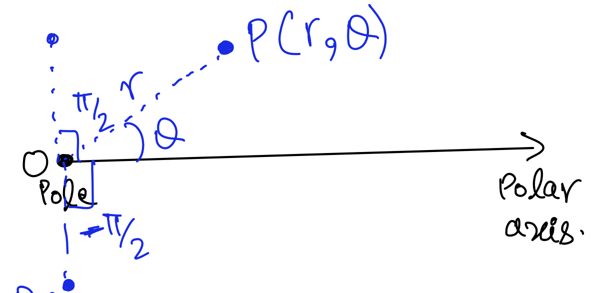
M16600 Lecture Notes

Section 10.3: Polar Coordinates

■ **Section 10.3** textbook exercises, *page 706*: #1, 3, 5, 21, 25, 29, 31

Polar Coordinates:

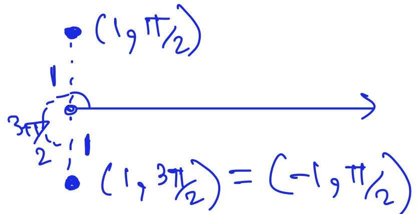


Pole has coordinates $(0, \theta)$ where θ can be anything.

Conventions:

- $\theta > 0$ if θ is measured in counterclockwise direction.
- $\theta < 0$ if θ is measured in clockwise direction.
- The polar coordinates for the pole is $(0, \theta)$ for any values of θ.
- For $r > 0$, to plot a point $(-r, \theta)$, i.e., a point with a negative radius, we plot the corresponding point of positive radius (r, θ) then reflect it about the pole.

$(-r, \theta)$ is the same point as $(r, \theta + \pi)$



$$\left(-1, \tfrac{\pi}{2}\right) = \left(1, \tfrac{\pi}{2} + \pi\right) = \left(1, \tfrac{3\pi}{2}\right)$$

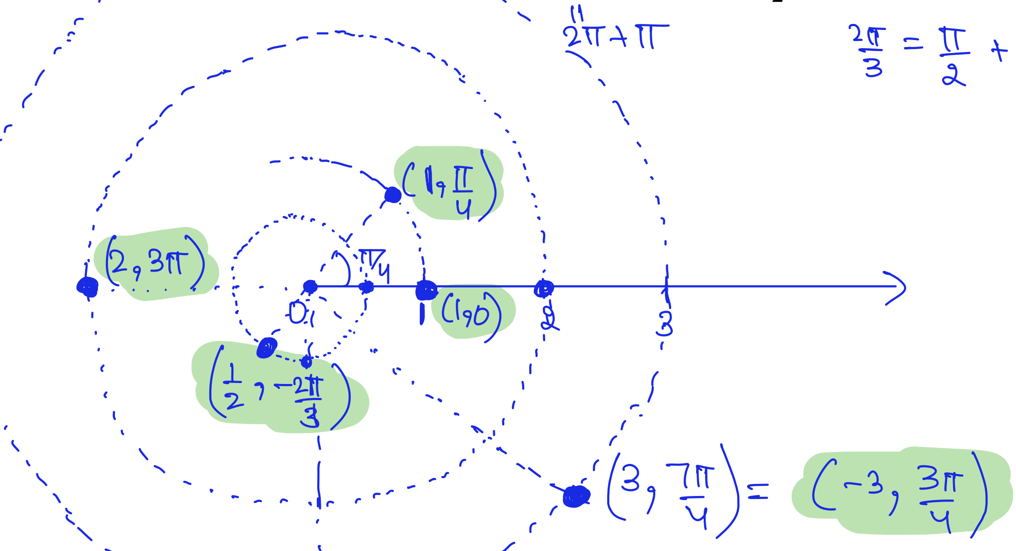
Example 1: Plot the points whose polar coordinates are given

(*a*) $(1, \pi/4)$ (*b*) $(2, 3\pi)$ (*c*) $(\frac{1}{2}, -2\pi/3)$ (*d*) $(-3, 3\pi/4)$ (*e*) $(1, 0)$

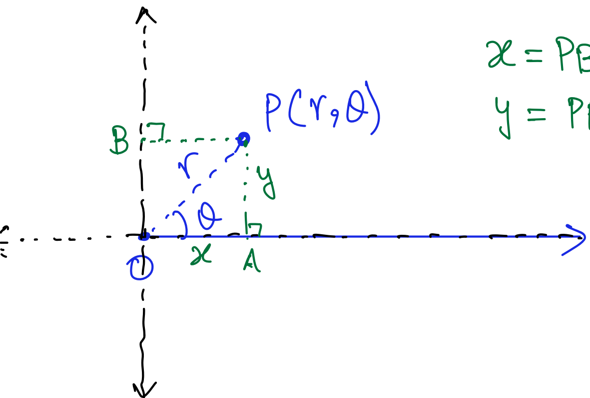
$3\pi = 2\pi + \pi$

$\frac{2\pi}{3} = \frac{\pi}{2} + \frac{\pi}{6}$

$$(-3, 3\pi/4) = \left(3, \pi + \frac{3\pi}{4}\right) = \left(3, \frac{7\pi}{4}\right) = \left(3, \frac{3\pi}{2} + \frac{\pi}{4}\right)$$



