Computational Complexity

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Exercise 1

a

$\sum_{i=2}^{n} \binom{i}{2} = \binom{n+i}{3}$

proof:

$\sum_{i=2}^{n} \binom{i}{2} = \sum_{i=1}^{n} \frac{2!}{0!2!} + \frac{3!}{1!2!} + ...$

$= \sum_{i=2}^{n} \frac{i!}{2!(i-2)!}$

$= \sum_{i=2}^{n} \frac{i(i-1)(!-2)!}{2!(i-2)!}$

$= \sum_{i=2}^{n} \frac{i(i-1)}{2}$

$= \sum_{i=2}^{n} \frac{i^2-i}{2}$

$= \sum_{i=1}^{n} \frac{i^2}{2} - 1 - \sum_{i=1}^{n} \frac{i}{2} - 1$

$= \frac{n(n+1)(2n+1)}{6*2} - \frac{n(n+1)}{2*2}$

$= \frac{n(n+1)(2n-2)}{6*2}$

$= \frac{n(n+1)(n-1)}{6}$

$= \frac{(n+1)n(n-1)}{3*2}$

multiply and divide by $(n-2)!$

$= \frac{(n+1)n(n-1)(n-2)!}{3*2(n-2)!}$

$= \frac{(n+1)n(n-1)(n-2)!}{3*2(n+1-3)!}$

$= \frac{(n+1)n(n-1)(n-2)!}{3!(n+1-3)!}$

$= \binom{n+i}{3}$

b

$\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2$

proof:

expanding $\sum_{i=1}^{n} N^4 - (N-1)^4$

$\sum_{i=1}^{n} N^4 - (N-1)^4 = n^4 - (n-1)^4 + (n-1)^4 - (n-2)^4 + ... + 3^4 - 2^4 + 2^4 - 1^4 + 1^4 - 0^4$

$\sum_{i=1}^{n} N^4 - (N-1)^4 = n^4 \ \ldots..$ (1) As remaining terms gets canceled with each others.

lets solve $N^4 - (N-1)^4$

$N^4 - (N-1)^4 = N^4 - [(N-1)^2(N-1)^2]$

$= 4N^3 - 6N^2 + 4N - 1$

$and, \sum_{i=1}^{n} N^4 - (N-1)^4 = \sum_{i=1}^{n} 4N^3 - \sum_{i=1}^{n} 6N^2 + \sum_{i=1}^{n} 4N - \sum_{i=1}^{n} 1$

$\sum_{i=1}^{n} N^4 - (N-1)^4 = 4\sum_{i=1}^{n} N^3 - \frac{6n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} - n$

by (1)

$4\sum_{i=1}^{n} N^3 - \frac{6n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} - n = n^4$

$4\sum_{i=1}^{n} N^3 = n^4 + \frac{6n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$

$4\sum_{i=1}^{n} N^3 = n^2(n+1)^2$

$\sum_{i=1}^{n} N^3 = \frac{n^2(n+1)^2}{4}$

$\sum_{i=1}^{n} N^3 = (\frac{n(n+1)}{2})^2$

Exercise 2

a

$(x^2 + 3x + 1)^3 = o(x^6)$

for $f(n) = o(g(n))$

$\lim_{n \to \inf} \frac{f(n)}{g(n)} = 0$

for $f(n)$ expand $(x^2 + 3x + 1)^3$

$f(n) = x^6 + 9x^5 + 30x^4 + 45^x3 + 30x^2 + 9x + 1$

$g(n) = x^6$

since $\lim_{x \to \inf} (\frac{x^6 + 9x^5 + 30x^4 + 45^x3 + 30x^2 + 9x + 1}{x^6}) \neq 0$

