INTRODUCTION TO DATA STRUCTURES and ALGORITHMS

Rutgers University

1B. UNION–FIND

- *union–find data type*
- *quick-find*
- *quick-union*
- *improvements*
- *applications*

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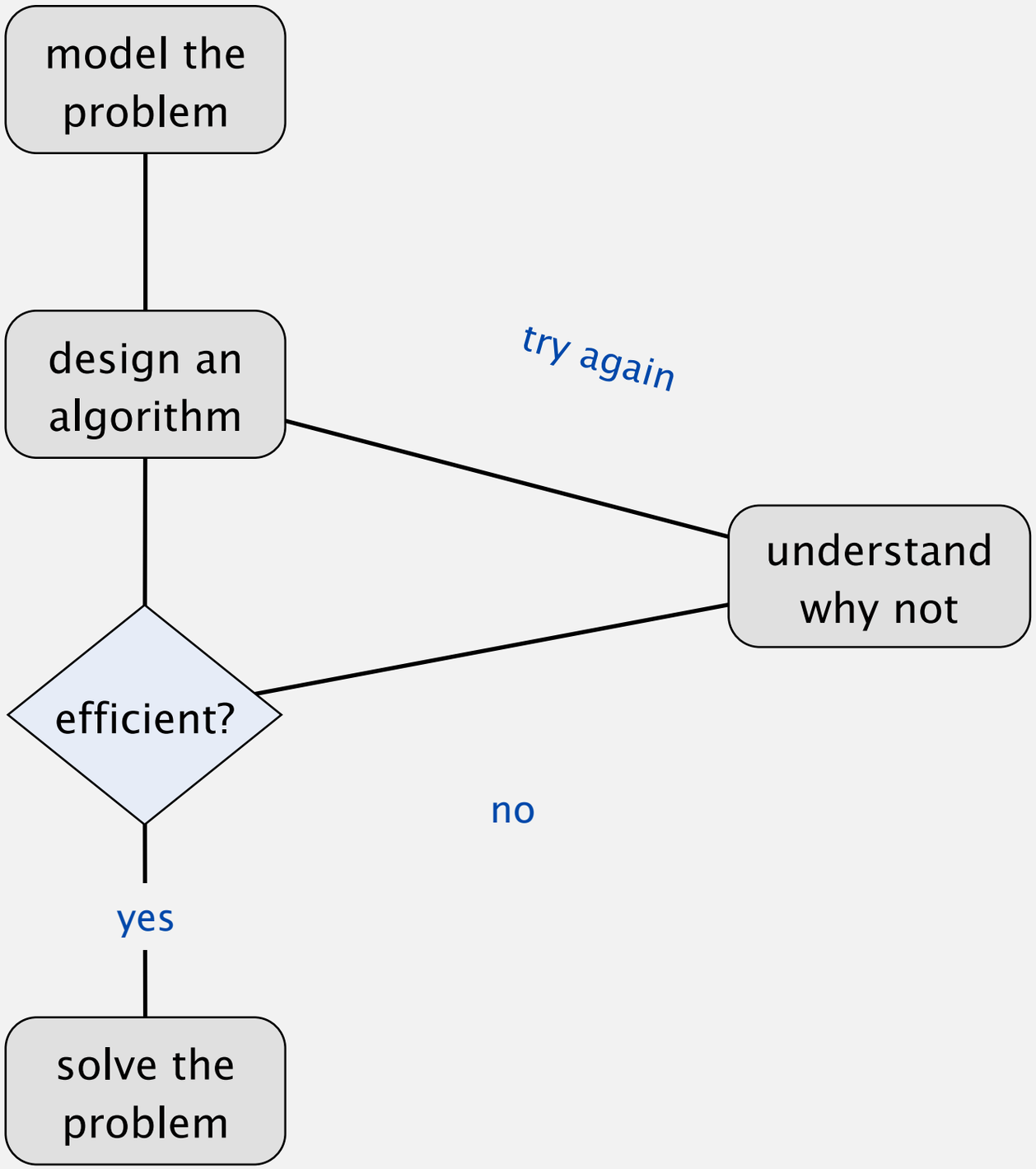
Last updated on 2/11/18 12:12 PM

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm to solve a computational problem.



UNION–FIND

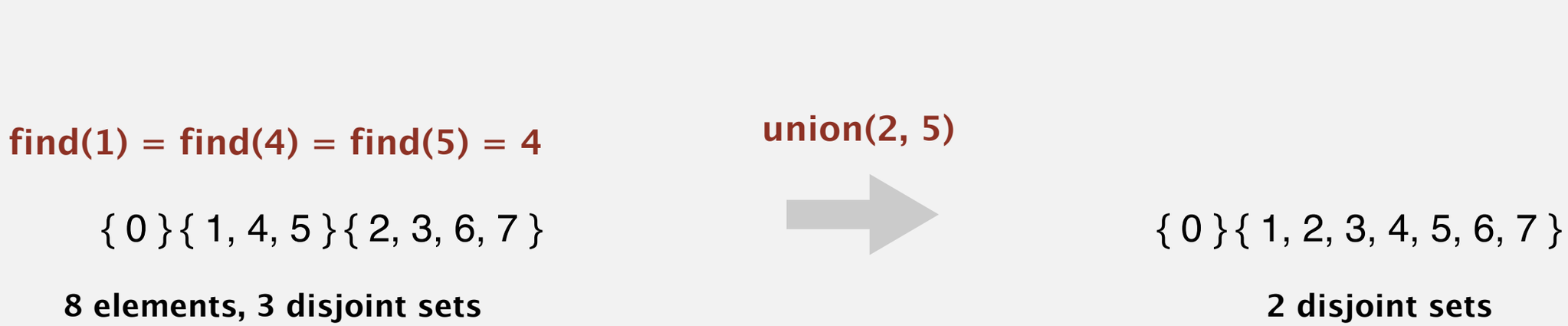
- *union–find data type*
- *quick-find*
- *quick-union*
- *improvements*
- *applications*

Union–find data type

Disjoint sets. A collection of sets; each element in exactly one set.

Find. Return a "canonical" element in the set containing p ?

Union. Merge the set containing p with the set containing q.



Simplifying assumption. The n elements are named 0, 1, …, n – 1.

Union–find data type (API)

Goal. Design an efficient union–find data type.

- Number of elements n can be huge.
- Number of operations m can be huge.
- Union and find operations can be intermixed.

public class UF

	UF(int n)	*initialize union–find data structure with n singleton sets* (0 *to* $n-1$)
void	union(int p, int q)	*merge sets containing elements p and q*
int	find(int p)	*canonical element in set containing p* (0 *to* $n-1$)

An application: dynamic connectivity

Given n vertices, support two operations:

- Add edge: directly connect two vertices with an edge.
- Connection query: is there a path connecting two vertices?

add edge 4–3

add edge 3–8

add edge 6–5

add edge 9–4

add edge 2–1

are 8 *and* 9 *connected?* ✓

are 5 *and* 7 *connected?* ☐

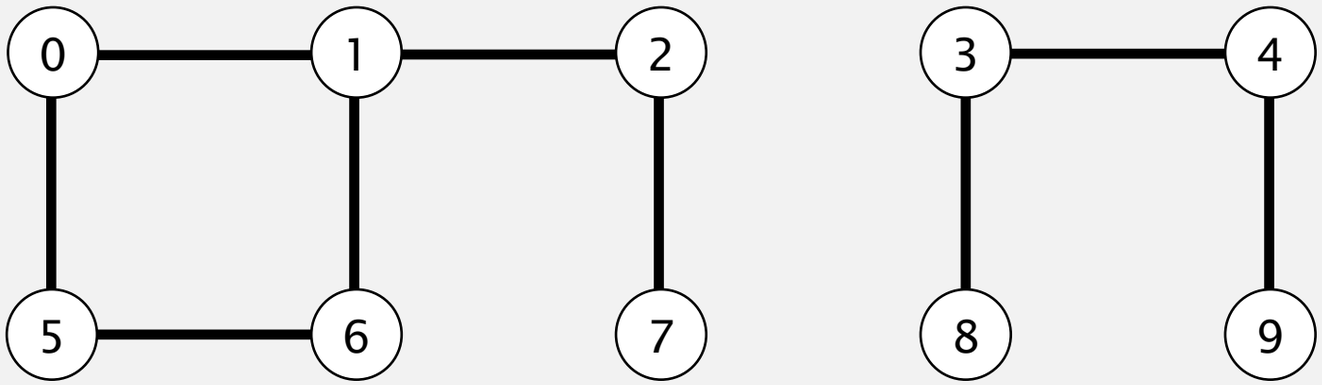
add edge 5–0

add edge 7–2

add edge 6–1

add edge 1–0

are 5 *and* 7 *connected?* ✓

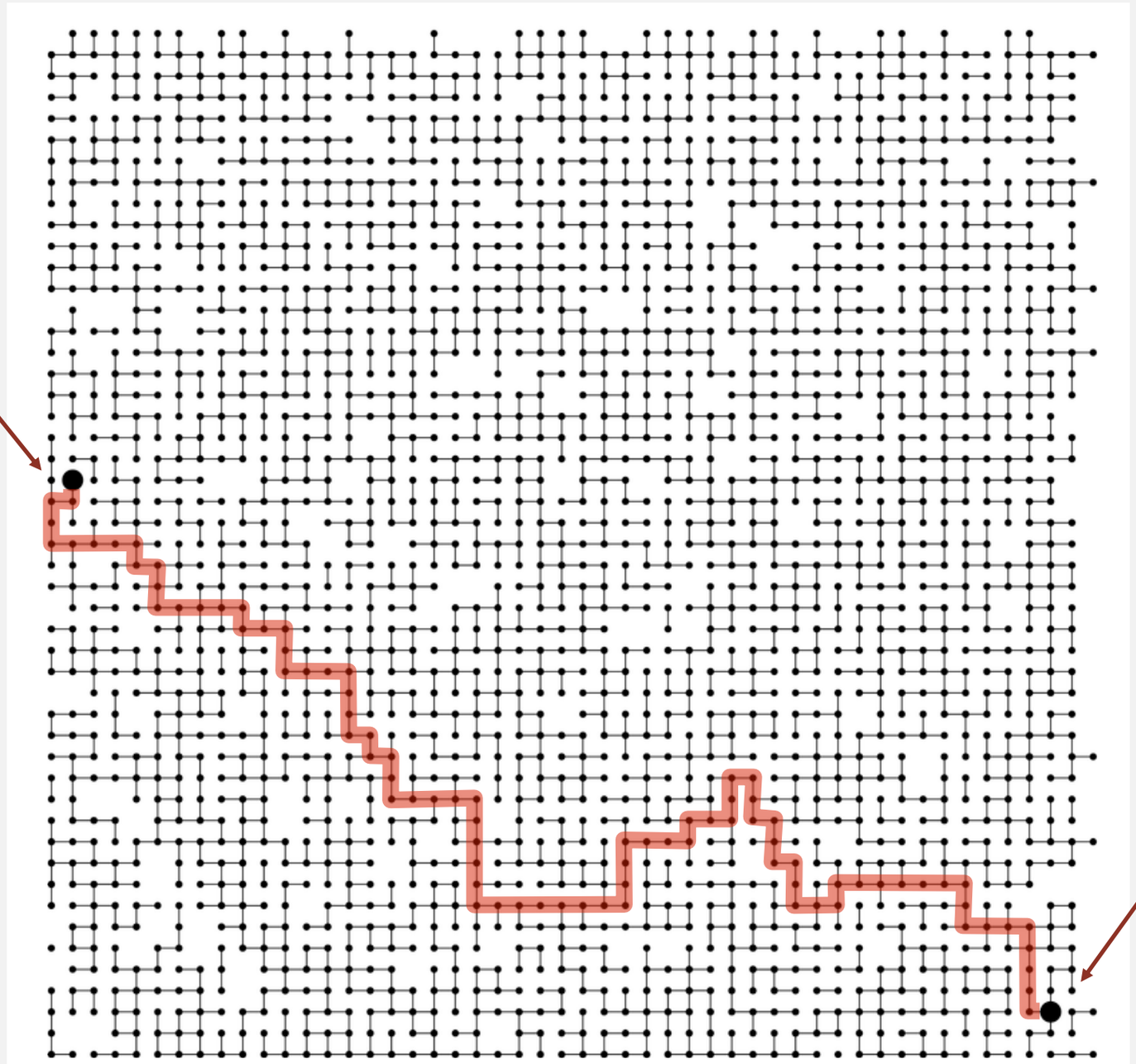


A larger connectivity example

Q. Is there a path connecting vertices v and w ?

finding a path is a slightly harder problem

(stay tuned for graph algorithms in Chapter 4)



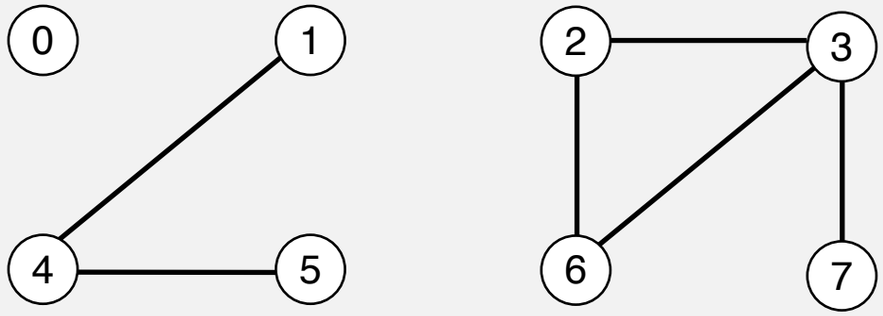
A. Yes.

