Binary Search Trees
Red-Black Trees

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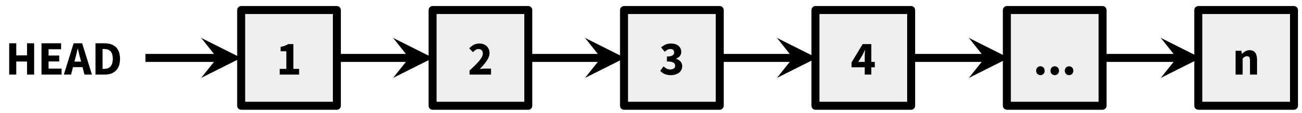
Binary Search Trees

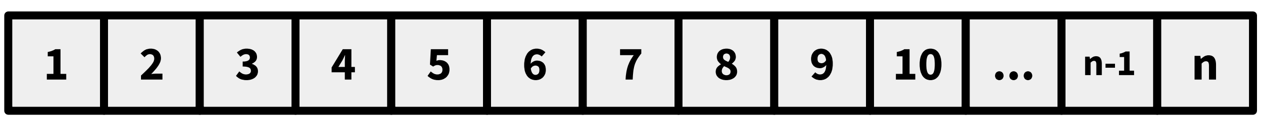
Why BSTs?

Good data structures help us to design more efficient algorithms!



Sorted linked lists: $O(n)$ search/select
$O(1)$ insert/delete

Assuming we already have a pointer to the location of the insert/delete



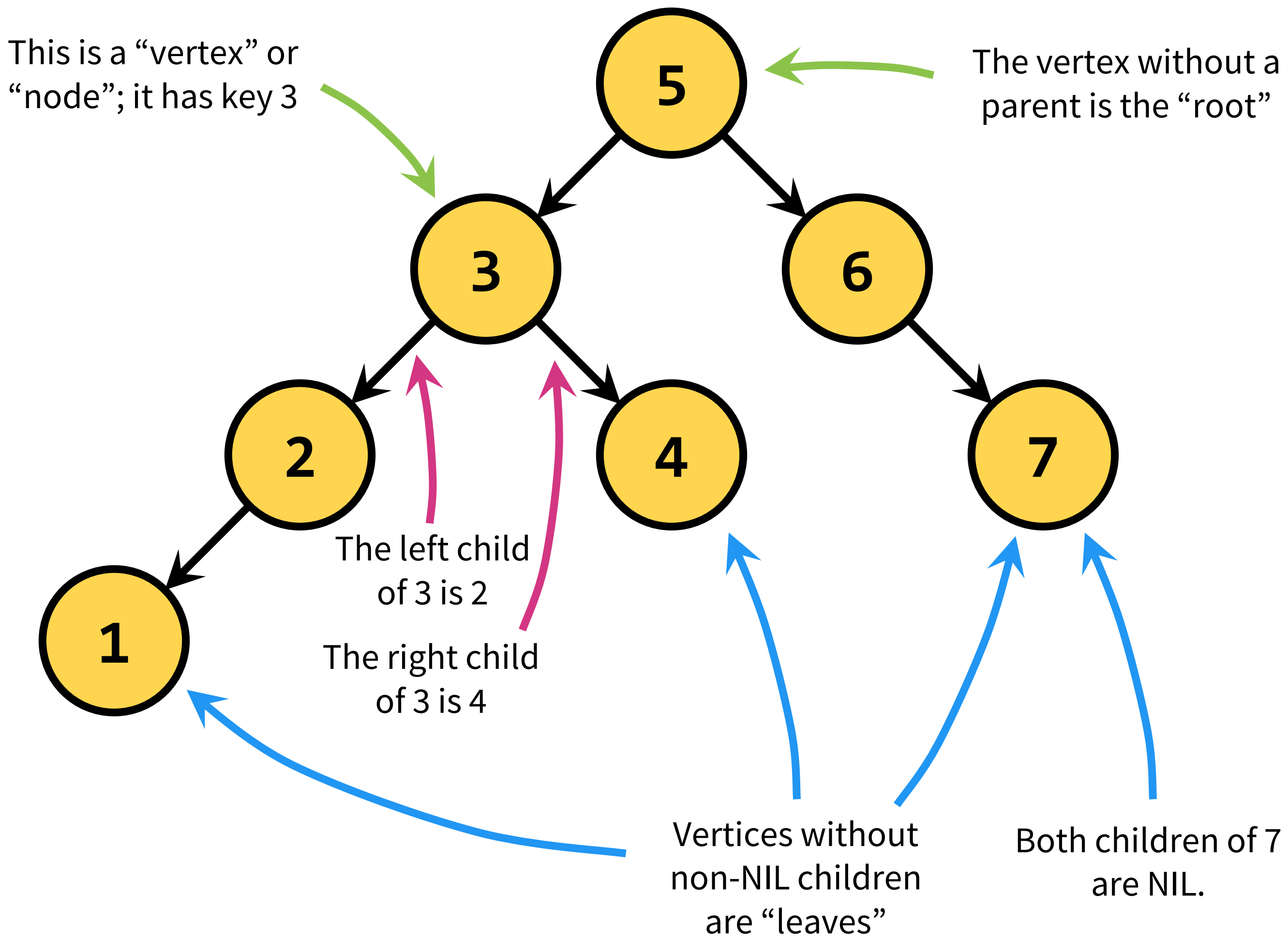
Sorted arrays: $O(\log n)$ search
$O(1)$ select
$O(n)$ insert/delete

"Get the k^{th} smallest element"

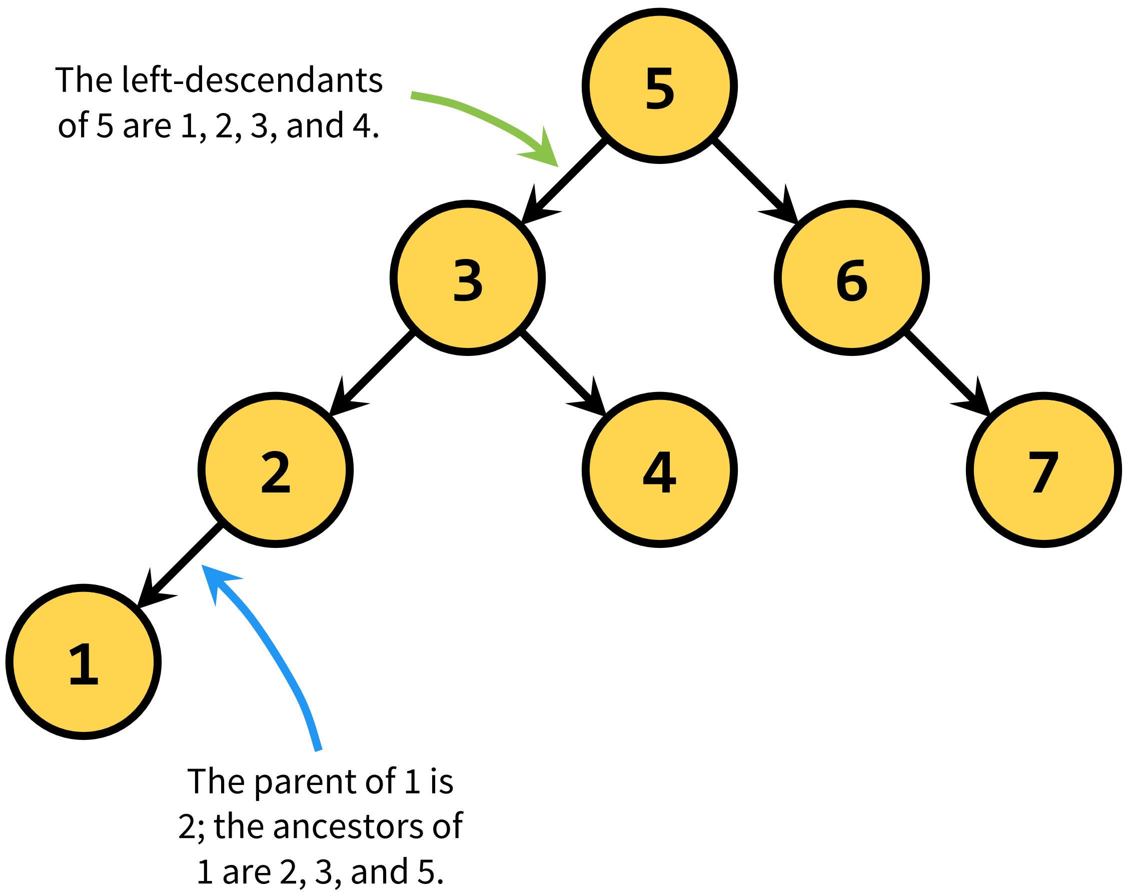
Why BSTs?

