■ **Section 7.3** exercises, *page 531*: #1, 2, 5, 6, 8, 12, 14, 9, 22, 17, 11.

Trigonometric Substitution is a new method which oftentimes are useful in solving integrals that involves the following radicals. We will also give the appropriate trig substitution for each type of radical:

$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

$\sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta$

$\sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2\sec^2\theta} = a\sec\theta$

$\sqrt{a^2\sec^2\theta - a^2} = a\tan\theta$

We might need these two formulas for integrals in this section:

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Example 1: Evaluate $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$ $\Rightarrow x = 3\sin\theta \Rightarrow \sin\theta = \frac{x}{3}$

$\Rightarrow dx = 3\cos\theta\,d\theta$ diff.

$$I = \int \frac{(3\sin\theta)^2}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,d\theta = \int \frac{(9\sin^2\theta)(3\cos\theta)}{\sqrt{9\cos^2\theta}}\,d\theta$$

$$= \int \frac{(9\sin^2\theta)\cancel{(3\cos\theta)}}{\cancel{3\cos\theta}}\,d\theta = \int 9\sin^2\theta\,d\theta$$

$$= 9\int \sin^2\theta\,d\theta = 9\int \frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= \int \left[\frac{9}{2} - \frac{9}{2}\cos(2\theta)\right]d\theta$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \int \frac{9}{2}\, d\theta - \int \frac{9}{2}\cos(2\theta)\, d\theta$$

$$= \frac{9}{2}\theta - \frac{9}{2}\int \cos(2\theta)\, d\theta = \frac{9}{2}\theta - \frac{9}{2}\,\frac{\sin(2\theta)}{2} + C$$

$$= \frac{9}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + C$$

$\sin\theta = \frac{x}{3}$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{x^2}{9}} = \frac{\sqrt{9-x^2}}{3}$$

$$\| $$

$$\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\left(\frac{x}{3}\right)\frac{\sqrt{9-x^2}}{3}$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2}$$

Example 2: Compute $\displaystyle\int_2^3 \frac{1}{\sqrt{x^2-1}}\, dx$

$$I = \int \frac{1}{\sqrt{x^2-1}}\, dx$$

$x = \sec\theta \ni dx = \sec\theta\tan\theta\, d\theta$

$$I = \int \frac{1}{\sqrt{\sec^2\theta - 1}}\sec\theta\tan\theta\, d\theta = \int \frac{1}{\sqrt{\tan^2\theta}}\sec\theta\tan\theta\, d\theta$$

$$= \int \frac{\sec\theta\cancel{\tan\theta}}{\cancel{\tan\theta}}\, d\theta = \int \sec\theta\, d\theta$$

$$\int \frac{\sec\theta(\sec\theta+\tan\theta)}{(\sec\theta+\tan\theta)}\, d\theta$$

$$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{(\sec\theta+\tan\theta)}\, d\theta$$

$u = \sec\theta + \tan\theta$

$$= \int \frac{du}{u} = \ln|u|$$

$$= \ln|\sec\theta + \tan\theta|$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{x^2-1}$$

$$I = \ln\left|x + \sqrt{x^2-1}\right| + C$$

$$\int_2^3 \frac{1}{\sqrt{x^2-1}}\, dx = \ln\left|x+\sqrt{x^2-1}\right|\Big]_2^3$$

$$= \ln(3+\sqrt{8}) - \ln(2+\sqrt{3})$$

$= \ln \frac{3+\sqrt{8}}{2+\sqrt{3}}$

Example 3: Find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$ $\quad \leftarrow a^2$