Modeling the dynamic-connectivity problem

Q. How to model the dynamic-connectivity problem using union–find?

A. Maintain disjoint sets that correspond to connected components.

Connected component. Maximal set of vertices that are mutually connected.



3 connected components

{ 0 } { 1, 4, 5 } { 2, 3, 6, 7 }

3 disjoint sets

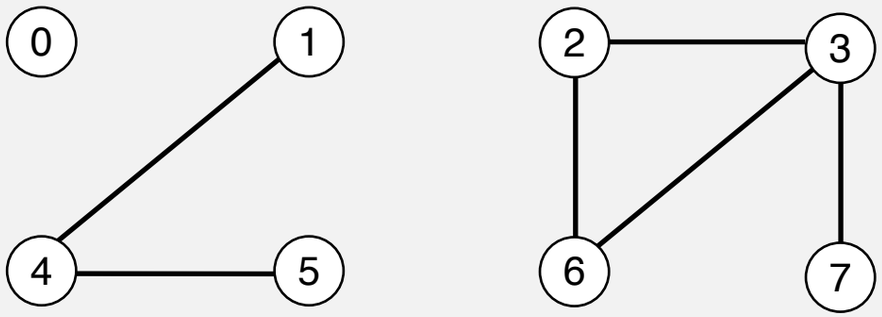
Modeling the dynamic-connectivity problem

Q. How to model the dynamic-connectivity problem using union–find?

A. Maintain disjoint sets that correspond to connected components.

- Add edge between vertices v and w.
- Are vertices v and w connected?

add edge 2–5



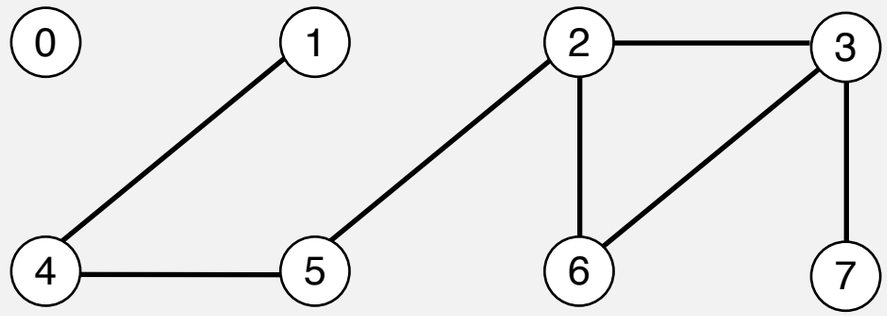
3 connected components

union(2, 5)

{ 0 }{ 1, 4, 5 }{ 2, 3, 6, 7 }

3 disjoint sets

are vertices 5 and 6 connected?



2 connected components

find(5) == find(6) ✓

{ 0 }{ 1, 2, 3, 4, 5, 6, 7 }

2 disjoint sets

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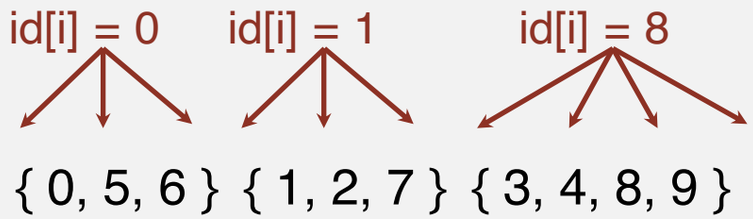
Quick-find [eager approach]

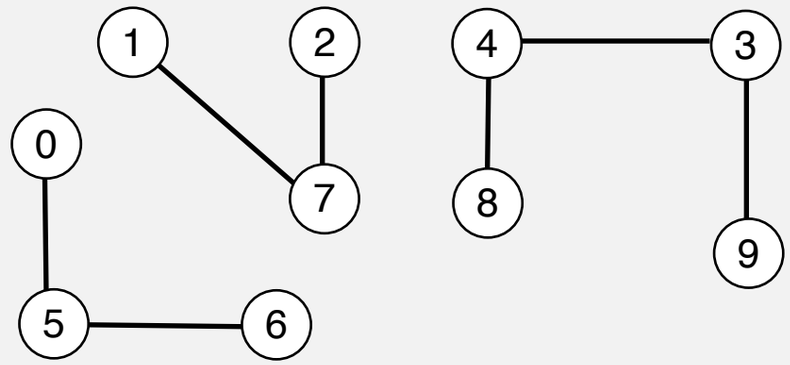
Data structure.

- Integer array id[] of length n.
- Interpretation: id[p] is canonical element in the set containing p.

	0	1	2	3	4	5	6	7	8	9
id[]	0	1	1	8	8	0	0	1	8	8



3 disjoint sets



3 connected components

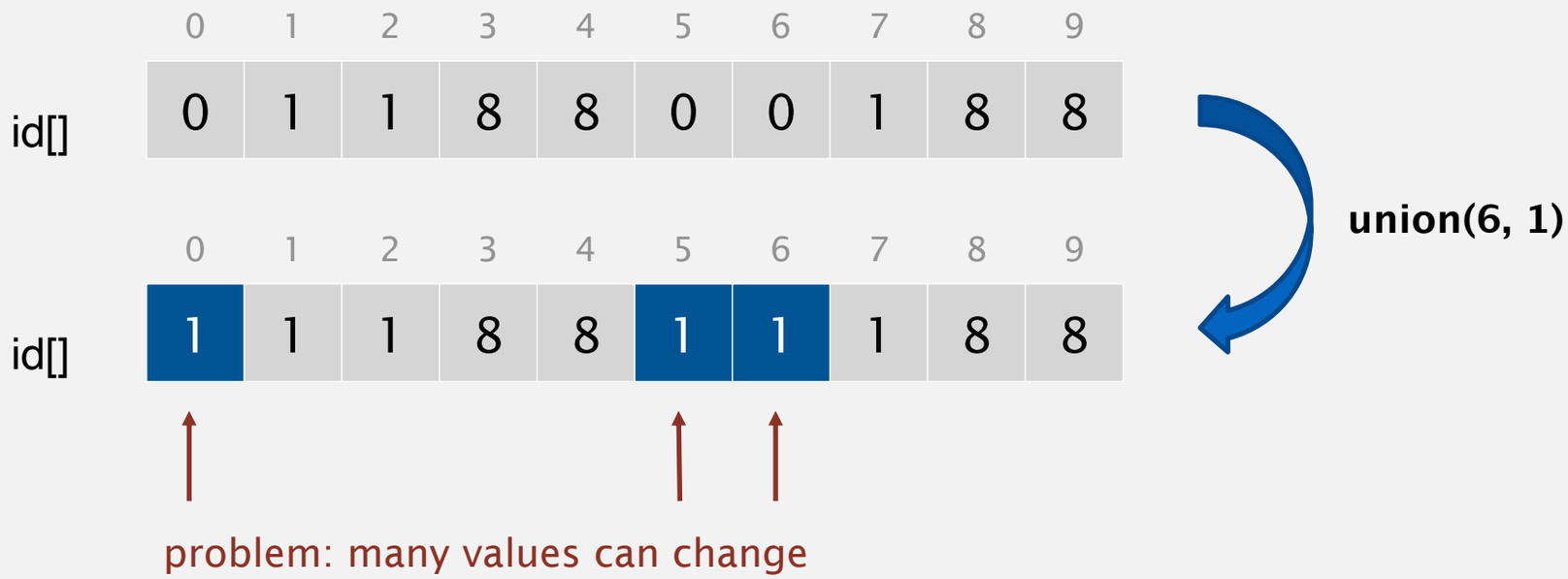
Q. How to implement find(p)?

A. Easy, just return id[p].

Quick-find [eager approach]

Data structure.

- Integer array id[] of length n.
- Interpretation: id[p] is canonical element in the set containing p.



Q. How to implement union(p, q)?

A. Change all entries whose identifier equals id[p] to id[q].

Quick-find: Java implementation

```
public class QuickFindUF
{
   private int[] id;
   public QuickFindUF(int n)
   {
      id = new int[n];
      for (int i = 0; i < n; i++)
         id[i] = i;
   }

   public int find(int p)
   {  return id[p];  }

   public void union(int p, int q)
   {
      int pid = id[p];
      int qid = id[q];
      for (int i = 0; i < id.length; i++)
         if (id[i] == pid) id[i] = qid;
   }
}
```

set id of each element to itself

(n array accesses)

return the id of p

(1 array access)

change all entries with id[p] to id[q]

(n+2 to 2n+2 array accesses)

Quick-find is too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find
quick-find	n	n	1

number of array accesses (ignoring leading constant)

Union is too expensive. Processing a sequence of n union operations on n elements takes more than n^2 array accesses.

quadratic

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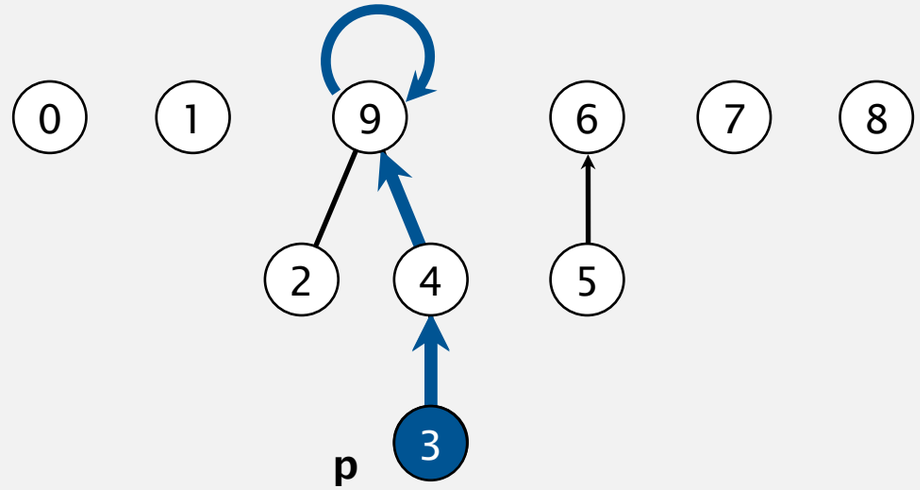
Quick-union [lazy approach]

Data structure.

- Integer array parent[] of length n, where parent[i] is parent of i in tree.
- Interpretation: elements in one tree correspond to one set.

0	1	2	3	4	5	6	7	8	9
0	1	9	4	9	6	6	7	8	9



{ 0 } { 1 } { 2, 3, 4, 9 } { 5, 6 } { 7 } { 8 }

6 disjoint sets (6 trees)

Q. How to implement find(p) operation?

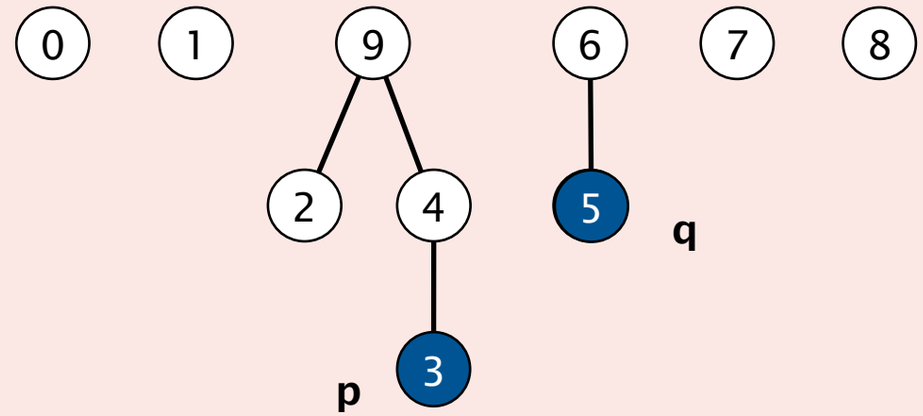
A. Return root of tree containing p.

Quick-union quiz

Data structure.

- Integer array parent[] of length n, where parent[i] is parent of i in tree.
- Interpretation: elements in one tree correspond to one set.

0	1	2	3	4	5	6	7	8	9
0	1	9	4	9	6	6	7	8	9



How to implement union(3, 5) ?

A. Set parent of 3 to 5.

B. Set parent of 9 to 5.

C. Set parent of 9 to 6.

D. Set parents of 2, 3, 4, and 9 each to 6.

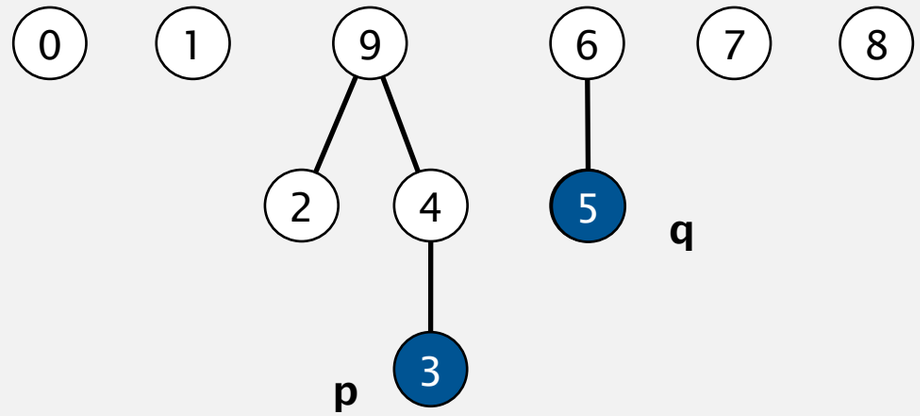
Quick-union [lazy approach]

Data structure.