	Sorted linked lists	Sorted arrays	Binary search trees
Search	`O(n)`	`O(log n)`	`O(log n)`
Insert/Delete	`O(n)` If we already find the place to insert or delete, then O(1)	`O(n)`	`O(log n)`

Tree Terminology

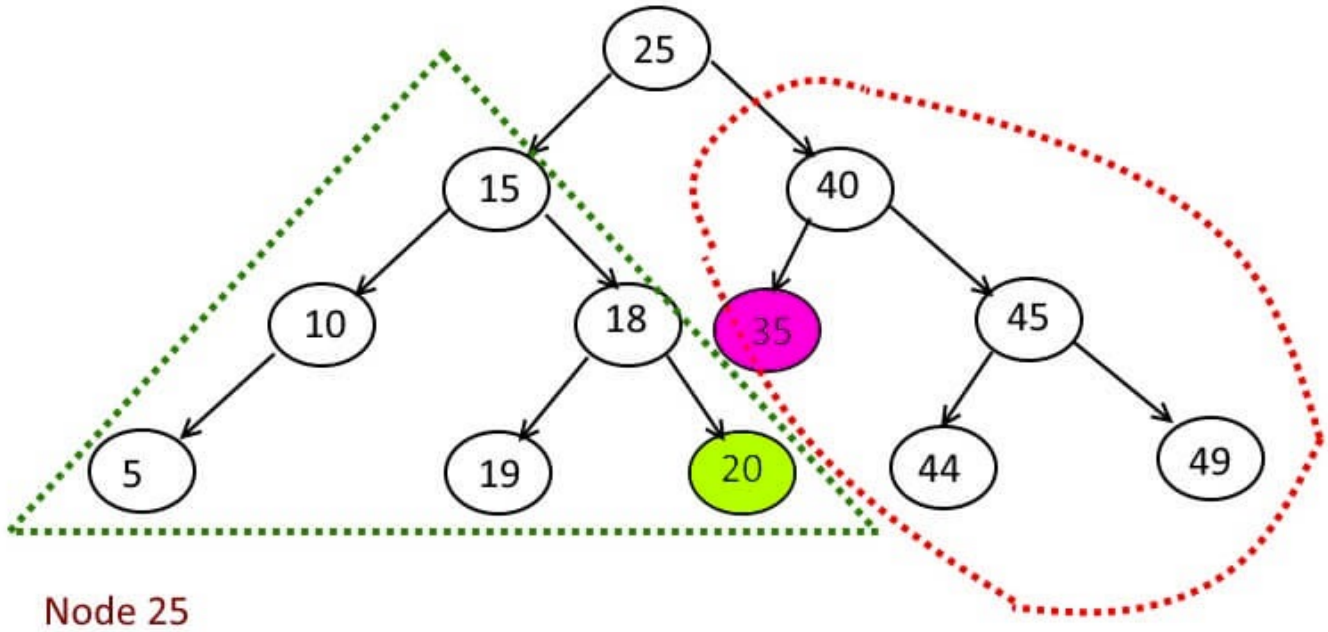


Tree Terminology



Tree Terminology

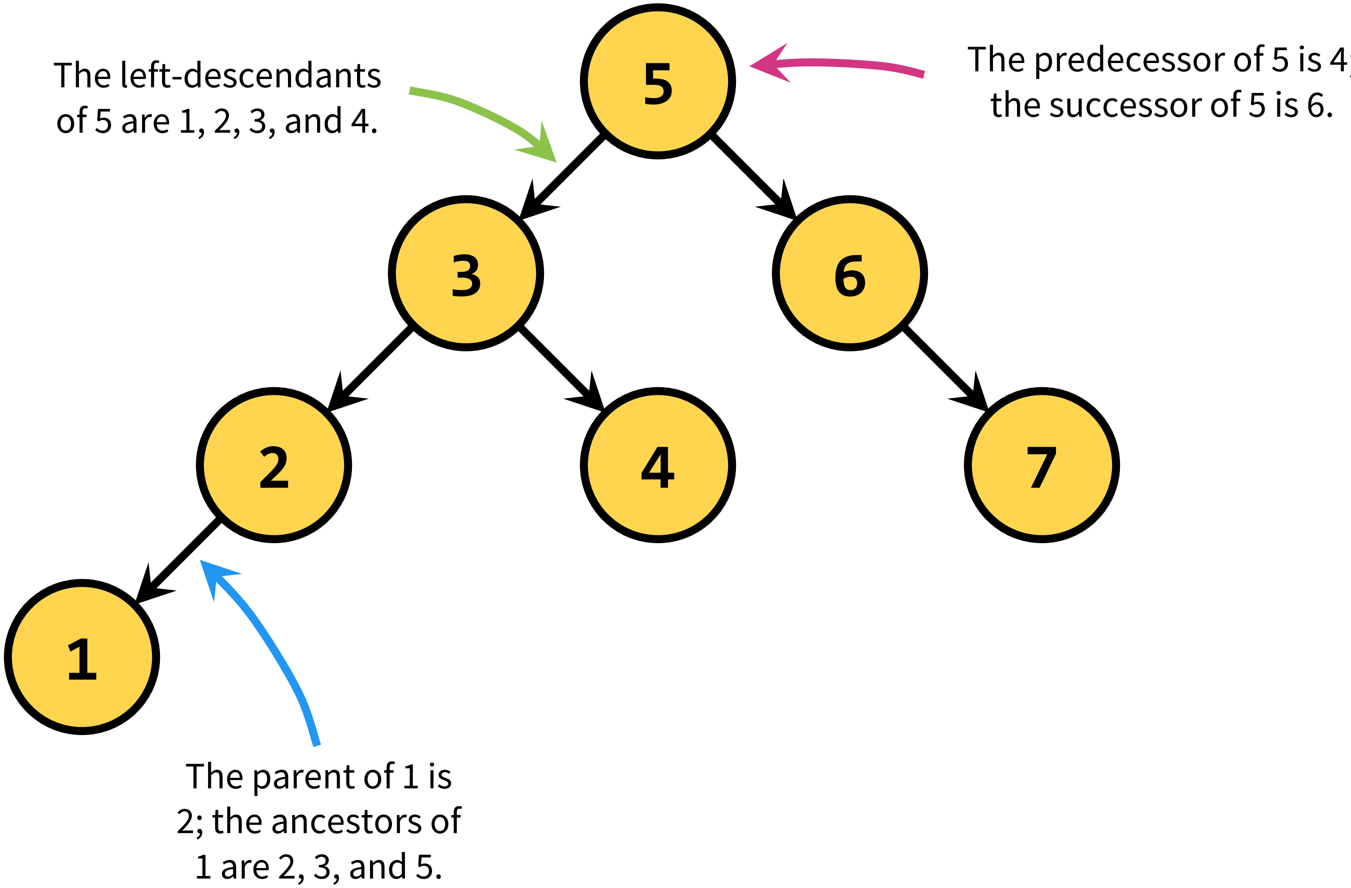
The **Predecessor** of a node is the right most element in its left subtree.
The **Successor** of a node is the left most element in its right subtree.



The predecessor of node 25 is the right most node in its left subtree, which is 20

The successor of node 25 is the left most node in its right subtree, which is 35

Tree Terminology



Binary Search Trees

Binary Trees are trees such that each vertex has at most 2 children.

Binary Search Trees are Binary Trees such that:

- **Every** left descendent of a vertex has key smaller than that vertex.
- **Every** right descendent of a vertex has key greater than that vertex.

Take care: Not only the left and right child, but all the left and right descendants!

Binary Search Trees

Binary Search Trees are Binary Trees such that:

