

**Problem 1:** Sketch the following parametric curves.

1.  $x = 2t + 1, y = t^2 + 1, t \in \mathbb{R}$ .
2.  $x = 1 + \sin \theta, y = -1 + 2 \cos \theta, 0 \leq \theta \leq 2\pi$ .
3.  $x = 2 + 2 \sec \theta, y = 1 + 4 \tan \theta, \theta \in (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2)$ .
4.  $x = t, y = 4 - t, 0 \leq t \leq 4$ .

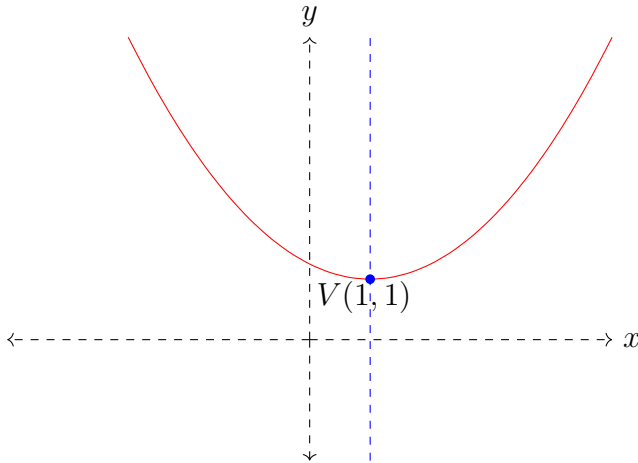
*Solutions.* (1)

$$x = 2t + 1 \Rightarrow t = \frac{x-1}{2} \text{ and } y = t^2 + 1 \Rightarrow t^2 = y - 1$$

Therefore,

$$y - 1 = t^2 = \left(\frac{x-1}{2}\right)^2 \Rightarrow (y-1) = \frac{1}{4}(x-1)^2 \text{ or } (x-1)^2 = 4(y-1)$$

Shifting the origin at  $(1, 1)$  we get  $X^2 = 4Y$ , which is the equation of a parabola.



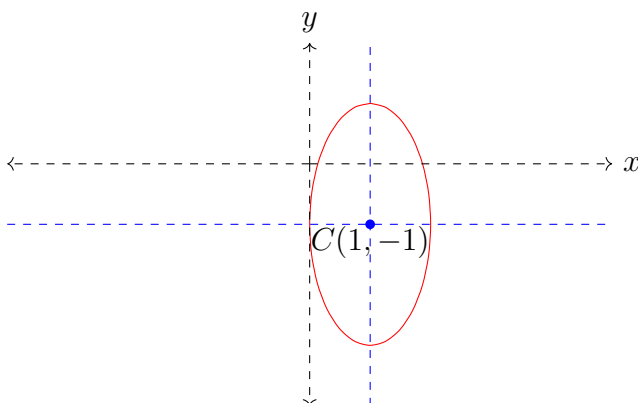
(2)

$$x = 1 + \sin \theta \Rightarrow \sin \theta = (x - 1) \text{ and } y = -1 + 2 \cos \theta \Rightarrow \cos \theta = \frac{y + 1}{2}$$

Since  $\sin^2 \theta + \cos^2 \theta = 1$  we have

$$(x-1)^2 + \frac{(y+1)^2}{4} = 1$$

which is the equation of an ellipse in standard form 2 (major axis parallel to  $y$ -axis) with center at  $(1, -1)$ ,  $a = 1$  and  $b = 2$ .



