Indiana University, Indianapolis

Spring 2025 Math-I 165

Practice Test 3a

Instructor: Keshav Dahiya

Name:____

Instructions:

- No cell phones, calculators, watches, technology, hats - stow all in your bags.
- **Write your name** on this cover page.
- This test is **closed book and closed notes**.
- All work must be clearly shown for partial credit.
- If you wish for something not to be graded, please strike it out neatly.
- Box, circle, or otherwise clearly indicate your final answer.
- When you finish, return your test to the proctor, and leave the classroom.
- There are a total of **12 problems including 2 bonus problems**.
  - Problems 1-10 are each worth 10 points.
  - The bonus problems are each worth 5 points.
- You can score a **maximum of 110 points out of 100**.
- There are a total of **7 pages** including the cover page.

Problem 1. Write an expression as limit of a sum for the integral $\int_{\pi}^{5\pi} \sqrt{\sin \pi x}\, dx$. [10 pts]

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\frac{b-a}{n}}_{\Delta x} \sum_{i=1}^{n} f(x_i) \quad \text{where } x_i = a + i\underbrace{\left(\frac{b-a}{n}\right)}_{\Delta x}$$

$$a = \pi,\quad b = 5\pi,\quad \Delta x = \frac{5\pi - \pi}{n} = \frac{4\pi}{n}$$

$$x_i = \pi + i\frac{4\pi}{n} = \pi\left(1 + \frac{4i}{n}\right)$$

$$\int_{\pi}^{5\pi} \sqrt{\sin \pi x}\, dx = \lim_{n\to\infty} \frac{4\pi}{n} \sum_{i=1}^{n} \sqrt{\sin \pi x_i}$$

$$= \lim_{n\to\infty} \frac{4\pi}{n} \sum_{i=1}^{n} \sqrt{\sin\left(\pi^2\left(1+\frac{4i}{n}\right)\right)}$$

Problem 2. Let f be an even continuous function. Suppose $\int_0^6 f(x)\,dx = 10$ and $\int_4^6 f(x)\,dx = 4$. Find $\int_{-4}^4 f(x)\,dx$. [10 pts]

$$\int_{-4}^{4} f(x)\,dx = 2\underbrace{\int_0^4 f(x)\,dx}_{} \quad \text{(1)}$$

$$[0,6] = [0,4] \cup [4,6]$$

$$\underbrace{\int_0^6 f(x)\,dx}_{10} = \int_0^4 f(x)\,dx + \underbrace{\int_4^6 f(x)\,dx}_{4}$$

$$\Rightarrow \int_0^4 f(x)\,dx = 10 - 4 = 6$$

From (1) :-

$$\int_{-4}^{4} f(x)\,dx = 2(6) = 12$$

Problem 3. Find the derivative of the function $f(x) = \int_{x^2-1}^{x^3-1} \tan(\theta+1)\, d\theta$. [10 pts]

$$f(x) = \int_{v(x)}^{u(x)} g(t)\, dt = G(u(x)) - G(v(x))$$

G → antiderivate of g

$$f'(x) = g(u(x))\, u'(x) - g(v(x))\, v'(x)$$

$$f'(x) = \operatorname{Tan}(\underbrace{x^3-1}_{u(x)}+1) \cdot (x^3-1)' - \operatorname{Tan}(\underbrace{x^2-1}_{v(x)}+1) \cdot (x^2-1)'$$

$$= 3x^2 \operatorname{Tan}(x^3) - 2x \operatorname{Tan}(x^2)$$

Problem 4. A particle moves in a straight line with velocity varying as a function of time such that $v(t) = 5\sin(2t+\pi)$. Find the distance travelled from $t = 0$ to $t = \pi$ seconds. [10 pts]

$$\text{distance from } 0 \text{ to } \pi = \int_0^{\pi} \text{speed}(t)\, dt$$

$$= \int_0^{\pi} |v(t)|\, dt = \int_0^{\pi} |5 \sin(2t+\pi)|\, dt$$

$$\sin(2t+\pi) = \begin{cases} -\text{ve} & \text{when } 0 \le t \le \pi/2 \\ +\text{ve} & \text{when } \frac{\pi}{2} \le t \le \pi \end{cases}$$

