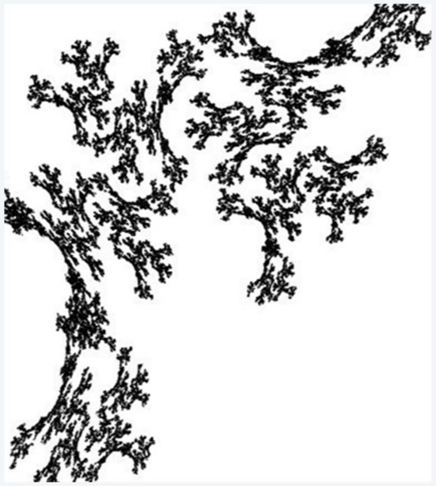


INTRODUCTION TO
DISCRETE STRUCTURES

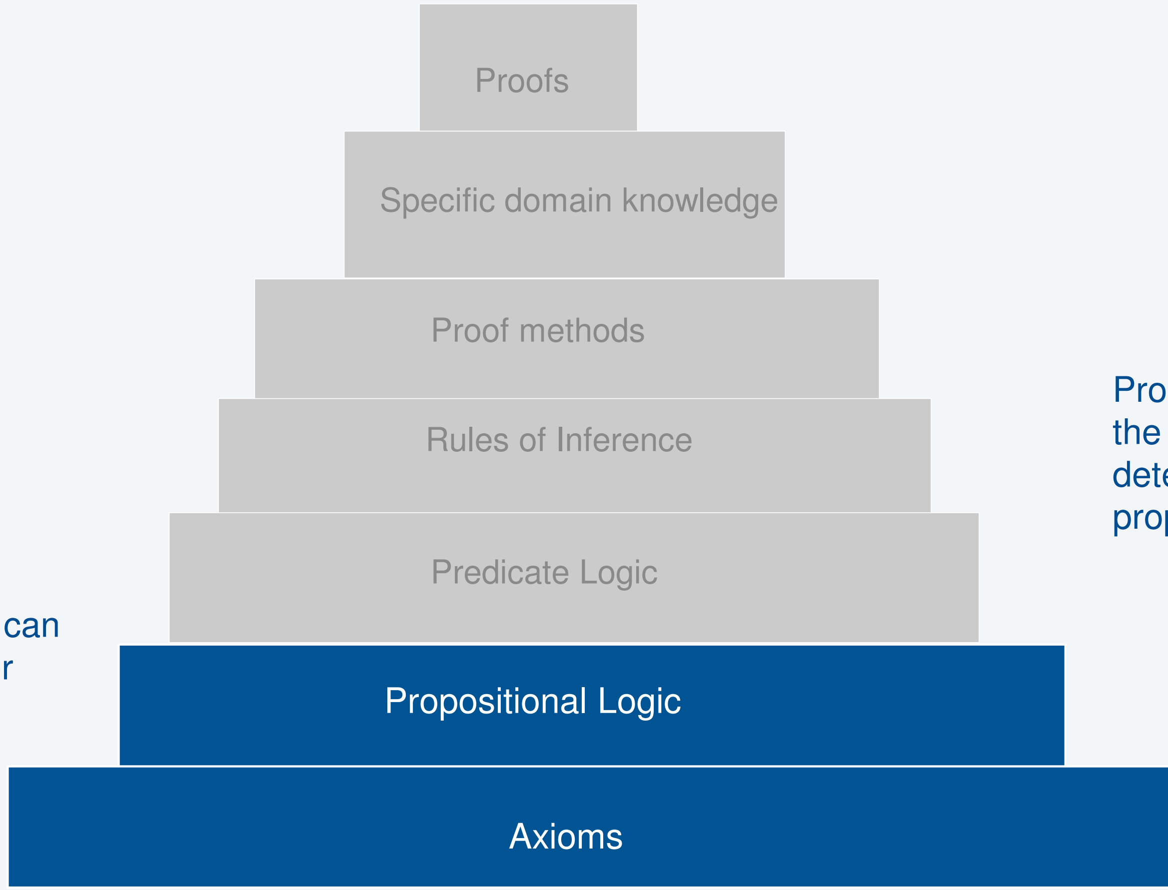
A. Propositional Logic

1.1–1.2

A. Propositional Logic

- Atomic and compound propositions
- Logical Operators: AND, OR, Exclusive OR, Negation
- Satisfiability
- Implications

The foundation for writing proofs



Propositional logic is the steps we use to determine the truth of a proposition

Propositions are statements that we can determine as true or false.

Axioms are assumptions we may use without proof

Propositions

- A proposition is a statement that can be determined to be true or false.
- Propositions do not contain variables that need to be defined.
- If a statement, such as an imperative or question, cannot be determined true or false, it is not a proposition.

Example: 7 is prime.

This is a proposition since we can verify that 7 is prime.

Example: 2 is odd.

This is a proposition since we can verify that 2 is even.

Example: Please walk to the metro.

This is not a proposition. It is an imperative.

Example: If the temperature is 200F, you win one million dollars.

This is a vacuously true proposition (sadly we cannot collect that money).

Example: n+1 is even.

This is not a proposition. Unless we know what value n is we cannot determine its truth.

Example: What is your favorite color?

This question is not a proposition as we cannot determine if it is true or false.

Atomic and Compound Propositions

- A single proposition may be represented as a propositional variable.
- An atomic proposition is a single proposition.
- A compound proposition is a proposition that combines atomic propositions using logical operators.

