Indiana University - Purdue University, Indianapolis

Math16600

Practice Test 3

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Name:____ [2 pts]

Instructions:

- No cell phones, calculators, watches, technology, hats - stow all in your bags.
- **Write your name** on this cover page. It carries 2 points.
- This test is closed book and closed notes.
- All work must be clearly shown for partial credit.
- If you wish for something not to be graded, please strike it out neatly.
- Box, circle, or otherwise clearly indicate your final answer.
- When you finish, return your test to the proctor, and leave the classroom.
- There are a total of **16 problems** including bonus problem.
  - Problems 1-10 are each worth 6 points.
  - Problems 11-15 are each worth 8 points.
  - The bonus problem is worth 8 points.
- You can score a **maximum of 110 points out of 100**.
- There are a total of **9 pages** including the cover page.

Problem 1: Determine whether the following sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{2n^2 + \ln n}{n^2 + n + 1}$$

[6 pts]

$$\lim_{n\to\infty} \frac{2n^2+\ln n}{n^2+n+1} = \lim_{n\to\infty} \frac{2n^2}{n^2} = 2$$

$\Rightarrow$ converges

Problem 2: Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \frac{1}{48} + \cdots$$

[6 pts]

$$\frac{1}{3\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{3\cdot 8} + \frac{1}{3\cdot 16} + \cdots$$

$$= \frac{1}{3}\left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right]$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{2^n}$$

$$r = \frac{a_{n+1}}{a_n} = \frac{1}{3}\,\frac{1}{2^{n+1}} \cdot 3 \cdot 2^n = \frac{1}{2}$$

geometric series with $r = \frac{1}{2} \Rightarrow \left|\frac{1}{2}\right| < 1$

$\Rightarrow$ series converges

$$S = \frac{\frac{1}{3}}{1-\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Problem 3: Determine whether the series is convergent or divergent:

$$\sum_{n=2}^{\infty} \frac{n}{\ln n}$$

Hint: Use Test for Divergence. [6 pts]

$$\lim_{n\to\infty} \frac{n}{\ln n} = \frac{\text{faster}}{\text{slower}} = \infty \neq 0$$

By TD, the series diverges.

Problem 4: Determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

Hint: Use Limit Comparison Test. [6 pts]

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \sim \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$$

p-series with p=1
$\Rightarrow$ diverges.

By LCT, $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ also diverges

Problem 5: Determine whether the series is convergent or divergent:

$$\frac{\ln 2}{\ln 3} - \frac{\ln 3}{\ln 4} + \frac{\ln 4}{\ln 5} - \frac{\ln 5}{\ln 6} + \frac{\ln 6}{\ln 7} \mp \cdots$$

Hint: Use Alternating Series Test. [6 pts]

$n=1 \Rightarrow \frac{\ln 1+1}{\ln 1+2}$, $n=2 \Rightarrow -\frac{\ln(2+1)}{\ln(2+2)}$, $n=3 \Rightarrow +\frac{\ln(3+1)}{\ln(3+2)}$, ...

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln(n+1)}{\ln(n+2)}$, $b_n = \frac{\ln(n+1)}{\ln(n+2)}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln(n+1)}{\ln(n+2)} = \lim_{n\to\infty} \frac{\ln(n)}{\ln(n)} = 1 \neq 0$$

$\Rightarrow$ By AST, given series is divergent.

Problem 6: Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

[6 pts]

Absolute Convergence

$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{n+1} \sim \sum_{n=1}^{\infty} \frac{1}{n} \to$ P-series with $P=1$

$\Rightarrow$ diverges.

$\Downarrow$

By LCT, diverges

$\Rightarrow$ Not absolutely convergent

Conditional convergence

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} \Rightarrow b_n = \frac{1}{n+1} \Rightarrow \lim_{n\to\infty} b_n = 0$

$b_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = b_n$

$\Rightarrow$ By AST, given series converges

Alternating series

$\Rightarrow$ Conditionally convergent.

Problem 7: Determine whether the series is convergent or divergent:

$$\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$$

Hint: If a series is absolutely convergent, then it is convergent. [6 pts]

$$\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2} \Rightarrow 0 \le \sum_{k=1}^{\infty} \frac{|\sin k|}{k^2} \le \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$0 \le |\sin k| \le 1$

p-series with $p=2$ $\Rightarrow$ Convergent

By the CT, $\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$ is convergent

By the Hint, the given series is convergent.