use graph
or
check that
$\sin\left(2\frac{\pi}{4}+\pi\right) = -1$
$\sin\left(2\frac{3\pi}{4}+\pi\right) = +1$

$$= \int_0^{\pi/2} -5\sin(2t+\pi)\, dt + \int_{\pi/2}^{\pi} 5\sin(2t+\pi)\, dt$$

$$= -\frac{5}{2}\left[-\cos(2t+\pi)\right]\Big|_0^{\pi/2} + \frac{5}{2}\left[-\cos(2t+\pi)\right]\Big|_{\pi/2}^{\pi}$$

$$= -\frac{5}{2}\left[-\cos 2\pi - (-\cos\pi)\right] + \frac{5}{2}\left[-\cos(3\pi) - (-\cos 2\pi)\right]$$

$$= -\frac{5}{2}\left[-1-(1)\right] + \frac{5}{2}\left[-(-1)-(-1)\right] = 5+5 = 10$$

Problem 5. Evaluate the indefinite integral $\int \frac{x+2}{\sqrt{x^2+4x}}\,dx$. [10 pts]

Substitute $u = x^2+4x \Rightarrow \frac{du}{dx} = 2x+4$

$$\Rightarrow \frac{du}{dx} = 2(x+2) \Rightarrow du = 2(x+2)\,dx$$

$$\Rightarrow \frac{1}{2}du = (x+2)\,dx$$

$$I = \int \frac{x+2}{\sqrt{x^2+4x}}\,dx = \int \frac{1}{\sqrt{x^2+4x}} \underbrace{(x+2)\,dx}_{\frac{1}{2}du}$$

$$= \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2}\,du = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

$$= \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\,\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \cancel{\frac{1}{2}}\,\frac{u^{1/2}}{\cancel{1/2}} + C = \sqrt{u} + C$$

$$= \sqrt{x^2+4x} + C$$

Problem 6. Evaluate definite integral $\int_0^{\pi/4} (1+\tan t)^3 \sec^2 t\,dt$. [10 pts]

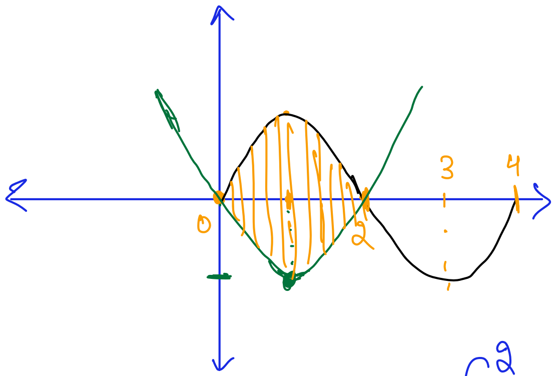
$u = 1+\tan t \Rightarrow \frac{du}{dt} = \sec^2 t \Rightarrow du = \sec^2 t\,dt$

$$\int_0^{\frac{\pi}{4}} (\underbrace{1+\tan t}_{u})^3 \underbrace{\sec^2 t\,dt}_{du}$$

$$= \int_{1+\tan 0}^{1+\tan\frac{\pi}{4}} u^3\,du = \int_1^2 u^3\,du = \left.\frac{u^4}{4}\right|_1^2$$

$$= \frac{2^4 - 1^4}{4} = \frac{15}{4}$$

Problem 7. Find area of the region bounded by the curves $y = \sin(\pi x/2)$ and $y = x^2 - 2x$. [10 pts]