- Integer array parent[] of length n, where parent[i] is parent of i in tree.
- Interpretation: elements in one tree correspond to one set.

union(3, 5)

0	1	2	3	4	5	6	7	8	9
0	1	9	4	9	6	6	7	8	9



Q. How to implement union(p, q)?

A. Set parent of p's root q's root.

Quick-union [lazy approach]

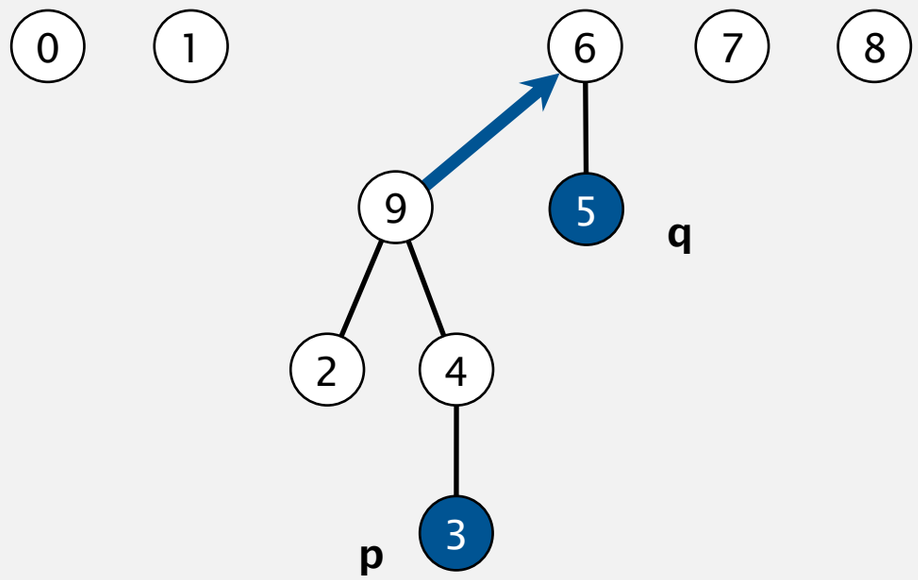
Data structure.

- Integer array parent[] of length n, where parent[i] is parent of i in tree.
- Interpretation: elements in one tree correspond to one set.

union(3, 5)

0	1	2	3	4	5	6	7	8	9
0	1	9	4	9	6	6	7	8	6

only one value changes



Q. How to implement union(p, q)?

A. Set parent of p's root to q's root.

Quick-union demo

0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

Quick-union: Java implementation

```
public class QuickUnionUF
{
   private int[] parent;
   public QuickUnionUF(int n)
   {
      parent = new int[n];
      for (int i = 0; i < n; i++)
          parent[i] = i;
   }

   public int find(int p)
   {
      while (p != parent[p])
         p = parent[p];
      return p;
   }

   public void union(int p, int q)
   {
      int rootP = find(p);
      int rootQ = find(q);
      parent[rootP] = rootQ;
   }
}
```

set parent of each element to itself

(n array accesses)

chase parent pointers until reach root

(depth of p array accesses)

change root of p to point to root of q

(depth of p and q array accesses)

Quick-union is also too slow

Cost model. Number of array accesses (for read or write).

algorithm	initialize	union	find
quick-find	n	n	1
quick-union	n	n	n

← worst case

number of array accesses (ignoring leading constant)

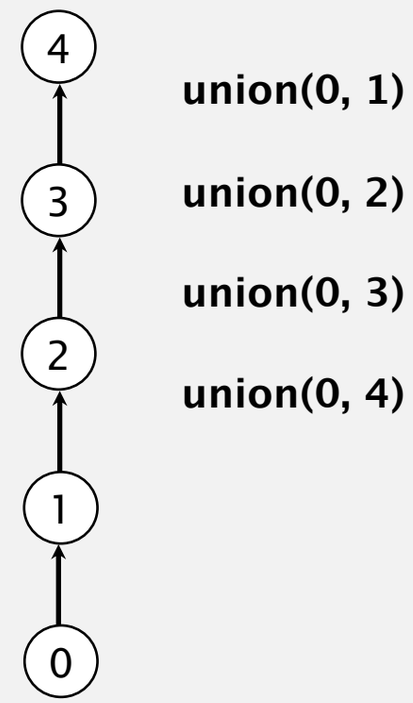
Quick-find defect.

- Union too expensive (more than n array accesses).
- Trees are flat, but too expensive to keep them flat.

Quick-union defect.

- Trees can get tall.
- Find too expensive (could be more than n array accesses).

worst-case input



1B. UNION–FIND

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Weighted quick-union quiz

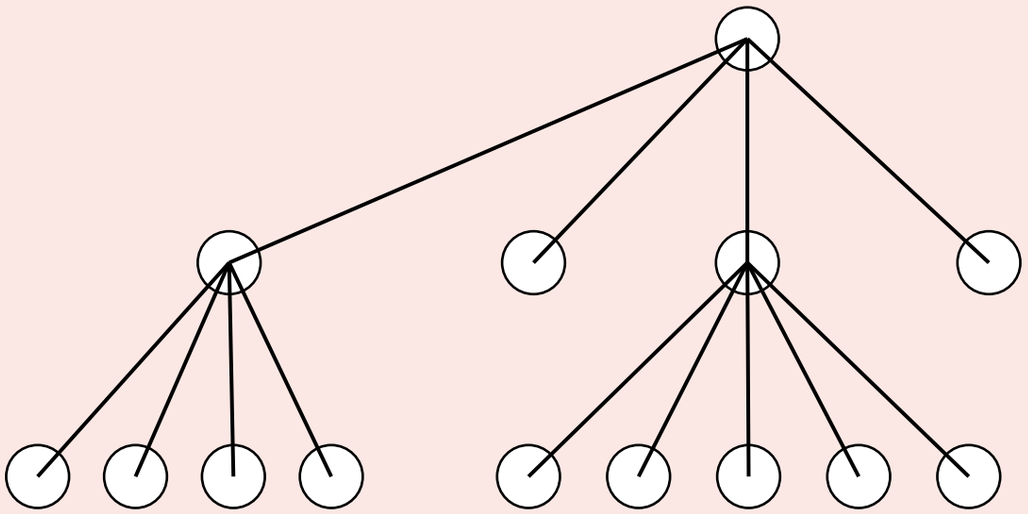
When merging two trees, which strategy is most effective?

A. Link the root of the smaller tree to the root of the larger tree.

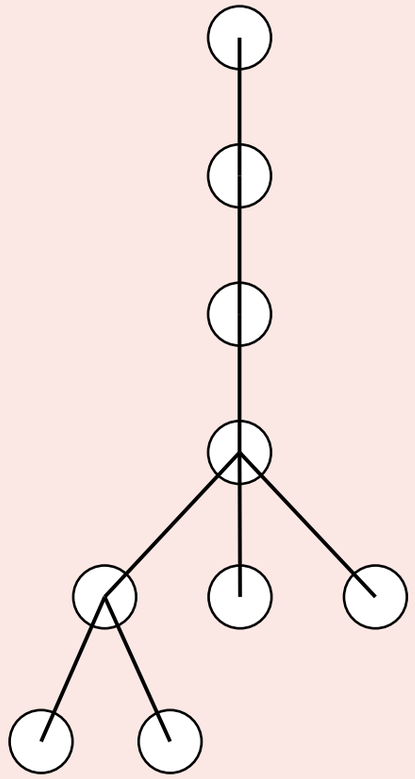
B. Link the root of the larger tree to the root of the smaller tree.

C. Link the root of the shorter tree to the root of the taller tree.

D. Link the root of the taller tree to the root of the shorter tree.



shorter and larger tree
(height = 2, size = 14)



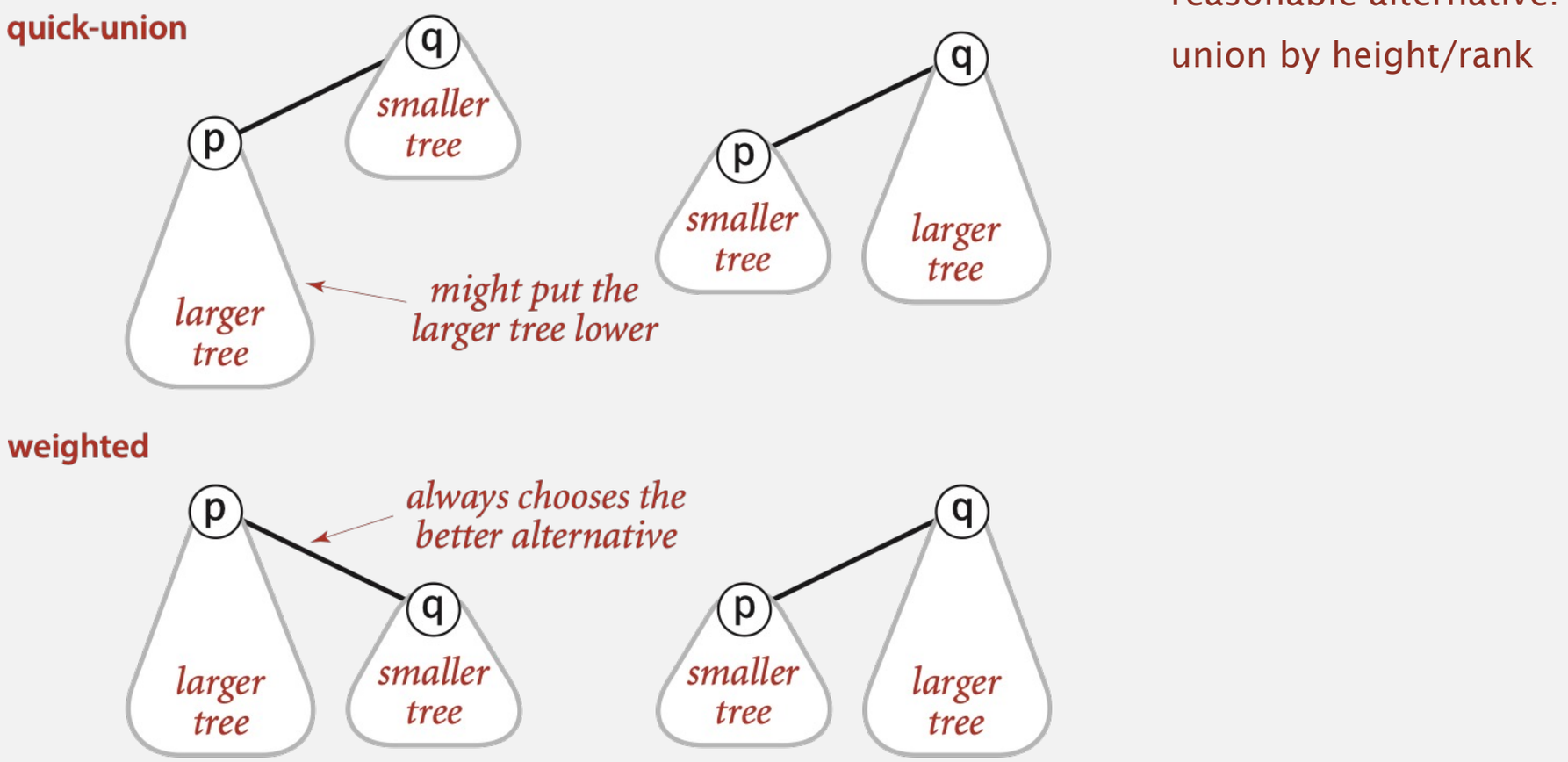
taller and smaller tree
(height = 5, size = 9)

Improvement 1: weighting

Weighted quick-union.

- Modify quick-union to avoid tall trees.
- Keep track of size of each tree (number of elements).
- Always link root of smaller tree to root of larger tree.

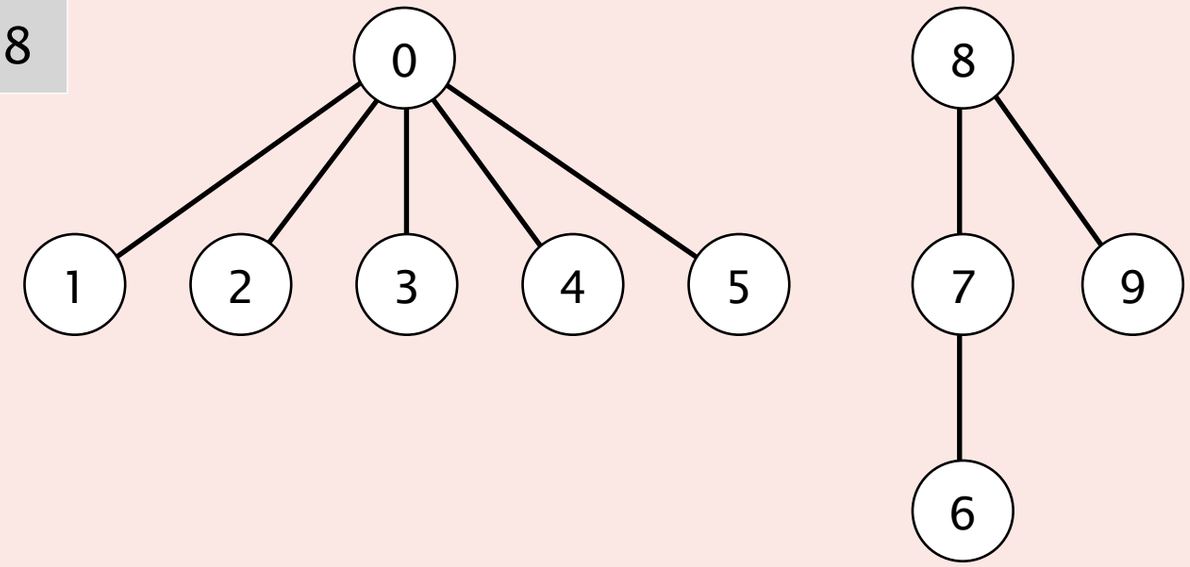
reasonable alternative:
union by height/rank



Weighted quick-union quiz

Suppose that the parent[] array during weighted quick-union is:

	0	1	2	3	4	5	6	7	8	9
parent[]	0	0	0	0	0	0	7	8	8	8



Which parent[] entry changes during union(2, 6)?

