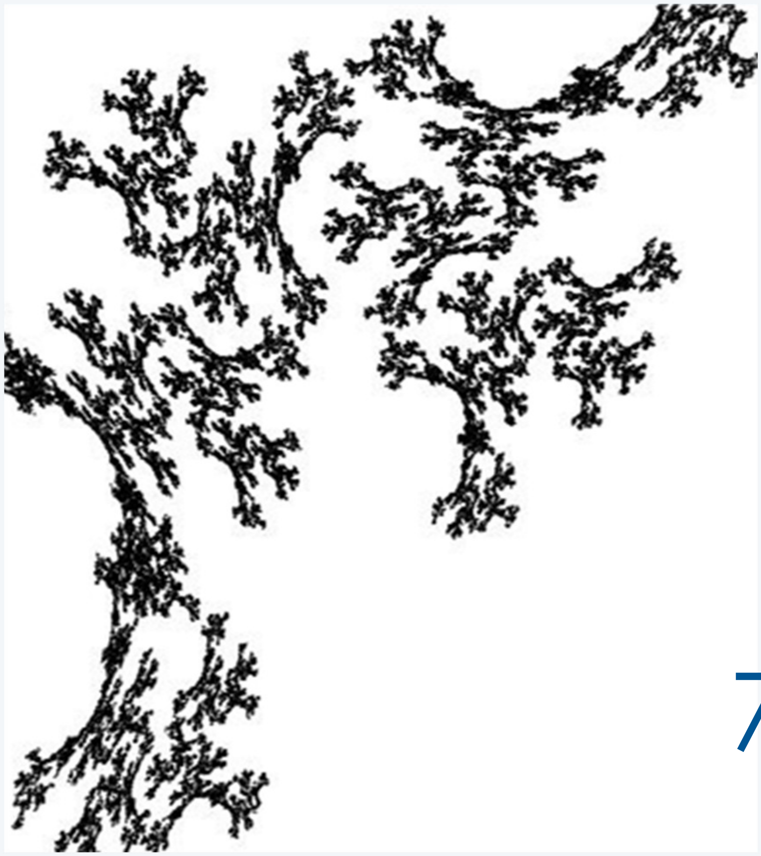
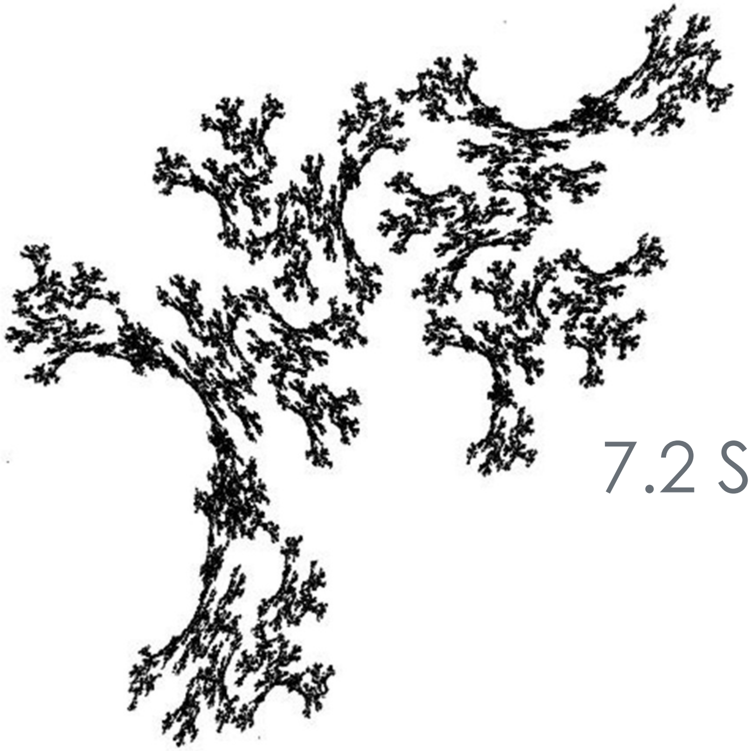


INTRODUCTION TO DISCRETE STRUCTURES

# 7.2 Strong Induction

1.1–1.2



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# 7.2 Strong induction

- Well-ordering principle
- Proofs using well-ordering principle
- What is strong induction?
- Proofs using strong induction

# Well-ordering principle

Well-ordering principle. Every non-empty subset of non-negative integers has a least element.

## Rules for using well-ordering principle to prove propositions

To prove that "$P(n)$ is true for all $n \in \mathbb{N}$" using the Well Ordering Principle:

- Define the set, $C$, of *counterexamples* to $P$ being true. Namely, define[1]

$$C ::= \{n \in \mathbb{N} \mid P(n) \text{ is false}\}.$$

- Assume for proof by contradiction that $C$ is nonempty.
- By the Well Ordering Principle, there will be a smallest element, $n$, in $C$.
- Reach a contradiction (somehow) —often by showing how to use $n$ to find another member of $C$ that is smaller than $n$. (This is the open-ended part of the proof task.)
- Conclude that $C$ must be empty, that is, no counterexamples exist. QED

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## Example Proof using Well-ordering principle

Prove that a fraction m/n for any positive integers can always be written in lowest form.