$x^2 - 2x = 0 \Rightarrow x(x-2) = 0$

$\Rightarrow x = 0,\ x = 2$

At $x = 1$, $x^2 - 2x = 1^2 - 2(1) = -1$

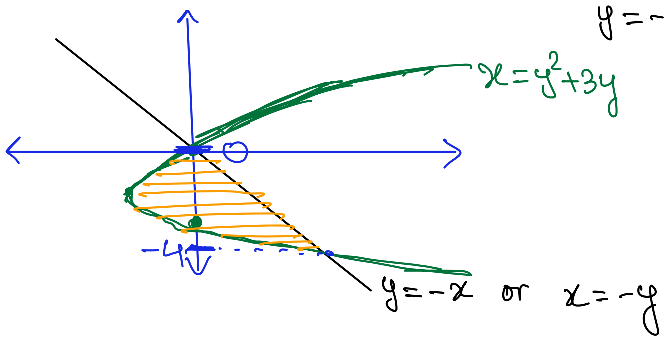
$$A = \int_0^2 (\text{upper curve} - \text{lower curve})\, dx$$

$$= \int_0^2 \left[\sin\frac{\pi x}{2} - (x^2 - 2x)\right] dx$$

$$= \frac{2}{\pi}\left(-\cos\frac{\pi}{2}x\right)\Big|_0^2 - \left(\frac{x^3}{3} - x^2\right)\Big|_0^2$$

$$= \frac{2}{\pi}\left[-\cos\pi - (-\cos 0)\right] - \left(\frac{2^3}{3} - 2^2\right) = \frac{4}{\pi} - \left(\frac{8}{3} - 4\right) = \frac{4}{\pi} + \frac{4}{3}$$

Problem 8. Find area of the region bounded by $\underbrace{x + y = 0}_{y=-x}$ and $x = y^2 + 3y$. [10 pts]



$y^2 + 3y = 0$

$y(y+3) = 0 \Rightarrow y = 0,\ y = -3$

At $y = \frac{-3}{2}$, $x = \left(\frac{-3}{2}\right)^2 + 3\left(\frac{-3}{2}\right)$

$= \frac{-9}{4}$

Pts of intersection

$x = -y$ and $x = y^2 + 3y \Rightarrow -y = y^2 + 3y \Rightarrow y^2 + 4y = 0$

$\Rightarrow y(y+4) = 0$

$\Rightarrow y = 0,\ y = -4$

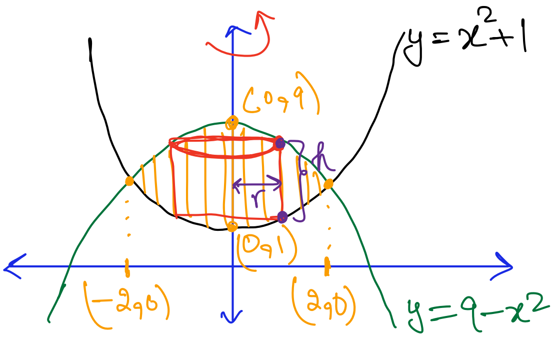
$$A = \int_{-4}^0 (\text{right curve} - \text{left curve})\, dy$$

$$= \int_{-4}^0 \left[-y - (y^2 + 3y)\right] dy = \int_{-4}^0 (-4y - y^2)\, dy$$

$$= \left(-2y^2 - \frac{y^3}{3}\right)\Big|_{-4}^0 = 2(-4)^2 + \frac{(-4)^3}{3}$$

$= \frac{32}{3}$

Problem 9. Find the volume of the solid obtained by rotating the region bounded by $y = x^2 + 1$ and $y = 9 - x^2$ about the y-axis. [10 pts]



Pts. of intersection

$x^2+1 = 9-x^2 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4$
$\Rightarrow x = \pm 2$

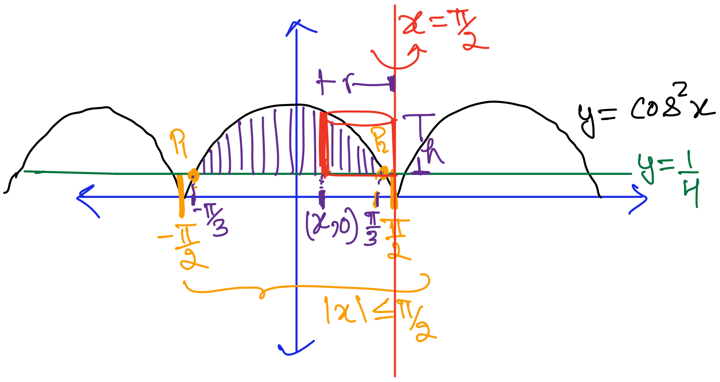
Curves can be expressed as fns. of x and axis of rotation is verticle
$\Downarrow$
use shell method.