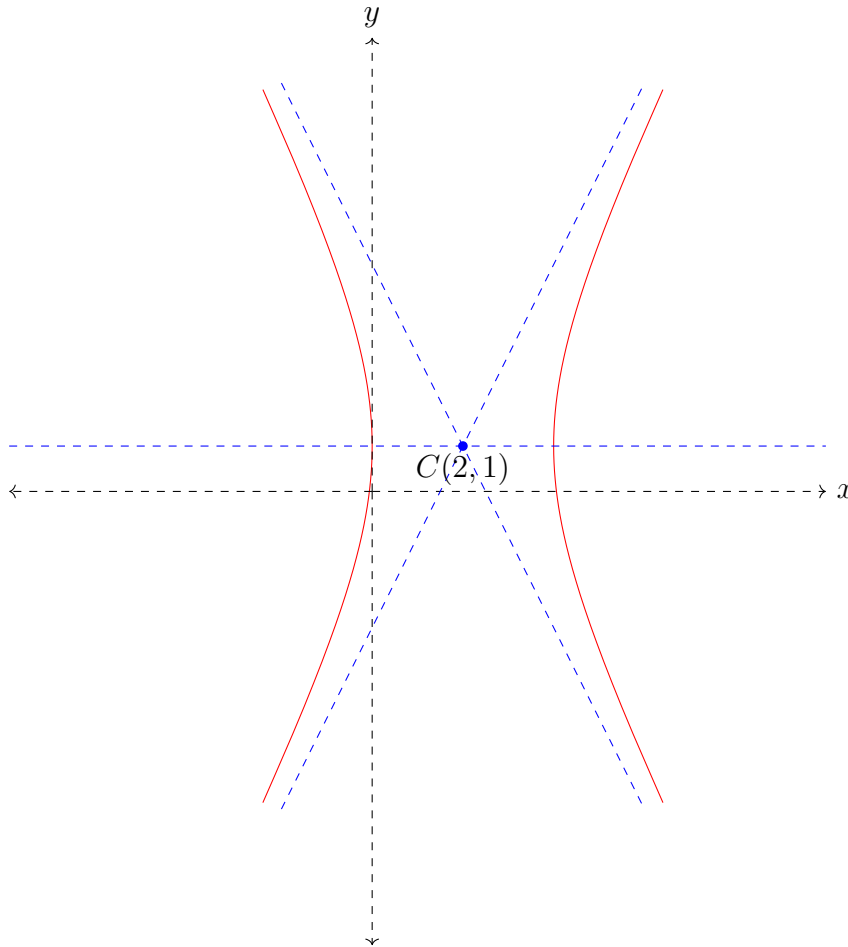
(3)

$$x = 2 + 2 \sec \theta \Rightarrow \sec \theta = \frac{x-2}{2} \text{ and } y = 1 + 4 \tan \theta \Rightarrow \tan \theta = \frac{y-1}{4}$$

Since  $\sec^2 \theta - \tan^2 \theta = 1$  we have

$$\frac{(x-2)^2}{4} - \frac{(y-1)^2}{16} = 1$$

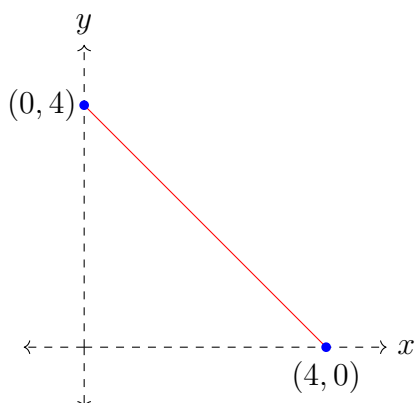
which is the equation of a hyperbola in standard form 1 (axis parallel to  $x$ -axis) with center at  $(2, 1)$ ,  $a = 2$  and  $b = 4$ .



(4)

$$x = t \text{ and } y = 4 - t \Rightarrow y = 4 - x \text{ or } x + y = 4$$

which is cartesian equation of a straight line. But since the parameter  $t$  varies between  $0 \leq t \leq 4$ , the given parametric equation represents a line segment as shown in the graph:-



**Problem 2:** Eliminate the parameter to find the cartesian equation for the following parametric curves.

1.  $x = \sqrt{t}, y = 1 - t$ .

2.  $x = t^2, y = \ln t$ .

3.  $x = t^2, y = t^3$ .

*Solutions.* (1)

$$x = \sqrt{t} \Rightarrow t = x^2$$

Substituting the value of  $t$  in  $y = 1 - t$  we get  $y = 1 - x^2$  or  $x^2 + y = 1$  which is the required cartesian equation.

