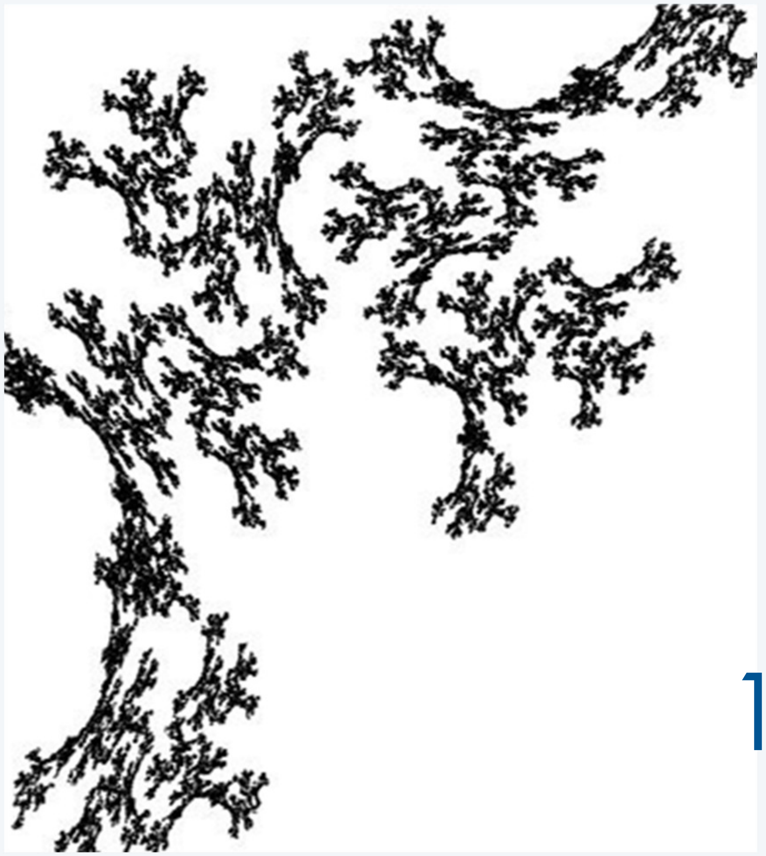
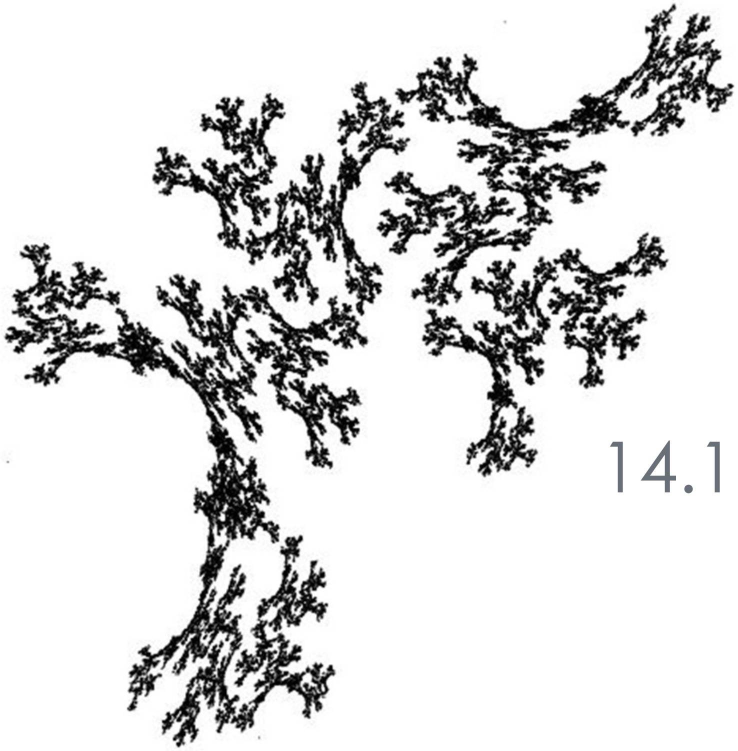