A. parent[0]

B. parent[2]

C. parent[6]

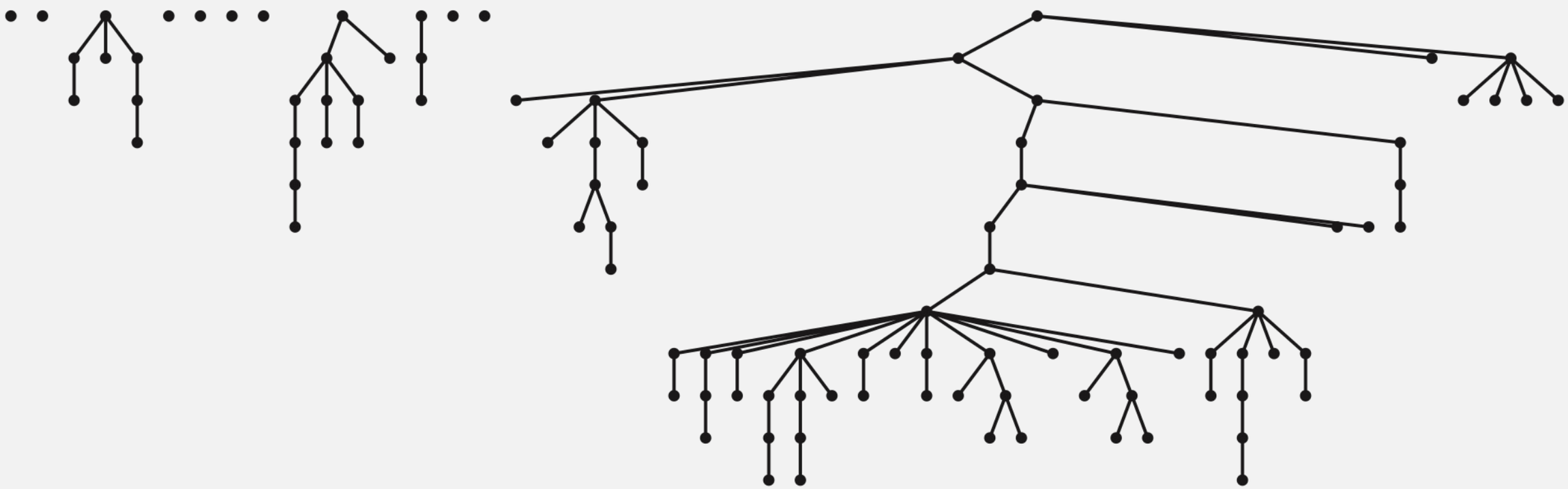
D. parent[8]

Weighted quick-union demo

	0	1	2	3	4	5	6	7	8	9
parent[]	0	1	2	3	4	5	6	7	8	9

Quick-union vs. weighted quick-union: larger example

quick-union



average distance to root: 5.11

weighted

average distance to root: 1.52

Quick-union and weighted quick-union (100 sites, 88 `union()` operations)

Weighted quick-union: Java implementation

Data structure. Same as quick-union, but maintain extra array size[i] to count number of elements in the tree rooted at i, initially 1.

- Find: identical to quick-union.
- Union: link root of smaller tree to root of larger tree; update size[].

```
public void union(int p, int q)
{
  int root1 = find(p);
  int root2 = find(q);
  if (root1 == root2) return;

  if (size[root1] >= size[root2])
  { int temp = root1; root1 = root2; root2 = temp; }

  parent[root1] = root2;
  size[root2] += size[root1];
}
```

root1 is root of smaller tree

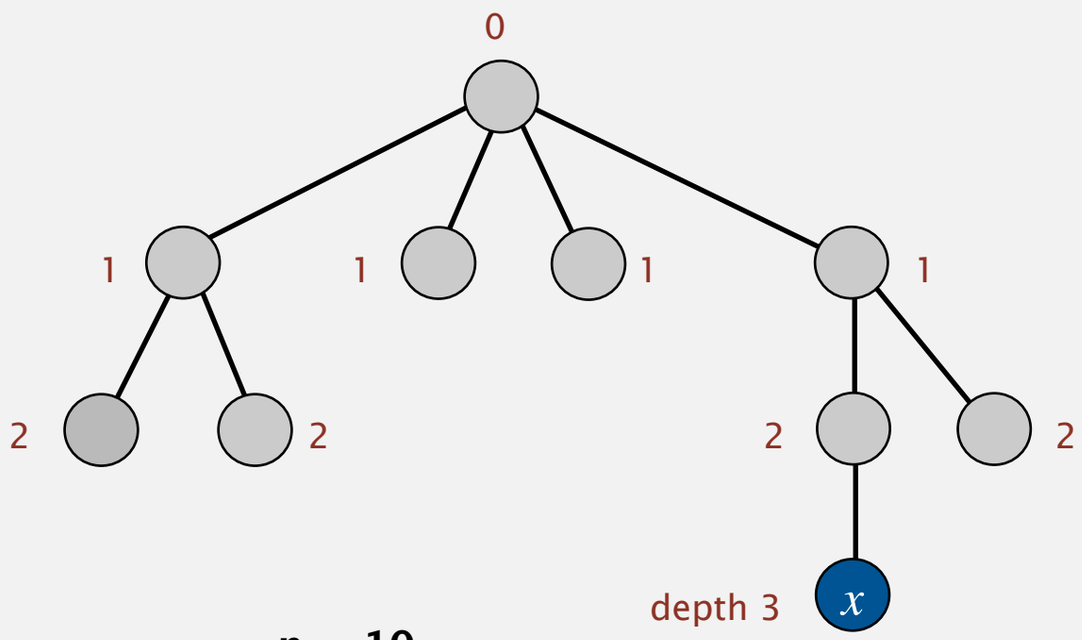
link root of smaller tree to root of larger tree

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given two roots.

Proposition. Depth of any node x is at most $\lg n$. ← in computer science, lg means base-2 logarithm



n = 10

depth(x) = 3 ≤ lg n

Weighted quick-union analysis

Running time.

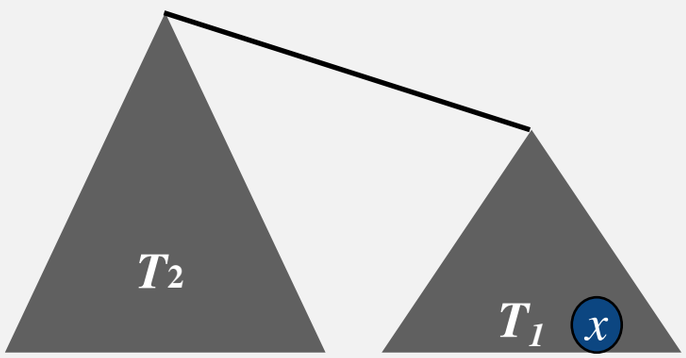
- Find: takes time proportional to depth of p.
- Union: takes constant time, given two roots.

Proposition. Depth of any node x is at most $\lg n$. ← in computer science, lg means base-2 logarithm

Pf. What causes the depth of element x to increase?

Increases by 1 when root of tree T_1 containing x is linked to root of tree T_2.

- The size of the tree containing x at least doubles since $|T_2| \geq |T_1|$.
- Size of tree containing x can double at most $\lg n$ times. Why?



1
2
4
8
16
⋮
n

lg n

Weighted quick-union analysis

Running time.

- Find: takes time proportional to depth of p.
- Union: takes constant time, given two roots.

Proposition. Depth of any node x is at most $\lg n$.

algorithm	initialize	union	find
quick-find	n	n	1
quick-union	n	n	n
weighted quick-union	n	$\log n$	$\log n$

← log mean logarithm, for some constant base

number of array accesses (ignoring leading constant)

Summary

Key point. Weighted quick-union makes it possible to solve problems that could not otherwise be addressed.

algorithm	worst-case time
quick-find	$m\ n$
quick-union	$m\ n$
weighted quick-union	$n + m \log n$
QU + path compression	$n + m \log n$
weighted QU + path compression	$n + m \log^* n$

order of growth for m union-find operations on a set of n elements

Ex. [10^9 unions and finds with 10^9 elements]

- Weighted quick-union reduces time from 30 years to 6 seconds.
- Supercomputer won't help much; good algorithm enables solution.

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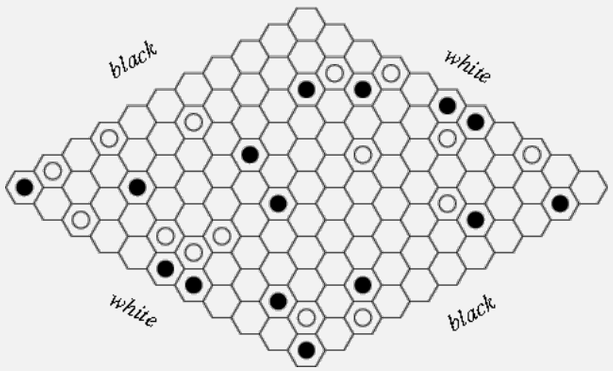
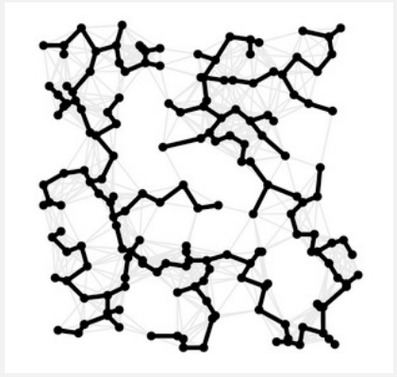
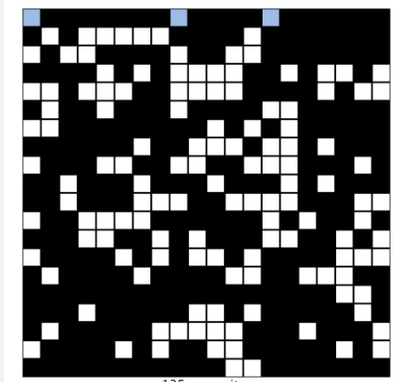
Union–find applications

- Percolation. ← first programming assignment
- Terrain analysis.
- Dynamic-connectivity problem. ← earlier, this lecture
- Least common ancestors in trees.
- Games (Go, Hex, maze generation).
- Minimum spanning tree algorithms.
- Equivalence of finite state automata.
- Hoshen–Kopelman algorithm in physics.
- Hindley–Milner polymorphic type inference.
- Compiling equivalence statements in Fortran.
- Matlab's bwlabel() function in image processing.





