■ **Section 7.5** exercises, *page 547*: 1, 3, 5, 7, 9, 11, 13, 15, 21, 20, 2, 4, 6, 12, 16, 18, 37, 38, 8, 14, 17, 26, .

As we have seen, integration is more challenging than differentiation. No hard and fast rules can be given as to which integration method applies in a given situation, but you can think about these steps as a guideline.

- Do we need to use algebra or trigonometric identities to **rewrite** the integrand so that we can apply basic integration formulas?
- What about an obvious u-**substitution**?
- If the integrand is a *rational function* but the above two steps couldn't solve the integral, think about **integration by partial fractions** (section 7.4).
- If the integrand is a *product* of a polynomial with a transcendental function (such as a trigonometric function, exponential, or logarithmic function), then you can try **integration by parts**.
- If the integrand involves radicals couldn't be solved by an obvious u-sub, you can think about using **trigonometric substitution** (section 7.3).
- Try again.

Obviously, the first step of integration is to remember basic integral formulas. See next page for the **Table of Integration Formulas**.

Table of Integration Formulas

$$\int x^n\,dx = \left(\frac{1}{n+1}\right)x^{n+1} + C \quad (n \neq -1)$$

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C$$

$$\int \csc^2(x)\,dx = -\cot(x) + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$ → To Prove, substitute $x = au$ in

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \sec(x)\,dx = \ln|\sec x + \tan x| + C$$

$$\int \tan(x)\,dx = \ln|\sec x| + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

$$\int \left(x + 1 + \ln x + \frac{\ln x}{x}\right)dx$$

① $$\int (x+1)\left(1+\frac{\ln x}{x}\right)dx$$

$$u = 1+\frac{\ln x}{x}$$

$$\frac{du}{dx} = \frac{x\frac{1}{x}-\ln x}{x^2}$$

$$= \frac{1-\ln x}{x^2}$$

$$= \int (x+1)\,\frac{(x+\ln x)}{x}\,dx$$

$$= \int \frac{x+1}{x}(x+\ln x)\,dx = \int\left(\frac{x}{x}+\frac{1}{x}\right)(x+\ln x)\,dx$$

$$= \int\left(1+\frac{1}{x}\right)(x+\ln x)\,dx$$

$$u = x+\ln x$$

$$\Rightarrow \frac{du}{dx} = 1+\frac{1}{x}$$

$$= \int \underbrace{(x+\ln x)}_{u}\,\underbrace{\left(1+\frac{1}{x}\right)dx}_{du}$$

$$= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(x+\ln x)^2 + C$$

$$\int\left(x+1+\ln x+\frac{\ln x}{x}\right)dx = \int x\,dx + \int 1\,dx + \int \ln x\,dx$$

$$+\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}$$

$$u = \ln x \Rightarrow du = \frac{dx}{x}$$

$$\int \underbrace{\ln x}_{u} \, \underbrace{dx}_{dv}$$

$u = \ln x \Rightarrow du = \frac{1}{x} dx$

$dv = dx \Rightarrow v = x$

$$= uv - \int v \, du = (\ln x) x - \int \cancel{x} \frac{1}{\cancel{x}} dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

Test 1

A radioactive material decayed to 25% of original mass in 10 years. Find the half-life.

Let Half-life be x years.

$$m(0) \xrightarrow{x} \frac{m(0)}{2} \xrightarrow{x} \frac{1}{4} m(0)$$

25% of original mass

$2x = 10$

$\Rightarrow x = 5$ years

$$\frac{10 \ln 2}{\ln 4}$$

$//$

$$\frac{10 \cancel{\ln 2}}{2 \cancel{\ln 2}} = 5 \text{ yrs.}$$

$\ln 4 = \ln 2^2$
$= 2 \ln 2$

(1) $\int \sqrt{1-x^2}\, dx$

Substitute $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$$\int \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta = \int \cos\theta\cos\theta\, d\theta = \int \cos^2\theta\, d\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\, d\theta = \int \frac{1}{2}\, d\theta + \int \frac{1}{2}\cos 2\theta\, d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\int \cos 2\theta\, d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\frac{\sin 2\theta}{2} + C$$

$$= \frac{1}{2}\theta + \frac{1}{2}\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}} + C$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$$

