $\Rightarrow a_n = \frac{1}{n}$

There is no quick test of convergence/divergence of telescoping series. To test the **Convergence/Divergence for Telescoping Series**, we must use the **definition of convergent and divergent series** on page 1.

Construct Partial sums.

$$s_1 = 1 - \frac{1}{2}$$

$$s_2 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$s_3 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \frac{1}{4} = 1 - \frac{1}{4}$$

$$\vdots$$

$$s_n = \underbrace{1 - \cancel{\frac{1}{2}}}_{1st} + \underbrace{\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}}_{2nd} + \underbrace{\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}}_{3rd} + \cdots + \underbrace{\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}}_{(n-1)st} + \underbrace{\cancel{\frac{1}{n}} - \frac{1}{n+1}}_{nth} = 1 - \frac{1}{n+1}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 - \lim_{n\to\infty}\frac{1}{n+1} = 1 - 0$$

$$= 1$$

$\Rightarrow$ the given series converges to 1

$$S = a + ar + ar^2 + \cdots\cdots$$

$$rS = \quad ar + ar^2 + \cdots\cdots$$

$$(1-r)S = a \quad \Rightarrow S = \frac{a}{1-r}$$

$|r| < 1$

Here is a very useful tool to see whether a series is **divergent**

Test for Divergence (TD). Given a series $\sum a_n$. If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$ then the series is divergent.

$\sum_{n=1}^{\infty} \frac{n^2}{5^n} \to \lim_{n\to\infty} \frac{n^2}{5^n} = \frac{\text{slower}}{\text{faster}} = 0$

$\Rightarrow$ TD fails

Example 4: Show that $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$a_n = \frac{n^2}{5n^2+4} \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{\cancel{n^2}}{5\cancel{n^2}} = \frac{1}{5}$

$\lim_{n\to\infty} a_n = \frac{1}{5} \neq 0$

By TD, the given series diverges

Warning: If $\lim_{n\to\infty} a_n = 0$, the series $\sum a_n$ could be convergent or divergent. We don't know! **Never** conclude that a series is convergent if you use the Test for Divergence.

Example 5: Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

Note: We know a series is a geometric series if the term a_n can be rewritten as $(\text{constant})(r)^{\text{exponent in terms of } n}$.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$ $\Rightarrow a_n = \frac{(-3)^{n-1}}{4^n} \Rightarrow a_{n+1} = \frac{(-3)^{n+1-1}}{4^{n+1}} = \frac{(-3)^n}{4^{n+1}}$

$r = \frac{a_{n+1}}{a_n}$

$= \frac{(-3)^n}{4^{n+1}} \cdot \frac{4^n}{(-3)^{n-1}} = \frac{(-3)^{n-(n-1)}}{4^{\cancel{n}+1-\cancel{n}}} = \frac{(-3)^{n-n+1}}{4^1} = \frac{-3}{4}$

$\Rightarrow r = \frac{-3}{4} \Rightarrow |r| = \left|\frac{-3}{4}\right| = \frac{3}{4} < 1 \Rightarrow$ the series converges

$a = \frac{(-3)^{1-1}}{4^1} = \frac{(-3)^0}{4^1} = \frac{1}{4} \Rightarrow S = \frac{a}{1-r} = \frac{\frac{1}{4}}{1-\left(\frac{-3}{4}\right)} = \frac{\frac{1}{4}}{1+\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{7}{4}}$

$= \frac{1}{7}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{2n+1}}{(-2)^n}$ $\Rightarrow a_n = \dfrac{3^{2n+1}}{(-2)^n} \Rightarrow a_{n+1} = \dfrac{3^{2(n+1)+1}}{(-2)^{n+1}} = \dfrac{3^{2n+2+1}}{(-2)^{n+1}} = \dfrac{3^{2n+3}}{(-2)^{n+1}}$

$$\Rightarrow r = \frac{a_{n+1}}{a_n} = \frac{3^{2n+3}}{(-2)^{n+1}} \cdot \frac{(-2)^n}{3^{2n+1}} = \frac{3^{\cancel{2n}+3-\cancel{2n}-1}}{(-2)^{\cancel{n}+1-\cancel{n}}} = \frac{3^{3-1}}{(-2)^1} = -\frac{9}{2}$$

If we knew that series is geometric:

$\Rightarrow n=0 \Rightarrow a = \dfrac{3^1}{(-2)^0} = 3$ $\qquad 3r = -\dfrac{27}{2} \Rightarrow r = -\dfrac{9}{2}$

$n=1 \Rightarrow ar = \dfrac{3^{2(1)+1}}{(-2)^1} = \dfrac{3^3}{-2} = -\dfrac{27}{2}$ $\qquad \Rightarrow |r| = \dfrac{9}{2} > 1 \Rightarrow$ diverges

Example 6: Determine whether the series is convergent or divergent.

Hint: Determine whether each series is a geometric series or a p-series first. If a series is neither one of those, think about using the Test of Divergence.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k^3+1}{k^2+2k+5}$

$$\Rightarrow \lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^3+1}{k^2+2k+5} = \lim_{k\to\infty} \frac{k^3}{k^2} = \lim_{k\to\infty} k = \infty$$

$$\Rightarrow \lim_{k\to\infty} a_k = \infty \neq 0$$

$\Rightarrow$ By TD, the given series diverges.

(b) $\sum_{n=1}^{\infty} 4^{-n}3^{n+1}$ $\Rightarrow a_n = 4^{-n}\,3^{n+1} = \frac{3^{n+1}}{4^n}$ $\Rightarrow a_{n+1} = \frac{3^{n+1+1}}{4^{n+1}} = \frac{3^{n+2}}{4^{n+1}}$

$\Rightarrow r = \frac{a_{n+1}}{a_n} = \frac{3^{n+2}}{4^{n+1}} \cdot \frac{4^n}{3^{n+1}} = \frac{3^{\cancel{n}+2-\cancel{n}-1}}{4^{\cancel{n}+1-\cancel{n}}} = \frac{3^{2-1}}{4^1} = \frac{3}{4}$

$\Rightarrow$ the given series is geometric.

$|r| = \frac{3}{4} < 1 \Rightarrow$ the series converges

$a = 4^{-1}3^{1+1} = \frac{3^2}{4} = \frac{9}{4}$, $S = \frac{a}{1-r} = \frac{\frac{9}{4}}{1-\frac{3}{4}} = \frac{\frac{9}{4}}{\frac{1}{4}} = 9$

(c) $\sum_{n=1}^{\infty} \frac{1}{e^{-n}+2}$ $\Rightarrow a_n = \frac{1}{e^{-n}+2}$ $\Rightarrow a_{n+1} = \frac{1}{e^{-n-1}+2}$

$\Rightarrow r = \frac{a_{n+1}}{a_n} = \frac{1}{e^{-n-1}+2} \cdot \frac{e^{-n}+2}{} = \frac{e^{-n}+2}{e^{-n-1}+2}$ $\rightarrow$ depends on n

$\Rightarrow$ not geometric

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{e^{-n}+2}$

$= \frac{1}{\lim_{n\to\infty}(e^{-n}+2)} = \frac{1}{\underbrace{\lim_{n\to\infty} e^{-n}}_{0} + 2} = \frac{1}{0+2} = \frac{1}{2} \neq 0$

$\Rightarrow$ By TD, the given series diverges

(d) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\downarrow$

P- series with $P = 2 > 1 \Rightarrow$ the given series converges.