INTRODUCTION TO
DISCRETE STRUCTURES

7.1 Mathematical Induction

1.1–1.2

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7.1 Mathematical induction

- Introduction
- The core idea
- Examples of basic inductive proofs
- Buggy inductive proofs
- Guidelines for proofs by induction
- More challenging proof

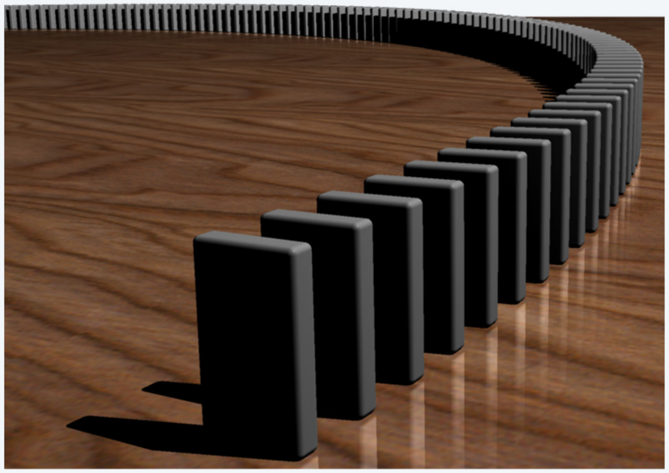
Introduction

Many propositions are stated as : P(n): $n^2 > n$ for all $n \geq 2$, a property that is true for all numbers. In some cases, proving such propositions require a proof technique called induction.

In general, prove that a property P(n) holds for all-natural numbers. In general induction is a powerful proof technique for any countable set.

The history of inductive proofs seems to go back to 300 BC where Euclid may have tried to prove that the number of primes is infinite.

Around 1000 AD there is evidence that induction has been used to prove binomial theorem $(a+b)^n$ and Facts about arithmetic sequences.



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What is mathematical induction?

Given a proposition P(n), prove the following rules.

Rule 1: P(0) is true

Rule 2: if P(k) then P(k+1)

Then we deduce that P(n) is true for all-natural numbers.

More formally

The Principle of Induction.

Let P be a predicate on nonnegative integers. If

- $P(0)$ is true, and
- $P(n)$ IMPLIES $P(n+1)$ for all nonnegative integers, n,

then

- $P(m)$ is true for all nonnegative integers, m.

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Proofs about sequences

Induction is a great tool to prove results about arithmetic sequences.

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Note. this formula, 1 + 2 + … + n = n(n+1)/2 can also be proved using different methods.

Workshop

Show that $2^n > n$ for all $n \geq 1$

Proof.

workshop

Show the following identities are true for any natural number n using induction

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \, .$$

workshop

Consider the following propositional statement

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n},$$

(a) Find the first five values P(1), P(2),…P(5)

(b) What values of n, you think is the proposition true?

(c) Prove your statement by induction

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Buggy inductive proof

Theorem : Every set of n dogs are the same color

Proof:

Let P(n) be the proposition "all sets of n dogs are the same color"

Then P(1) is obviously true since a set of 1 dog is all the same color

Assume P(n) is true for some $n \geq 1$.