(2)

$$y = \ln t \Rightarrow t = e^y$$

Substituting the value of  $t$  in  $x = t^2$  we get:-

$$x = e^{2y}.$$

(3)

$$x = t^2 \Rightarrow x^3 = t^6 \text{ and } y = t^3 \Rightarrow y^2 = t^6$$

Therefore,

$$x^3 = y^2$$

is the required cartesian equation. □

**Problem 3:** Find the parametric equation of the following conic sections.

1. A parabola with vertex at  $(2, 2)$  and focus at  $(3, 2)$ .

2. An ellipse with center at  $(-1, 4)$ , a vertex at  $(-1, 0)$  and a focus at  $(-1, 6)$

3. A hyperbola with foci at  $(2, 0)$ ,  $(2, 8)$  and asymptotes  $y = 3 + \frac{1}{2}x$ ,  $y = 5 - \frac{1}{2}x$ .

*Solutions.* (1) Shift the origin to the vertex  $(2, 2)$  to obtain new coordinates

$$X = x - 2 \text{ and } Y = y - 2$$

In the new coordinates, the focus is at  $(3 - 2, 2 - 2) = (1, 0)$ . Thus,  $p = 1$ . Since the focus lie on +ve  $X$ -axis, the parametric equation is given by

$$X = 1t^2, Y = 2(1)t \text{ or } x - 2 = t^2, y - 2 = 2t$$

that is

$$x = 2 + t^2, y = 2 + 2t \text{ where } t \in \mathbb{R}.$$

(2) Shift the origin at the center  $(-1, 4)$  to obtain new coordinates

$$X = x + 1 \text{ and } Y = y - 4$$

In the new coordinates, a vertex is at  $(-1 + 1, 0 - 4) = (0, -4)$   
and a focus is at  $(-1 + 1, 6 - 4) = (0, 2)$ .

Therefore,  $a = 4$  and  $c = 2$ . This implies,

$$b^2 = a^2 - c^2 = 4^2 - 2^2 = 16 - 4 = 12 \Rightarrow b = \sqrt{12} = 2\sqrt{3}$$

Since focus and vertex lie on  $Y$ -axis, the parametric equation of this ellipse is given by:-

$$X = b \cos \theta, Y = a \sin \theta \Rightarrow x + 1 = 2\sqrt{3} \cos \theta, y - 4 = 4 \sin \theta$$

that is

$$x = -1 + 2\sqrt{3} \cos \theta, y = 4 + 4 \sin \theta \text{ where } \theta \in [0, 2\pi).$$

(3) We first compute intersection of the asymptotes  $y = 3 + \frac{1}{2}x$ ,  $y = 5 - \frac{1}{2}x$  to find center.

$$3 + \frac{1}{2}x = 5 - \frac{1}{2}x \Rightarrow x = 5 - 3 = 2 \Rightarrow y = 3 + \frac{1}{2} \times 2 = 4$$

Therefore, center of the hyperbola lies at  $(2, 4)$ . Shift the origin at  $(2, 4)$  to obtain new coordinates

$$X = x - 2 \text{ and } Y = y - 4$$

The foci in new coordinates are at

$$(2 - 2, 0 - 4) = (0, -4) \text{ and } (2 - 2, 8 - 4) = (0, 4)$$

Therefore,

$$c = 4 \Rightarrow c^2 = a^2 + b^2 = 16 \dots\dots\dots (*)$$

Since the foci lie on  $Y$ -axis, the slope of asymptotes is given by  $\pm \frac{a}{b} = \pm \frac{1}{2}$ . Thus,  $b = 2a$ .

Substituting  $b = 2a$  in  $(*)$  we have

$$a^2 + (2a)^2 = 16 \Rightarrow a^2 = \frac{16}{5} \Rightarrow a = \frac{4}{\sqrt{5}} \text{ and } b = \frac{8}{\sqrt{5}}$$

Since the foci are on  $Y$ -axis, the parametric equation of the hyperbola is given by:-

$$Y = a \sec \theta, X = b \tan \theta \Rightarrow y - 4 = \frac{4}{\sqrt{5}} \sec \theta, x - 2 = \frac{8}{\sqrt{5}} \tan \theta$$

Thus, the parametric equation of the given hyperbola is:-

$$x = 2 + \frac{8}{\sqrt{5}} \tan \theta, y = 4 + \frac{4}{\sqrt{5}} \sec \theta \text{ where } \theta \in [0, 2\pi) \setminus \{\pi/2, 3\pi/2\}.$$

□