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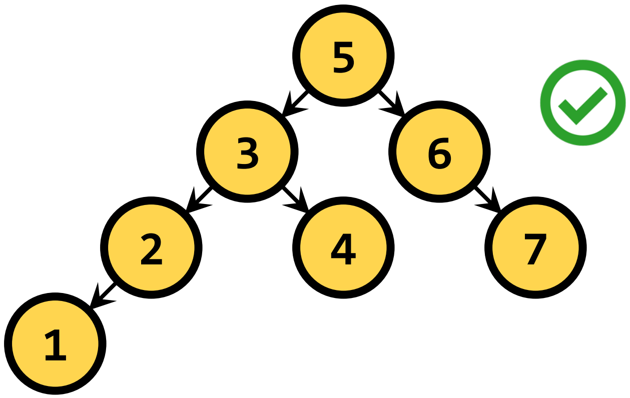
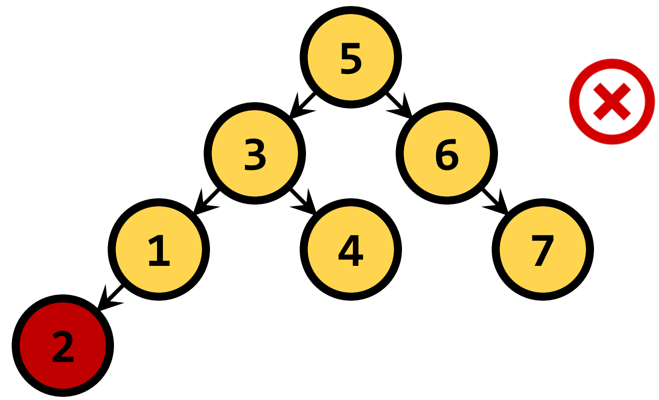
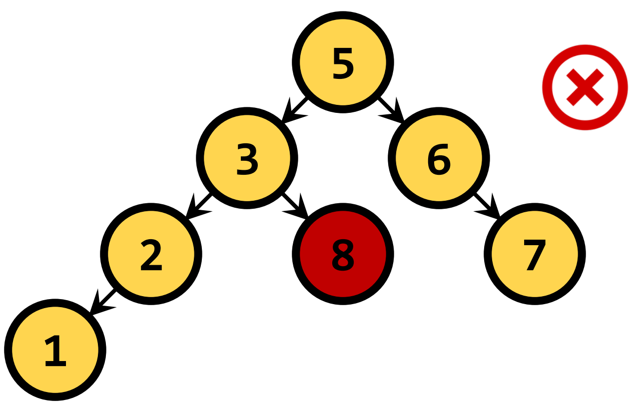
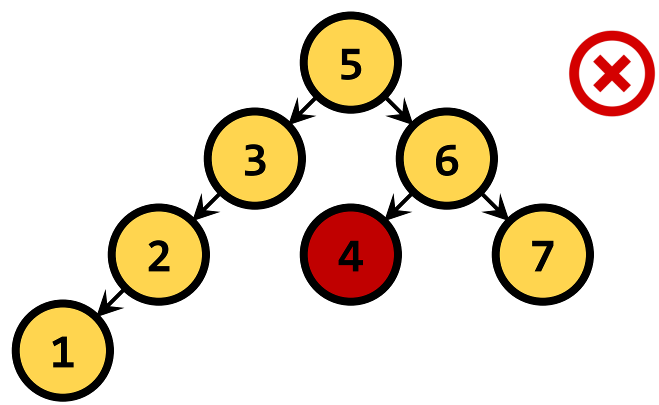
Take care: Not only the left- and right-child, but all the left and right descendants!







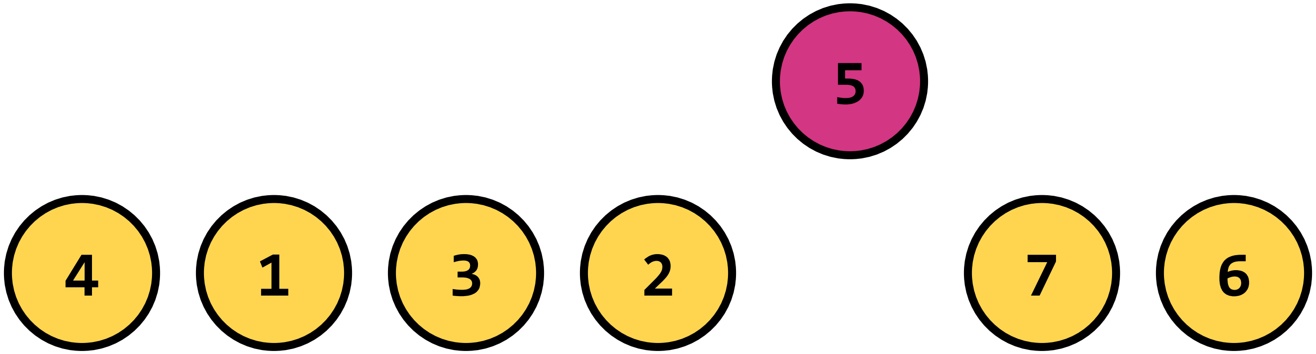


Building BSTs by Example

Building BSTs by Example

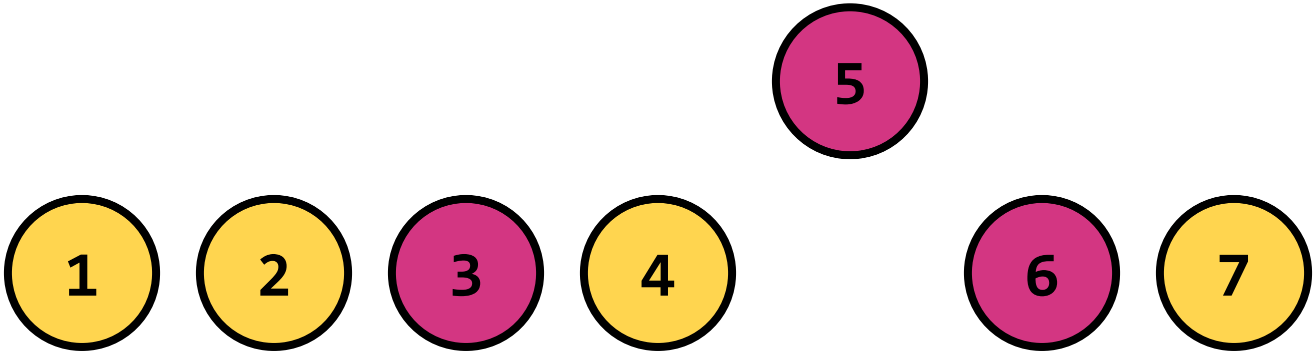
Let's partition about one of the vertices . . .

Building BSTs by Example



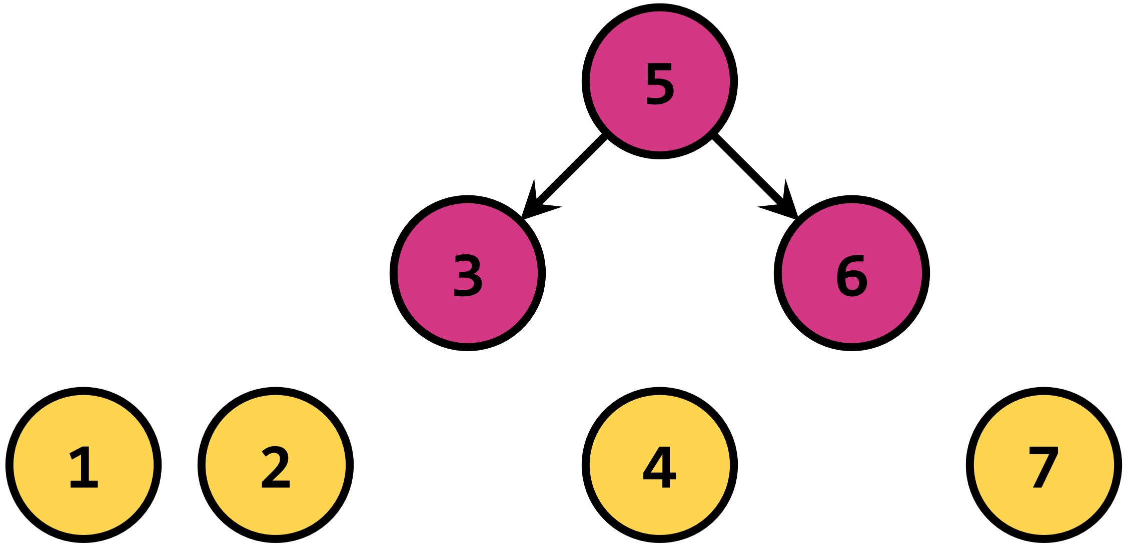
… and build a tree with that vertex as the root.

Building BSTs by Example



Then, recursively build trees out of its descendants.

Building BSTs by Example



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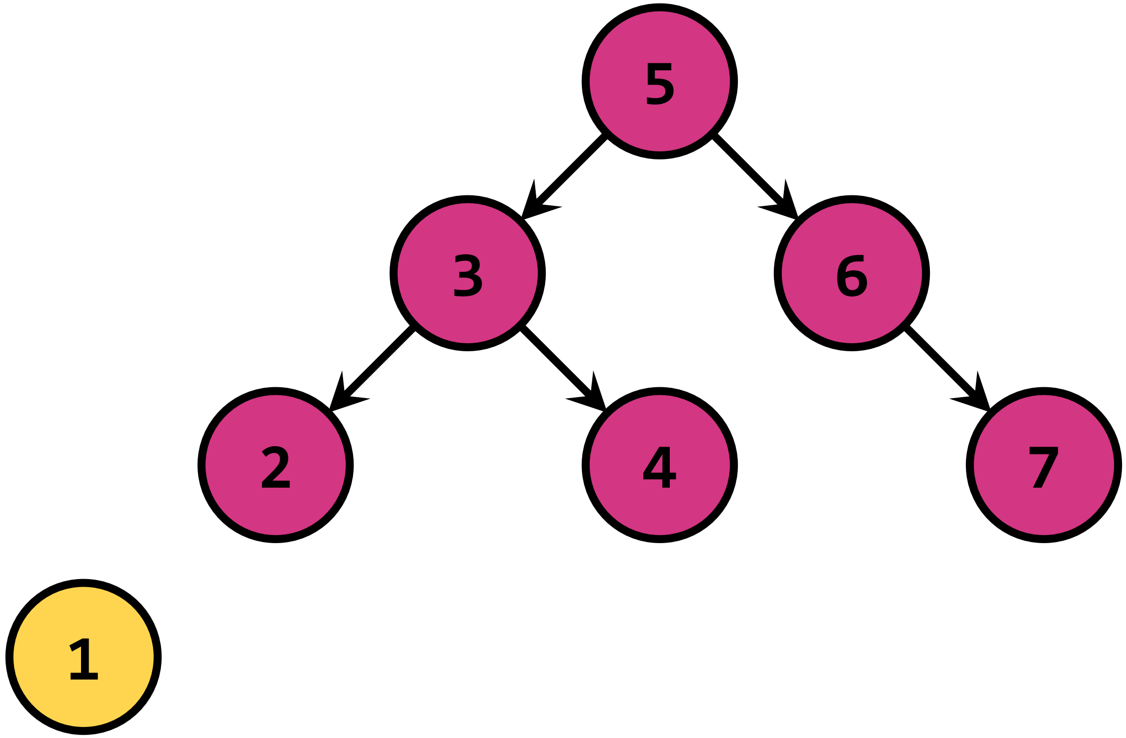
Building BSTs by Example



