INTRODUCTION TO DISCRETE STRUCTURES

6.2 Sequences, recurrences & countability

1.1–1.2

INTRODUCTION TO
DISCRETE STRUCTURES

6.2 sequences, recurrences and countability

- sequences
- recurrences
- countability

Definition of a sequence

A sequence is a function from a subset of integers (usually N or N+) to a set S. We use the notation a_n to denote the image of n. We call a_n , the n^{th} term of the sequence.

Examples of sequences.

1. $a_n = 2n$

2. $a_n = 1/n$

3. $a_n = (-1)^n$

Some useful sequences

Many algorithm analysis tasks end up in describing the number of operations performed in terms on the data size n. In those cases, it is good to know how the summations look like.

Here are some standard sequences define by $a_n = f(n)$ where f(n) is the function n^2, n^3, …

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Geometric sequences

Often it is helpful to know the sum of a geometric series as given below.

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term* a and the *common ratio* r are real numbers.

The sum is given by

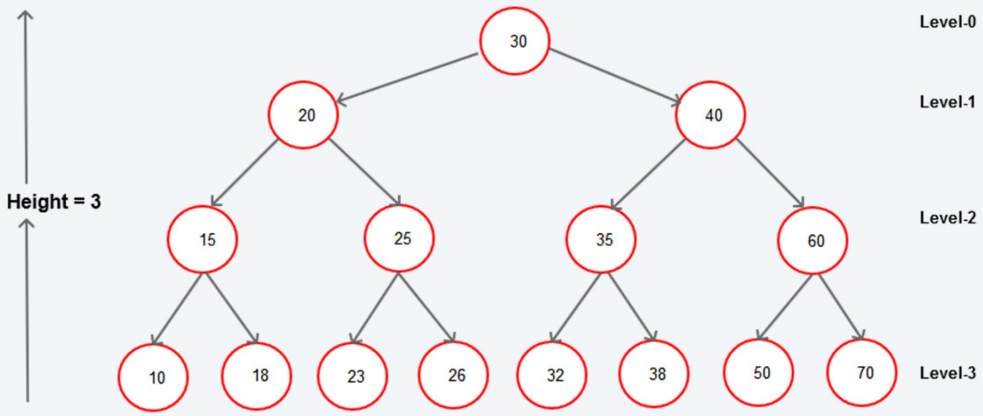
If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Applications

A binary tree is a tree structure, where each node has at most 2 children.

Show that the max number of nodes in the lowest level of a binary tree is almost equal to the max number of nodes in all prior levels combined.



Notation for describing summations

All of the following notations describe sum of a sequence within certain bounds.

$$\sum_{j=m}^{n} a_j , \qquad \textstyle\sum_{j=m}^{n} a_j , \qquad \text{or} \qquad \textstyle\sum_{m \le j \le n} a_j$$

Here are some useful summations.

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Similarly we can define double summations.

workshop

1. Count the total number of operations in insertion sort (worst case)

2. Count the total number of operations in binary search

6.2 sequences, recurrences and countability

- sequences
- recurrences
- countability

Recurrence relation

A recurrence relation for the sequence an is an equation that expresses an in terms of one or more of the previous terms of the sequence. Namely, a0, a1, …., a(n-1) for all integers n with n >= n0 where n0 is a non-negative integer. A sequences is called a solution of a recurrence relation, if it satisfies the recurrence relation. A recurrence relation is a recursively defined sequence.

Example. Newton's formula for finding square root of a.