Problem 8: Find the radius of convergence of the power series:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$$

[6 pts]

$$a_n = \frac{(x-2)^n}{n^2+1}, \quad a_{n+1} = \frac{(x-2)^{n+1}}{(n+1)^2+1} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n}$$

$$= \frac{(x-2)(n^2+1)}{(n+1)^2+1}$$

$$\Rightarrow \lim_{n\to\infty} \left| \frac{(x-2)(n^2+1)}{(n+1)^2+1} \right| = \lim_{n\to\infty} \frac{|x-2|(n^2+1)}{(n+1)^2+1}$$

$$= |x-2| \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = |x-2| \underbrace{\lim_{n\to\infty} \frac{n^2+1}{n^2+1}}_{=1} = |x-2|$$

By Ratio test, the series converges if $|x-2|<1$ and diverges if $|x-2|>1$ $\Rightarrow$ $R=1$

Problem 9: Find a power series representation for the function $f(x) = \frac{x}{1-x}$. [6 pts]

$$f(x) = \frac{x}{1-x} = x \cdot \frac{1}{1-x}$$

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

$$\underbrace{\frac{1}{1-x}}_{r=x} = \sum_{n=0}^{\infty} x^n$$

$$f(x) = x \cdot \frac{1}{1-x} = x \cdot \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}$$

Problem 10: Find Maclaurin series for the function $f(x) = \cosh x$. [6pts]

$$f(x) = \cosh x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$\left[\begin{aligned} (\sinh x)' &= \cosh x \\ (\cosh x)' &= \sinh x \\ \sinh(0) &= 0 \\ \cosh(0) &= 1 \end{aligned}\right]$$

n	$f^{(n)}(0)$
0	1
1	0
2	1
3	0
4	1

$f(x) = \cosh x \Rightarrow f(0) = \cosh(0) = 1$

$f'(x) = \sinh x \Rightarrow f'(0) = \sinh(0) = 0$

$f''(x) = \cosh x \Rightarrow f''(0) = \cosh(0) = 1$

$f'''(x) = \sinh x \Rightarrow f'''(0) = \sinh(0) = 0$

$f^{(IV)}(x) = \cosh x \Rightarrow f^{IV}(0) = \cosh(0) = 1$

$$f^{(n)}(0) = \begin{cases} 1 & \text{if } n = 0, 2, 4, \dots \\ 0 & \text{if } n = 1, 3, 5, \dots \end{cases}$$

$$\cosh(x) = \frac{x^0}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Problem 11: Find the radius of convergence and interval of convergence of the power series:

$$\sum_{n=2}^{\infty} \frac{(x-1)^n}{2^n \ln n}$$

[8 pts]

$a_n = \dfrac{(x-1)^n}{2^n \ln(n)}$

$a_{n+1} = \dfrac{(x-1)^{n+1}}{2^{n+1} \ln(n+1)} \Rightarrow \dfrac{a_{n+1}}{a_n} = \dfrac{(x-1)^{n+1-n}}{2^{n+1-n} \ln(n+1)} \dfrac{\cancel{2^n} \ln(n)}{\cancel{(x-1)^n}}$

$\dfrac{a_{n+1}}{a_n} = \dfrac{(x-1)\ln(n)}{2 \ln(n+1)} \Rightarrow \left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{|x-1| \ln(n)}{2 \ln(n+1)}$

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{|x-1| \ln n}{2 \ln(n+1)} = \lim\limits_{n\to\infty} \dfrac{|x-1| \cancel{\ln n}}{2 \cancel{\ln(n)}} = \dfrac{|x-1|}{2}$

$\dfrac{|x-1|}{2} < 1 \Rightarrow |x-1| < 2 \Rightarrow \boxed{R=2} \Rightarrow -2 < x-1 < 2$

$\Rightarrow -2+1 < x < 2+1$

$\Rightarrow -1 < x < 3$

$x=-1 \Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{(-1-1)^n}{2^n \ln(n)}$

Problem 12: Find a power series representation of the function $f(x) = \ln(1-x)$ and determine its radius of convergence. [8 pts]

$\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n}{2^n \ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n \cancel{2^n}}{\cancel{2^n} \ln(n)} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)} \to$ A.S. with $b_n = \dfrac{1}{\ln(n)}$

$\lim\limits_{n\to\infty} \dfrac{1}{\ln(n)} = 0$, $\dfrac{1}{\ln(n+1)} < \dfrac{1}{\ln(n)} \Rightarrow$ By A.S.T, Convergence

$\underline{x=3}$ $\displaystyle\sum_{n=2}^{\infty} \frac{(3-1)^n}{2^n \ln(n)} = \sum_{n=2}^{\infty} \frac{2^n}{2^n \ln(n)} = \sum_{n=2}^{\infty} \frac{1}{\ln(n)} > \sum_{n=2}^{\infty} \frac{1}{n}$

P-series with $p=1$ $\Rightarrow$ diverges

$n > \ln(n) \Rightarrow \dfrac{1}{n} < \dfrac{1}{\ln(n)}$

$\Rightarrow$ By CT, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges.

