MATH 16600 Practice Final Exam, *Version 1*

1 Given a one-to-one function $f(x) = \dfrac{x}{x^2-4}$, $-2 < x < 2$. find $f^{-1}(0)$ and $(f^{-1})'(0)$.

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))}$$

Let $f^{-1}(0) = x$

$\Rightarrow f(x) = 0$

$\Rightarrow \dfrac{x}{x^2-4} = 0 \Rightarrow x = 0$

$\Rightarrow f^{-1}(0) = 0$

$$= \frac{1}{f'(0)} = -4$$

$$f(x) = \frac{x}{x^2-4}$$

$$\Rightarrow f'(x) = \frac{(x^2-4) - x(2x)}{(x^2-4)^2} = \frac{x^2-4-2x^2}{(x^2-4)^2} \Rightarrow f'(0) = \frac{0-4-0}{(0-4)^2} = \frac{-1}{4}$$

2 A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 400. Find an expression for the number of bacteria after t hours.

$N(0) = 100$, $\quad N(t) = \underbrace{N(0)\, e^{kt}} = 100\, e^{kt}$

$N(1) = 400$

$$\Downarrow$$

$$N(1) = 100\, e^{k}$$

$\Rightarrow 100\, e^{k} = 400 \Rightarrow e^{k} = 4 \Rightarrow \ln e^{k} = \ln 4$

$\Rightarrow k \ln e = \ln 4$

$\Rightarrow k = \ln 4$

$$\Rightarrow N(t) = 100\, e^{(\ln 4)t} = 100\, (e^{\ln 4})^{t} = 100\,(4)^{t}$$

3 Find the limit. $\lim_{x \to 0} \frac{x^2}{1 - \cos x}$. $= \frac{0^2}{1 - \cos 0} = \frac{0}{0}$

$||$

$\lim_{x \to 0} \frac{2x}{\sin x} = \frac{0}{0} \Rightarrow \lim_{x \to 0} \frac{2}{\cos x} = 2$

4 Evaluate the integral $\int \frac{x}{x^2 - 9} dx$

$u = x^2 - 9 \Rightarrow du = 2x\, dx \Rightarrow x\, dx = \frac{1}{2} du$

5 Evaluate the integral. $\int \frac{dx}{x^2\sqrt{4-x^2}}$.

$\sqrt{4-x^2} \Rightarrow x = 2\sin\theta$

$\sqrt{a^2-x^2}$ with $a=2$

$x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$

$$\int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2 \sqrt{\underbrace{4-4\sin^2\theta}_{4(1-\sin^2\theta)}}} = \int \frac{2\cos\theta}{4\sin^2\theta \sqrt{4\cos^2\theta}}\, d\theta$$

$$= \int \frac{\cancel{2\cos\theta}}{4\sin^2\theta\, \cancel{2\cos\theta}}\, d\theta = \int \frac{1}{4\sin^2\theta}\, d\theta = \frac{1}{4}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

$\sin\theta = \frac{x}{2} = \frac{P}{H}$

$$= -\frac{1}{4}\,\frac{\sqrt{4-x^2}}{x} + C$$

Let $P = x$

$H = 2$

$\cot\theta = \frac{B}{P}$

$\Rightarrow \cot\theta = \frac{\sqrt{4-x^2}}{x}$

$\Rightarrow B = \sqrt{4-x^2}$

6 Evaluate the integral. $\int 2x \ln x\, dx$.

↳ By Parts

ILATE

⟵ Preference for u increases.