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

Other examples

Mortgage rates are calculated as a recurrence relation given by

P_0 = initial balance

$P_n = P_{n-1} + r*P_{n-1}$

Fibonacci sequence is also given as a recurrence formula

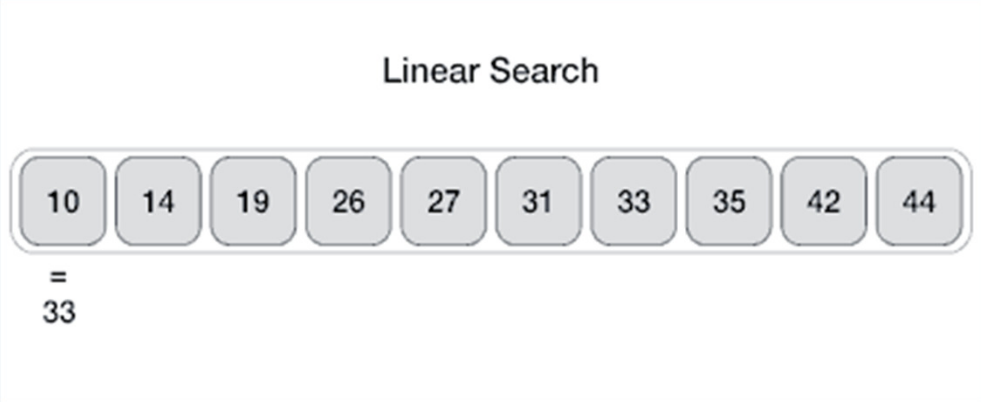
The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $f_0 = 0$, $f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \ldots$.

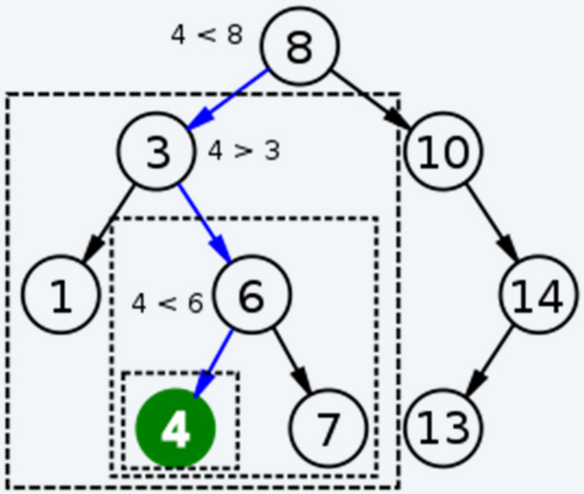
Solving recurrences

1. Linear search



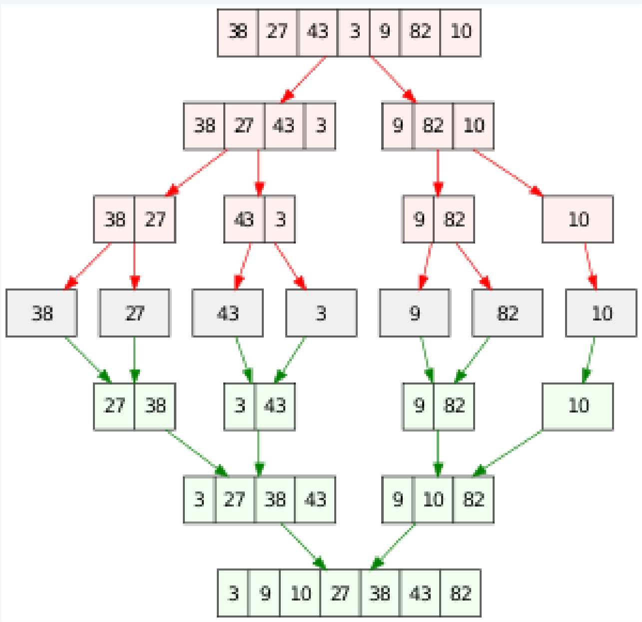
Solving recurrences

Binary search



Solving recurrences

Merge sort



6.2 sequences, recurrences and countability

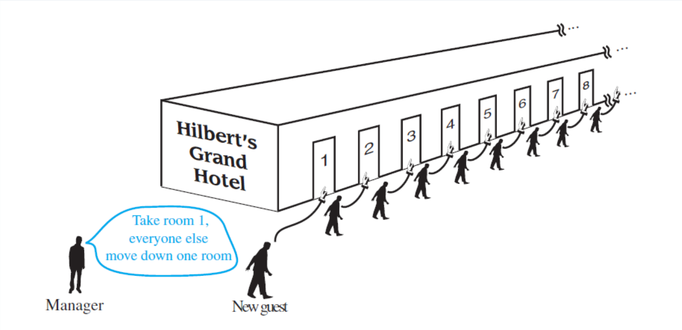
- sequences
- recurrences
- countability