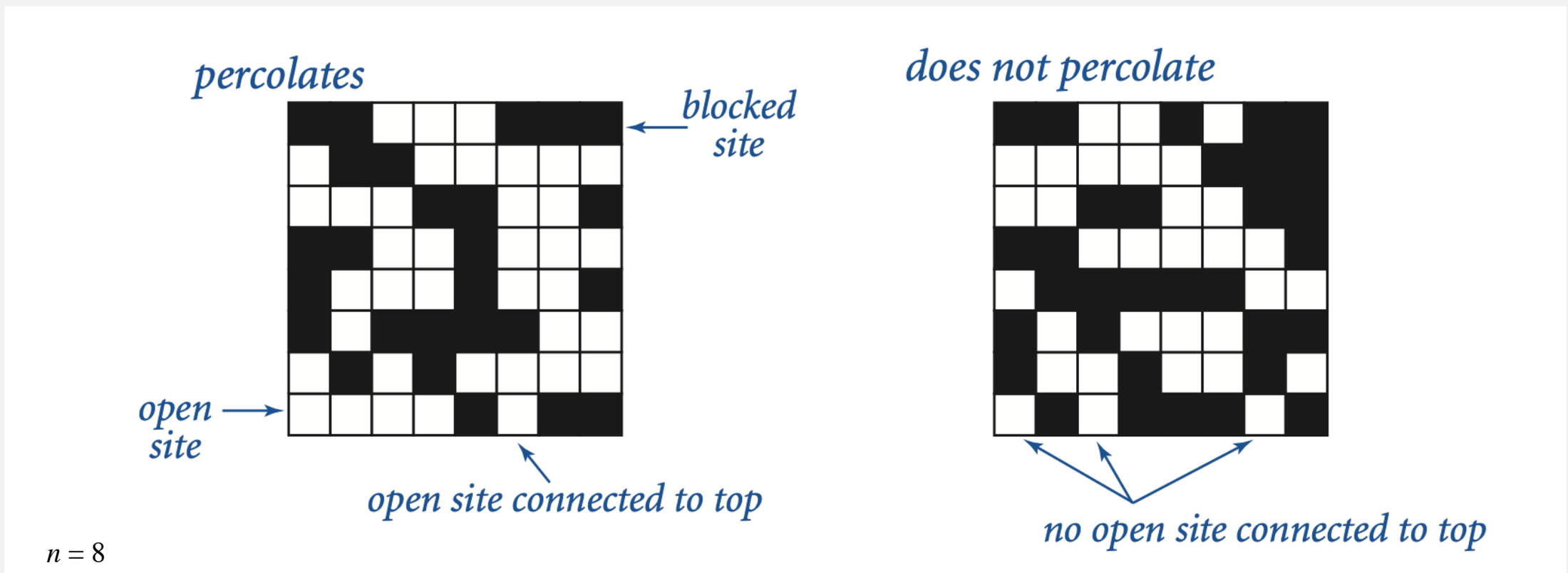


Percolation

An abstract model for many physical systems:

- n-by-n grid of sites.
- Each site is open with probability p (and blocked with probability $1 - p$).
- System percolates iff top and bottom are connected by open sites.

if and only if



$n = 8$

Percolation

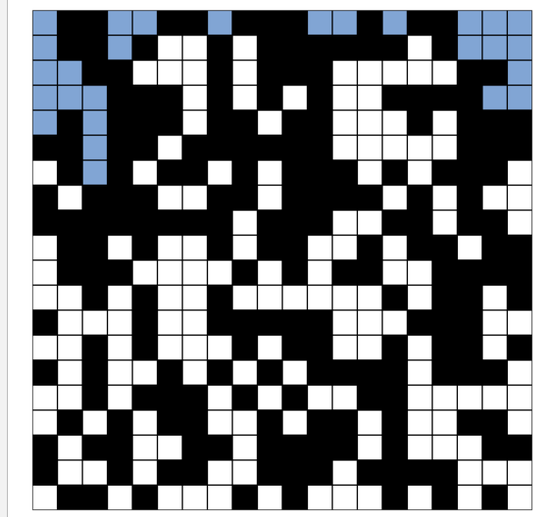
An abstract model for many physical systems:

- n-by-n grid of sites.
- Each site is open with probability p (and blocked with probability $1 - p$).
- System percolates iff top and bottom are connected by open sites.

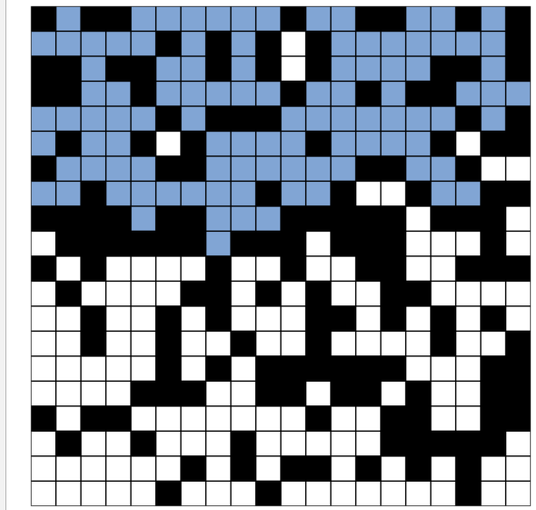
model	system	vacant site	occupied site	percolates
electricity	material	conductor	insulated	conducts
fluid flow	material	empty	blocked	porous
social interaction	population	person	empty	communicates

Likelihood of percolation

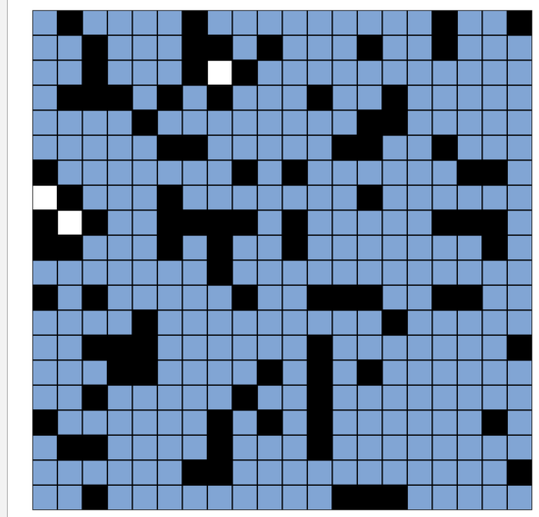
Depends on grid size n and site vacancy probability p.



p low (0.4)
does not percolate



p medium (0.6)
percolates?



p high (0.8)
percolates

empty open site
(not connected to top)

full open site
(connected to top)

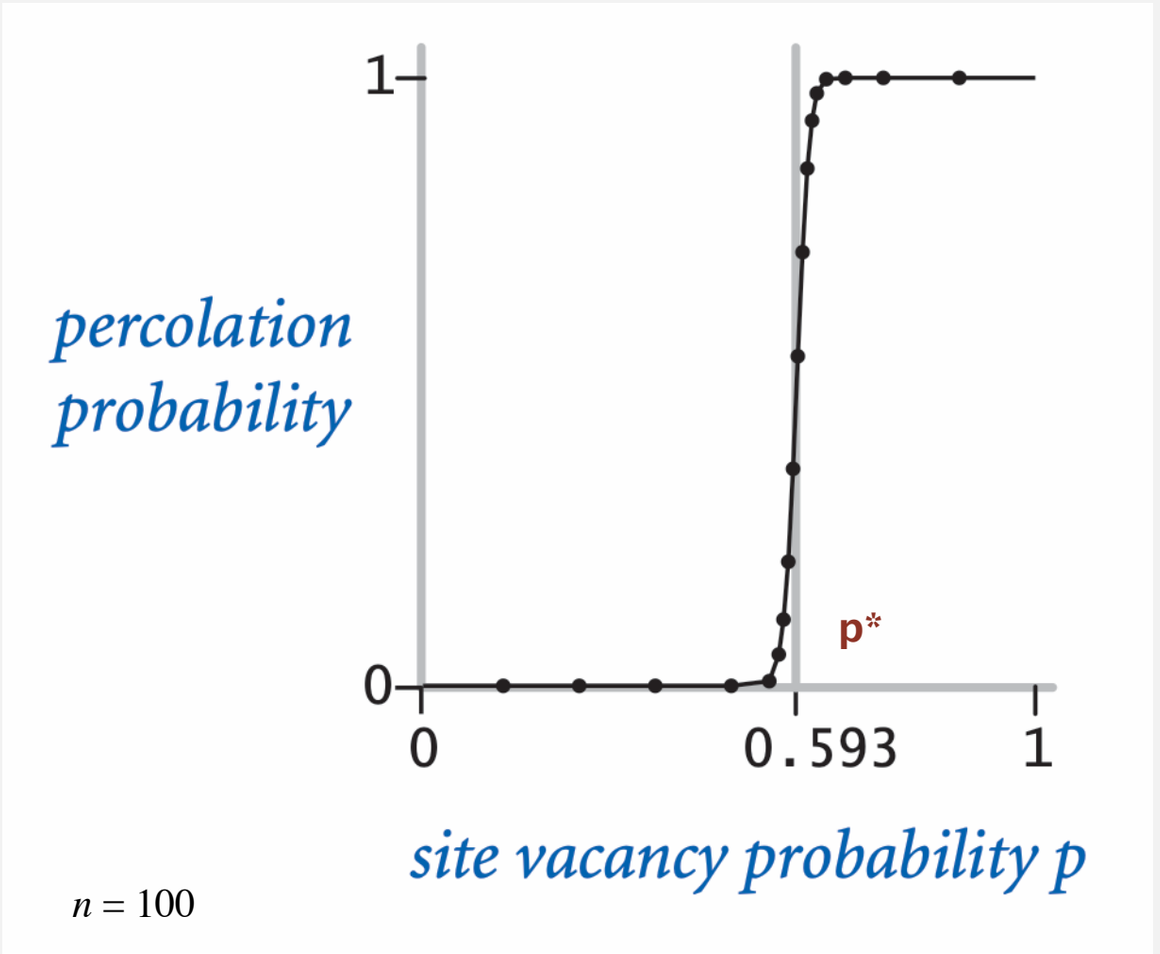
blocked site

Percolation phase transition

When n is large, theory guarantees a sharp threshold p^*.

- $p > p^*$: almost certainly percolates.
- $p < p^*$: almost certainly does not percolate.

Q. What is the value of p^* ?



Monte Carlo simulation

Barrier. Determining the exact threshold p^* is beyond mathematical reach.

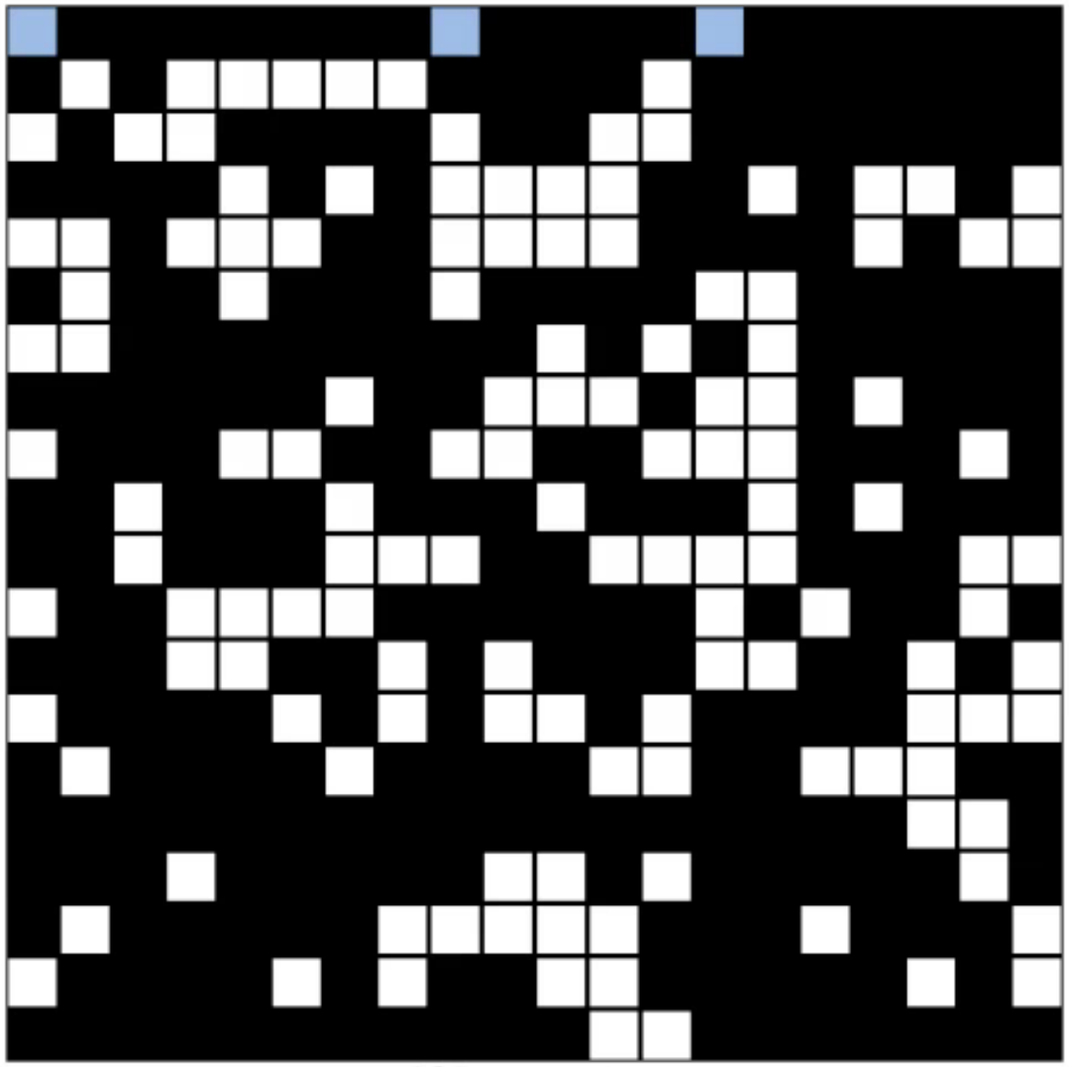
Computational approach.

- Conduct many random experiments.
- Compute statistics.
- Obtain estimate of p^*.

