Learning Objectives:

1. Understand the intuitive definition of the limit of a function at a given point.
2. The left hand and right hand limits of a function at a given point.
3. Intuitive definition of an infinite limit.
4. What are vertical asymptotes to the graph of a function?

Consider the expression

$$\lim_{x\to 4} \frac{x^2}{x+4}\ .$$

x:	4.1	4.01	4.001	4.0001	3.9	3.99	3.999	3.9999
$\frac{x^2}{x+4}$:	2.1	2.01	2.001	2.0001	1.9	1.99	1.999	1.9999

We see that the values of $f(x) = \dfrac{x^2}{x+4}$ are getting closer and closer to 2 as x approaches 4. We write this as

$$\lim_{x\to 4} \frac{x^2}{x+4} = 2\ .$$

Notice that $f(4) = 2$.

Intuitive definition of a limit

Let f be a function defined on both sides of a except possibly at a itself. Suppose that $f(x)$ becomes arbitrarily close to the number L (written as $f(x) \to L$) as x approaches a ($x \to a$). Then we say that the limit of $f(x)$ as x approaches a is L and we write

$$\lim_{x\to a} f(x) = L\ .$$

Note that in general:

1. The number a may or may not be in the domain of the function f.
2. We may not always have $\lim\limits_{x\to a} f(x) = f(a)$.

Example 1.

Guess the value of $\lim\limits_{x \to 1} \dfrac{x-1}{x^2-1}$.

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

$\downarrow$	$\downarrow$
1	0.5

Example 2 Estimate the value of $\lim_{t\to 0} \dfrac{\sqrt{t^2+9}-3}{t}$.

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t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	0.162277 . . .
± 0.5	0.165525 . . .
± 0.1	0.166620 . . .
± 0.05	0.166655 . . .
± 0.01	0.166666 . . .

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 0.001	0.166667
± 0.0001	0.166670
± 0.00001	0.167000
± 0.000001	0.000000

One-sided limits

Right hand limit: When x approaches a from the right, that is, through values larger than a, the limit obtained is called right-hand limit and is written as

$$\lim_{x \to a^+} f(x) = L .$$

Left hand limit: When x approaches a from the left, that is, through values smaller than a, the limit obtained is called left-hand limit and is written as

$$\lim_{x \to a^-} f(x) = L .$$

$$\boxed{\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L .}$$

Example 3.

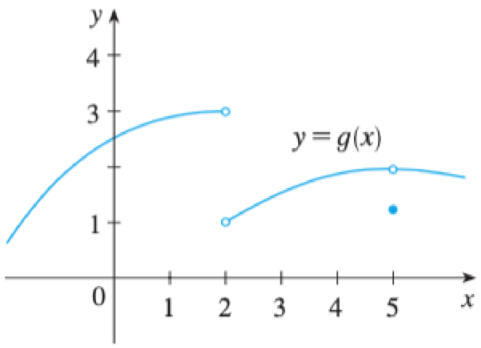
The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 , \\ 1 & \text{if } t > 0 . \end{cases}$$

Guess the value of $\lim_{t \to 0} H(t)$.

Example 4.

The graph of a function g is shown below.



Use it to state the values:

1. $\lim_{x\to 2^-} g(x)$.
2. $\lim_{x\to 2^+} g(x)$.
3. $\lim_{x\to 2} g(x)$.
4. $\lim_{x\to 5^-} g(x)$.
5. $\lim_{x\to 5^+} g(x)$.
6. $\lim_{x\to 5} g(x)$.

Intuitive Definition of Infinite Limits Let f be a function defined on both sides of a except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a,

and

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a,

Example 5.

Find $\lim_{x \to 0} \dfrac{1}{x^2}$ if it exists.

Vertical Asymptote

The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

1. $\lim_{x \to a} f(x) = \infty$
2. $\lim_{x \to a^-} f(x) = \infty$
3. $\lim_{x \to a^+} f(x) = \infty$
4. $\lim_{x \to a} f(x) = -\infty$
5. $\lim_{x \to a^-} f(x) = -\infty$
6. $\lim_{x \to a^+} f(x) = -\infty$

Example 6.

Find $\lim_{x \to 3^-} \frac{2x}{x-3}$, $\lim_{x \to 3^+} \frac{2x}{x-3}$ and $\lim_{x \to 3} \frac{2x}{x-3}$.

Is $x = 3$ a vertical asymptote of $f(x) = \frac{2x}{x-3}$?