$u = \ln x$

$dv = 2x\, dx$

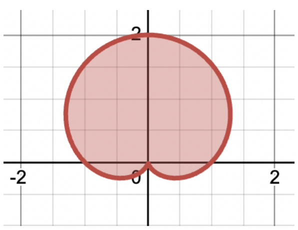
7 Set up an integral that represents the length of the curve $y = \sin x$, $0 \le x \le 2\pi$.

$$L = \int_0^{2\pi} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$A = \int_a^b 2\pi r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$r = \begin{cases} x & \text{if rotating about } y\text{-axis} \\ y & \text{if rotating about } x\text{-axis} \end{cases}$$

8 Find the area of the region bounded by the cardioid $r = 1 + \sin\theta$.



$$\int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} (1 + \sin\theta)^2 \, d\theta$$

9 Find an equation of the tangent line to the curve at the point corresponding to the given value of the parameter. $x = 2\sin t$, $y = \frac{1}{2}\cos t$; $t = \pi/4$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\frac{1}{2}(-\sin t)}{2\cos t}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{\frac{1}{2}(-\sin \pi/4)}{2\cos \pi/4} = \frac{-1}{4}$$

$$x_0 = 2\sin\frac{\pi}{4} = \sqrt{2}$$

$$y_0 = \frac{1}{2}\cos\frac{\pi}{4} = \frac{1}{2\sqrt{2}}$$

$$y - y_0 = m(x - x_0)$$

$$\downarrow$$

$$y - \frac{1}{2\sqrt{2}} = \frac{-1}{4}(x - \sqrt{2})$$

$$\Rightarrow 4y - \frac{4}{2\sqrt{2}} = -1(x - \sqrt{2})$$

$$\Rightarrow 4y - \sqrt{2} = -x + \sqrt{2}$$

$$\Rightarrow x + 4y - 2\sqrt{2} = 0$$

$$\boxed{ax + by + c = 0}$$

$$\frac{\cancel{(2)}\,\cancel{(\sqrt{2})}(\sqrt{2})}{\cancel{2\sqrt{2}}}$$

10 If $f(x) = \ln\left(x\sqrt{\sin x}\right)$. Use the properties of logarithmic functions to decompose $f(x)$ completely then find $f'(x)$.

$$\Rightarrow f(x) = \ln\left(x\sqrt{\sin x}\right) = \ln x + \ln\sqrt{\sin x}$$

$$= \ln x + \ln(\sin x)^{1/2}$$

$$= \ln x + \frac{1}{2}\ln(\sin x)$$

$$f'(x) = \frac{1}{x} + \frac{1}{2}\frac{1}{\sin x}(\sin x)' = \frac{1}{x} + \frac{\cos x}{2\sin x}$$

$$= \frac{1}{x} + \frac{1}{2}\cot x$$

11 Determine whether $\int_0^\infty \frac{1}{e^{2x}}\,dx$ is convergent or divergent. Evaluate the integral if it is convergent.

$$\Rightarrow \lim_{t\to\infty} \int_0^t \frac{1}{e^{2x}}\,dx$$

$$\int_0^t \frac{1}{e^{2x}}\,dx = \int_0^t e^{-2x}\,dx = \left.\frac{e^{-2x}}{-2}\right|_0^t$$

$$= \frac{e^{-2t}}{-2} - \frac{e^{0}}{-2} = -\frac{1}{2}e^{-2t} + \frac{1}{2}$$

$$\lim_{t\to\infty}\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}\right) = \lim_{t\to\infty} -\frac{1}{2}e^{-2t} + \frac{1}{2}$$

$$= \lim_{t\to\infty} -\frac{1}{2}\frac{1}{e^{2t}} + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$$

12 Determine whether the series $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$ is convergent or divergent.

similar

$$\sum_{n=2}^{\infty} \frac{n}{n^3} = \sum_{n=2}^{\infty} \frac{1}{n^2}$$

Converges because its P-series with $P = 2 > 1$

$$\lim_{t\to\infty} t e^{-t} = \infty \cdot 0$$

$$\lim_{t\to\infty} \frac{t}{e^t} = \frac{\infty}{\infty}$$

$$= \lim_{t\to\infty} \frac{1}{e^t} = \frac{1}{\infty} = 0$$

$\Rightarrow$ By LCT, $\sum_{n=2}^{\infty} \frac{n}{n^3-1}$ is also convergent.

13 Determine whether the series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$ is convergent or divergent.

AST: $\sum_{n=1}^{\infty}(-1)^n b_n$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n} = 1 \neq 0$$

For convergence: $\lim_{n\to\infty} b_n = 0$, $b_{n+1} < b_n$

By AST, the given series diverges.

