Test 3 : Tue 13Jun to Tue 20Jun

Applications of Logarithmic Functions

Tue 20Jun → Last Class

Wed–Thu. Ju21–Jun22 Final Exam

ESSENTIALS

Example

- The loudness L, in decibels (dB), of a sound is given by $L = 10 \cdot \log \frac{I}{I_0}$, where I is the intensity of the sound, in watts per square meter $\left(\frac{\text{W}}{\text{m}^2}\right)$, and $I_0 = 10^{-12}$ W/m². I_0 is approximately the intensity of the softest sound that can be heard by the human ear. The average maximum intensity of sound on a construction site is 2.9×10^{-3} W/m². How loud, in decibels, is this sound level? Round to the nearest whole number.

$\boxed{\log_a a^m = m}$

$$L = 10 \cdot \log \frac{I}{I_0}$$

$$= 10 \cdot \log \frac{2.9 \times 10^{-3}}{10^{-12}} \quad \text{Substituting}$$

$$\frac{10^{-3}}{10^{-12}} = 10^{-3-(-12)} = 10^{-3+12} = 10^9$$

$$= 10 \cdot \log\left(2.9 \times 10^9\right) \quad \text{Subtracting exponents}$$

$$= 10\left[\log 2.9 + \log 10^9\right]$$

$$= 10\left[\log 2.9 + 9\right] = 10\left[0.46 + 9\right] = 10\left[9.46\right] = 94.6$$

$$\approx 95$$

The volume of sound on the construction site is about 95 decibels.

GUIDED LEARNING: **Textbook** **Instructor** **Video**

EXAMPLE 1	YOUR TURN 1
In chemistry, the pH of a liquid is a measure of its acidity and is calculated as follows: $\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles per liter. The hydrogen ion concentration of lemonade is about 3.1×10^{-4} moles per liter. Find the pH of lemonade. Round to the nearest tenth.	The hydrogen ion concentration of a solution is 2.5×10^{-6}. Use the formula from Example 1 to find the pH of the solution. Round to the nearest tenth.
$\text{pH} = -\log[\text{H}^+]$ $= -\log\left[3.1 \times 10^{-4}\right] = -1\left[\log 3.1 + \log 10^{-4}\right]$ $\approx -(-3.5086) \quad = -1\left[0.49 - 4\right]$ $\approx \boxed{3.51} \quad = -0.49 + 4$ The pH of lemonade is about $\boxed{3.51}$. $= 3.51$	$\text{pH} = -\log\left(2.5 \times 10^{-6}\right)$ $= -1\left[\log 2.5 + \log 10^{-6}\right]$ $= -1\left[0.39 - 6\right]$ $= -0.39 + 6$ $= 5.61$

EXAMPLE 2	YOUR TURN 2
The pH of saliva is 6.2. Using the formula from Example 1, find the hydrogen ion concentration of saliva. $\mathrm{pH} = -\log[\mathrm{H}^+]$ $6.2 = -\log[\mathrm{H}^+]$ Substituting 6.2 for pH $\boxed{-6.2} = \log[\mathrm{H}^+]$ Dividing both sides by -1 $10^{-6.2} = [\mathrm{H}^+]$ Converting to an exponential equation $6.31 \times 10^{-7} \approx [\mathrm{H}^+]$ Using a calculator; writing scientific notation The hydrogen ion concentration of saliva is about $\boxed{6.3 \times 10^{-7}}$ moles per liter.	The pH of blood is 7.4. Using the formula from Example 1, find the hydrogen ion concentration of blood. $7.4 = -\log[H^+]$ $\Rightarrow \log[H^+] = -7.4$ $\Rightarrow [H^+] = 10^{-7.4}$ $= 3.98 \times 10^{-8}$

YOUR NOTES Write your questions and additional notes.