ANS (a): **FALSE**

b

$\frac{\sqrt{x}+1}{2} = o(1)$

for $f(n) = o(g(n))$

$\lim_{n \to \inf} \frac{f(n)}{g(n)} = 0$

since $\lim_{x \to \inf} (\frac{\sqrt{x}+1}{2*1}) \neq 0$

ANS (b): **FALSE**

c

$e^{\frac{1}{x}} = o(1)$

for $f(n) = o(g(n))$

$\lim_{n \to \inf} \frac{f(n)}{g(n)} = 0$

since $\lim_{x \to \inf} \frac{e^{\frac{1}{x}}}{1} \neq 0$

$e^{\frac{1}{\inf}} = e^0 = 1$

ANS (c): **FALSE**

d

$\frac{1}{x} = o(1)$

for $f(n) = o(g(n))$

$\lim_{n \to \inf} \frac{f(n)}{g(n)} = 0$

since $\lim_{x \to \inf} \frac{\frac{1}{x}}{1} = 0$

ANS (d): **TRUE**

e

$x^3(\log(\log x))^2 = o(x^3 \log x)$

for $f(n) = o(g(n))$

$\lim_{n \to \inf} \frac{f(n)}{g(n)} = 0$

$\lim_{x \to \inf} \frac{x^3(\log(\log x))^2}{x^3 \log x}$

let us consider $y = \log x$

then

$\lim_{y \to \inf} \frac{(\log y)^2}{y} = 0$

ANS (e): **TRUE**

f

$\sqrt{\log x + 1} = \Theta(\log \log x)$

for $\Theta(n)$

$c1g(n) \leq f(n) \leq c2g(n)$

let us consider $y = \log x$

$\sqrt{y+1} = \Theta(\log y)$

$\sqrt{y+1} \approx \sqrt{y}$

since $\frac{d\sqrt{y}}{dy} > \frac{d \log y}{dy}$

$\sqrt{\log x + 1} > c2(\log \log x)$

ANS (f): **FALSE**

g

$2 + sinx = \Omega(1)$

for $\Omega(n)$

$f(n) \geq cg(n)$

since $\sin(x)$ ranges from -1 to 1

$2 + sinx \geq 1$ and $2 + sinx \geq c(1)$

ANS (g): **TRUE**

h

$\frac{\cos x}{x} = O(1)$

for $O(n)$

$f(n) \leq cg(n)$

since $\lim_{x \to \inf} \frac{\cos(x)}{x} \approx 0, x \neq 0$

$\frac{\cos(x)}{x} \leq c * 1$

ANS (h): **TRUE**

i

$\int_4^x \frac{dt}{t} = O(\ln x)$

for $O(n)$

$f(n) \leq cg(n)$

$\int_4^x \frac{dt}{t} = \ln t|_4^x = \ln x - \ln 4 \approx \ln x$

$\int_4^x \frac{dt}{t} \leq c(\ln x)$

ANS (i): **TRUE**

j

$\sum_{j=1}^{x} \frac{1}{j^2} = O(1)$

for $O(n)$

$f(n) \leq cg(n)$

$\sum_{j=1}^{\text{inf}} \frac{1}{j^2} = \frac{\pi^2}{6}$

$\sum_{j=1}^{x} \frac{1}{j^2} < \frac{\pi^2}{6}$

$\sum_{j=1}^{x} \frac{1}{j^2} \leq c * (1)$

ANS (j): **TRUE**

k

$\sum_{j=1}^{n} 1 = \Theta(x)$

for $\Theta(n)$

$c1g(n) \leq f(n) \leq c2g(n)$

$\sum_{j=1}^{x} 1 = x$

$c1(x) \leq \sum_{j=1}^{x} 1 \leq c2(x)$

ANS (k): **TRUE**

l

$\int_0^x e^{-t^2} dt = O(1)$

for $O(n)$

$f(n) \leq cg(n)$

$\int_{-\inf}^{\inf} e^{-t^2} dt = \sqrt{\pi}$

$\int_0^x e^{-t^2} dt < \sqrt{\pi}$

$\int_0^x e^{-t^2} dt < c * (1)$

ANS (k): **TRUE**

Exercise 3

a

ANS: $e^{\log n^3}, n^{3.01}, 2^{\sqrt{n}}, 2^{n^2}$

b

ANS: $1 + \log^3 n, n^{1.6}, n^{\log n}, \sqrt{n!}$

c

ANS: $(\log\log n)^2, n^3 \log n, (n+4)^9, 2^{n\sqrt{n}}$

d

ANS: $(\frac{1}{3})^n, 17, \log\log n, \log n, 2^{\sqrt{\log n}}, \sqrt{n}, \sqrt{n}(\log n), \frac{n}{logn}, 2n, (\frac{3}{2})^n, (\frac{n}{2})^{\log n}$