14 Determine whether the series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{2^n}$ is absolutely convergent, conditionally convergent, or divergent.

Absolute Convergence $\sum_{n=1}^{\infty} \left|(-1)^n \frac{n}{2^n}\right| = \sum_{n=1}^{\infty} \frac{n}{2^n}$

check convergence

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = \lim_{n\to\infty} \frac{n}{2n}$$

$$= \frac{1}{2} < 1$$

$\Rightarrow$ the given series is absolutely convergent

15 Set up an integral that represents the area of the surface obtained by rotating the curve $y = x^2$, $0 \le x \le 2$, about the x-axis.

$$A = \int_0^2 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$= \int_0^2 2\pi x^2 \sqrt{1+4x^2}\, dx$$

16 Find the radius of convergence and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x^{n+1}|}{(n+1)^2} \frac{n^2}{|x^n|}$$

$$= \lim_{n\to\infty} |x^{n+1-n}| \frac{n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty} |x| \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} |x| \frac{n^2}{n^2}$$

$$= |x|$$

$$|x| < 1 \Rightarrow R = 1$$

$$-1 < x < 1$$

↳ Check conv. at endpoints

$\underline{\underline{x=1}}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^2} \longrightarrow$ converges because p-series with $p=2>1$

$x=-1$ $\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

↳ converges because it converges absolutely

$$I = [-1, 1]$$

17 Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ is convergent or divergent.

$$f(x) = \frac{1}{x(\ln x)^3}$$

- f is positive for $x > 2$
- f is continuous for $x > 2$
- $f'(x) = \left(\frac{1}{x}\right)' \frac{1}{(\ln x)^3} + \frac{1}{x}\left(\frac{1}{(\ln x)^3}\right)'$

$$= \underbrace{\frac{-1}{x^2(\ln x)^3}}_{-ve} + \underbrace{\left(\frac{1}{x}\right)\frac{-3}{(\ln x)^4}\,\frac{1}{x}}_{-ve}$$

$\Rightarrow f'(x) < 0$

$\Rightarrow f$ is dec.

$du = \frac{dx}{x}$

$\Uparrow$

$u = \ln x$

$$\int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \int_{\ln 2}^{\infty} \frac{du}{u^3} = \left.\frac{u^{-3+1}}{-3+1}\right|_{\ln 2}^{\infty} = \left.\frac{-1}{2u^2}\right|_{\ln 2}^{\infty} = \frac{-1}{\infty} - \frac{-1}{2(\ln 2)^2}$$

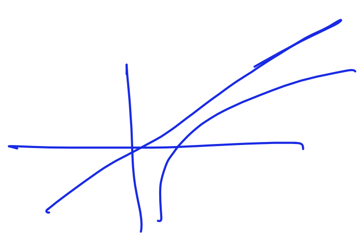
$0 + \frac{1}{2(\ln 2)^2} < \infty$

The integral converges

$\Downarrow$

By IT, series converges

18 Use the definition of Taylor series to find the first **four** nonzero terms of the series for $f(x) = \ln x$ centered at $a = 1$.



$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

<u>Hint</u> $\ln(n) < n$

$$\frac{a_{n+1}}{a_n} = \frac{\ln n}{\ln(n+1)} \xrightarrow{n\to\infty} 1$$

$$\Rightarrow \frac{1}{\ln(n)} > \frac{1}{n}$$

$$\sum_{n=2}^{\infty} \frac{1}{\ln(n)} > \sum_{n=2}^{\infty} \frac{1}{n}$$

$\hookrightarrow$ diverges

$p = 1$

$\downarrow$

also diverges by CT

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \longrightarrow \text{Alternating}$$

$$b_n = \frac{1}{\ln n} \quad \Rightarrow \quad b_{n+1} < b_n$$

$$\lim_{n\to\infty} \frac{1}{\ln(n)} = \frac{1}{\infty} = 0$$

By AST, it conv.

Conditionally

Conv.