$\sin\theta = x \rightarrow \theta = \sin^{-1}(x)$

$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$

$$\Rightarrow \int \sqrt{1-x^2}\, dx = \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2} + C$$

$$\int \sqrt{a^2-x^2}\, dx = \int \sqrt{a^2-(au)^2}\; a\, du$$

$x = au \Rightarrow dx = a\, du$

* a is +ve real number

$$\Rightarrow \int \sqrt{a^2-x^2}\, dx = \int \sqrt{a^2(1-u^2)}\quad a\, du$$

$$= \int a\sqrt{1-u^2}\quad a\, du$$

$$= a^2 \int \sqrt{1-u^2}\, du$$

$$= a^2\left[\frac{1}{2}\sin^{-1}(u) + \frac{1}{2}u\sqrt{1-u^2}\right] + C$$

$x = au$

$\Rightarrow u = \frac{x}{a}$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}a^{\cancel{2}}\frac{x}{\cancel{a}}\sqrt{1-\frac{x^2}{a^2}}$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\cancel{a}\,x\,\frac{\sqrt{a^2-x^2}}{\cancel{\sqrt{a^2}}}$$

$$\boxed{\int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2-x^2} + C}$$

(2) $\int \sqrt{1+x^2}\, dx$

$x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\Rightarrow I = \int \sqrt{1+\tan^2\theta}\ \sec^2\theta\, d\theta = \int \sqrt{\sec^2\theta}\ \sec^2\theta\, d\theta$$

$$= \int \sec\theta\ \sec^2\theta\, d\theta$$

$$\Rightarrow I = \int \sec^3\theta \, d\theta = \int \frac{1}{\cos^3\theta} d\theta$$

$$= \int \frac{\cos\theta}{\cos^4\theta} d\theta$$

$$\cos^4\theta = (\cos^2\theta)^2$$
$$= (1-\sin^2\theta)^2$$

$$= \int \frac{\overbrace{\cos\theta \, d\theta}^{du}}{(1-\sin^2\theta)^2}$$

$u = \sin\theta \Rightarrow du = \cos\theta \, d\theta$

$$= \int \frac{1}{(1-u^2)^2} du$$

$$\Rightarrow I = \int \frac{1}{(u^2-1)^2} du = \int \frac{1}{(u-1)^2(u+1)^2} du$$

$$(u^2-1)^2 = ((u-1)(u+1))^2 = (u-1)^2(u+1)^2$$

$$\left[\frac{1}{(u-1)^2(u+1)^2} = \frac{a}{u-1} + \frac{b}{(u-1)^2} + \frac{c}{u+1} + \frac{d}{(u+1)^2}\right] (u-1)^2(u+1)^2$$

$$1 = a(u-1)(u+1)^2 + b(u+1)^2 + c(u-1)^2(u+1) + d(u-1)^2$$

$u=1 \Rightarrow 1 = 0 + b(1+1)^2 + 0 + 0 \Rightarrow 1 = 4b \Rightarrow b = \frac{1}{4}$

$u=-1 \Rightarrow 1 = 0 + 0 + 0 + d(-1-1)^2 \Rightarrow 1 = 4d \Rightarrow d = \frac{1}{4}$

$u=0 \Rightarrow 1 = a(-1)(1)^2 + b(1)^2 + c(0-1)^2(0+1) + d(0-1)^2$

$\Rightarrow 1 = -a + b + c + d \Rightarrow 1 = -a + \frac{1}{4} + c + \frac{1}{4}$

$$\Rightarrow 1 = \frac{1}{2} - a + c \Rightarrow -a + c = \frac{1}{2} \quad \text{— (I)}$$

$$u = 2 \Rightarrow 1 = a(2-1)(2+1)^2 + b(2+1)^2 + c(2-1)^2(2+1) + d(2-1)^2$$

$$\Rightarrow 1 = 9a + 9b + 3c + d$$

$$1 = 9a + \frac{9}{4} + 3c + \frac{1}{4} \Rightarrow 1 = 9a + 3c + \frac{5}{2}$$

$$\Rightarrow -\frac{3}{2} = 9a + 3c$$

$$-a + c = \frac{1}{2} \qquad \Rightarrow \frac{-1}{2} = 3a + c \quad \text{— (II)}$$

