M16600 Lecture Notes

Section 11.6: Absolute Convergence and the Ratio and Root Tests

■ **Section 11.6** textbook exercises, *page 782*: #1, 2, 3, 4, 5, 7, 8, 9, 11, 14, 25, 26, 27, 31, 33, 35.

DEFINITION OF ABSOLUTE CONVERGENCE. A series $\sum a_n$ is called ***absolutely convergent*** if the series of absolute values $\sum |a_n|$ is convergent.

Example 1: Test for absolute convergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ $\Rightarrow$ Test the convergence of $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{|(-1)^{n-1}|}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$: P-series with $P=2>1$

$\Rightarrow$ convergent

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ $\Rightarrow$ Test the convergence of $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{|(-1)^{n-1}|}{|n|} = \sum_{n=1}^{\infty} \frac{1}{n}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$: P-series with $P=1\leq 1$

$\Rightarrow$ divergent. $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is not absolutely convergent

DEFINITION OF CONDITIONAL CONVERGENCE. A series $\sum a_n$ is called ***conditionally convergent*** if

- it is **<u>not</u>** absolutely convergent, but
- it is convergent.

For example, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent because

$\rightarrow$ not absolutely convergent.

Alternating series

with $b_n = \frac{1}{n}$ $\Rightarrow$ $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{n+1} \leq \frac{1}{n}$

$\Rightarrow$ By AST, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is convergent

$\Rightarrow$ Conditionally convergent

Example 2: Determine whether the series is absolutely convergent or conditionally convergent.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}+1}$$

For absolute convergence, check the convergence of

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{\sqrt{n}+1}\right| = \sum_{n=1}^{\infty}\left|(-1)^{n-1}\right|\left|\frac{1}{\sqrt{n}+1}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+1}$$

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+1} \sim \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} \longrightarrow \text{P-series with } P=\frac{1}{2} \Rightarrow \text{diverges.}$$

By LCT, $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+1}$ also diverges.

$\Rightarrow \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}+1}$ is not absolutely convergent. (1)

Now we want to check its convergence

$b_n = \frac{1}{\sqrt{n}+1} \Rightarrow \lim_{n\to\infty}\frac{1}{\sqrt{n}+1} = \frac{1}{\infty} = 0 \longrightarrow$ (i) of AST is true

$b_{n+1} = \frac{1}{\sqrt{n+1}+1} \leq \frac{1}{\sqrt{n}+1} = b_n \Rightarrow$ (ii) of AST is true

By AST, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\sqrt{n}+1}$ is Convergent. (11)

Using (1) & (11), the given series is Cond. Conv.

THEOREM. If a series $\sum a_n$ is absolutely convergent then $\sum a_n$ is convergent.

For example, we know the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ is absolutely convergent from example 1. Therefore, by the theorem above, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ is automatically ___Convergent___ without using Alternating Series Tests.

1) Absolutely Convergent
2) Conditionally Convergent
3) Divergent.

divergent $\Rightarrow$ not absolutely Convergent.

Example 3: Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent or divergent.

$\cos n \to$ +ve / −ve depending on n and moreover its not alternating.

↓ Cannot use comparison tests.

↓ Cannot use AST

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

$-1 \le \cos n \le 1$

$0 \le |\cos n| \le 1$

$$|\cos n| \le 1 \Rightarrow \frac{|\cos n|}{n^2} \le \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$$

↓ P-series with P=2 ⇒ Convergent

By CT, $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ is convergent.

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent.

By the theorem $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent.

The following test is very useful in determining whether a given series is absolutely convergent

The Ratio Test. Given $\sum a_n$. First, we compute $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

(i) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =$ a number < 1, then the series $\sum a_n$ is absolutely convergent (and therefore $\sum a_n$ is convergent).

(ii) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ (or $= \infty$), then the series $\sum a_n$ is divergent.

(iii) If $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive; that is no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

Example 4: Use the Ratio Test to determine whether the series is convergent or divergent

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{(n+1)^3\,5^n}{3^n}$ $\Rightarrow a_n = (-1)^n \dfrac{(n+1)^3\,5^n}{3^n}$ $\Rightarrow a_{n+1} = (-1)^{n+1} \dfrac{(n+1+1)^3\,5^{n+1}}{3^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{\cancel{(-1)^{n+1}}(n+2)^3\,\cancel{5^{n+1}}}{\cancel{3^{n+1}}} \times \frac{\cancel{3^n}}{\cancel{(-1)^n}(n+1)^3\,\cancel{5^n}} = (-1)^{n+1-n}\frac{(n+2)^3\,5^{n+1-n}}{3^{n+1-n}\,(n+1)^3}$$

$$= \frac{(-1)\,5}{3}\,\frac{(n+2)^3}{(n+1)^3} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)5}{3}\,\frac{(n+2)^3}{(n+1)^3}\right| = \frac{5}{3}\,\frac{(n+2)^3}{(n+1)^3}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{5}{3}\,\frac{(n+2)^3}{(n+1)^3} = \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty}\frac{5}{3}\,\frac{n^3}{n^3} = \frac{5}{3} > 1$$

$\Rightarrow$ the given series is divergent.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n-1}}{n!}$ $\Rightarrow a_n = \dfrac{2^{n-1}}{n!}$ $\Rightarrow a_{n+1} = \dfrac{2^{n+1-1}}{(n+1)!} = \dfrac{2^n}{(n+1)!}$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{2^n}{(n+1)!} \times \frac{n!}{2^{n-1}} = \frac{2^{n-(n-1)}\,n!}{(n+1)!} = \frac{2\,n!}{(n+1)!}$$

$$= \frac{2\,(\cancel{n}\cdot\cancel{(n-1)}\cdot\cancel{(n-2)}\cdots\cdots\cancel{1})}{(n+1)\cancel{n}\,\cancel{(n-1)}\,\cancel{(n-2)}\cdots\cdots\cancel{1}} = \frac{2}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2}{n+1}\right| = \lim_{n\to\infty}\frac{2}{n+1} = \frac{2}{\infty} = 0 < 1$$

$\Rightarrow$ the given series is absolutely convergent, hence convergent.

The following test is convenient to apply when nth powers occur.

THE ROOT TEST. Given $\sum a_n$.

$\sqrt[n]{|a_n|} = (|a_n|)^{1/n}$

nth root

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ = a number < 1, then the series $\sum a_n$ is absolutely convergent (and therefore $\sum a_n$ is convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ (or $= \infty$), then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.

Example 5: Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n \Rightarrow \sqrt[n]{|a_n|} = \left|\left(\frac{2n+3}{3n+2}\right)^n\right|^{\frac{1}{n}}$$

$$= \left[\left(\left|\frac{2n+3}{3n+2}\right|\right)^n\right]^{\frac{1}{n}} = \left|\frac{2n+3}{3n+2}\right|^{n\times\frac{1}{n}}$$

$$= \left|\frac{2n+3}{3n+2}\right| = \frac{2n+3}{3n+2}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{2n}{3n} = \frac{2}{3} < 1$$

$\Rightarrow$ the given series is absolutely convergent.