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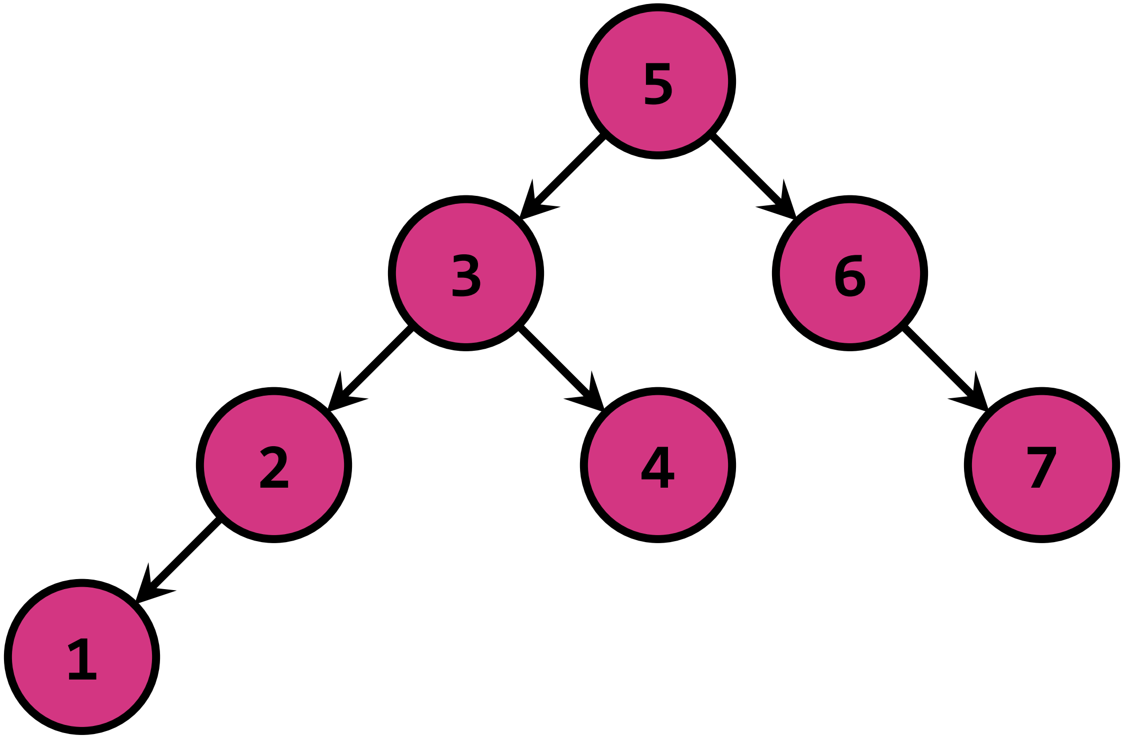
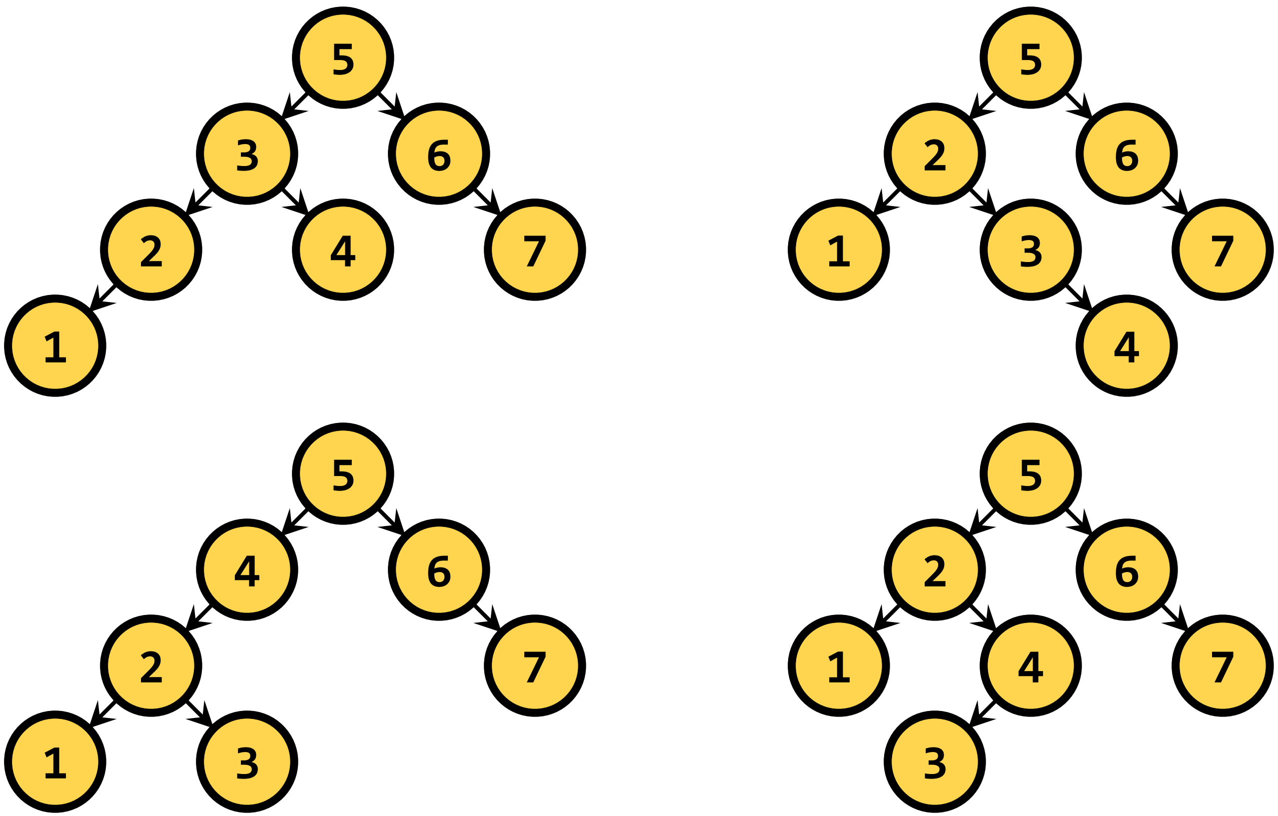
Building BSTs by Example



Then, recursively build trees out of its descendants.

There Exist Many Valid BSTs

Explanation on board: construction of another BST by selecting 4 as the root.



… and many more.

How many? 

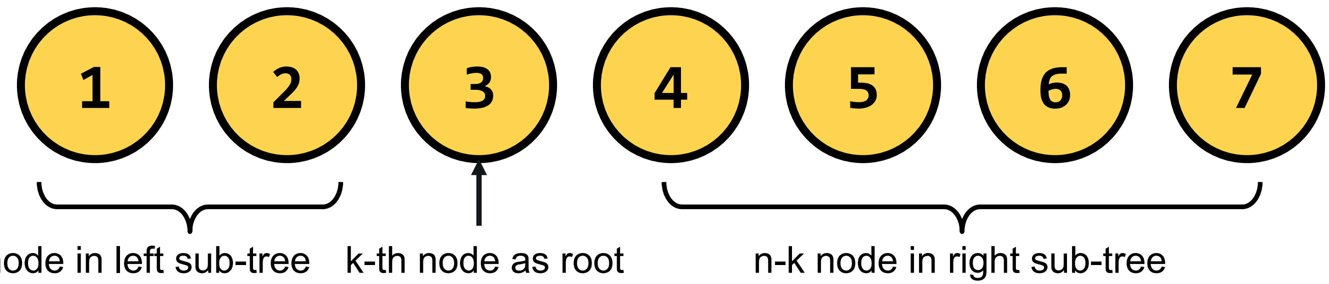
Catalan number:

$$C(n) = \frac{(2n)!}{n!\,(n+1)!}$$

There Exist Many Valid BSTs

Given *n* vertices, how many valid BSTs can we possibly build?

