M16600 Lecture Notes

Section 11.5: Alternating Series

■ **Section 11.5** textbook exercises, *page 776*: # 4, 5, 7, 9, 6, 14.

Definition. An ***alternating series*** is a series whose terms are alternately positive and negative.

$- \; + \; - \; + \; \cdots$

$+ \; - \; + \; - \; \cdots$

E.g., $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$

As a convention, we write an alternating series as $\sum(-1)^n b_n$, where $b_n > 0$ for all n.

$\hookrightarrow -b_1 + b_2 - b_3 + b_4 \mp \cdots$

For the example above, $b_n = \frac{1}{n}$

Convergence/Divergence for Alternating Series $\sum(-1)^n b_n$

- **Alternating Series Test (AST):** The alternating series $\sum(-1)^n b_n$ **converges** if these two conditions are satisfied:
  - (i) $\displaystyle\lim_{n\to\infty} b_n = 0$
  - (ii) $b_{n+1} \le b_n$ (the terms b_n are decreasing)
- The alternating series $\sum(-1)^n b_n$ **diverges** if $\displaystyle\lim_{n\to\infty} b_n \neq 0$.

Example 1: Use the Alternating Series Test to show that the alternating series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

$b_n = \frac{1}{n}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = \frac{\text{slower}}{\text{faster}} = 0$$

$\Rightarrow$ (i) is satisfied.

(ii) $n+1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n}$

$\Rightarrow b_{n+1} < b_n \Rightarrow b_n$'s are dec. $\Rightarrow$ By AST, the given

(*) $\displaystyle\sum_{n=1}^{\infty}(-1)^{2n-1}\frac{1}{n}$ is not an alternating series.

$$(-1)^{2n-1} = (-1)^{2n}(-1)^{-1} = \left[(-1)^2\right]^n \frac{1}{-1} = [1]^n(-1) = -1$$

series is Convergent

Example 2: Test the series for convergence or divergence

Hint: The first step in determining convergence or divergence for an **alternating series** is to compute $\lim_{n\to\infty} b_n = 0$.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2\sqrt{n}+5}$

$$\Rightarrow \quad b_n = \frac{1}{2\sqrt{n}+5} \quad \Rightarrow \lim_{n\to\infty} \frac{1}{2\sqrt{n}+5} = \frac{\text{slower}}{\text{faster}} = 0$$

$$b_{n+1} = \frac{1}{2\sqrt{n+1}+5} < \frac{1}{2\sqrt{n}+5} = b_n$$

$$\sqrt{n+1} > \sqrt{n} \Rightarrow 2\sqrt{n+1} > 2\sqrt{n} \Rightarrow 2\sqrt{n+1}+5 > 2\sqrt{n}+5$$

$$\frac{1}{2\sqrt{n+1}+5} < \frac{1}{2\sqrt{n}+5} \Rightarrow b_n\text{'s are decreasing.}$$

$\Rightarrow$ By AST, the given series converges.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3n^4+n}{4n^4-n^3+n^2+2}$

$$b_n = \frac{3n^4+n}{4n^4-n^3+n^2+2} \quad \Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n^4}{4n^4} = \lim_{n\to\infty} \frac{3}{4}$$

$$= \frac{3}{4}$$

$$\Rightarrow \lim_{n\to\infty} b_n = \frac{3}{4} \neq 0$$

$\Rightarrow$ the given series diverges.