INTRODUCTION TO
DISCRETE STRUCTURES

14 Boolean Algebra

1.1–1.2



INTRODUCTION TO
DISCRETE STRUCTURES

14.1 Boolean Algebra

- Boolean Functions
- Representations
- Logic Gates*
- Minimization of Circuits*

Boolean Algebra

Boolean algebra are rules for working with the Boolean numbers {0,1}

The operators on the Boolean numbers

- Boolean sum (+)
- Boolean Product (.)
- Complement (bar ~)

+	0	1
0	0	1
1	1	1

.	0	1
0	0	0
1	0	1

-	0	1
	1	0

Examples

- Find the value of $1 \cdot 0 + \overline{(0 + 1)}$

- Find the value of $(1 + 0) + \overline{(0 \,.\, 1)}$

Boolean Expressions and Functions

Definition: Let $B = \{0, 1\}$. Then $B^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in B \text{ for } 1 \leq i \leq n\}$ is the set of all possible n-tuples of 0s and 1s.

Question. How many tuples exists, if n = 3?

The variable x is called a *Boolean variable* if it assumes values only from B, that is, if its only possible values are 0 and 1. A function from B^n to B is called a *Boolean function of degree n*.

Example: The function $F(x, y) = x$ from the set of ordered pairs of Boolean variables to the set $\{0, 1\}$ is a Boolean function of degree 2.

TABLE 1

x	y	$F(x, y)$
1	1	1
1	0	0
0	1	0
0	0	1

Boolean Expressions and Boolean Functions (*continued*)

Exercise: Find the values of the Boolean function represented by $F(x, y, z) = xy + \bar{z}$.

TABLE 2

x	y	z	xy	$\bar{z}$	$F(x, y, z) = xy + \bar{z}$
1	1	1	1	0	
1	1	0	1	1	
1	0	1	0	0	
1	0	0	0	1	
0	1	1	0	0	
0	1	0	0	1	
0	0	1	0	0	
0	0	0	0	1	

Equality of Boolean Functions

Definition: Boolean functions F and G of n variables are equal if and only if

$$F(b_1, b_2, \ldots, b_n) = G(b_1, b_2, \ldots, b_n)$$

whenever $b_1, b_2, \ldots, b_n$ belong to B.

Two different Boolean expressions that represent the same function are *equivalent*.

Complement of a Boolean function

Definition: The complement of the Boolean function F is the function $\bar{F}$,

where $\bar{F}(x_1, x_2, \ldots, x_n) = \overline{F(x_1, x_2, \ldots, x_n)}$.

Exercise. Find the complement of $F(x, y) = (x \cdot y) + \overline{(x + y)}$

Boolean Sum and Boolean Product of Functions

Definition: Let F and G be Boolean functions of degree n.

The Boolean sum $F + G$ and the are defined by

$(F + G)(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n) + G(x_1, x_2, \ldots, x_n)$

Boolean product FG *is defined by*

$(FG)(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)G(x_1, x_2, \ldots, x_n)$

Question. How many different Boolean functions of degree n exists?

TABLE 3 The 16 Boolean Functions of Degree Two.

x	y	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
1	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

Slide Credits: **Rosen Textbook**

Workshop

How many Boolean functions of degree 3 exists?

Write three of those functions

Boolean Identities

TABLE 5 Boolean Identities.

Identity	*Name*
$\overline{\overline{x}} = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 1 = x$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Domination laws
$x + y = y + x$ $xy = yx$	Commutative laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$\overline{(xy)} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \overline{x}\,\overline{y}$	De Morgan's laws
$x + xy = x$ $x(x + y) = x$	Absorption laws
$x + \overline{x} = 1$	Unit property
$x\overline{x} = 0$	Zero property

Duality of Boolean Expressions

The dual of a Boolean expression is obtained by interchanging **+ and .** and interchanging **0 and 1**

E.g**.** Consider the identity $x + 1 = 1$ its dual is $x.0 = 0$

Exercise. Find the dual of $x + xy = x$

Workshop

Prove that absorption law : $x + xy = x$ is true

x	y	x + xy	x	

Formal Definition of Boolean Algebra

Definition: A *Boolean algebra* is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and a unary operation $\bar{}$ such that for all x, y, and z in B:

$x \vee 0 = x$ $x \wedge 1 = x$	*identity laws*
$x \vee \bar{x} = 1$ $x \wedge \bar{x} = 0$	*complement laws*
$(x \vee y) \vee z = x \vee (y \vee z)$ $(x \wedge y) \wedge z = x \wedge (y \wedge z)$	*associative laws*
$x \vee y = y \vee x$ $x \wedge y = y \wedge x$	*commutative laws*
$x \vee (y \wedge z) = (x \vee y) \wedge (y \vee z)$ $x \wedge (y \vee z) = (x \wedge y) \vee (y \wedge z)$	*distributive laws*

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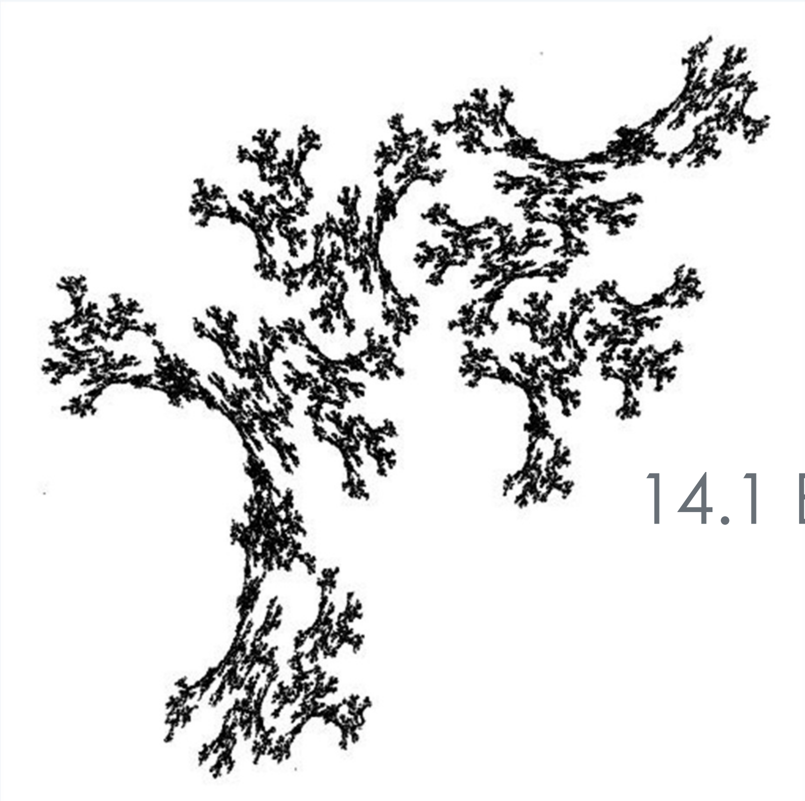

14.1 Boolean Algebra

- Boolean Functions
- Representations
- Logic Gates*
- Minimization of Circuits*

Sum-of-Products Expansion

Definition: A *literal* is a Boolean variable or its complement. A *minterm* of the Boolean variables $x_1, x_2, \ldots, x_n$ is a Boolean product $y_1 y_2 \cdots y_n$, where $y_i = x_i$ or $y_i = \overline{x}_i$. Hence, a minterm is a product of n literals, with one literal for each variable.

The minterm $y_1 y_2 \ldots y_n$ has value has value 1 if and only if each x_i is 1.

This occurs if and only if $x_i = 1$ when $y_i = x_i$ and $x_i = 0$ when $y_i = \overline{x}_i$.

Example. The minterm $xy\overline{z}$ =1 if and only if x = y = z =

Definition: The sum of minterms that represents the function is called the *sum-of-products expansion* or the *disjunctive normal form* of the Boolean function.

Sum-of-Products Expansion

Example: Find Boolean expressions that represent the functions

(i) $F(x, y)$

(ii) $G(x, y, z)$

TABLE 1

x	y	z	F	G
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

Workshop

Find the sum-of-products expansion of the Boolean function $F(x,y,z) = x + y + z$ using a table

Sum-of-Products Expansion

Find the sum-of-products expansion of the Boolean function F(x,y,z) = x.(y + z) using Boolean identities

TABLE 5 **Boolean Identities.**

Identity	*Name*
$\overline{\overline{x}} = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 1 = x$	Identity laws
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$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$\overline{(xy)} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \overline{x}\,\overline{y}$	De Morgan's laws
$x + xy = x$ $x(x + y) = x$	Absorption laws
$x + \overline{x} = 1$	Unit property
$x\overline{x} = 0$	Zero property

Example

Find the sum-of-products expansion for the function $F(x,y,z) = (x + y)\,\bar{z}$.