Proof. (by contradiction)

# Example proof using well-ordering principle

Prove that $2^n > n$ for all $n \geq 1$

Proof:

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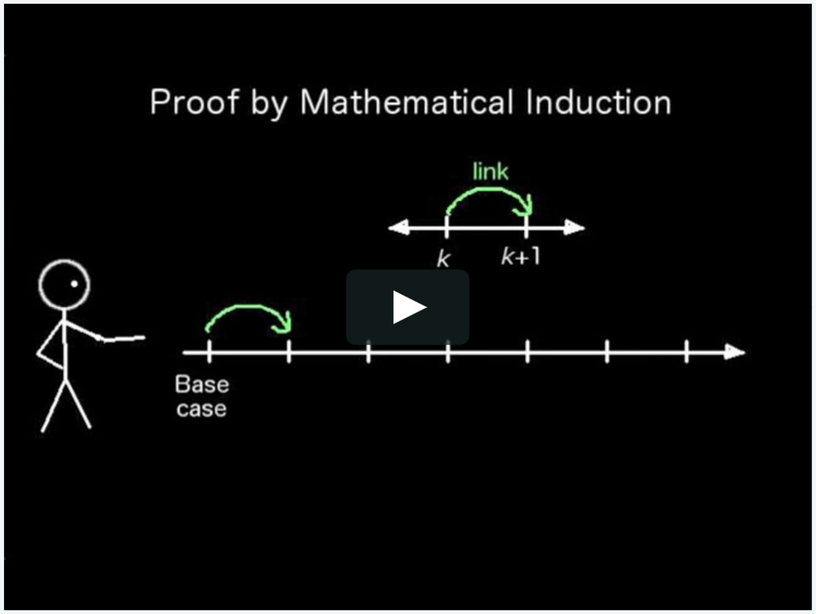
# Ordinary induction vs strong induction

**Ordinary/weak induction**

- Rule 1: P(0) (or any other base case)
- Rule 2: P(n) ➔ P(n+1)

**Strong induction**

- Rule 1: P(0) (or any other base case)
- Rule 2: P(1),P(2), P(3),….P(n) ➔ P(n+1)



# Equivalence of ordinary induction and strong induction

Strong induction is a variant of weak induction. In fact the two ideas can be shown to be equivalent.

Then why strong induction?

Because in some problems, having a stronger assumption help prove the proposition P(n+1)

The general rule.

1. If P(n+1) can be proven from P(n) only, then weak/ordinary induction is sufficient
2. If P(n+1) requires other propositions prior to P(n) (e.g. P(n-1) or P(n-2)) then strong induction may be appropriate.

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## A proof using strong induction

Prove that any integer n greater than 1 can be expressed as a product of primes.

That is, given $n \geq 2$, we can express $n = p_1.p_2.p_3....p_k$ where each $p_i$ is prime.

**Proof:**

# Workshop

Suppose we have the following sequence:

- $a_1 = 1$
- $a_2 = 3$
- $a_k = a_{k-2} + 2a_{k-1}$, for all integers $k \geq 3$

For all integers $n \geq 1$, $P(n)$: Given the sequence $a_1$, $a_2$, ..., $a_k$ as defined above, $a_n$ is odd.

Prove that every term in this sequence is odd.

## Solution to workshop

We know the base cases P(1) and P(3) are true since 1 and 3 are odd numbers

Assume that P(k) is true for k=1,2….n (inductive hypothesis)

We want to show that P(n+1) is then true

**P(n+1) :**

- show $a_{(n+1)}$ is odd
- But $a_{(n+1)} = a_{(n-1)} + 2.\ a_{(n)}$ (by definition)
- by induction hypothesis, $a_{(n-1)}$ and $a_{(n)}$ are odd and so we can express them as 2k+1 and 2l+1.
- Hence $a_{(n+1)} = 2k + 4l + 2 + 1 = 2(k + 2l + 1) + 1 = 2p + 1$
- Implies $a_{(n+1)}$ is also odd.

## Postage problem

Theorem. Every amount of postage that is at least 12 cents can be made from 4-cent and 5-cent stamps

For example, 12 cents uses three 4-cent stamps. 13 cents of postage uses two 4-cent stamps plus a 5-cent stamp. 14 uses one 4-cent stamp plus two 5-cent stamps. If you experiment with small values, you quickly realize that the formula for making k cents of postage depends on the one for making k – 4 cents of postage. That is, you take the stamps for k – 4 cents and add another 4-cent stamp. We can make this into an inductive proof as follows:

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