Atomic Proposition using propositional variable p: p = "7 is prime. "

Atomic Proposition using propositional variable q: q = "25 is divisible by 2."

Compound proposition using and operator:

$$p \wedge q$$

7 is prime and 25 is divisible by 2.

Compound proposition using or operator:

$$p \vee q$$

7 is prime or 25 is divisible by 2.

1.2 Propositional Logic

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Logical Operator: AND

- A truth table assigns T or F to each proposition and calculates the truth assignment of the entire compound proposition.
- Logical and is also called conjunction.
- Given two atomic propositions, p, q, let's construct truth table for the conjunction.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Logical Operator: OR

- A truth table assigns T or F to each proposition and calculates the truth assignment of the entire compound proposition.
- Logical or is also called disjunction.
- Given two atomic propositions, p, q, let's construct truth table for the disjunction.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical Operator: Exclusive OR

- A truth table assigns T or F to each proposition and calculates the truth assignment of the entire compound proposition.
- Exclusive OR is true if exactly one of the propositions is true.
- Given two atomic propositions, p, q, let's construct truth table for the exclusive OR.

True Example: p = "2 is prime", q = "25 is divisible by 3"

False Example: p = "2 is prime", q = "25 is divisible by 5"

p	q	$p \otimes q$	
T	T	F	False Example
T	F	T	True Example
F	T	T	
F	F	F	

Logical Operator: Negation

- Observe either a proposition is true or its negation is true.
- Let's construction the truth table for negation and double negation.
- We'll see to prove a statement is false, show its negation is true. We do this when we give counterexamples.

False Example: p = "All prime numbers are odd."

Negation of p = "There is a prime number that is not odd."
Counterexample: 2 is prime and even.

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

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Satisfiability

- A compound proposition is satisfiable if and only if there exists at least one assignment to the propositional variables that makes the entire proposition true
- In a truth table this is a line that is true.
- For example, exclusive OR had two such assignments.

p	q	$p \otimes q$
T	T	F
T	F	T
F	T	T
F	F	F

Satisfiability

- A compound proposition is satisfiable if and only if there exists at least one assignment to the propositional variables that makes the entire proposition true
- A tautology is true for every assignment to the variables
- For any proposition, the following is a tautology:

Example: p = "There was a blizzard on April 1, 1997 in Cambridge, MA."

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Satisfiability

- A compound proposition is satisfiable if and only if there exists at least one assignment to the propositional variables that makes the entire proposition true
- There exists compound propositions that are unsatisfiable.
- For any proposition, the following is unsatisfiable:

Example: p = “There was a blizzard on April 1, 1997 in Cambridge, MA.”

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Consistency

- A set of compound propositions are consistent if there exists truth
- assignment such that all propositions are satisfied at the same time.
- First, we will show how two compound propositions using the atomic propositions, p and q, are consistent.
- Then we will introduce a third proposition to show when they are not longer consistent.

Example: p = “There was a heatwave in Los Angeles in July 2019. ”
q = “There was a heatwave in London in July 2019.”

p	q	$p \vee q$	$p \vee \neg q$	$\neg p$
T	T	T	T	F
T	F	T	T	F
F	T	T	F	T
F	F	F	T	T

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Implications

- An implication is a compound proposition using propositions, p and q, stated as if p then q or p implies q.
- p is called the premise or sufficient condition for q.
- q is called the conclusion or necessary condition for p.
- Let's draw the truth table for the implication:

Example: p = "There was a day in London in July 2019 with a temperature of 110 F. "
q = "There was a record high temperature in London in July 2019."

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Record London temperature July 2019: 101F

Vacuously true!

Implications: Converse

- Give an implication, p implies q, the converse is q implies p.
- If an implication is true, its converse need not be true.
- Let's consider the true table for each and an example

Example: Let x and y be real numbers with y not equal to 0.
p = "x is rational and y is rational"
q = "x/y is rational."

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

$$x = y = \sqrt{2}$$

Implications: Contrapositive

- Give an implication, p implies q, the contrapositive is not q implies not p.
- If an implication is true, its contrapositive is true and vice versa.
- Let's consider the true table for each and an example

Example: Let x and y be real numbers with y not equal to 0.
p = "x is rational and y is rational" negation of p: "x is not rational or y is not rational."
q = "x/y is rational." negation of q: "x/y is not rational."

p	q	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Implications: Inverse

- Give an implication, p implies q, the inverse is not p implies not q.
- If an implication is true, its inverse need not be true.
- If the converse is true, the inverse is true and vice versa.

Example: Let x and y be real numbers with y not equal to 0.

p = "x is rational and y is rational" negation of p: "x is not rational or y is not rational."

q = "x/y is rational." negation of q: "x/y is not rational."

p	q	$p \Rightarrow q$	$\neg p \Rightarrow \neg q$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

$$x = y = \sqrt{2}$$

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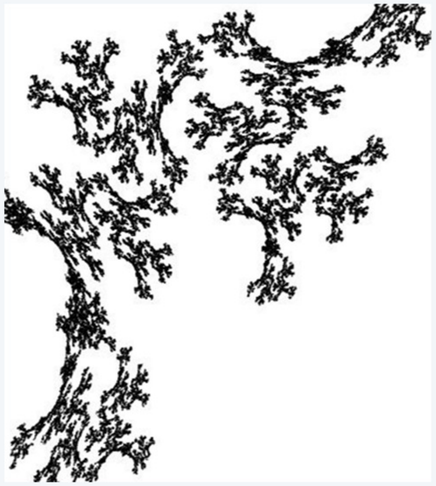

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