Let C(n) be the number of valid BSTs using n nodes.



C(0) = C(1) = 1
Rewrite C(n) by construction: $C(n) = \sum_{k=1}^{n} C(k-1)C(n-k)$

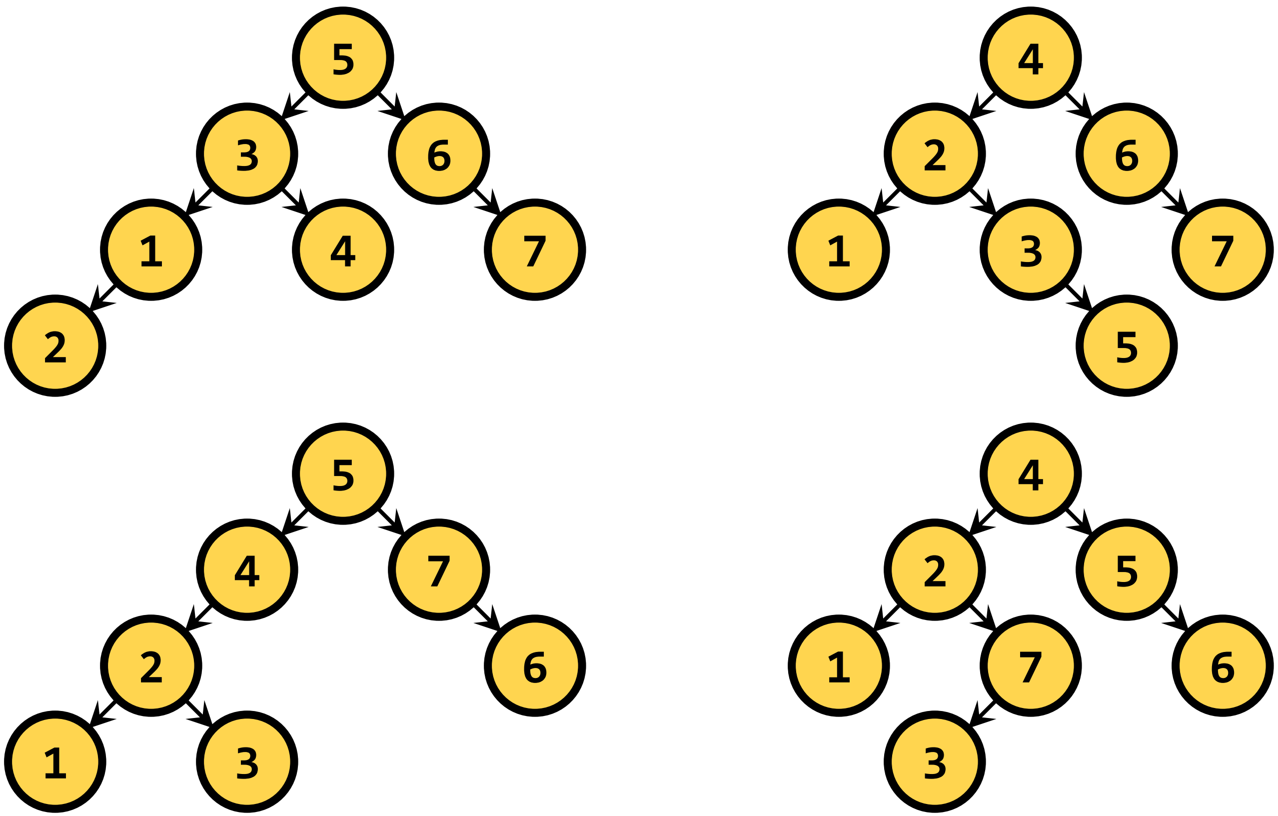
C(n) is a series of numbers defined by the following recurrence relation:

$$C(0) = C(1) = 1$$
$$C(n) = C(0)C(n-1) + C(1)C(n-2) + \cdots + C(n-1)C(0)$$

This is the Catalan series: 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, …
Answer to this series is:

$$C(n) = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}$$

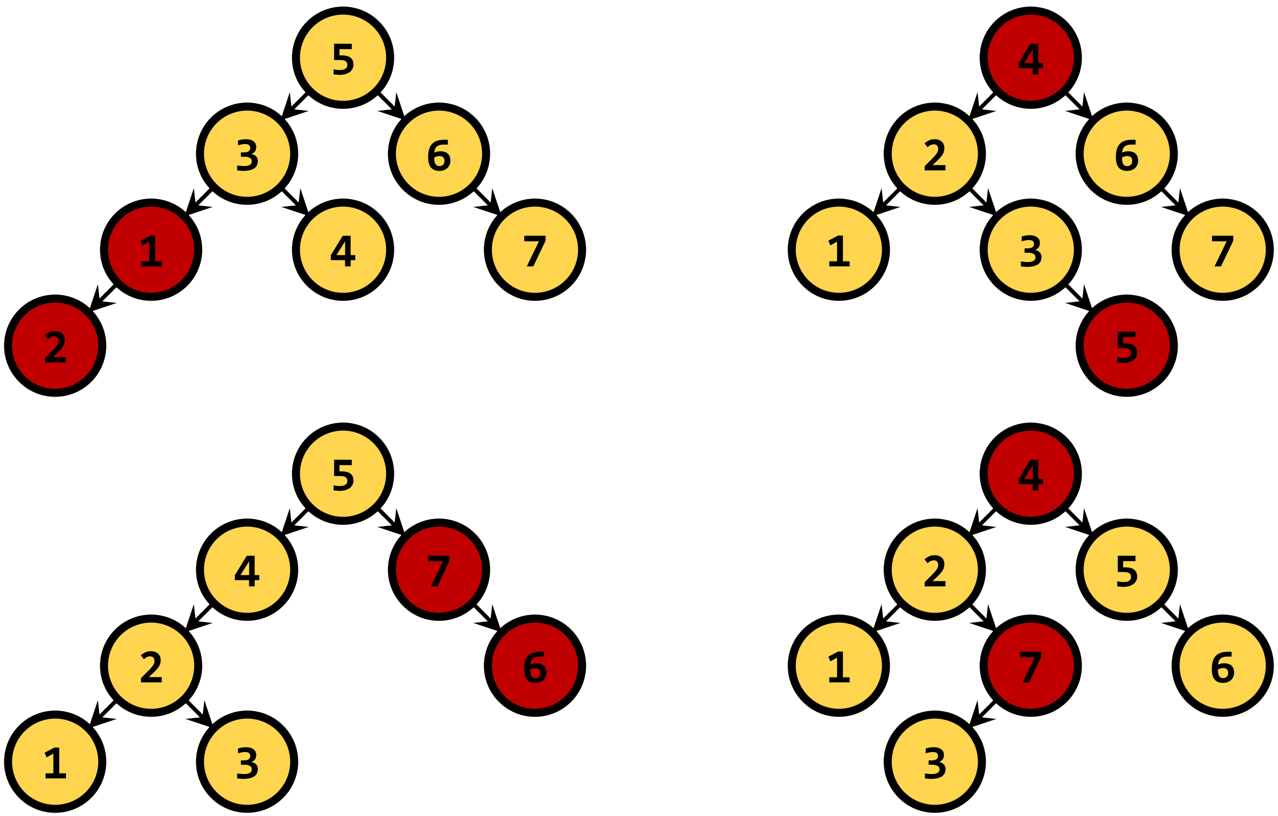
There Exist Many Invalid BSTs



… and many more.

How many? 

There Exist Many Invalid BSTs



… and many more.

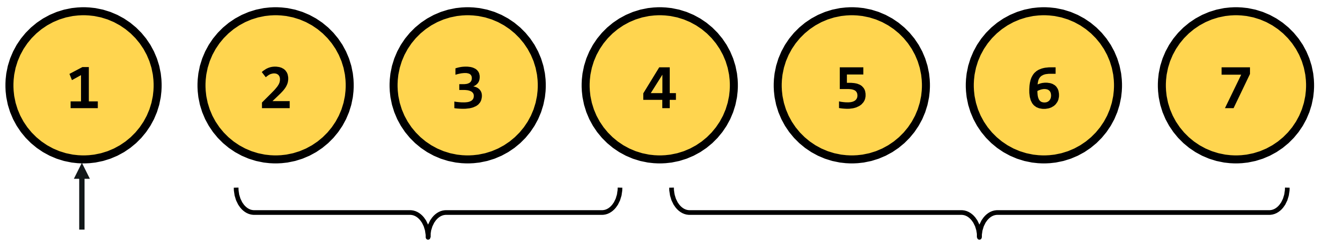
How many? 

$$\frac{(2n)!}{(n+1)!} - \frac{(2n)!}{n!\,(n+1)!}$$

There Exist Many Invalid BSTs

Given *n* vertices, how many BTs can we possibly build?

Let T(n) be the number of binary trees (BTs) using n nodes, no matter being valid or invalid BST.



