Math17100 Section 22866 Quiz 12
Spring 2023, April 19

Name: [1 pt]

Problem 1: Find all the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. [10 pts]

Solution: Find the eigenvalues first.

$$A - \lambda I_2 = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix}$$

$$c_A(\lambda) = \det(A - \lambda I_2) = (2-\lambda)(3-\lambda)$$

$$c_A(\lambda) = 0 \Rightarrow (2-\lambda) = 0 \text{ or } (3-\lambda) = 0 \Rightarrow \lambda = 2, 3$$

Thus, the eigenvalues of the given matrix are 2 and 3.

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to the eigenvalue 2. Then

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} 2x \\ 3y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow 2x = 2x \text{ and } 3y = 2y$$

The first equation is true for any real number x. The second equation gives $y = 0$.

Thus, the eigenvectors corresponding to the eigenvalue of 2 are of the form $\begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Therefore, an eigenvector corresponding to the eigenvalue of 2 is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector corresponding to the eigenvalue 3. Then

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} 2x \\ 3y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix} \Rightarrow 2x = 3x \text{ and } 3y = 3y$$

The second equation is true for any real number y. The first equation gives $x = 0$.

Thus, the eigenvectors corresponding to the eigenvalue of 3 are of the form $\begin{bmatrix} 0 \\ y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Therefore, an eigenvector corresponding to the eigenvalue of 3 is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Problem 2: Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Find whether v_1, v_2 are in the image of A or not. [9 pts]

Solution: Form the augmented matrix $[A|v_1|v_2]$ and get A into reduced row-echelon form.

$$[A|v_1|v_2] = \left[\begin{array}{ccc|cc} 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|cc} 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|cc} 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

From the last row we have the left hand side to be always 0. Thus, the right hand side should also be zero. But it is zero only for the first vector, which means the system $Ax = v_1$ has a solution while the system $Ax = v_2$ has no solution.

Therefore, the vector v_1 is in the image of A but the vector v_2 is not in the image of A.

Bonus Problem: Find the determinant of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. [2 pts]

Solution:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_2 - R_1} \begin{vmatrix} 3 & 3 & 3 \\ 4 & 5 & 6 \\ 3 & 3 & 3 \end{vmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{vmatrix} 0 & 0 & 0 \\ 4 & 5 & 6 \\ 3 & 3 & 3 \end{vmatrix}$$

Since we have an all 0 row, the determinant of the given matrix is 0.