$\Rightarrow$ Interval of convergence $[-1, 3)$

(12) $\ln(1-x)$ Power series rep.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \longrightarrow |x|<1, \text{ that is, } R=1$$

Integrate both the sides :-

$$\int \frac{1}{1-x}\,dx = \int \sum_{n=0}^{\infty} x^n\,dx$$

$$\frac{1}{-1}\ln|1-x| = \sum_{n=0}^{\infty}\int x^n\,dx = C + \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

$$\downarrow\; x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \xrightarrow{x=0} 0$$

Put $x=0$ on both sides :-

$$-1\ln|1-0| = C + 0$$

$$-1\cdot 0 = C \Rightarrow C=0$$

$$-\ln|\underbrace{1-x}_{+ve}| = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} \qquad \downarrow R=1$$

valid only if $|x|<1 \Rightarrow -1<x<1$

$\Rightarrow x<1$

$\Rightarrow 1-x>0$

$$\Rightarrow -\ln(1-x) = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

$$\Rightarrow \ln(1-x) = \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}$$

Problem 13: Find the Taylor series of $f(x) = \cos x$ about the point $x = \pi$. [8 pts]

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\pi)}{n!}(x-\pi)^n = \frac{-1}{0!}(x-\pi)^0 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 \pm \cdots$$

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\cos x$	$\cos\pi = -1$
1	$-\sin x$	$-\sin\pi = 0$
2	$-\cos x$	$-\cos\pi = 1$
3	$\sin x$	$\sin\pi = 0$
4	$\cos x$	$\cos\pi = -1$
5	$-\sin x$	$-\sin\pi = 0$
	$\vdots$	

$$= \frac{(-1)^{0+1}}{0!}(x-\pi)^0 + \frac{(-1)^{1+1}}{2!}(x-\pi)^2 + \frac{(-1)^{2+1}}{4!}(x-\pi)^4 \pm \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!}(x-\pi)^{2n}$$

$$\Rightarrow f^{(n)}(\pi) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{m+1} & \text{if } n = 2m \end{cases}$$

Problem 14: Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$$

[8 pts]

Absolute convergence

$$\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$

$$\ln(n) < n \Rightarrow \frac{1}{\ln(n)} > \frac{1}{n}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{\ln(n)} > \sum_{n=2}^{\infty} \frac{1}{n}$$

P-series with $P = 1$ $\Rightarrow$ diverges.

By Comparison test, $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges $\Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ is not absolutely convergent. — (I)

Conditional convergence

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)} \rightarrow$ alternating series with $b_n = \frac{1}{\ln(n)}$

$\Rightarrow \lim_{n\to\infty} \frac{1}{\ln(n)} = 0$ and $\frac{1}{\ln(n+1)} < \frac{1}{\ln(n)}$

$\Rightarrow$ By AST, $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ is convergent — (II)

Using (I) and (II), we have that the given series is Conditionally Convergent

Problem 15: Determine whether the series is convergent or divergent:

$$\lim_{x\to 0}\frac{\tan x}{x}=1$$

$$\sum_{n=2}^{\infty} n\tan(1/n)$$

[8 pts]

$$\lim_{n\to\infty} n\tan\left(\frac{1}{n}\right)=\lim_{n\to\infty}\frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}}=\frac{0}{0}$$

By LH Rules,

$$=\lim_{n\to\infty}\frac{\sec^2\left(\frac{1}{n}\right)\cancel{\left(\frac{-1}{n^2}\right)}}{\cancel{\left(\frac{-1}{n^2}\right)}}$$

$$=\sec^2(0)=1\neq 0$$

$\Rightarrow$ By TD, the given series diverges.

Bonus Problem: Find the radius of convergence of the Maclaurin series of the function $f(x)=2^x$. [8 pts].

Maclaurin series : $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	2^x	$2^0=1$
1	$2^x\ln 2$	$2^0\ln 2=\ln 2$
2	$2^x(\ln 2)^2$	$2^0(\ln 2)^2=(\ln 2)^2$
3	$2^x(\ln 2)^3$	$2^0(\ln 2)^3=(\ln 2)^3$
⋮	⋮	⋮

diff. at each step

$\Rightarrow f^{(n)}(0)=(\ln 2)^n$

$$\Downarrow$$

$$\sum_{n=0}^{\infty}\frac{(\ln 2)^n}{n!}x^n$$

Now find radius of convergence

$$a_n=\frac{(\ln 2)^n}{n!}x^n \Rightarrow a_{n+1}=\frac{(\ln 2)^{n+1}}{(n+1)!}x^{n+1}$$

$$\Rightarrow \frac{a_{n+1}}{a_n}=\frac{(\ln 2)^{\cancel{n+1}}}{(n+1)!}x^{\cancel{n+1}}\frac{n!}{\cancel{(\ln 2)^n}\cancel{x^n}}$$

$$=(\ln 2)x\frac{n!}{(n+1)!}=\frac{(\ln 2)x}{n+1}$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{|x|\ln 2}{n+1}=0<1$$

for every value of x

$\Rightarrow$ Power series converges for every real x.

$\Rightarrow R=\infty$, $I=(-\infty,\infty)$