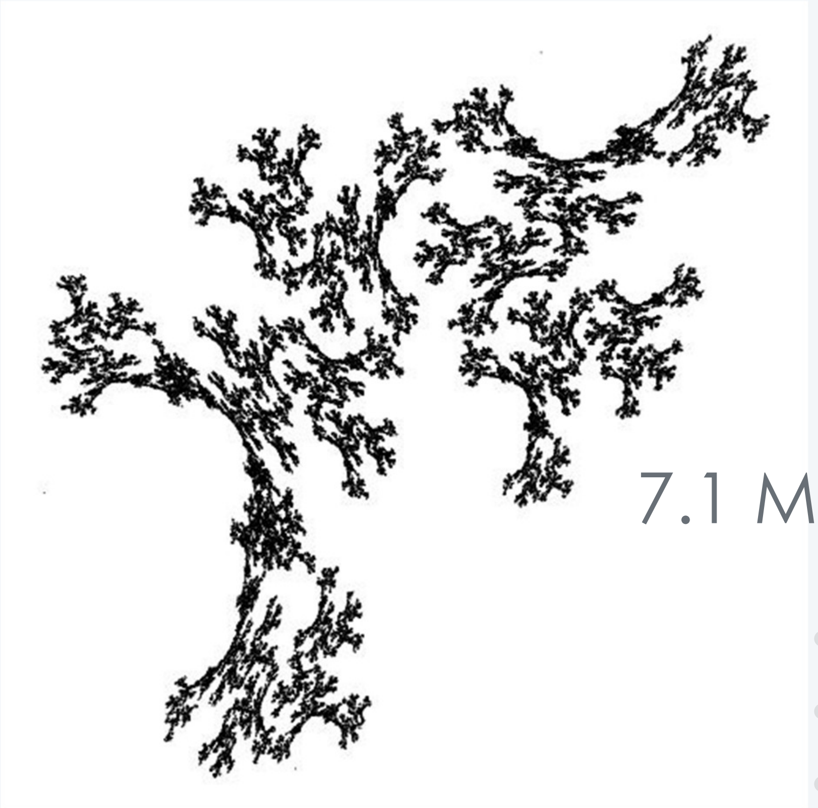
This means all sets of n dogs are the same color

Let us consider a set of (n+1) dogs $d_1, d_2, \ldots, d_{n+1}$

By induction assumption, the last n dogs $d_2, \ldots, d_{n+1}$ are the same color

Also first n dogs, $d_1, \ldots, d_n$ are the same color by induction assumption

This implies all (n+1) dogs $d_1, \ldots, d_{n+1}$ are the same color. ????



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Steps in an inductive proof

Step 1: State that the proof is done using induction

Always start with: “we prove this statement by induction”

Step 2: Define P(n)

Eg. P(n) : P(n) : n – sqrt(n) > 0 for n ≥ 2

Step 3: Show P(0) or some base case is true

Step 4: Assume P(k) and deduce P(k+1)

That is, if k – sqrt(k) > 0 ,

then show (k+1) – sqrt(k+1) > 0

Step 5: Conclude that P(n) is true for all n

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workshop

Prove or disprove. 5 is a factor of $n^5 - 1$ for all $n >= 1$

workshop

Prove that 3 is a factor of $n^3 + 2n$ for all $n >= 1$

workshop

Prove that 21 divides $4^{n+1} + 5^{2n-1}$ for all positive n

More challenging exercises

Construct an inductive argument to prove the following propositions.

Prove that $3^n < n!$ if n is an integer greater than 6.

Prove that $2^n > n^2$ if n is an integer greater than 4.

For which nonnegative integers n is $n^2 \leq n!$? Prove your answer.

For which nonnegative integers n is $2n + 3 \leq 2^n$? Prove your answer.

Prove that $1/(2n) \leq [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)]/(2 \cdot 4 \cdot \cdots \cdot 2n)$ whenever n is a positive integer.

Dijkstra’s algorithm

The key ideas of the algorithm are

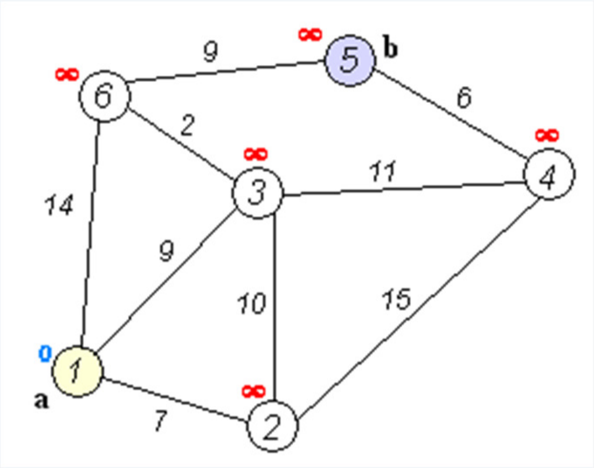
Maintain a set of shortest distances from source to every node found so far

Suppose we have added n nodes to the shortest distance list

We need to show when we add the (n+1)-th node to the list (according to algorithm), then it should be the shortest path.

We need to show that there is no other shorter path possible

This is done using induction



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