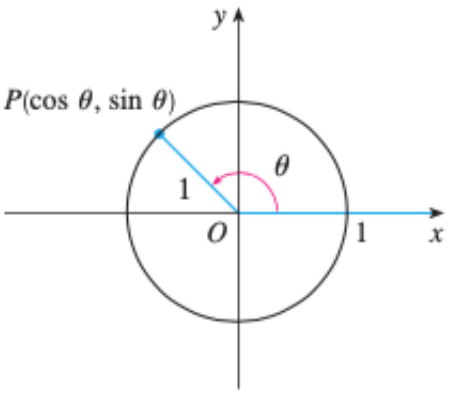
Learning objectives:

1. Derivatives of trigonometric functions.
2. Derivatives of combinations of trigonometric functions.

Trigonometric Functions

$\sin\theta$ is the length of perpendicular and $\cos\theta$ is the length of base in the triangle formed OPA inside a unit circle.

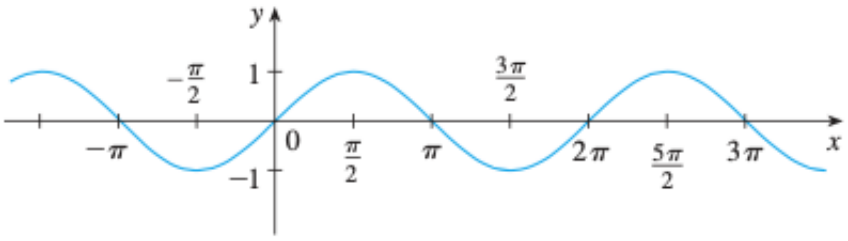


x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin x$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos x$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0

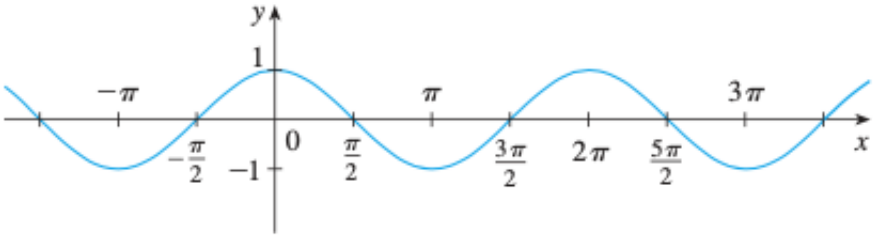
We have

$$\sin(\pi-x) = \sin x\,, \quad \sin(\pi+x) = -\sin x\,, \quad \sin(2\pi-x) = -\sin x\,, \quad \sin(-x) = -\sin x\,.$$

$$\cos(\pi-x) = -\cos x\,, \quad \cos(\pi+x) = -\cos x\,, \quad \cos(2\pi-x) = \cos x\,, \quad \cos(-x) = \cos x\,.$$



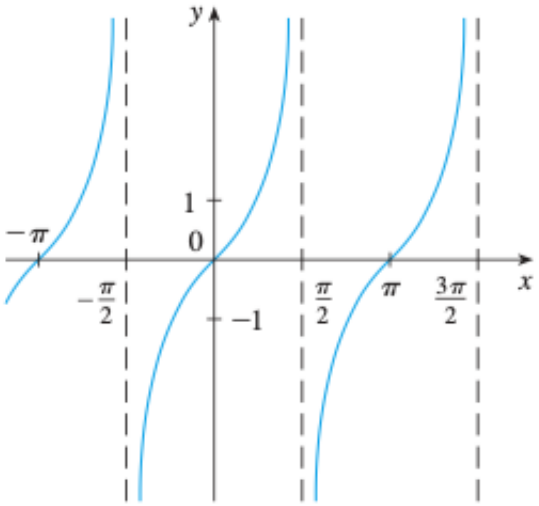
(a) $f(x) = \sin x$



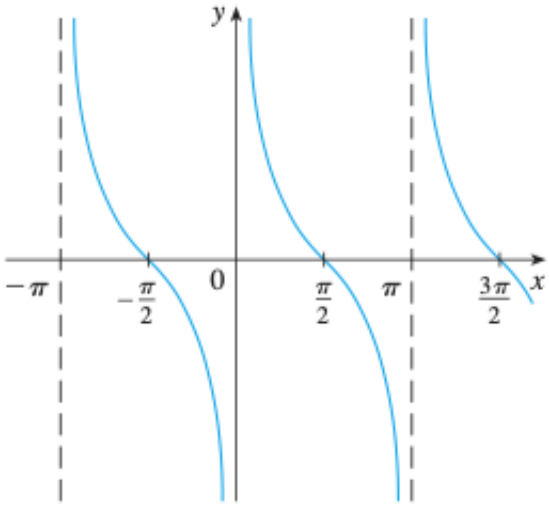
(b) $g(x) = \cos x$

The other trigonometric functions are defined as:

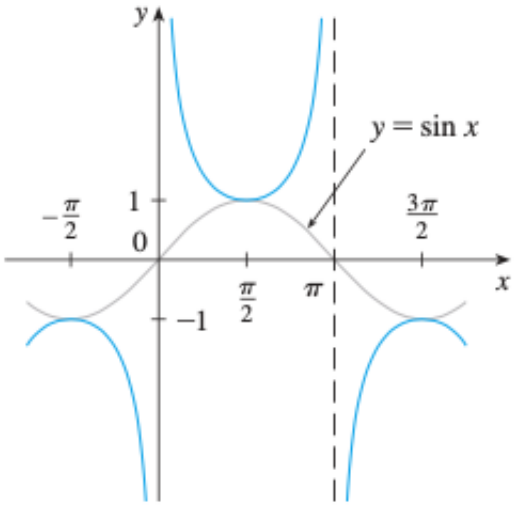
$$\tan x = \frac{\sin x}{\cos x}\,, \quad \cot x = \frac{\cos x}{\sin x}\,, \quad \sec x = \frac{1}{\cos x}\,, \quad \csc x = \frac{1}{\sin x}\,.$$



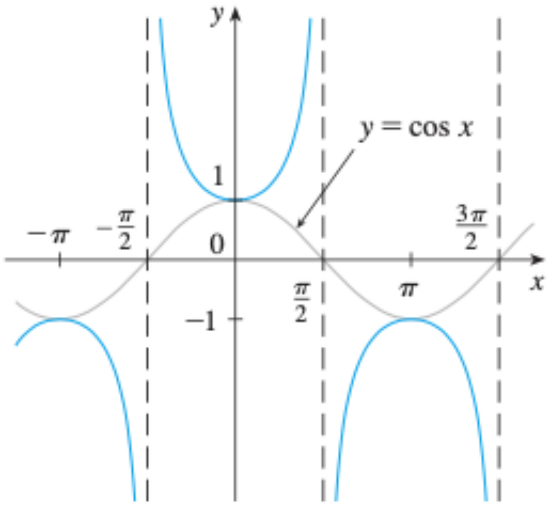
(a) $y = \tan x$



(b) $y = \cot x$



(c) $y = \csc x$



(d) $y = \sec x$

Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1\,, \quad 1 + \tan^2 x = \sec^2 x\,, \quad 1 + \cot^2 x = \csc^2 x\,.$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y\,, \quad \cos(x+y) = \cos x \cos y - \sin x \sin y\,.$$

$$\sin x \cos x = \frac{1}{2}\sin 2x\,, \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)\,, \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)\,.$$

Two important limits

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\,,\qquad\qquad \lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}=0\,.$$

Example 1. Evaluate the following limits:

1. $\lim\limits_{x\to 0}\dfrac{\sin 7x}{4x}$.
2. $\lim\limits_{x\to 0}\dfrac{\sin 10x}{\sin 5x}$.

Example 2. Evaluate $\lim_{x\to 0} x\cot x$.

Derivatives of trigonometric functions

$$\frac{d}{dx}(\sin x) = \cos x\,, \qquad \frac{d}{dx}(\cos x) = -\sin x\,,$$

$$\frac{d}{dx}(\tan x) = \sec^2 x\,, \qquad \frac{d}{dx}(\cot x) = -\csc^2 x\,,$$

$$\frac{d}{dx}(\sec x) = \sec x\,\tan x\,, \qquad \frac{d}{dx}(\csc x) = -\csc x\,\cot x\,.$$

Example 3. Differentiate $f(x) = \dfrac{\sec x}{1 + \tan x}$.

Example 4. An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t = 0$. Fixing the downward direction to be positive, its position at time t is given by $s(t) = 4 \cos t$. Find the velocity and acceleration at time t. Find the time instants at which the velocity and acceleration have greatest and smallest magnitudes.

Example 5. Find the 97-the derivative of $f(x) = \cos x$.

Example 6. Find the derivative of $r(\theta) = \theta \cos \theta$.

Example 7. Find the second derivative of $\csc x$.