(i) Using the table

(ii) Using identities

TABLE 2

x	y	z	$x + y$	$\bar{z}$	$(x + y)\bar{z}$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

Functional Completeness

Definition: Every Boolean function can be represented using the Boolean operators $\cdot$, $+$, and $^{-}$, we say that the set $\{\cdot, +, ^{-}\}$ is *functionally complete*.

The set $\{\cdot, ^{-}\}$ is functionally complete since $x + y = \overline{\bar{x}\bar{y}}$.

The set $\{+, ^{-}\}$ is functionally complete since $xy = \overline{\bar{x} + \bar{y}}$.

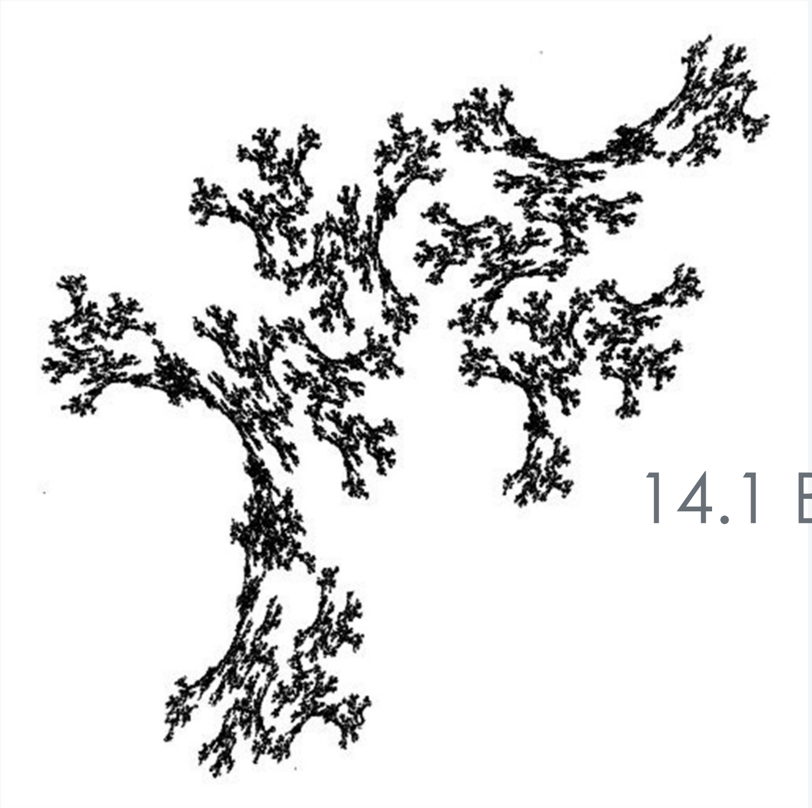
The *nand* operator, denoted by $|$, is defined by $1|1 = 0$, and $1|0 = 0|1 = 0|0 = 1$.

The set consisting of just the one operator nand $\{|\}$ is functionally complete.

Note that $\bar{x} = x \mid x$ and $xy = (x|y)|(x|y)$.

The *nor* operator, denoted by $\downarrow$, is defined by $0 \downarrow 0 = 1$, and $1 \downarrow 0 = 0 \downarrow 1 = 1 \downarrow 1 = 0$.

Prove that the set consisting of just the one operator nor $\{\downarrow\}$ is functionally complete.



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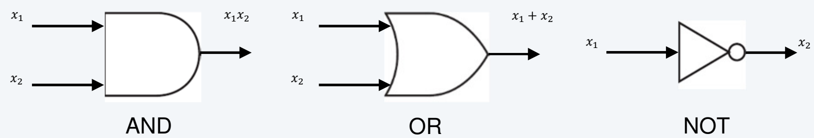
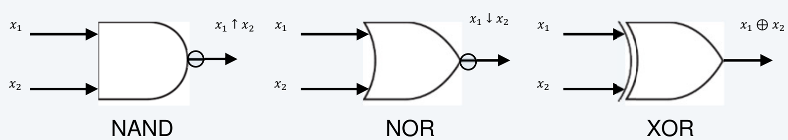
Logic Gates

NOT (invertor), OR, and AND gates.