Casino de Monte-Carlo

Monte Carlo simulation

- Initialize all sites in an n-by-n grid to be blocked.
- Declare random sites open until top connected to bottom.
- Vacancy percentage estimates p^*.
- Repeat many times to get more accurate estimate.



full open site
(connected to top)

empty open site
(not connected to top)

blocked site

$$\hat{p} = \frac{204}{400} = 0.51$$

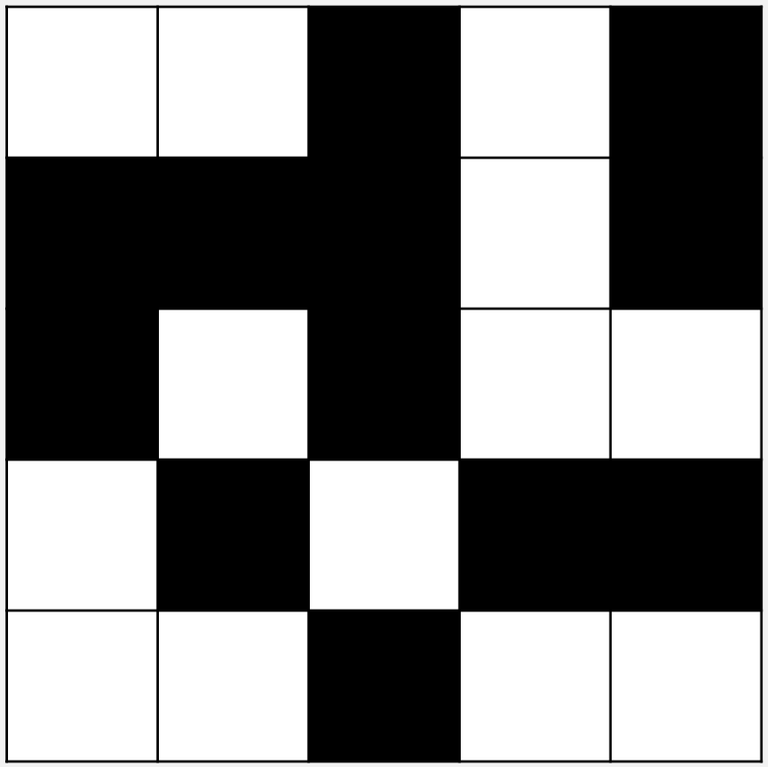
$$n = 20$$

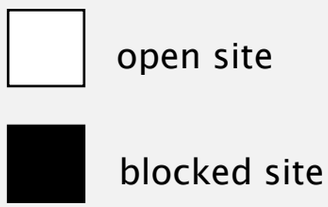
Dynamic-connectivity solution to estimate percolation threshold

Q. How to check whether an n-by-n system percolates?

A. Model as a dynamic-connectivity problem problem and use union–find.

$n = 5$

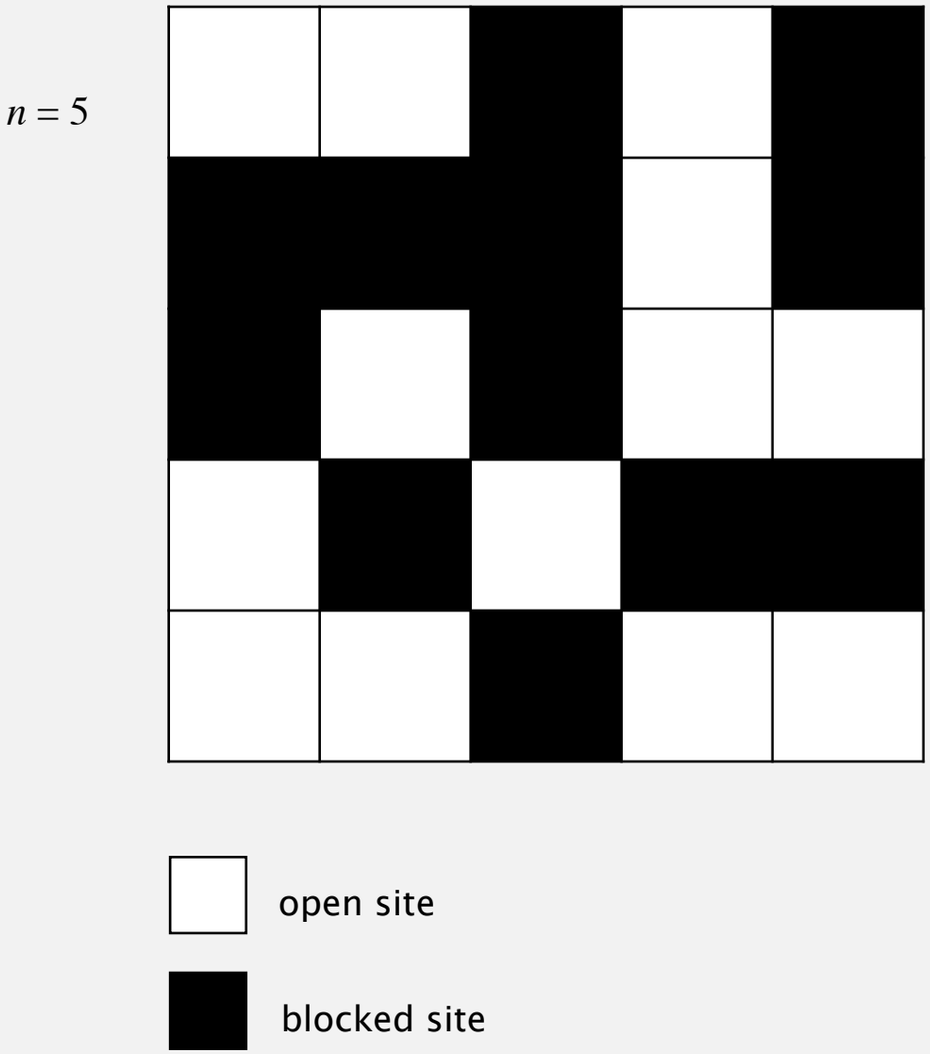


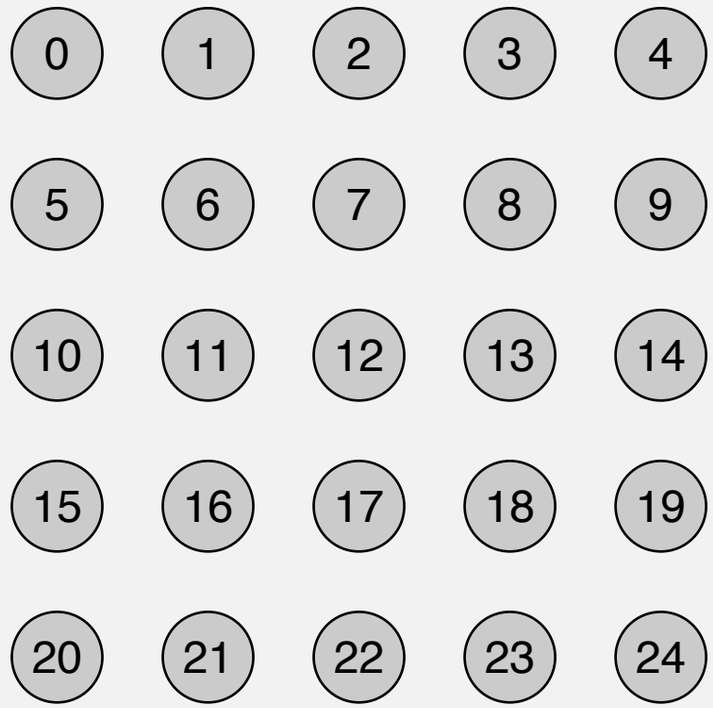


Dynamic-connectivity solution to estimate percolation threshold

Q. How to check whether an n-by-n system percolates?

- Create an element for each site, named 0 to $n^2 - 1$.



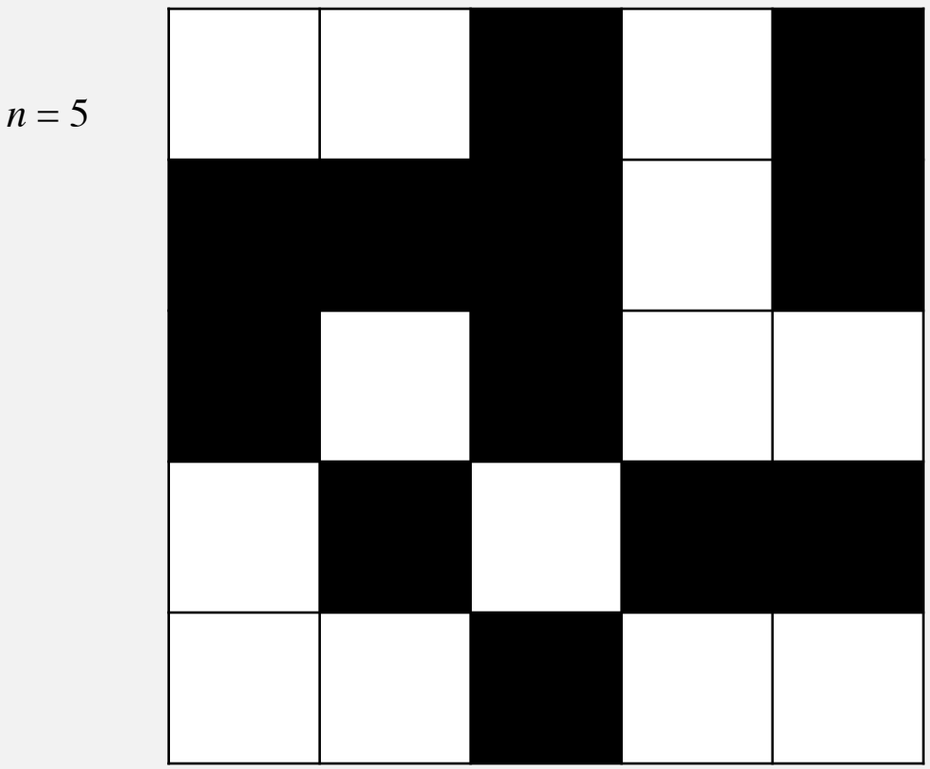


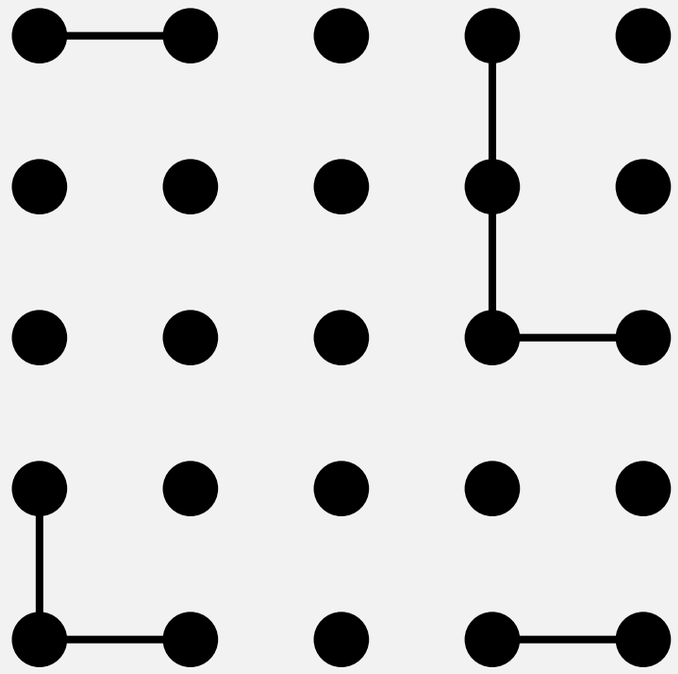
Dynamic-connectivity solution to estimate percolation threshold

Q. How to check whether an n-by-n system percolates?

- Create an element for each site, named 0 to $n^2 - 1$.
- Add edge between two adjacent sites if they both open.