$a^2 = 4 \Rightarrow a = 2$

$x = 2\tan\theta$

$\Rightarrow dx = 2\sec^2\theta\, d\theta$

$$I = \int \frac{1}{(2\tan\theta)^2\sqrt{4\tan^2\theta+4}}\, 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{4\tan^2\theta\sqrt{4(\tan^2\theta+1)}}\, 2\sec^2\theta\, d\theta = \int \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4\sec^2\theta}}\, d\theta$$

$$= \int \frac{\cancel{2}\sec^{\cancel{2}}\theta}{4\tan^2\theta\,\cancel{2}\cancel{\sec\theta}}\, d\theta = \int \frac{\sec\theta}{4\tan^2\theta}\, d\theta$$

$$= \int \frac{1/\cos\theta}{4\,\frac{\sin^2\theta}{\cos^2\theta}}\, d\theta = \int \frac{1}{4}\,\frac{1}{\cancel{\cos\theta}}\,\frac{\cos^{\cancel{2}}\theta}{\sin^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \frac{\cos\theta\, d\theta}{\sin^2\theta} \qquad u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$$

$$= \frac{1}{4}\int \frac{du}{u^2} = \frac{1}{4}\int u^{-2}\, du = \frac{1}{4}\,\frac{u^{-2+1}}{-2+1} + C$$

$$= \frac{-1}{4}u^{-1} + C = \frac{-1}{4u} + C = \frac{-1}{4\sin\theta} + C$$

$\tan\theta = \frac{x}{2} = \frac{P}{B}$

$P = x,\ B = 2$

$H^2 = x^2 + 2^2 \Rightarrow H = \sqrt{x^2+4}$

$\sin\theta = \frac{P}{H} = \frac{x}{\sqrt{x^2+4}}$

$\Uparrow$

$$\Rightarrow I = \frac{-1}{4\cdot\frac{x}{\sqrt{x^2+4}}} + C$$

$$\Rightarrow I = -\frac{\sqrt{x^2+4}}{4x} + C$$

Example 4 : $\int \sqrt{a^2 - x^2}\, dx$

$x = a \sin\theta$

$dx = a\cos\theta\, d\theta$

$$I = \int \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta$$

$$= \int \sqrt{a^2 \underbrace{(1-\sin^2\theta)}_{\cos^2\theta}}\; a\cos\theta\, d\theta$$

$$= \int a\cos\theta\; a\cos\theta\, d\theta = a^2 \int \cos^2\theta\, d\theta$$

$$= a^2 \int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\int 1\, d\theta + \frac{a^2}{2}\int \cos 2\theta\, d\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\,\frac{\sin 2\theta}{2} + C$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\,\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}} + C$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\sin\theta\cos\theta + C$$

$x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a} \Rightarrow \cos\theta = \sqrt{1-\sin^2\theta}$

$$= \sqrt{1 - \frac{x^2}{a^2}}$$

$$= \sqrt{\left(\frac{a^2-x^2}{a^2}\right)}$$

$$= \frac{\sqrt{a^2-x^2}}{a}$$

$$\Rightarrow I = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{\cancel{a^2}}{2}\cdot\frac{x}{\cancel{a}}\cdot\frac{\sqrt{a^2-x^2}}{\cancel{a}}$$

$$\Rightarrow I = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C$$

$$\boxed{\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+C}$$

$$\left.\begin{aligned} &\int \sqrt{a^2+x^2}\,dx = ??\\ &\int \sqrt{x^2-a^2}\,dx = ?? \end{aligned}\right\}$$

HW.

$$x = a\tan\theta \Rightarrow dx = a\sec^2\theta\, d\theta$$

$$\int \sqrt{a^2+x^2}\,dx = \int \sqrt{a^2+a^2\tan^2\theta}\; a\sec^2\theta\, d\theta$$

$$= \int (a\sec\theta)\, a\sec^2\theta\, d\theta = a^2\int \sec^3\theta\, d\theta$$

$$\int \sec^3\theta\, d\theta = \int \frac{1}{\cos^3\theta}\, d\theta = \int \frac{\cos\theta}{\cos^4\theta}\, d\theta$$

$$= \int \frac{du}{[\cos^2\theta]^2}$$

$$u = \sin\theta$$

$$\Rightarrow du = \cos\theta\, d\theta$$

$$= \int \frac{du}{[1-\sin^2\theta]^2} = \int \frac{du}{(1-u^2)^2}$$

$$= \int \frac{du}{(u^2-1)^2} = \int \frac{du}{(u-1)^2(u+1)^2}$$