radius of a shell $= x$

height of a shell = upper y-value at x − lower y-value at x
$= (9-x^2) - (x^2+1) = 8 - 2x^2$

$\Rightarrow dV = 2\pi r h \, dx$
$= 2\pi x (8-2x^2) dx \Rightarrow V = \int_0^2 2\pi x (8-x^2)\, dx$
$= 2\pi \left(4x^2 - \frac{x^4}{4}\right)\Big|_0^2 = 24\pi$

Problem 10. Set up an integral for the volume of the solid obtained by rotating the region bounded by $y = \cos^2 x$, $|x| \le \pi/2$, $y = 1/4$ about the axis $x = \pi/2$. [10 pts]



Find Pts. of intersections P_1, P_2

$\cos^2 x = \frac{1}{4} \Rightarrow \cos x = \pm \frac{1}{2}$

For $|x| \le \frac{\pi}{2}$, $\cos x > 0$

$\Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \pm \frac{\pi}{3}$

$\Rightarrow$ x-cord. of $P_1 = -\frac{\pi}{3}$

x-cord. of $P_2 = \frac{\pi}{3}$

Curves can be expressed as fns. of x and axis of rotation is verticle
$\Downarrow$
use shell method.

$r = \frac{\pi}{2} - x$, $h = \cos^2 x - \frac{1}{4}$

$$\Rightarrow V = \int_{-\pi/3}^{\pi/3} 2\pi \left(\frac{\pi}{2} - x\right)\left(\cos^2 x - \frac{1}{4}\right) dx$$

Bonus Problem 1. Evaluate the limit: $\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right]$. [5 pts]

Try to express this limit as an integral.

$$L = \lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right]$$

$$= \lim_{n\to\infty} \underbrace{\frac{1}{n}}_{\Delta x} \sum_{i=1}^{n} \underbrace{\left(\frac{i}{n}\right)^9}_{f(x_i)}$$

$\Rightarrow \Delta x = \frac{1}{n}$

$x_i = \frac{i}{n} = 0 + i\left(\frac{1}{n}\right)$

In general we have $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\left(\frac{b-a}{n}\right)$

$\Rightarrow \frac{b-a}{n} = \frac{1}{n}$ and $a + i\left(\frac{b-a}{n}\right) = \frac{i}{n} \Rightarrow a + \frac{i}{n} = \frac{i}{n}$

$\Rightarrow \boxed{a=0}$

$\Rightarrow \frac{b-0}{n} = \frac{1}{n} \Rightarrow \boxed{b=1}$

$f(x_i) = \left(\frac{i}{n}\right)^9 = x_i^9$

$\Rightarrow \boxed{f(x) = x^9}$

Therefore, $L = \int_0^1 x^9\,dx = \left.\frac{x^{10}}{10}\right|_0^1 = \frac{1}{10}$

Bonus Problem 2. Evaluate the integral $\int_{-1}^{1}\left(x + \sqrt{1-x^2}\right)dx$. [5 pts]

$$I = \int_{-1}^{1}\left(x + \sqrt{1-x^2}\right)dx = \int_{-1}^{1} x\,dx + \int_{-1}^{1}\sqrt{1-x^2}\,dx$$

odd function even function

$\Rightarrow \int_{-1}^{1} x\,dx = 0$

and $\int_{-1}^{1}\sqrt{1-x^2}\,dx = 2\underbrace{\int_0^1 \sqrt{1-x^2}\,dx}_{\text{area under a quarter of the unit circle}} = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

$\Rightarrow I = 0 + \frac{\pi}{2} = \frac{\pi}{2}$