$10^{-7.4} = 10^{-8+0.6} = 10^{-8} \times 10^{0.6}$

$-6.2 = \log_{10}[H^+]$

$10^{-6.2} = 10^{-7+0.8} = 10^{-7} \times 10^{0.8}$

Applications of Exponential Functions

ESSENTIALS

Exponential Growth

An **exponential growth** model is a function of the form

$$P(t) = P_0 e^{kt},\ k > 0,$$

where P_0 is the population at time 0, $P(t)$ is the population at time *t*, and *k* is the **exponential growth** rate for the situation. The **doubling time** is the amount of time necessary for the population to double in size.

↳ t for which $P(t) = 2P_0$

Exponential Decay

An **exponential decay** model is a function of the form

$$P(t) = P_0 e^{-kt},\ k > 0,$$

where P_0 is the quantity present at time 0, $P(t)$ is the amount present at time *t*, and *k* is the **decay rate.** The **half-life** is the amount of time necessary for half the quantity to decay.

↳ t for which $P(t) = \frac{1}{2}P_0$

Example

- In 2012, the population of a country was 104 million and the exponential growth rate was 0.946% per year.
  a) Find the exponential growth function.
  b) Predict the country's population in 2020.

$t \rightarrow$ number of years after 2012

a) At $t = 0$, the population, P_0, is 104 million. The growth rate, *k*, is 0.946% or 0.00946. So, $P(t) = 104e^{0.00946t}$, where $P(t)$ is the population, in millions, *t* years after 2012.

$0.946\% = \frac{0.946}{100}$

b) 2020 is 8 years after 2012, so we have

$$P(8) = 104e^{0.00946(8)} \approx 112 \text{ million.}$$

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EXAMPLE 1	YOUR TURN 1
Suppose that \$12,000 is invested at 3%, compounded annually. In *t* years it will grow to the amount *A* given by $A(t) = 12{,}000(1.03)^t$. How long will it take to have \$18,000 in the account? Round to the nearest tenth. (continued)	Suppose that \$30,000 is invested at 5%, compounded annually. In *t* years it will grow to the amount *A* given by $A(t) = 30{,}000(1.05)^t$. How long will it take to have \$75,000 in the account? Round to the nearest tenth.

Set $A(t)=18{,}000$ and solve for t.

$$A(t)=12{,}000(1.03)^t$$
$$18{,}000=12{,}000(1.03)^t$$
$$1.5=(1.03)^t$$
$$\log 1.5=\log(1.03)^t$$
$$\log 1.5=t\cdot\boxed{}$$
$$\frac{\log 1.5}{\log 1.03}=t$$
$$\boxed{}\approx t$$

It will take about $\boxed{}$ years to have \$18,000 in the account.

Find t for which

$A(t)=75000$

$30\,000(1.05)^t=75000$

$(1.05)^t=\frac{\cancel{75\,000}^{75}}{\cancel{30\,000}_{30}}=\frac{\cancel{75}^{5}}{\cancel{30}_{2}}$

$=2.5$

$\log 1.05^t=\log 2.5$

$\Rightarrow t\log 1.05=\log 2.5\Rightarrow t=\frac{\log 2.5}{\log 1.05}$

$\Rightarrow t=18.7$ years.

EXAMPLE 2

The decay rate of a substance is 4.5% per year. What is its half-life? Round to the nearest tenth.

We must find the time, T, when $P(T)$ is half of P_0. The decay rate k is 4.5%, or 0.045, so

$$P(T)=P_0e^{-kT}$$
$$0.5P_0=P_0e^{-0.045T}$$
$$0.5=e^{-0.045T}$$
$$\ln 0.5=\ln e^{-0.045T}$$
$$\ln 0.5=-0.045T$$
$$\frac{\ln 0.5}{\boxed{}}=T$$
$$\boxed{}\approx T.$$

The half-life of the substance is about $\boxed{}$ years.

YOUR TURN 2

The decay rate of a substance is 7.8% per day. What is its half-life? Round to the nearest tenth.