4 possible neighbors: left, right, top, bottom





open site

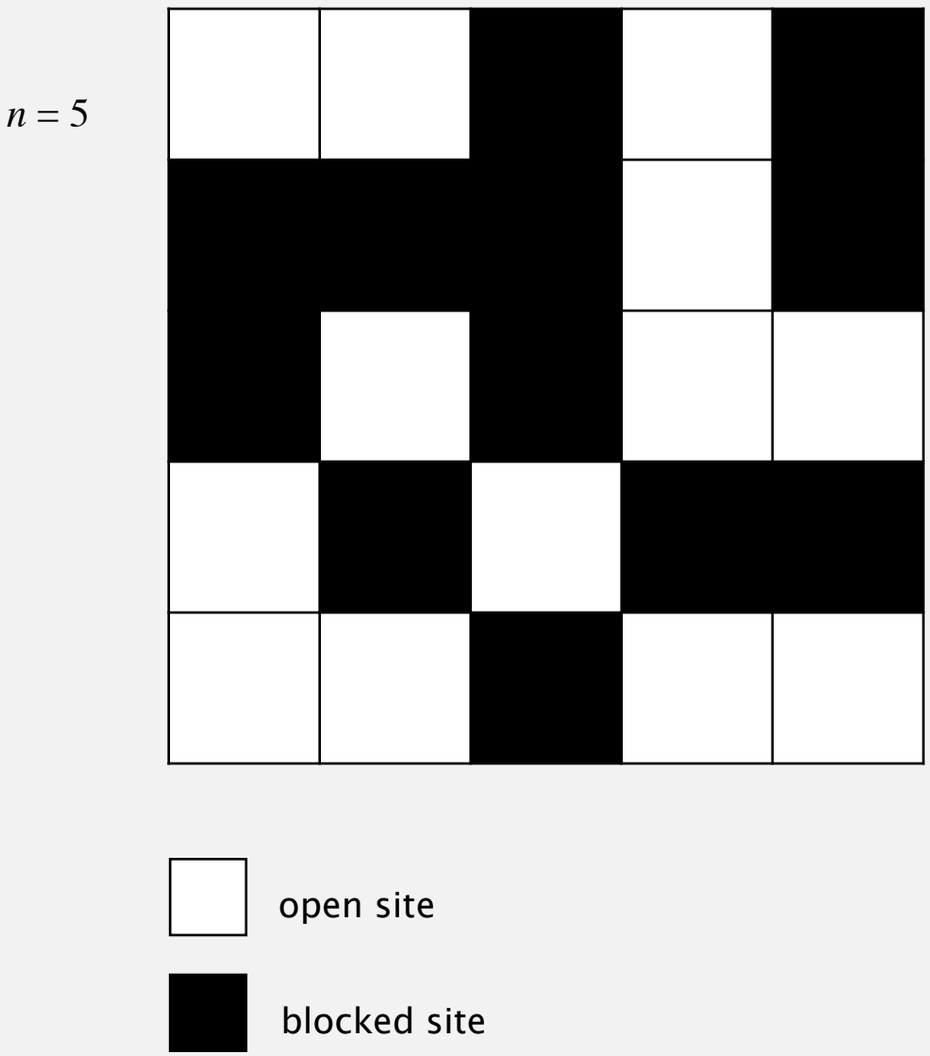
blocked site

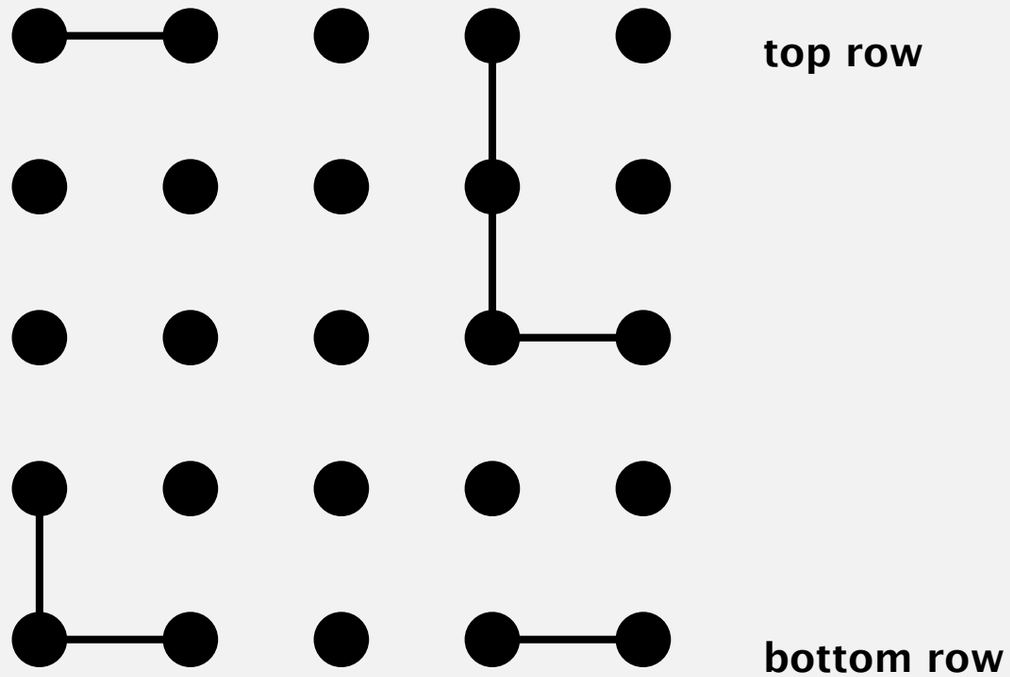
Dynamic-connectivity solution to estimate percolation threshold

Q. How to check whether an n-by-n system percolates?

- Create an element for each site, named 0 to $n^2 - 1$.
- Add edge between two adjacent sites if they both open.
- Percolates iff any site on bottom row is connected to any site on top row.

brute-force algorithm: n^2 connected queries



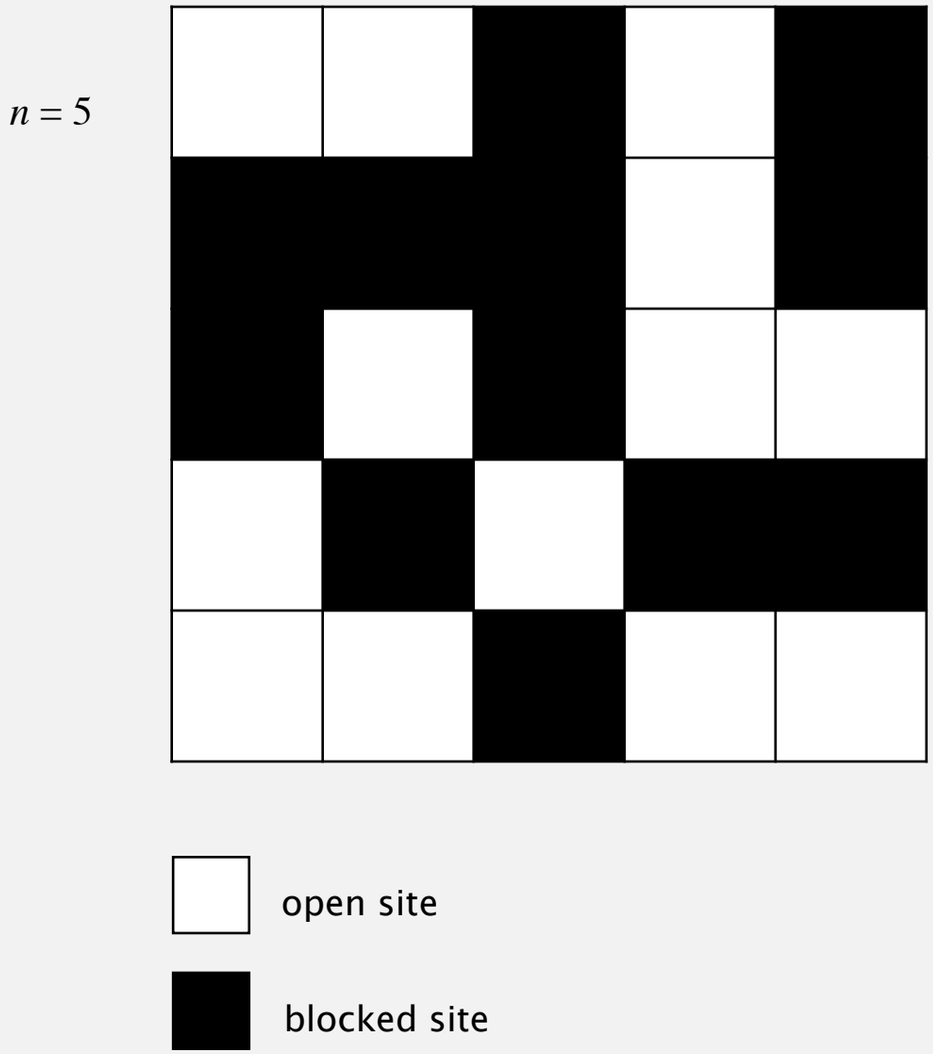


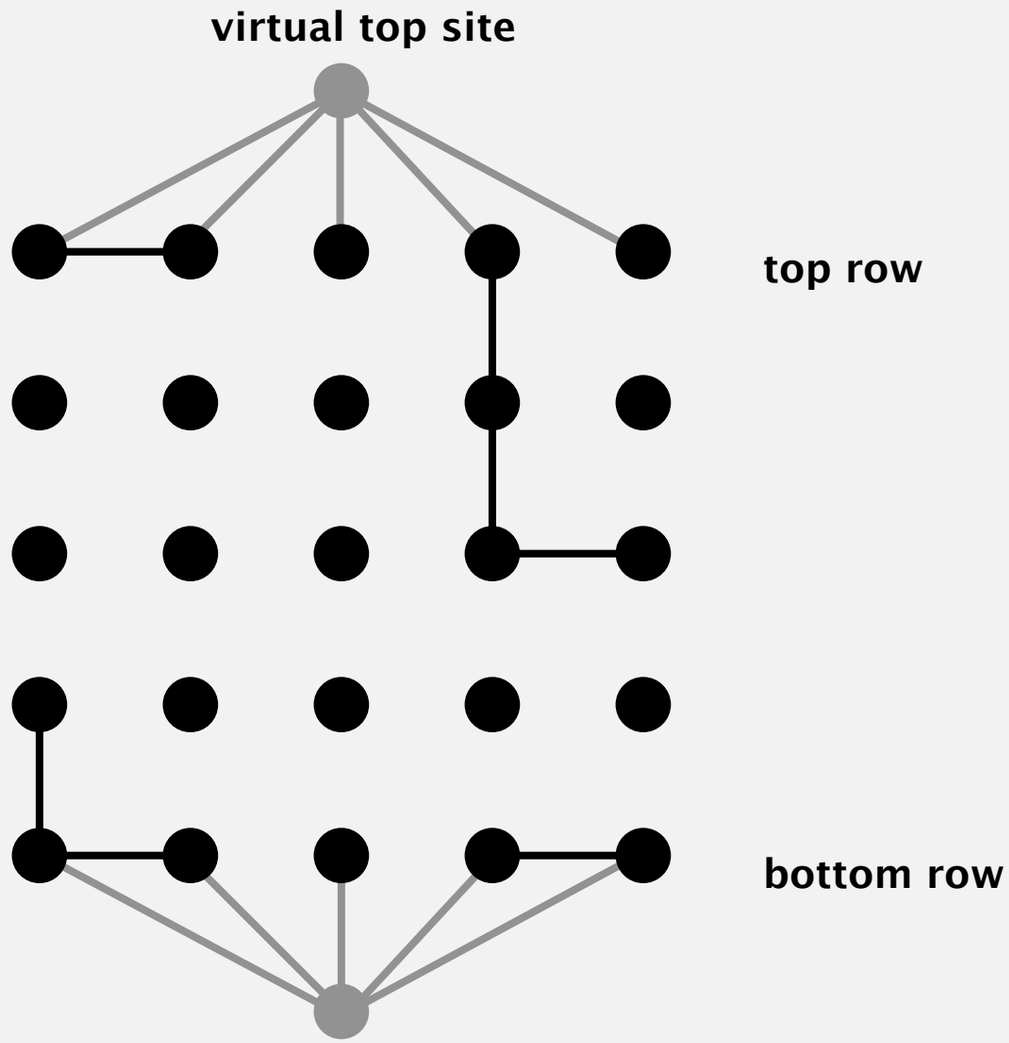
Dynamic-connectivity solution to estimate percolation threshold

Clever trick. Introduce 2 virtual sites (and edges to top and bottom).

- Percolates iff virtual top site is connected to virtual bottom site.

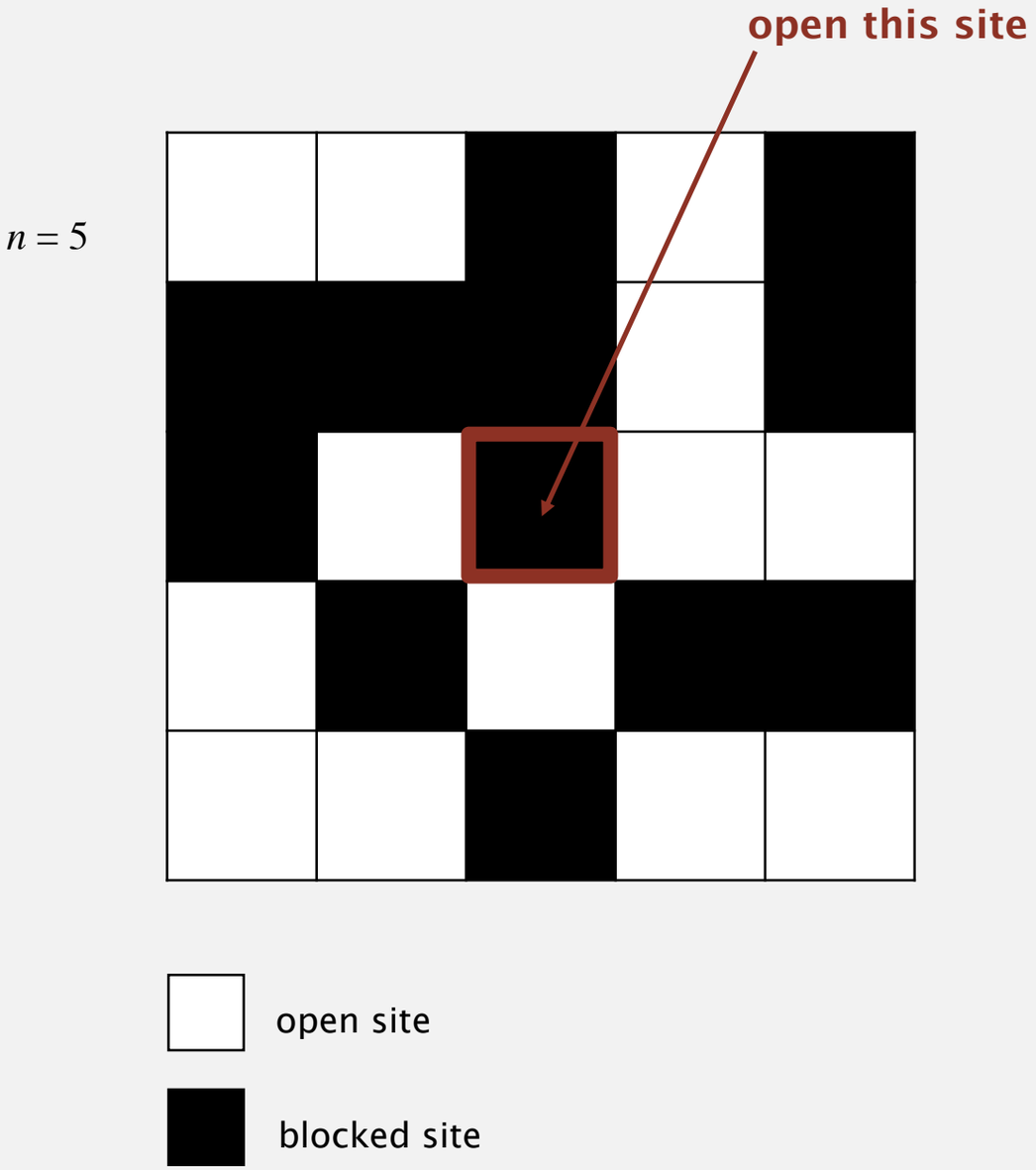
more efficient algorithm: only 1 connected query

