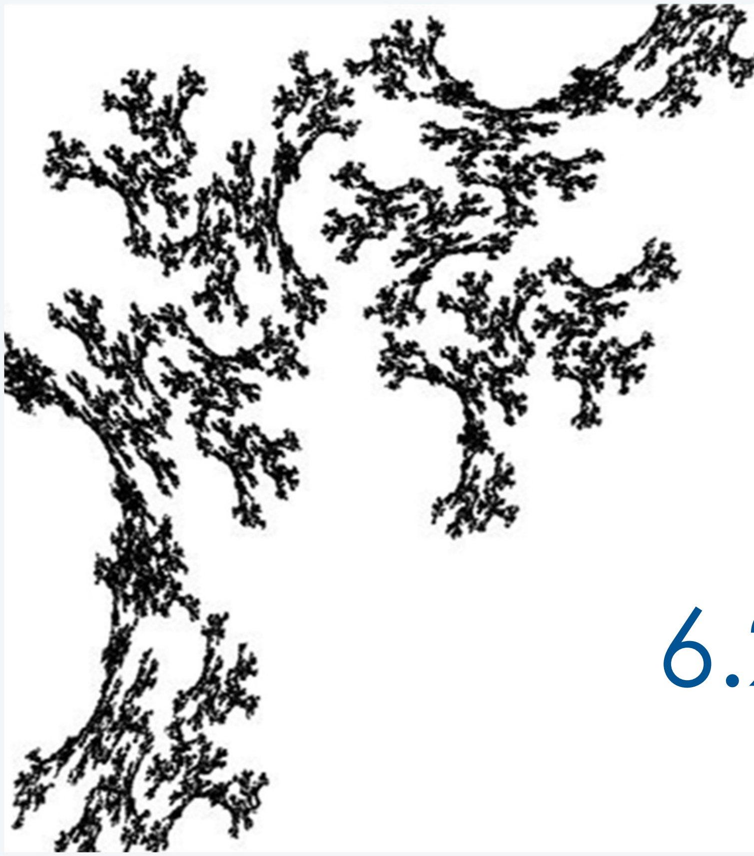
INTRODUCTION TO
DISCRETE STRUCTURES



6.2 Relations, functions and sequences

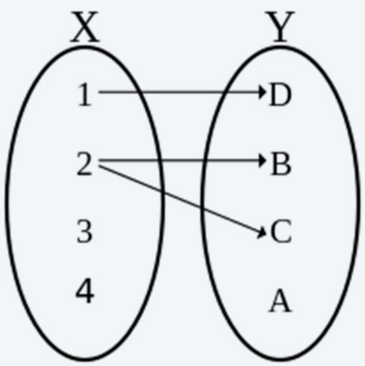
1.1–1.2

6.2 Relations, functions and sequences

- Relations as sets
- Functions as sets
- 1-1 and onto functions
- Increasing and decreasing functions
- sequences

Relations

Given two sets A and B, a relation R is a subset of the cross product of A x B



R = {(1,D), (2,B), (2,C)}

Notation. Let x, y are in sets A and B of a relation, then we write (x,y) in R or x R y to indicate that x and y are related by R

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Exercise. Write all possible relations from set A={0,1} to set B = {1}

Exercise. How many possible relations exists between two sets A and B?

Hint. Give the answer using the cardinality of A, B and the cross product.

Inverse relations

If R is a relation, then the inverse relation of R^{-1} is defined as

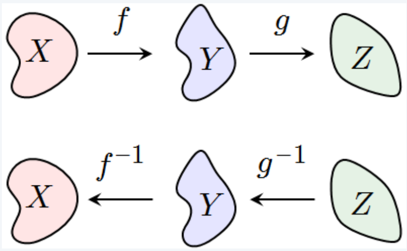
x R y if and only if y R^{-1}x

Exercise. Write the inverse relation R^{-1} of R = {(1,D), (2,B), (2,C)}

Composition of functions

Let g be a function from A to B and f be a function from B to C

The composition of the function f and g, denoted for all a in A as f o g is defined by: (f o g)(a) = f(g(a))



Exercise.

Given 3 sets, P(rofessors) = {p1, p2}, S(tudents)={s1, s2, s3} and C(ourses) = {c1, c2, c3}

Define two relations. R1 = “advise” and R2 = “taking course”

R1 = {(p1,s1), (p2, s3), (p1, s2)}

R2 = {(s1,c1), (s1,c2), (s2,c3), (s3,c1)}

Find the relation R1 o R2 (composition relation)

Exercise

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R1 = {(p1,s1), (p2, s3), (p1, s2)}

R2 = {(s1,c1), (s1,c2), (s2,c3), (s3,c1)}

Find the relation R2 o R1 (composition relation)

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Functions

The formal definition of a function

Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

Which of the following relations are functions?

f: {0,1} ➔ {1,2}, f(0) = 1, f(1) = 2

g: {0,1} ➔ {1,2}, g(0) = 1, g(0) = 2, g(1)=1

h: {0,1} ➔ {1,2}, h(0) = 1, h(1) = 1

Domain and co-domain of a function

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range*, or *image*, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f *maps* A to B.