$P(t)=P_0e^{-kt}$

$k=7.8\%=\frac{7.8}{100}=0.078$

$P(t)=P_0e^{-0.078t}$

t for which $P(t)=\frac{1}{2}P_0$

$P_0e^{-0.078t}=\frac{1}{2}P_0$

$e^{-0.078t}=\frac{1}{2}$

$\ln e^{-0.078t}=\ln\frac{1}{2}$

$\Rightarrow -0.078t=\ln 0.5$

$\Rightarrow t=\frac{\ln 0.5}{-0.078}=8.88\approx 9$

$\boxed{t_{1/2}\approx 9 \text{ years}}$

YOUR NOTES Write your questions and additional notes.

$\log_e e^x=x$

(*) If we have e^x involved then use ln, otherwise use log.

Practice Exercises

Readiness Check

For the exponential decay model $P(t) = P_0 e^{-kt}$, $k > 0$, *match each variable with its description.*

1. T, where $P(T) = 0.5P_0$

2. $P(t)$

3. k

4. P_0

A. Quantity present at time 0

B. Quantity present at time t

C. Half-life

D. Exponential decay rate

Applications of Exponential Functions and Logarithmic Functions

5. A college loan of \$42,000 is made at 4% interest, compounded annually. After t years, the amount due, A, is given by the function $A(t) = 42{,}000(1.04)^t$.

a) After what amount of time will the amount due reach \$50,000?

b) Find the doubling time.

6. The radioactive element carbon-14 has a half-life of 5750 years. Soil was found to have lost 15% of its carbon-14. How old was the soil? (Use the function for the decay of carbon-14: $P(t) = P_0 e^{-0.00012t}$. If the soil has lost 15% of its carbon-14, then $100\% - 15\%$, or 85% is still present.)

7. The Richter scale is used to measure earthquake magnitude. The Richter magnitude m of an earthquake is given by $m = \log \frac{A}{A_0}$ where A is the maximum amplitude of the earthquake and A_0 is a constant. What is the magnitude on the Richter scale of an earthquake with an amplitude of 5,406,125 times A_0? Round to the nearest tenth.

8. We calculate pH by the formula $\text{pH} = -\log\left[\text{H}^+\right]$, where $\left[\text{H}^+\right]$ is the hydrogen ion concentration in moles per liter.

a) The hydrogen ion concentration of a substance is about 2.98×10^{-6} moles per liter. Find the pH of the substance. Round to the nearest tenth.

b) The average pH of orange juice is 3.3. Find the hydrogen ion concentration.

Quiz 17

① $f(x) = x^2 - 2x + 2$

a) Find the vertex and axis of symmetry of the graph of f.

b) Find the domain and range of f.

x-coord. of vertex $= \dfrac{-b}{2a}$ $\quad (ax^2+bx+c)$

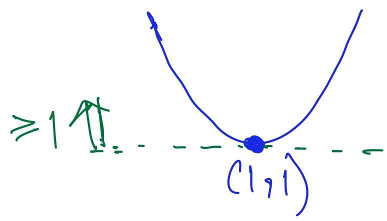
$a=1,\ b=-2,\ c=2$

x-coord of vertex $= \dfrac{-(-2)}{2(1)} = \dfrac{2}{2} = 1$

y-coord. of vertex $= f(1) = 1^2 - 2(1) + 2 = 1$

$\Rightarrow$ Vertex $= (1,1)$

Axis of symmetry : $x = 1$



$a = 1 > 0$

Domain $= (-\infty, \infty)$ and Range $= [1, \infty)$

② $\log_x 2 = \frac{1}{2}$. Find x.

$\Rightarrow 2 = x^{1/2}$

$\Rightarrow \sqrt{x} = 2$

$\Rightarrow x = 2^2 = 4$

$\Rightarrow \boxed{x=4}$

$$\boxed{\log_b a = c \iff a = b^c}$$

$10^3 = 1000$

$\underbrace{\log_{10}}_{\text{Common log.}} 3 = 0.477$

$\Rightarrow 10^{0.477} = 3$

$$\boxed{\log_a a^m = m}$$

$\log_x 3 = \frac{1}{2} \Rightarrow x^{1/2} = 3 \Rightarrow x = 9$

square both sides.