Learning objectives:

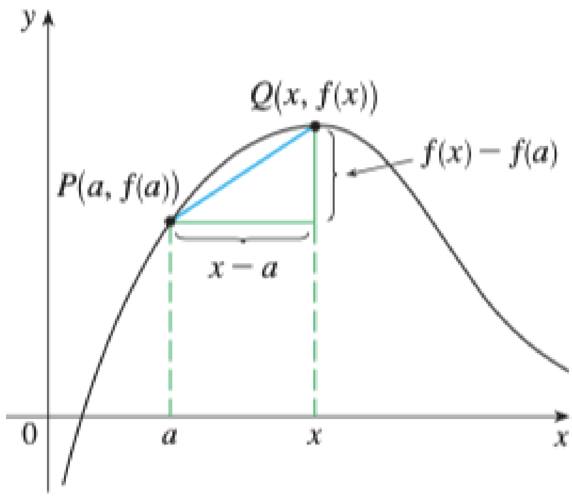
1. Using limits to find the slope of tangent line to a function at a given point.
2. Define the derivative of a function at a given point.
3. Interpret the derivative as an instantaneous rate of change of the dependent variable with respect to the independent variable.
4. Examples of rates of change: velocity and acceleration.

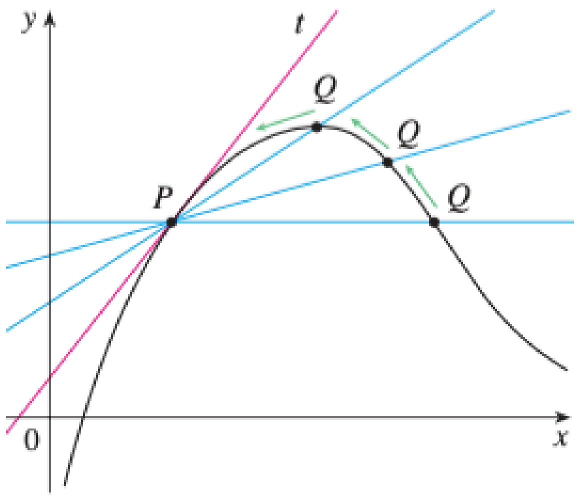
Slope of tangent line

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the this limit exists.





Example 1.

Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $P(3, 1)$.

The derivative of a function at a point

The derivative of a function f at a number a, denoted by $f'(a)$ is given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

If we write $x = a + h$, then we have $h = x - a$ so that $h \to 0$ as $x \to a$. Therefore,

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

Therefore, the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ is given by $f'(a)$, the derivative of f at a.

Example 2.

Find the derivative of the function $f(x) = \sqrt{x}$ at the number a.

Rates of Change

Let y depend on x via the function f, that is, $y = f(x)$.

If x changes from x_1 to x_2, the change (or increment) in x is $\Delta x = x_2 - x_1$.

The corresponding change in y is $\Delta y = f(x_2) - f(x_1)$.

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad \text{(slope of the secant line PQ)}$$

is called the *average rate of change of y with respect to x over the interval* $[x_1, x_2]$.

Taking limit $\Delta x \to 0$, we obtain

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim x_2 \to x_1 \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad \text{(slope of tangent line at P)}$$

the *instantaneous rate of change of y with respect to x at the instant* x_1. This is same as the derivative $f'(x_1)$.

$\boxed{\text{Thus, } f'(a) \text{ is the instantaneous rate of change of } y = f(x) \text{ w.r.t. } x \text{ at instant } a.}$

Examples of instantaneous rates of change

The velocity of a particle at a time instant t is the instantaneous rate of change of displacement of the particle with respect to time at t.

The acceleration of a particle at a time instant t is the instantaneous rate of change of velocity of the particle with respect to time at t.

Example 3. A particle moves along the x-axis with its displacement varying with time as $s(t) = t^2 - 3t + 1$. Find the velocity of the particle at the instant $t = 3$ seconds.

Example 4. A particle is moving along a straight line with its velocity varying with time as $v(t) = (t^2 + 1)/t$. Find the acceleration of the particle at $t = 1$ seconds.