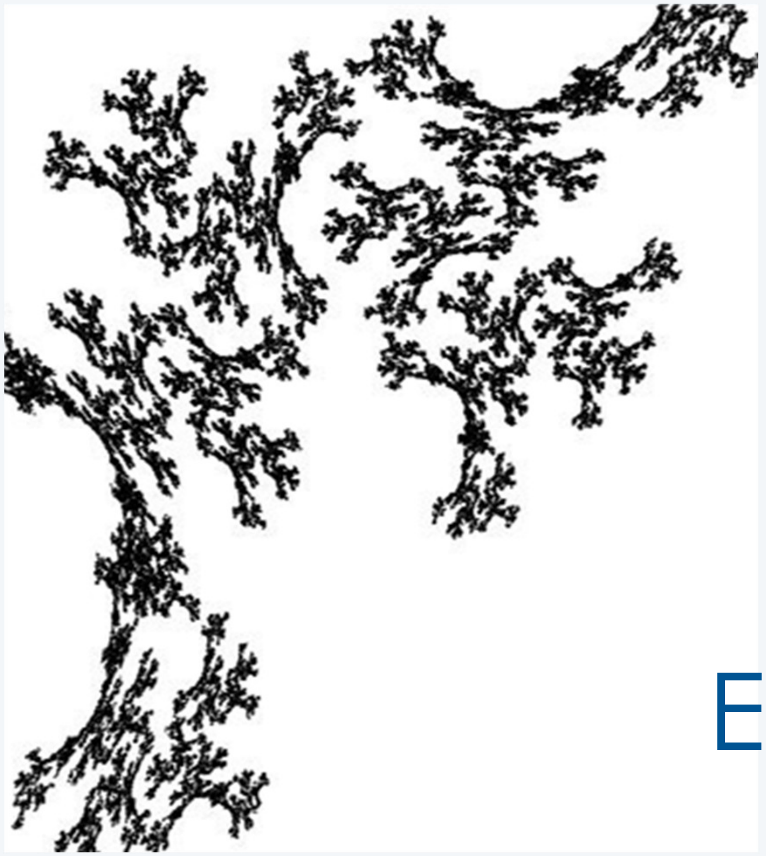

INTRODUCTION TO DISCRETE STRUCTURES

E. Nested Quantifiers

1.1–1.2

@2020 A.D. Gunawardena

3.2 Nested Quantifiers

- Universal and Existential Quantifiers
- Nested quantifiers
- From English to nested quantifiers
- Negating nested quantifiers
- Rules of inference

Universal and Existential Quantifiers

Quantifiers. Expresses the truth of a propositional statement over a domain.

There are two kinds of quantifiers.

Universal. expresses the truth of a predicate over an entire domain.
A proposition P(x) is true for all x in X

Existential. Expresses the truth of a predicate for at least one instance.
There exist some x, such that P(x) is true.

Precedence of quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus. For example, $\forall x\, P(x) \vee Q(x)$ is the disjunction of $\forall x\, P(x)$ and $Q(x)$. In other words, it means $(\forall x\, P(x)) \vee Q(x)$ rather than $\forall x(P(x) \vee Q(x))$.

INTRODUCTION TO DISCRETE STRUCTURES

3.2 Nested Quantifiers

- Universal and Existential Quantifiers
- Nested quantifiers
- From English to nested quantifiers
- Negating nested quantifiers
- Rules of inference

Restating nested quantifiers

Restate : ∀ x!=0, ∃y (x.y = 1)

Let P(x,y) : x.y=1

∃y P(x,y) is a statement that only involves x. Let Q(x) = ∃y P(x,y) .

∀ x!=0, ∃y (x.y = 1) can be expressed as ∀ x!=0, Q(x) where Q(x): ∃y P(x,y) and P(x,y) : x.y=1.

Quantification of two variables using nested quantifiers

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

3.2 Nested Quantifiers

- Universal and Existential Quantifiers
- Nested quantifiers
- From English to nested quantifiers
- Negating nested quantifiers
- Rules of inference

From nested quantifiers to English

$$\forall x \forall y ((x > 0) \land (y < 0) \rightarrow (xy < 0)),$$

Answer: For any given numbers x, y , the product of the numbers is negative if they have opposite signs
Note: There are other ways to say this as well.

workshop

Translate to symbolic notation

For every even integer n greater than 2, there exist primes p and q such that $n = p + q$.