NAND, NOR, and XOR gates



Each gate is a Boolean function

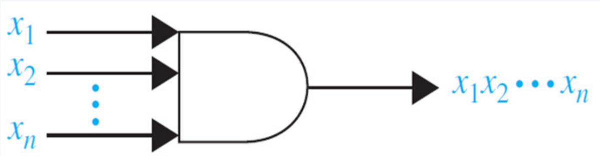
Boolean Circuits

These "circuit gates" act like actual gates, in a sense

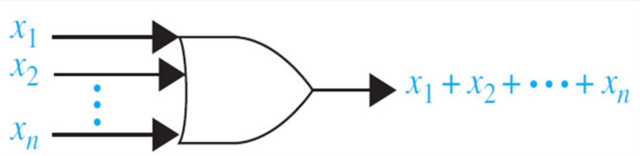
- There is a trigger that "opens" a gate periodically
- The specified operation (AND, OR, NOT, ...) then happens

In modern computers, these gates open/close a few trillion times a second, giving us GHz chips.

Multiple input AND and OR gates



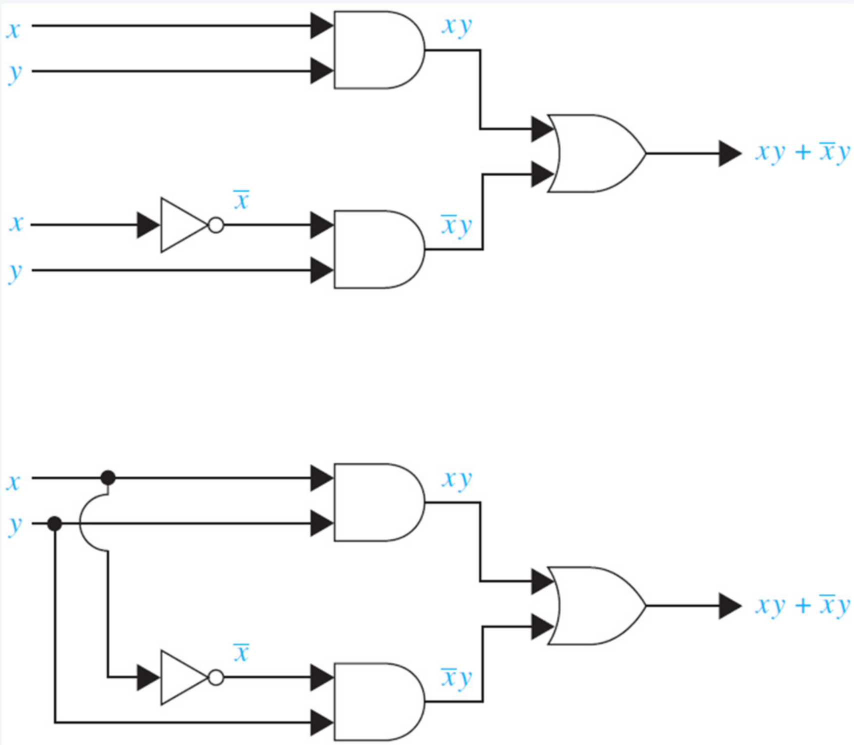
Multiple AND gate



Multiple OR gate

Credits: **Jingjin Yu**

Boolean Circuits are not unique



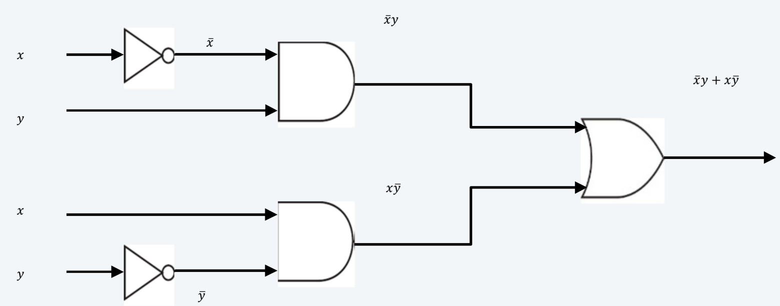
A Two-way Switch