The Connection between Polar and Cartesian Coordinates:



$x = PB = OA$

$y = PA = OB$

$\underbrace{\sin\theta = \frac{y}{r} \quad \text{and} \quad \cos\theta = \frac{x}{r}}_{\text{In } \Delta OAP}$

$\Rightarrow \boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}} \Rightarrow \quad x^2 + y^2 = r^2, \quad \frac{y}{x} = \tan\theta$

$\leftarrow$ Polar to Cartesian.

$\Downarrow$

$\boxed{\begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{array}}$

$\uparrow$ Cartesian to Polar

Example 2: Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates

$r = 2, \quad \theta = \frac{\pi}{3} \quad \Rightarrow x = 2\cos\pi/3, \quad y = 2\sin\pi/3$

$\Rightarrow x = 2 \times \frac{1}{2}, \quad y = 2 \times \frac{\sqrt{3}}{2} \quad \Rightarrow x = 1, \; y = \sqrt{3}$

$\Rightarrow (1, \sqrt{3})$

Example 3: Convert the point to polar coordinates.

(a) $(1, -1)$

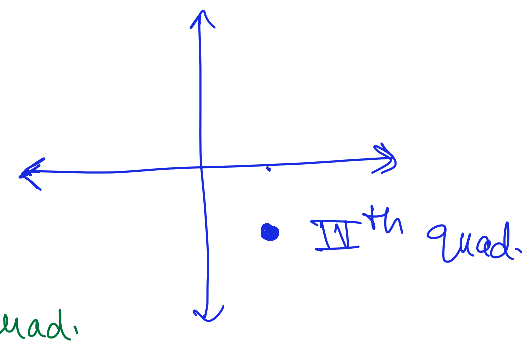
$x = 1, \; y = -1$

$r = \sqrt{x^2+y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{1}\right) = \underbrace{2\pi - \tan^{-1}(1)}_{\rightarrow \; 4^{th} \text{ quad.}}$

$= 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

$(1, -1) \equiv \left(\sqrt{2}, \frac{7\pi}{4}\right)$



3^{rd} quad. $\rightarrow \pi + \tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

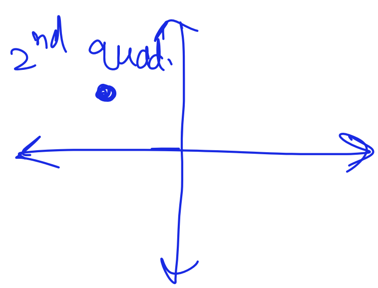
2^{nd} quad $\rightarrow \pi - \tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

1^{st} quad. $\rightarrow \tan^{-1}\left(\frac{y}{x}\right)$ both x and y are anyway +ve

(b) $(-\sqrt{3}, 1)$

$x = -\sqrt{3}, \quad y = 1$

$r = \sqrt{(-\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = 2$



$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$

$= \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$(-\sqrt{3}, 1) \equiv \left(2, 5\pi/6\right)$

Example 4: Find a polar equation for the curve represented by the Cartesian equation $x^2 + 2x + y^2 = 0$.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$(r\cos\theta)^2 + 2(r\cos\theta) + (r\sin\theta)^2 = 0$$