Translate to English

Translate these statements into English, where the domain for each variable consists of all real numbers.

a) $\forall x \exists y (x < y)$

b) $\forall x \forall y (((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0))$

c) $\forall x \forall y \exists z (xy = z)$

Workshop

. A discrete mathematics class contains 1 mathematics major who is a freshman, 12 mathematics majors who are sophomores, 15 computer science majors who are sophomores, 2 mathematics majors who are juniors, 2 computer science majors who are juniors, and 1 computer science major who is a senior. Express each of these statements in terms of quantifiers and then determine its truth value.

a) There is a student in the class who is a junior.
b) Every student in the class is a computer science major.
c) There is a student in the class who is neither a mathematics major nor a junior.
d) Every student in the class is either a sophomore or a computer science major.
e) There is a major such that there is a student in the class in every year of study with that major.

answer

We let $P(s, c, m)$ be the statement that student s has class standing c and is majoring in m. The variable s ranges over students in the class, the variable c ranges over the four class standings, and the variable m ranges over all possible majors.

a) The proposition is $\exists s \exists m P(s, \text{junior}, m)$. It is true from the given information.

b) The proposition is $\forall s \exists c P(s, c, \text{computer science})$. This is false, since there are some mathematics majors.

c) The proposition is $\exists s \exists c \exists m \big(P(s, c, m) \wedge (c \neq \text{junior}) \wedge (m \neq \text{mathematics})\big)$. This is true, since there is a sophomore majoring in computer science.

d) The proposition is $\forall s \big(\exists c P(s, c, \text{computer science}) \vee \exists m P(s, \text{sophomore}, m)\big)$. This is false, since there is a freshman mathematics major.

e) The proposition is $\exists m \forall c \exists s P(s, c, m)$. This is false. It cannot be that m is mathematics, since there is no senior mathematics major, and it cannot be that m is computer science, since there is no freshman computer science major. Nor, of course, can m be any other major.

Answer to workshop

3.2 Nested Quantifiers

- Universal and Existential Quantifiers
- Nested quantifiers
- From English to nested quantifiers
- Negating nested quantifiers
- Rules of inference

Negating quantifiers

Negating universal. Leads to a statement of existential

not (all humans are mammals) is equivalent to there is at least one human who is not mammal

Negating existential. Leads to a statement of universal

not (at least one human is not mammal) is equivalent all humans are mammals

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x).$$

De Morgan's Law and quantifiers

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

workshop

Let $Q(x, y)$ denote the statement "x is the capital of y." What are these truth values?

a) Q(Denver, Colorado)

b) Q(Detroit, Michigan)

c) Q(Massachusetts, Boston)

d) Q(New York, New York)

Express the negation of these propositions using quantifiers, and then express the negation in English.

a) Some drivers do not obey the speed limit.

b) All Swedish movies are serious.

c) No one can keep a secret.

d) There is someone in this class who does not have a good attitude.

Let $Q(x)$ be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

a) $Q(0)$ **b)** $Q(-1)$ **c)** $Q(1)$

d) $\exists x Q(x)$ **e)** $\forall x Q(x)$ **f)** $\exists x \neg Q(x)$

g) $\forall x \neg Q(x)$

INTRODUCTION TO
DISCRETE STRUCTURES

3.2 Nested Quantifiers

- Universal and Existential Quantifiers
- Nested quantifiers
- From English to nested quantifiers
- Negating nested quantifiers
- Rules of inference

Validity of simple arguments

p ^ (p ➔ q) ➔ q is a tautology

This is a rule of inference called Modus Ponens

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

INTRODUCTION TO
DISCRETE STRUCTURES

3.2 Nested Quantifiers

- Universal and Existential Quantifiers
- Nested quantifiers
- From English to nested quantifiers
- Negating nested quantifiers
- Rules of inference

INTRODUCTION TO
DISCRETE STRUCTURES

3.2 Nested Quantifiers

1.1–1.2

@2020 A.D. Gunawardena