Any of the n nodes could be selected as root node

Any k nodes from the remaining nodes could be selected for the left sub-tree ($0 \le k \le n-1$)

The remaining n-k-1 nodes will be selected for the right sub-tree

T(0) = T(1) = 1

Rewrite T(n) by construction: $T(n) = \binom{n}{1}\sum_{k=0}^{n-1}\binom{n-1}{k}T(k)T(n-k-1)$

Answer to this series is: $T(n) = \frac{(2n)!}{(n+1)!}$

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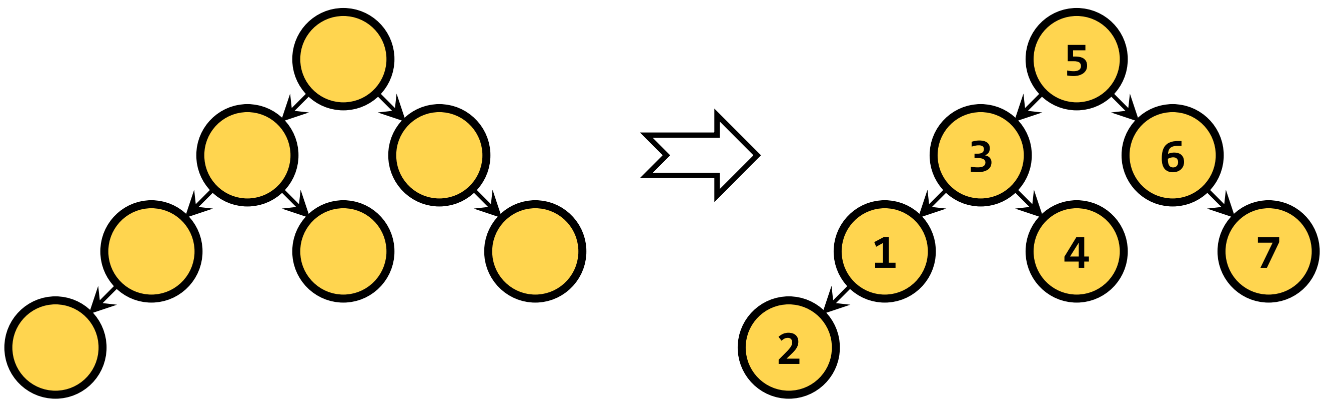
Answer to this series is: $T(n) = \dfrac{(2n)!}{(n+1)!}$

An easier way to get the answer:

Each unique BT structure corresponds to a unique valid BST, e.g., given the following structure, the root must be 5, because there are 4 nodes smaller than root and 4 greater than root. Similar for remaining nodes.

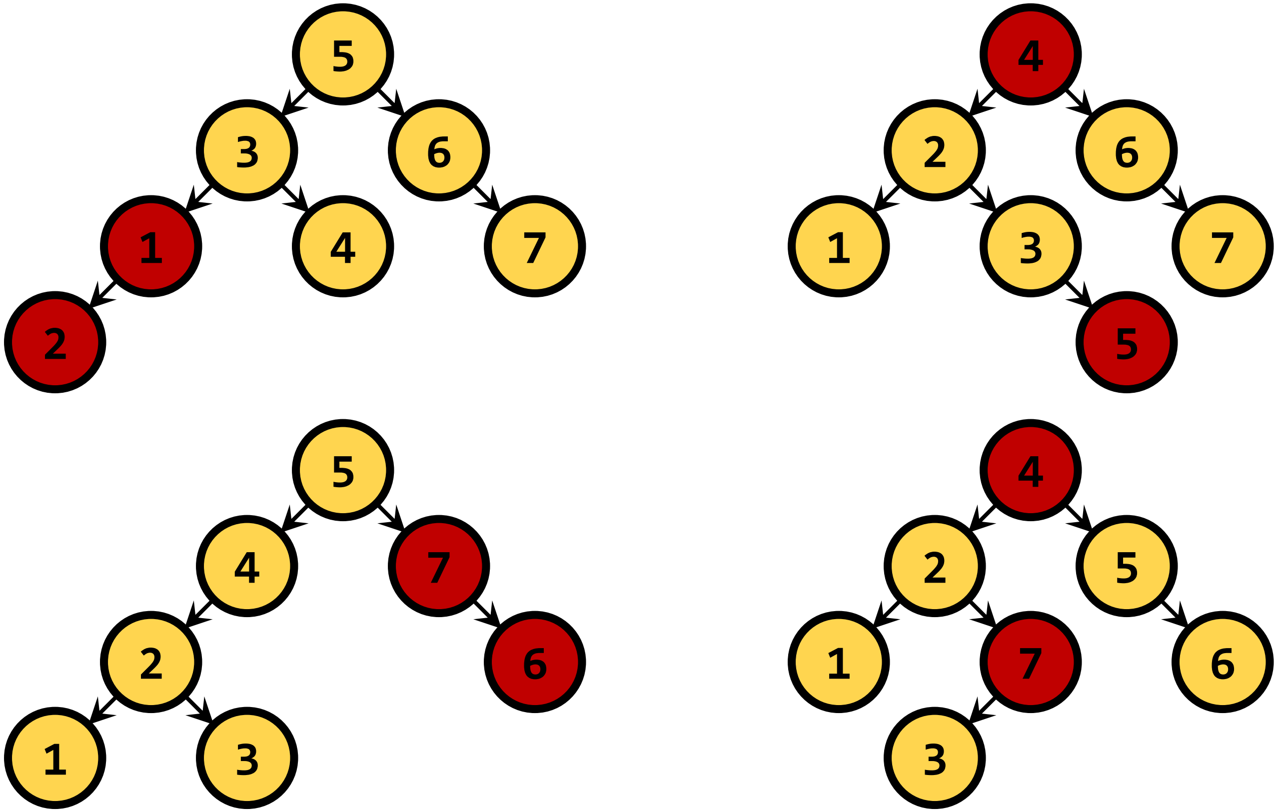


If we do not care about validity of BST, for each BST, we can permute n numbers arbitrarily to create BTs. n numbers will give n! permutations. As a result:

$$T(n) = C(n) \cdot n! = \frac{(2n)!}{n!\,(n+1)!} \cdot n! = \frac{(2n)!}{(n+1)!}$$

There Exist Many Invalid BSTs

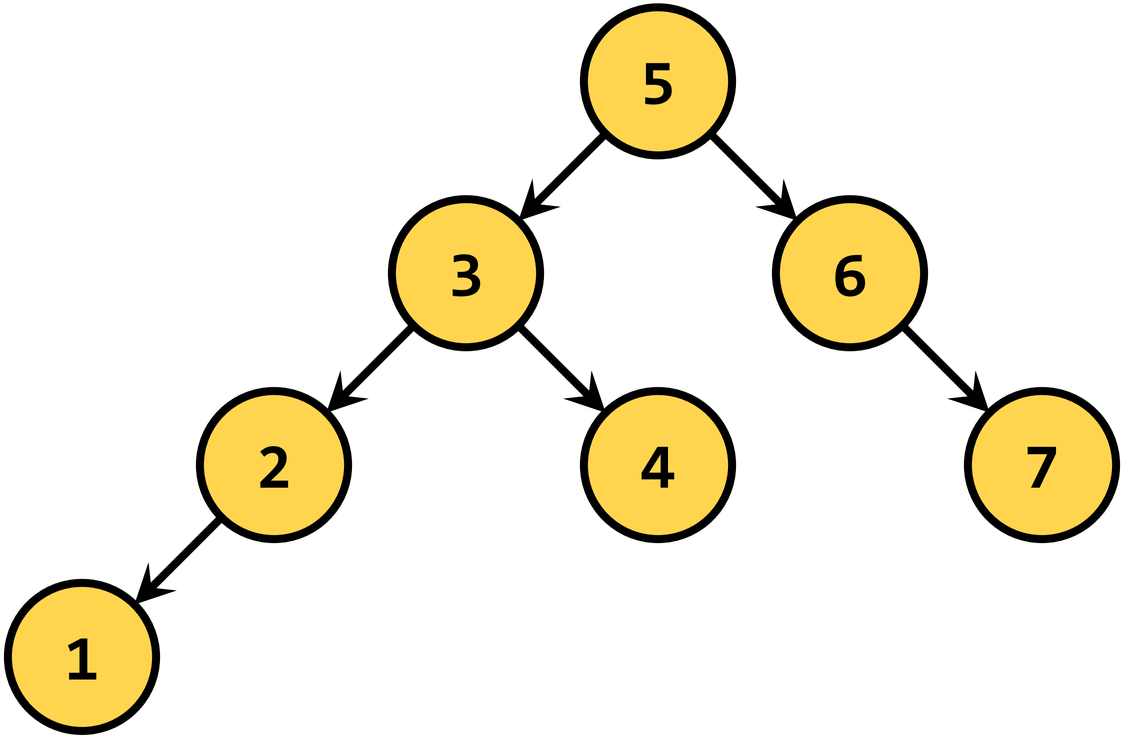
Given *n* vertices, how many invalid BSTs can we possibly build?