Task: Designing a two-way light switch

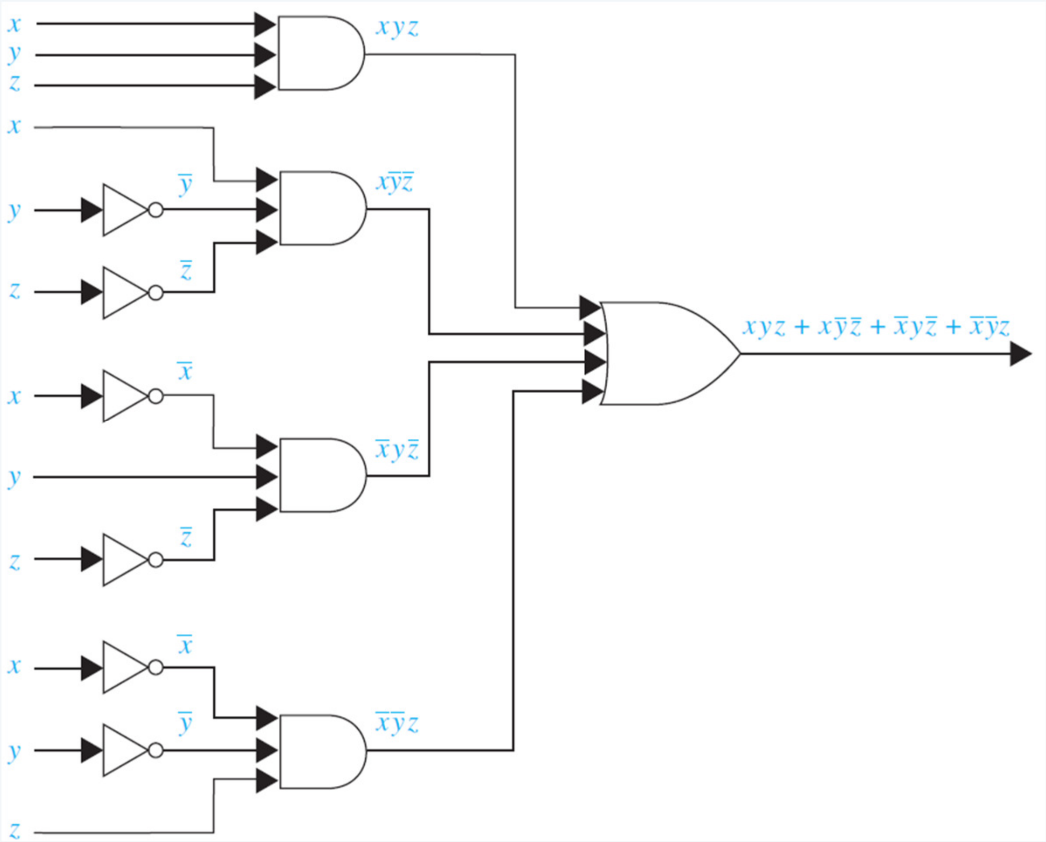
- Input: switches x, y, which can be on (1) or off (0)
- Output: light on/off as a Boolean function $F(x, y)$

Steps:

- Assume that $x = 0$ means switch x is off. Same for y. [different from the textbook]
- Assume when $x = y = 0$, $F(x, y) = 0$, light off.
- From here, two possibilities: $x = 0, y = 1$ or $x = 1, y = 0$, $F(x, y) = 1$, light on.
- From here, two possibilities: $x = y = 0$ or $x = y = 1$, light off.
- We get $F(x, y) = 1$ when $x = 0, y = 1$ or $x = 1, y = 0$.
- Using the sum-of-product construction, $F(x, y) = x\bar{y} + \bar{x}y$
- Circuit:

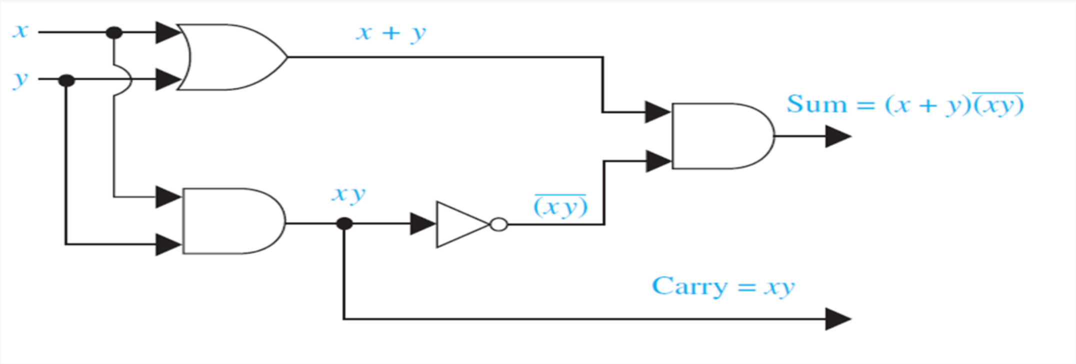


3-way Switch



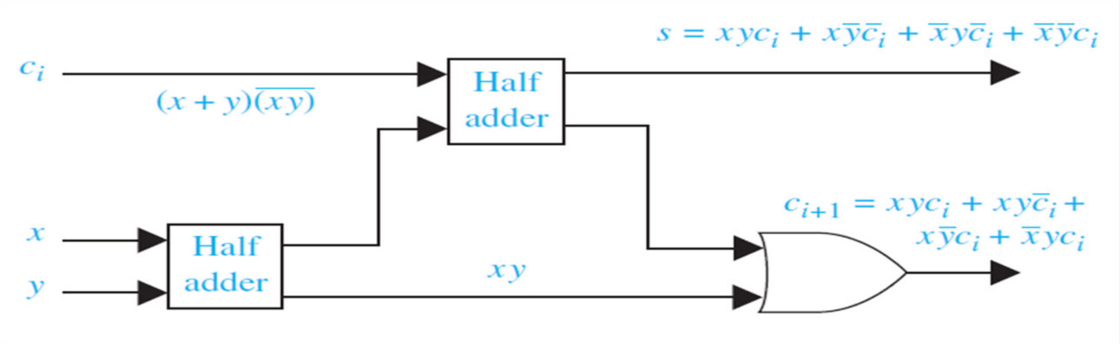
Exercise. Find the circuit that is needed to design a 3-way switch

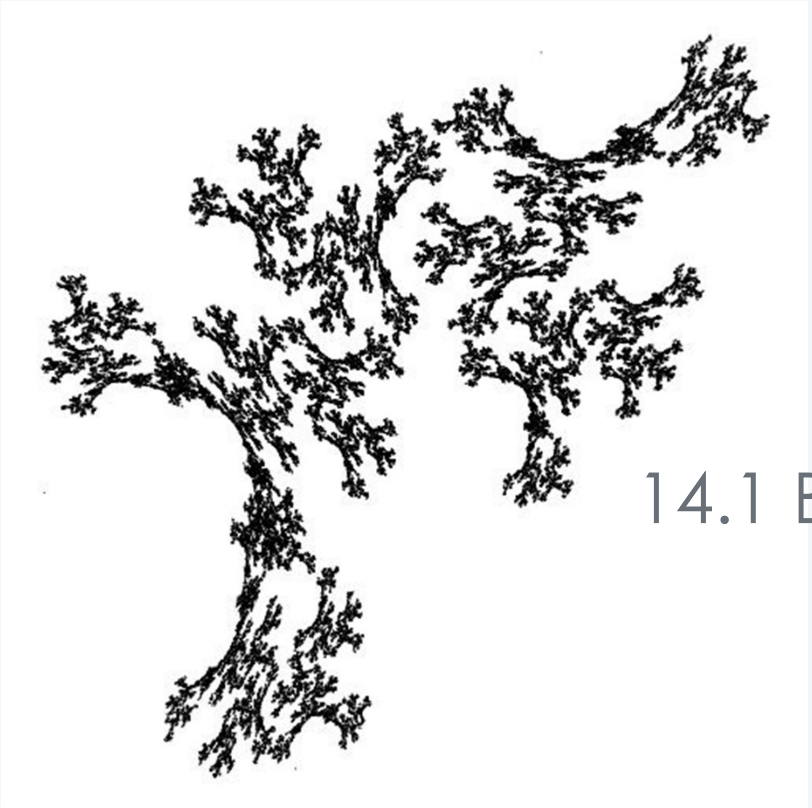
Half Adder and Full Adder



Half adder

Full adder





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Under construction

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