$$3a + c = \frac{-1}{2}$$

$$-4a = \frac{1}{2} - \left(-\frac{1}{2}\right) \Rightarrow -4a = 1 \Rightarrow a = \frac{-1}{4} \Rightarrow \frac{1}{4} + c = \frac{1}{2}$$

$$\Rightarrow c = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\Rightarrow c = \frac{1}{4}$$

$$\int \frac{1}{(u-1)^2 (u+1)^2} du = \frac{-1}{4} \int \frac{1}{u-1} du + \frac{1}{4} \int \frac{1}{(u-1)^2} du + \frac{1}{4} \int \frac{1}{u+1} du + \frac{1}{4} \int \frac{1}{(u+1)^2} du$$

$$= \frac{-1}{4} \ln|u-1| + \frac{1}{4} \frac{(u-1)^{-2+1}}{-2+1} + \frac{1}{4} \ln|u+1| + \frac{1}{4} \frac{(u+1)^{-2+1}}{-2+1} + C$$

$$= \frac{-1}{4} \ln|u-1| - \frac{1}{4(u-1)} + \frac{1}{4} \ln|u+1| - \frac{1}{4(u+1)} + C$$

$u = \sin\theta$

$$= \frac{-1}{4}\left[\frac{1}{u-1} + \frac{1}{u+1}\right] + \frac{1}{4}\left[\ln|u+1| - \ln|u-1|\right]$$

$x = \tan\theta$

$$= \frac{-1}{4}\left[\frac{u \cancel{+1} + u \cancel{-1}}{(u-1)(u+1)}\right] + \frac{1}{4} \ln\left|\frac{u+1}{u-1}\right| + C$$

$$= \frac{-1}{\cancel{4}_{2}} \frac{\cancel{2}u}{u^2-1} + \frac{1}{4} \ln\left|\frac{u+1}{u-1}\right| + C$$

$$= \frac{-u}{2(u^2-1)} + \frac{1}{4} \ln\left|\frac{u+1}{u-1}\right| + C$$

$$\| \ u = \sin\theta$$

$$\frac{-\sin\theta}{2(\sin^2\theta-1)} + \frac{1}{4} \ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right| + C$$

$$= \frac{\sin\theta}{2(1-\sin^2\theta)} + \frac{1}{4} \ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right| + C$$

$$= \frac{\sin\theta}{2\cos^2\theta} + \frac{1}{4} \ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right| + C$$

$$\boxed{\frac{\sin\theta}{\cos^2\theta} = \frac{\sin\theta}{\cos\theta} \frac{1}{\cos\theta}}$$

$$= \frac{1}{2} \tan\theta \sec\theta + \frac{1}{4} \ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right| + C$$

$$\left.\begin{array}{l} x = \tan\theta \\ \sqrt{1+x^2} = \sec\theta \end{array}\right\} \text{divide} \Rightarrow \frac{x}{\sqrt{1+x^2}} = \sin\theta$$

$$= \frac{1}{2} x\sqrt{1+x^2} + \frac{1}{4} \ln\left|\frac{\frac{x}{\sqrt{1+x^2}}+1}{\frac{x}{\sqrt{1+x^2}}-1}\right| + C$$

$$\int \sqrt{1+x^2}\, dx = \frac{1}{2} x\sqrt{1+x^2} + \frac{1}{4} \ln\left|\frac{x+\sqrt{1+x^2}}{x-\sqrt{1+x^2}}\right| + C$$

$$\int \sqrt{1+x^2}\, dx = \frac{1}{2} x \sqrt{1+x^2} + \frac{1}{2} \ln\left|x + \sqrt{1+x^2}\right| + C$$

$$\frac{x+\sqrt{1+x^2}}{x-\sqrt{1+x^2}} \left(\frac{x+\sqrt{1+x^2}}{x+\sqrt{1+x^2}}\right) = \frac{\left(x+\sqrt{1+x^2}\right)^2}{x^2-(1+x^2)}$$

$$= \frac{\left(x+\sqrt{1+x^2}\right)^2}{-1}$$

$$\ln\left|\frac{x+\sqrt{1+x^2}}{x-\sqrt{1+x^2}}\right| = \ln\left|\left(x+\sqrt{1+x^2}\right)^2\right| = 2\ln\left|x+\sqrt{1+x^2}\right|$$

HW:

$$\int \sqrt{x^2-1}\, dx = \frac{x}{2}\sqrt{x^2-1} + \frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right| + C$$