Invalid BSTs = # BTs - # Valid BSTs $= T(n) - C(n) = \frac{(2n)!}{(n+1)!} - \frac{(2n)!}{n!\,(n+1)!}$

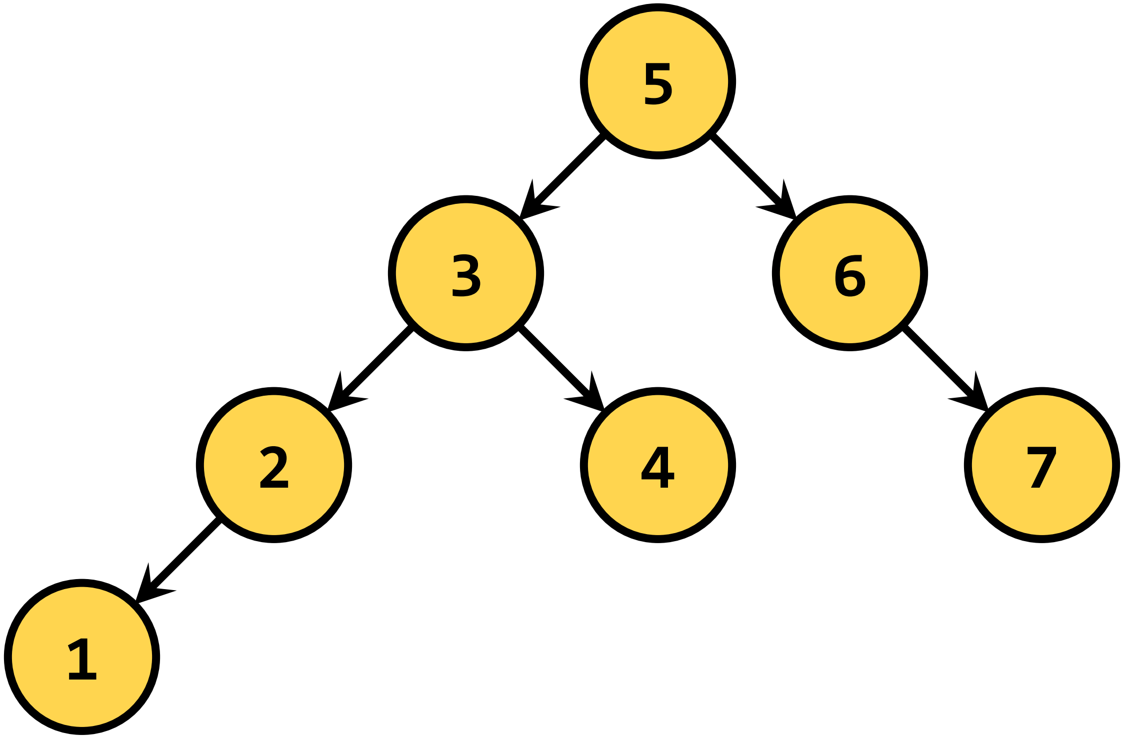
Operations on BSTs

`search` in BSTs



`search` compares the desired key to the current vertex, visiting left or right children as appropriate.

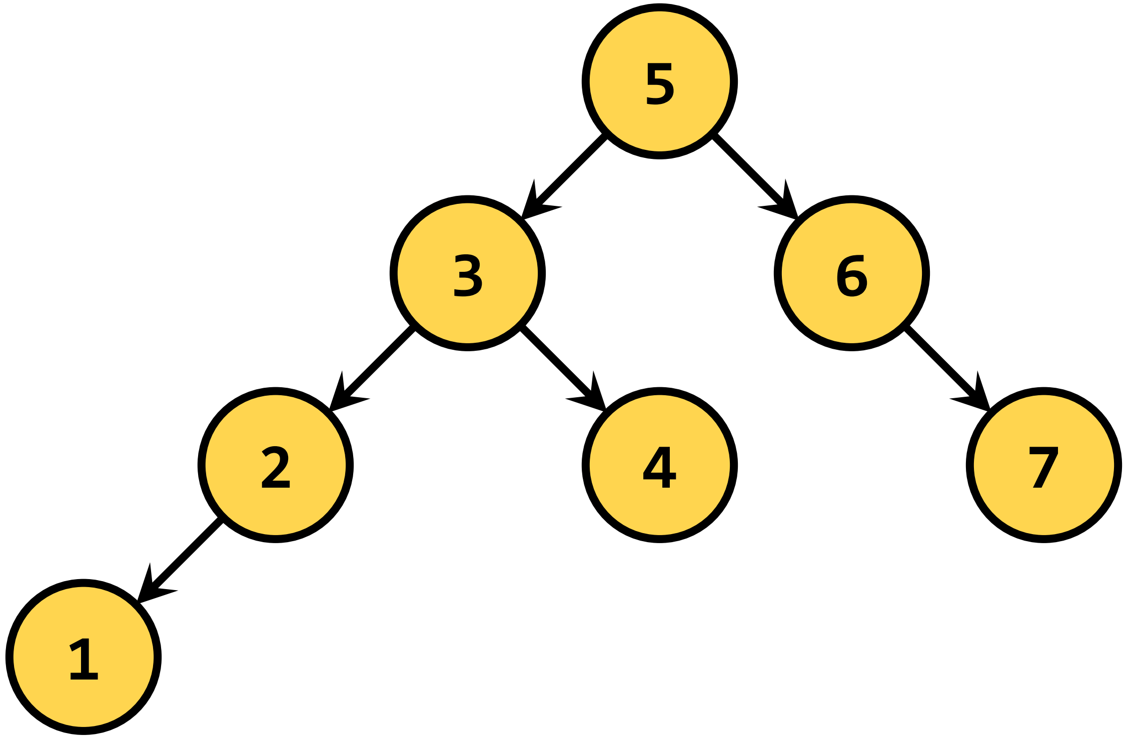
`search` in BSTs



For example, `search(4)` compares the 4 to the 5, then visits its left child of 3, then visits its right child of 4.

Write pseudocode to implement this algorithm! 

`search` in BSTs



If we desire a non-existent key, such as `search(4.5)`, we can either return the last seen node (in this case, 4) or we can throw an exception. For now, let's do the former (helpful to other algorithms that may use search() function, e.g., insert.

`insert` in BSTs

```
algorithm insert(root, key_to_insert):
  x = search(root, key_to_insert)
  v = new vertex with key_to_insert
  if key_to_insert > x.key:
    x.right = v
  if key_to_insert < x.key:
    x.left = v
  if key_to_insert == x.key:
    return
```

Explain on board: insert(4.5), insert(6.5), insert(4)

Runtime: `O(log n)` if balanced, `O(n)` otherwise

Explain on board: an extremely unbalanced BST

`delete` in BSTs

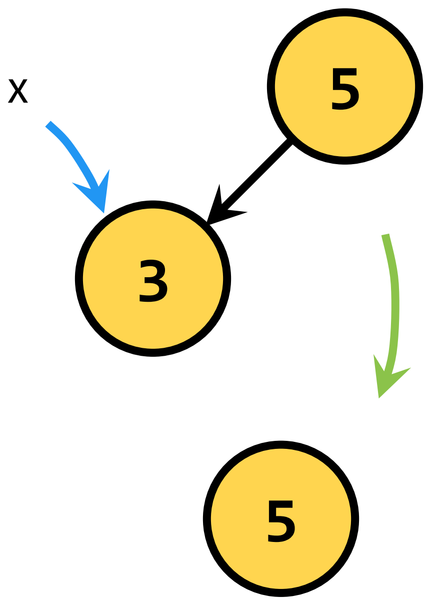
```
algorithm delete(root, key_to_delete):
  x = search(root, key_to_delete)
  if key_to_delete == x.key:
    delete x
```

This is somewhat complicated …

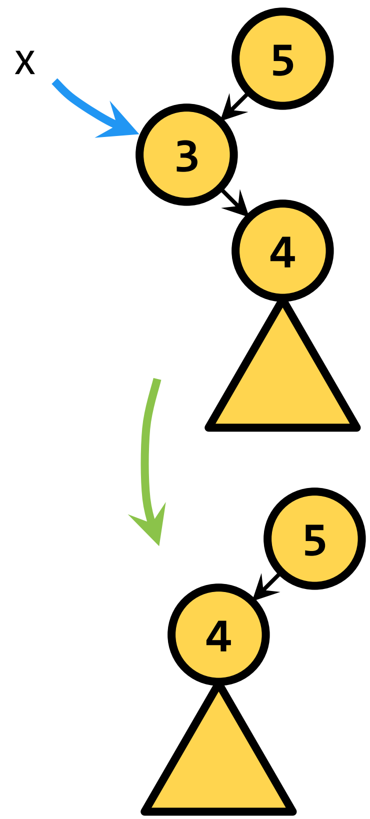
Runtime: `O(log n)` if balanced, `O(n)` otherwise