Exercise 4

a

$\binom{2}{3}^n + \sum_{i=1}^{n} \sin^2 n + n^2 + \ln\left(\sum_{i=1}^{n} \binom{n}{i}\right)$

$\frac{d\binom{2}{3}^n}{dn} = n * \binom{2}{3}^{n-1}$

$\frac{d\sum_{i=1}^{n} \sin^2 n}{dn} = \sum_{i=1}^{n} \sin 2n$

$\frac{dn^2}{dn} = 2n$

$\frac{d\ln\left(\sum_{i=1}^{n} \binom{n}{i}\right)}{dn} = \frac{d\ln 2^n}{dn} = \frac{d\frac{\log_2 2^n}{\log_2 e}}{dn} = \frac{d\frac{n}{log_2 e}}{dn} = \frac{1}{\log_2 e}$

ANS: $n * \binom{2}{3}^{n-1} + \sum_{i=1}^{n} \sin 2n + 2n + \frac{1}{\log_2 e}$

b

$\binom{n}{2} + \sum_{1}^{n} \log n + n^2 \sin n$

$\frac{d\binom{n}{2}}{dn} = \frac{d\frac{n(n-1)}{2}}{dn} = \frac{2n-1}{2}$

$\frac{d\sum_{1}^{n} \log n}{dn} = \sum_{1}^{n} \frac{1}{n}$

$\frac{dn^2 \sin n}{dn} = 2n \sin n + n^2 \cos n$

ANS: $\frac{2n-1}{2} + \sum_{1}^{n} \frac{1}{n} + 2n \sin n + n^2 \cos n$

Exercise 5

ANS: **d** $2^{\sqrt{n}}$ grows faster than n^2 and slower than $\sqrt{2^n}$

Exercise 6

Outer Loop: $0(\sqrt{n})$

Inner Loop: $0(\sqrt{n})$

ANS: $O(n)$

Exercise 7

Outer Loop: $0(n)$

1st Inner Loop: $0(n)$

2nd Inner Loop: $0(n)$

ANS: $O(n^3)$

Exercise 8

ANS

$2n$ multiplication operations need to be done in the worst case

n summations

Exercise 9

Outer Loop: $0(n-1) \approx O(n)$

1st Inner Loop: $0(n-1) \approx O(n)$

2nd Inner Loop: $0(n-1) \approx O(n)$

ANS: $O(n^3)$

Exercise 10

Outer Loop: $0(n-1) \approx O(n)$

1st Inner Loop: $0(n/2)$

2nd Inner Loop: $0(n/2)$

$ = O(n)*((O({n/2})+O(n/2)))$

ANS: $O(n^2)$

Exercise 11

Outer Loop: $0(n-1) \approx O(n)$

1st Inner Loop: $0(n/2)$

2nd Inner Loop: $0(n/2)$

$= O(n) * ((O(n/2) * O(n/2))) = O(n^3/4) \approx O(n^3)$

ANS: $O(n^3)$

Exercise 12

a $\Theta(n^2) summations$

```
M <- matrix(c(1,2,3,4,5,6,7,8,9), nrow = 3)

sum = 0

for (i in 1: nrow(M)){

    for (j in 1:i){
```

```
    sum = sum + M[i,j]

  }
}
sum
```

```
## [1] 26
```

b $\Theta(n) summations$

```
M <- matrix(c(1,2,3,4,5,6,7,8,9), nrow = 3)

v = 0

for (i in 1:nrow(M)){

 v = v + sum(M[i,1:i])

}

v
```

```
## [1] 26
```

c $O(1) summations$

```
M <- matrix(c(1,2,3,4,5,6,7,8,9), nrow = 3)

sum(M[lower.tri(M,diag = TRUE)])
```

```
## [1] 26
```

Exercise 13

a

ANS: $O(2^n - 1)$ and the function returns 2^{n-1} values for $n > 1$

b

ANS: $O(n)$ and the function returns2^{n-1} values for $n > 1$

c

ANS: $O(n)$ and the function returns 1

d

ANS: $O(n)$ and the function returns n

Exercise 14

ANS: $O(n^3)$

Exercise 15

ANS: $O(2^n)$ and the function returns n