$$r^2\cos^2\theta + 2r\cos\theta + r^2\sin^2\theta = 0$$

$$r^2(\cos^2\theta + \sin^2\theta) + 2r\cos\theta = 0$$

Pole

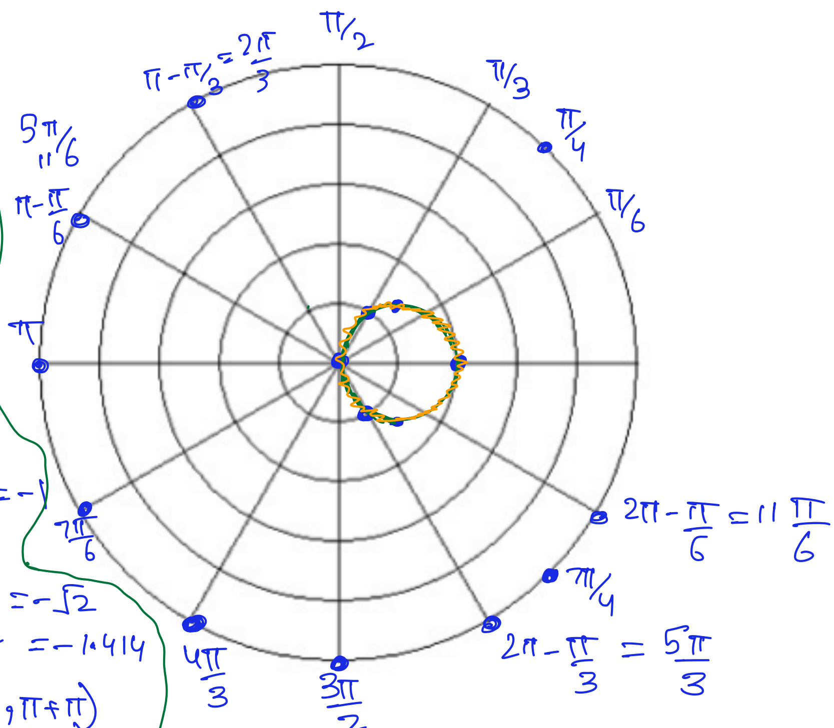
$$\Rightarrow r^2 + 2r\cos\theta = 0 \Rightarrow r(r + 2\cos\theta) = 0 \Rightarrow r = 0 \text{ ✗}$$
$$\text{or } r + 2\cos\theta = 0 \text{ ✓}$$

$\Rightarrow$ Polar eqn. of the given curve is $\boxed{r = -2\cos\theta}$

A Polar Curve is the **Graph of a Polar Equation** $r = r(\theta)$.

Example 5: Sketch the polar curve $r = 2\cos\theta$

θ	$r = 2\cos\theta$
0	$2\cos 0 = 2$ $\quad (2, 0)$
$\pi/4$	$2\cos\frac{\pi}{4} = 2\times\frac{1}{\sqrt{2}} = \sqrt{2} = 1.414$
$\pi/3$	$2\cos\frac{\pi}{3} = 2\times\frac{1}{2} = 1$
$\pi/2$	$2\cos\frac{\pi}{2} = 0$ $\quad \left(-1, \frac{2\pi}{3}\right)$
$2\pi/3$	$2\cos\frac{2\pi}{3} = -2\cos\frac{\pi}{3} = -2\times\frac{1}{2} = -1$
$3\pi/4$	$2\cos\frac{3\pi}{4} = -2\cos\frac{\pi}{4} = -2\times\frac{1}{\sqrt{2}} = -\sqrt{2} = -1.414$
π	$2\cos\pi = -2 = (-2, \pi) = (2, \pi+\pi) = (2, 2\pi)$



$$\left(-1, \frac{2\pi}{3}\right) = \left(1, \pi + \frac{2\pi}{3}\right) = \left(1, \frac{5\pi}{3}\right)$$

$$\left(-\sqrt{2}, \frac{3\pi}{4}\right) = \left(\sqrt{2}, \pi + \frac{3\pi}{4}\right) = \left(\sqrt{2}, \frac{7\pi}{4}\right)$$
$$\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$$

Example 6: Sketch the polar curve $r = 1 + \sin\theta$

θ	$r = 1 + \sin\theta$
0	$1 + \sin 0 = 1$
$\frac{\pi}{6}$	$1 + \sin \pi/6 = 3/2$
$\frac{\pi}{3}$	$1 + \sin \pi/3 = 1 + \sqrt{3}/2 = 1.866$
$\frac{\pi}{2}$	$1 + \sin \pi/2 = 1 + 1 = 2$
$\frac{2\pi}{3}$	$1 + \sin \frac{2\pi}{3} = 1 + \sqrt{3}/2 = 1.866$
$\frac{5\pi}{6}$	$1 + \sin \frac{5\pi}{6} = 1 + \frac{1}{2} = 3/2$
π	$1 + \sin \pi = 1 + 0 = 1$
$\pi + \frac{\pi}{6}$	$1 + \sin(\pi + \pi/6) = 1 - \sin \pi/6 = 1 - \frac{1}{2} = \frac{1}{2}$
$\pi + \frac{\pi}{3}$	$1 - \sin \pi/3 = 1 - \sqrt{3}/2 = 1 - 0.866 = 0.134$
$\frac{3\pi}{2}$	$1 - \sin \pi/2 = 1 - 1 = 0$
$2\pi - \pi/3$	$1 - \sin \pi/3 = 1 - \sqrt{3}/2 = 0.134$
$2\pi - \pi/6$	$1 - \sin \pi/6 = 1 - 1/2 = 1/2$
2π	$1 - \sin 2\pi = 1 - 0 = 1$