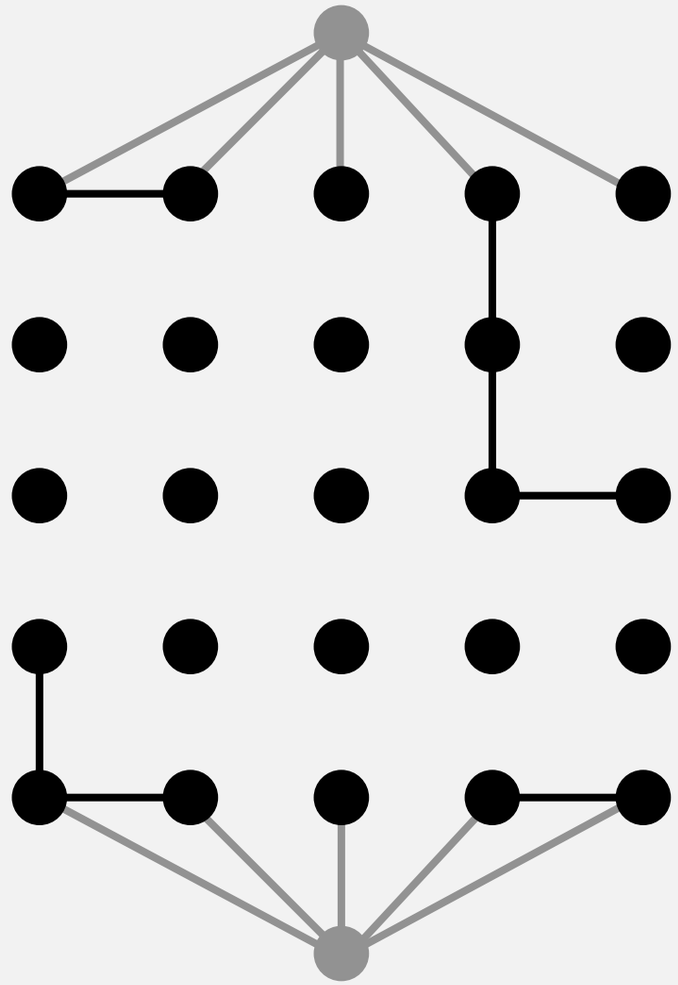




Dynamic-connectivity solution to estimate percolation threshold

Q. How to model opening a new site?



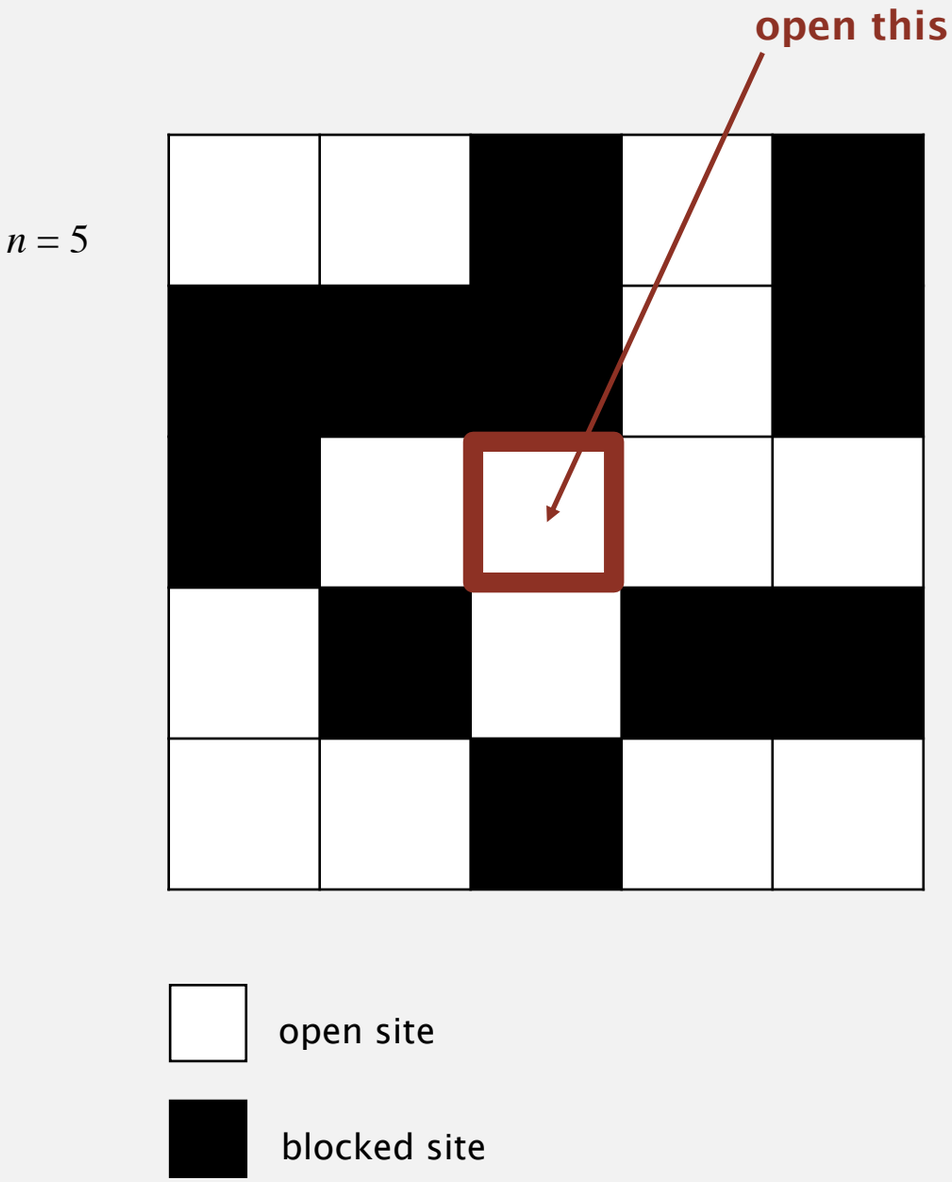


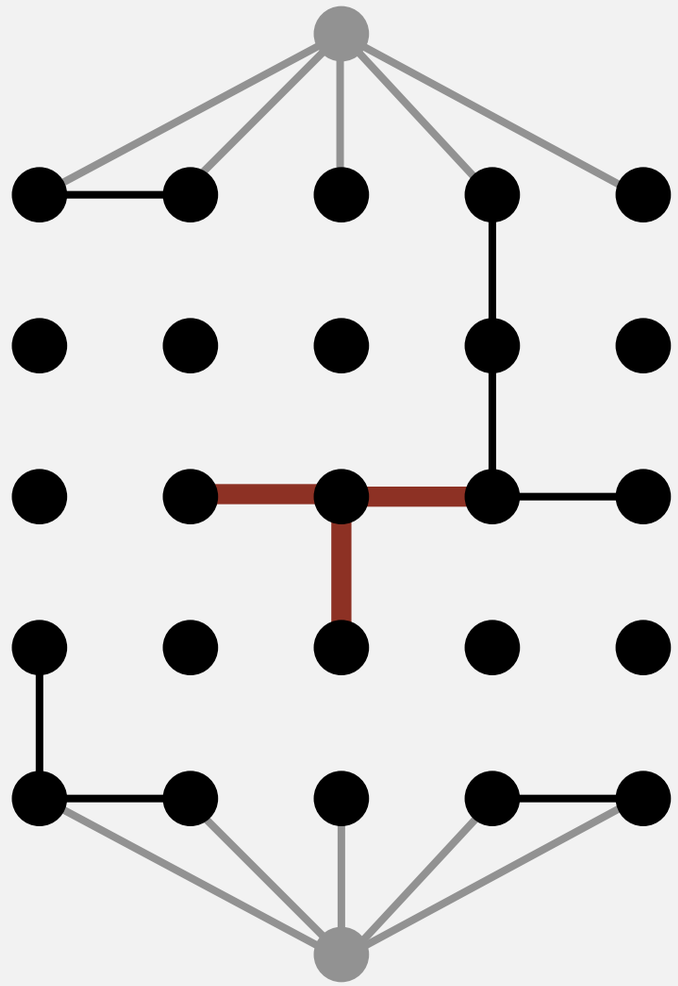
Dynamic-connectivity solution to estimate percolation threshold

Q. How to model opening a new site?

A. Mark new site as open; add edge to any adjacent site that is open.

adds up to 4 edges



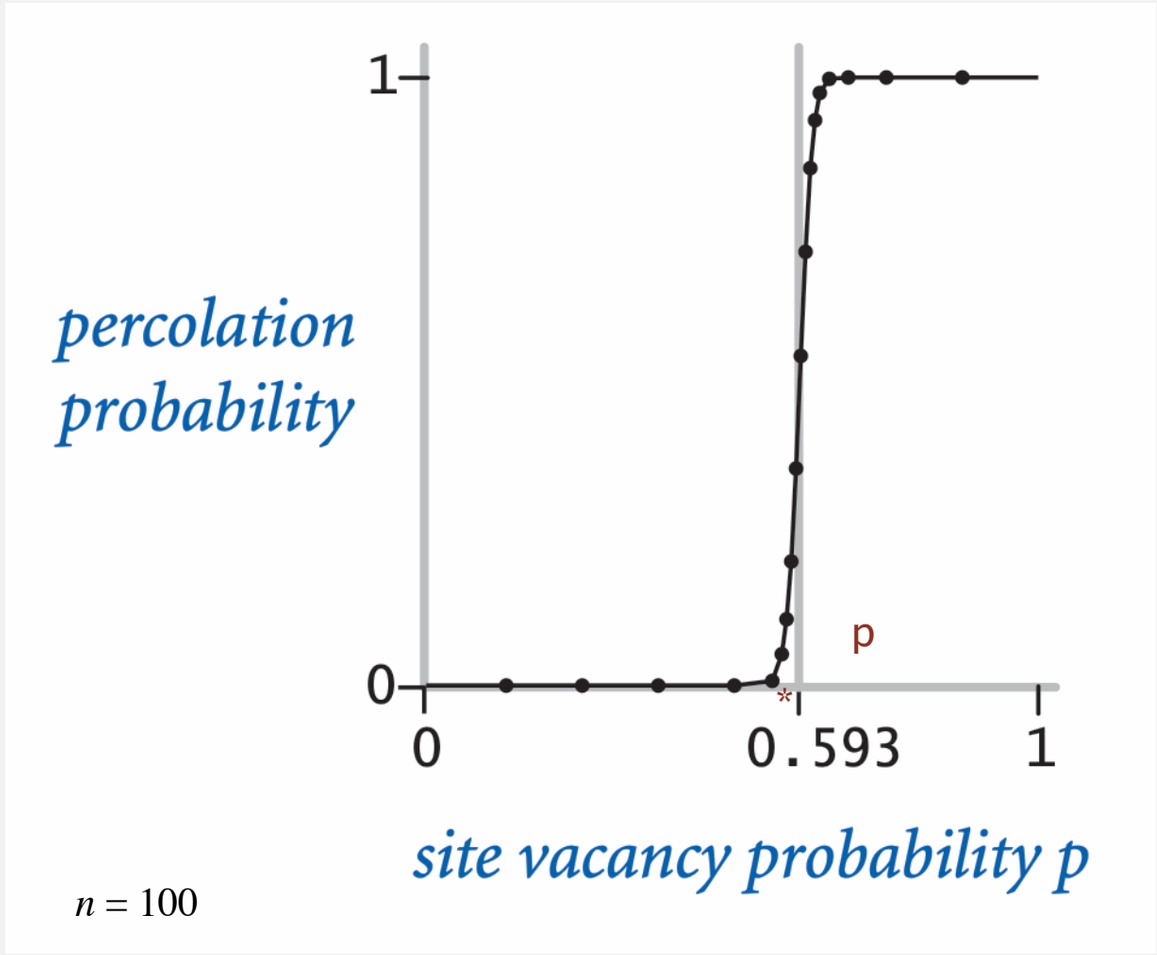


Percolation threshold

Q. What is percolation threshold p^* ?

A. About 0.592746 for large square lattices.

constant known only via simulation



Fast algorithm enables accurate answer to scientific question.

Subtext of today's lecture (and this course)

Steps to developing a usable algorithm.

- Model the problem.
- Find an algorithm to solve it.
- Fast enough? Fits in memory?
- If not, figure out why.
- Find a way to address the problem.
- Iterate until satisfied.

The scientific method.

Mathematical analysis.

INTRODUCTION TO DATA STRUCTURES and ALGORITHMS

Rutgers University

1B. UNION–FIND

- *union–find data type*
- *quick-find*
- *quick-union*
- *improvements*
- *applications*

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