Cardinality of a set

Cardinality of a set is defined as the number of elements in the set, when the set is finite

Cardinality of a set is called countable, if there exists a 1-1 and onto (bijection) between set and N+

Example. Set of odd positive numbers are countable



Example. Rational numbers are countable.

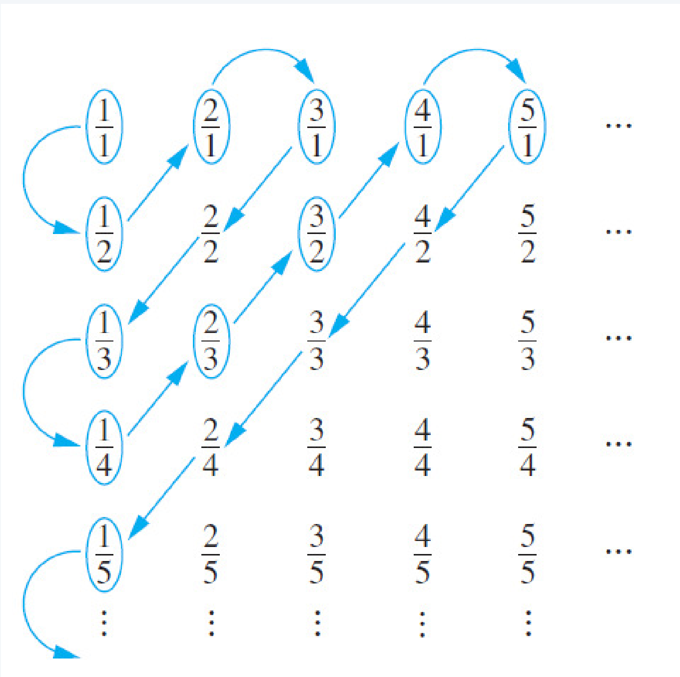
A set that is not countable is called uncountable. For example, R is uncountable.

Exericse

If A and B are countable sets, then $A \cup B$ is also countable

Proof (by cases)

Rational Numbers are Countable



Set of Real Numbers are uncountable

Suppose that the set of real numbers is countable

Then the set of all real numbers between 0 and 1 is also countable

Assume that the real numbers between 0 and 1 are given by r1, r2, r3, etc..

$$
\begin{aligned}
r_1 &= 0.d_{11}d_{12}d_{13}d_{14} \ldots \\
r_2 &= 0.d_{21}d_{22}d_{23}d_{24} \ldots \\
r_3 &= 0.d_{31}d_{32}d_{33}d_{34} \ldots \\
r_4 &= 0.d_{41}d_{42}d_{43}d_{44} \ldots \\
&\vdots
\end{aligned}
$$

Define

$r = 0.d_1d_2d_3d_4 \ldots$, where the decimal digits are determined by the following rule:

$$d_i = \begin{cases} 4 \text{ if } d_{ii} \neq 4 \\ 5 \text{ if } d_{ii} = 4. \end{cases}$$

exercise

Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

exercise

Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) the integers greater than 10

b) the odd negative integers

c) the integers with absolute value less than 1,000,000

d) the real numbers between 0 and 2

e) the set $A \times \mathbf{Z}^+$ where $A = \{2, 3\}$

f) the integers that are multiples of 10

INTRODUCTION TO DISCRETE STRUCTURES

6.2 sequences, recurrences and countability

- sequences
- recurrences
- countability

INTRODUCTION TO DISCRETE STRUCTURES

6.2 Sequences, recurrences & countability

1.1–1.2