Image of a set

Let f be a function from A to B and let S be a subset of A. The *image* of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S\ (t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

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Find the domain and range of these functions.

a) the function that assigns to each pair of positive integers the first integer of the pair

b) the function that assigns to each positive integer its largest decimal digit

c) the function that assigns to a bit string the number of ones minus the number of zeros in the string

workshop

Find the domain and range of these functions.

d) the function that assigns to each positive integer the largest integer not exceeding the square root of the integer

e) the function that assigns to a bit string the longest string of ones in the string

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One to one Functions

1-1 functions

A function f is said to be *one-to-one*, or an *injunction*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one.

onto functions

Onto functions

A function f from A to B is called *onto,* or a *surjection,* if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

Bijection

The function f is a *one-to-one correspondence,* or a *bijection,* if it is both one-to-one and onto. We also say that such a function is *bijective*.

Proving the 1-1 and onto properties of functions

To prove f(x) is 1-1, show the following. $f(x) = f(y) \Rightarrow x = y$

To prove f(x) is onto show the following. Given y such that $f(x) = y$, find the x that makes it work

Workshop on 1-1 functions

Which of the following functions are injective?

- $f(x) = 2x + 1$ {domain = R}
- $f(x) = x^2$ {domain = R}
- $f(x) = \text{sqrt}(x)$ {domain = R+}

Exercises

Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

a) $f(a) = b$, $f(b) = a$, $f(c) = c$, $f(d) = d$
b) $f(a) = b$, $f(b) = b$, $f(c) = d$, $f(d) = c$
c) $f(a) = d$, $f(b) = b$, $f(c) = c$, $f(d) = d$

Workshop on onto functions

Consider the following questions.

Determine whether $f: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = 2m - n$.

b) $f(m, n) = m^2 - n^2$.

c) $f(m, n) = m + n + 1$.

d) $f(m, n) = |m| - |n|$.

e) $f(m, n) = m^2 - 4$.

Workshop on bijections

. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = -3x + 4$

b) $f(x) = -3x^2 + 7$

c) $f(x) = (x + 1)/(x + 2)$

d) $f(x) = x^5 + 1$

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Increasing and decreasing functions

A function f whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and *strictly increasing* if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called *decreasing* if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f. (The word *strictly* in this definition indicates a strict inequality.)

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sequences

A sequence is a function from a subset of integers to a set S. We use the notation a_n to denote the image of the integer n. We call a_n, a term of the sequence. The summation notation is often used in analyzing sequences.

$$\sum_{j=m}^{n} a_j, \quad \textstyle\sum_{j=m}^{n} a_j, \quad \text{or} \quad \textstyle\sum_{m \le j \le n} a_j$$

Examples.

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Geometric sequence

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term* a and the *common ratio* r are real numbers.

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Useful summations

The following summations are highly useful in many situations.

TABLE 2 **Some Useful Summation Formulae.**

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

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Exercise. We know the sum of the first n integers. Can we find the sum of the first (n-1) integers.

Exercise. We know the sum of the first n squares. Can we find the sum of the next n squares?

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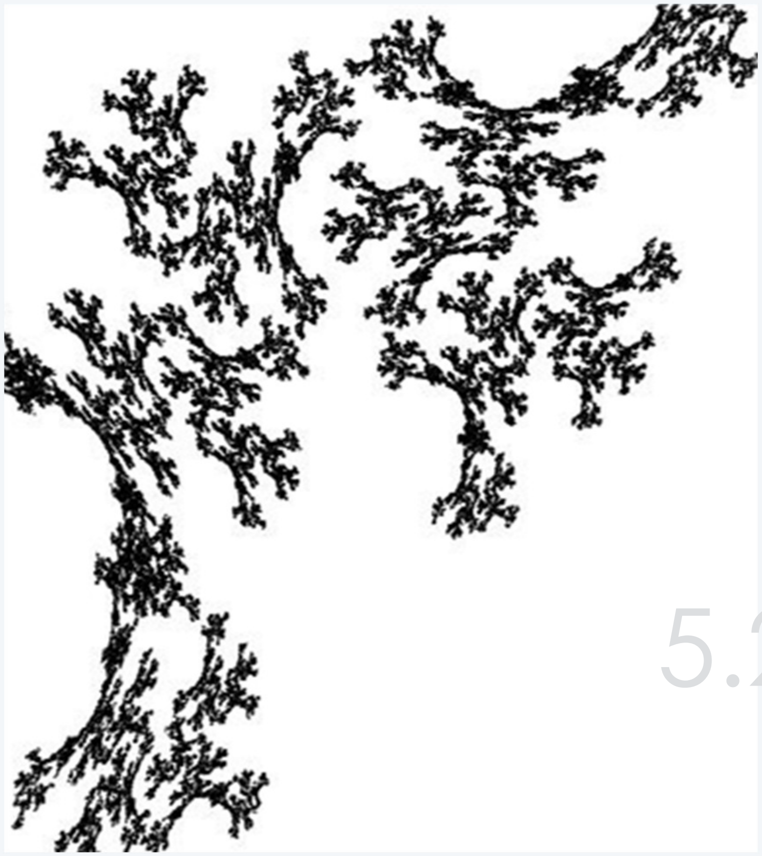

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