`delete` in BSTs

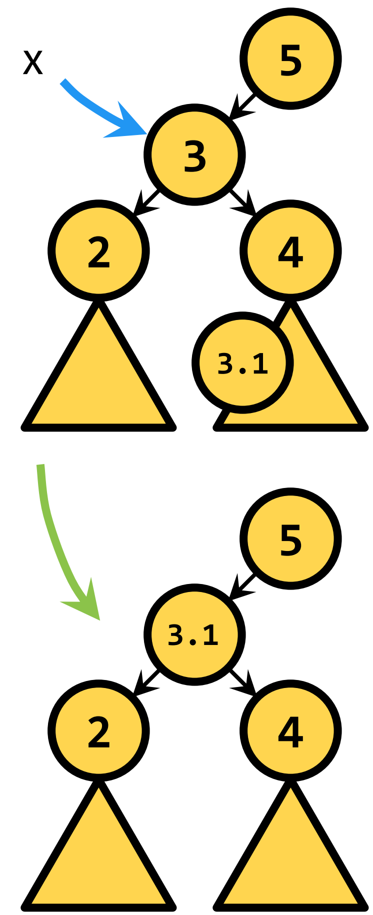
Case 1: x is a leaf
Just delete x



Case 2: x has 1 child
Move its child up

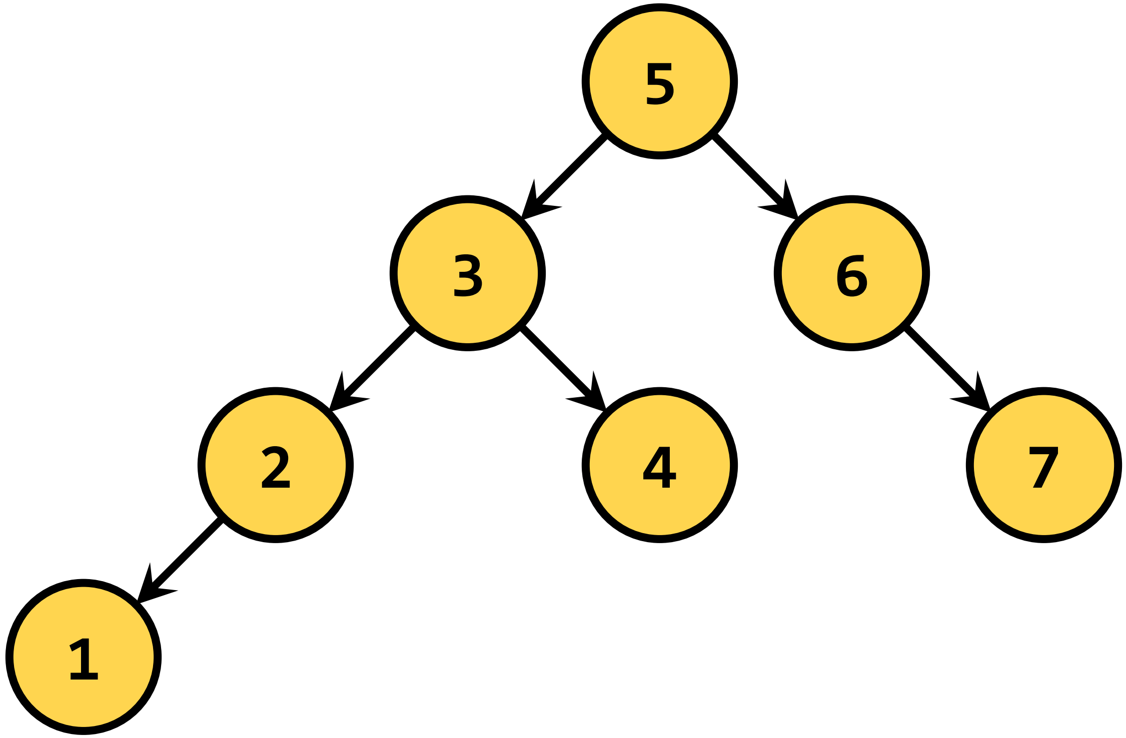


Case 3: x has 2 children
Replace x with its successor



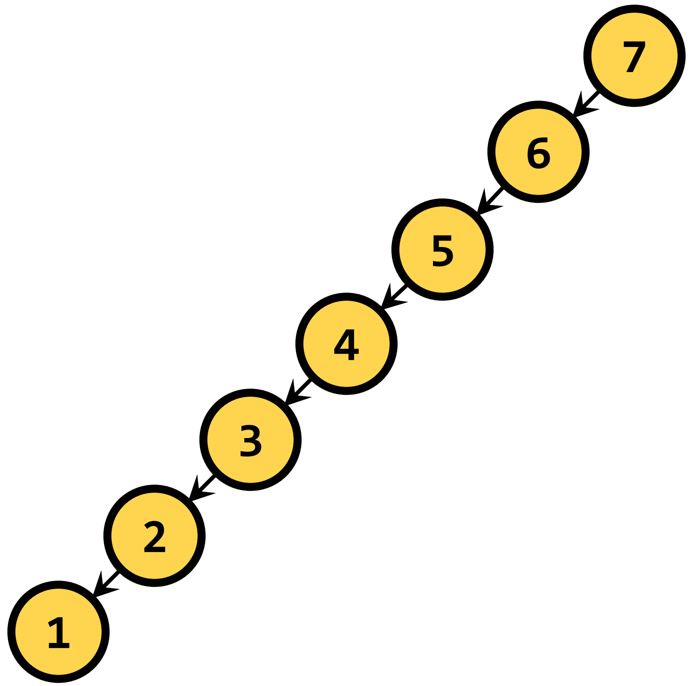
Explain on board: why the successor is selected in case 3.
It should be greater than 2 and all its descendants, while being less than 4 and all its descendants. So we choose the smallest value in the right-descendants of 3, i.e., successor.

Runtime Analysis



Runtime of `search` (which `insert` and `delete` both call) is **`O(depth of tree)`**.

Runtime Analysis



But this is a valid BST and the depth of the tree is n, resulting in a runtime of `O(n)` for search.

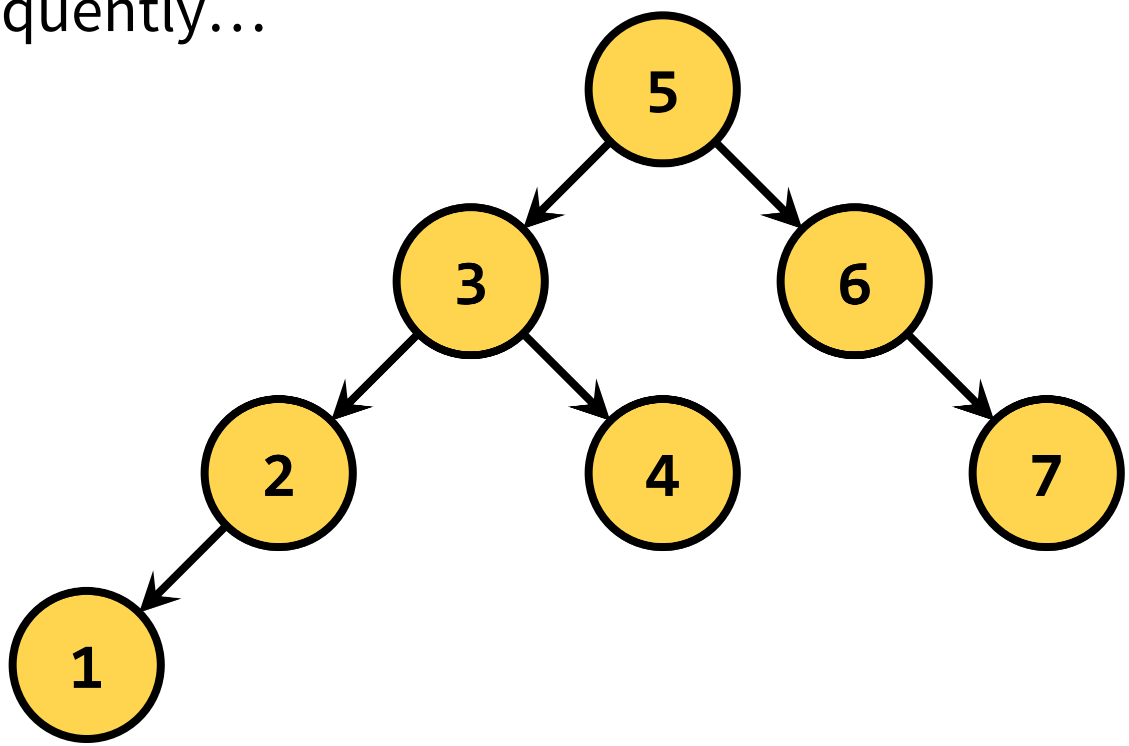
In what order would we need to insert vertices to generate this tree?

What To Do?

We could keep track of the depth of the tree. If it gets too tall, re-do everything from scratch.

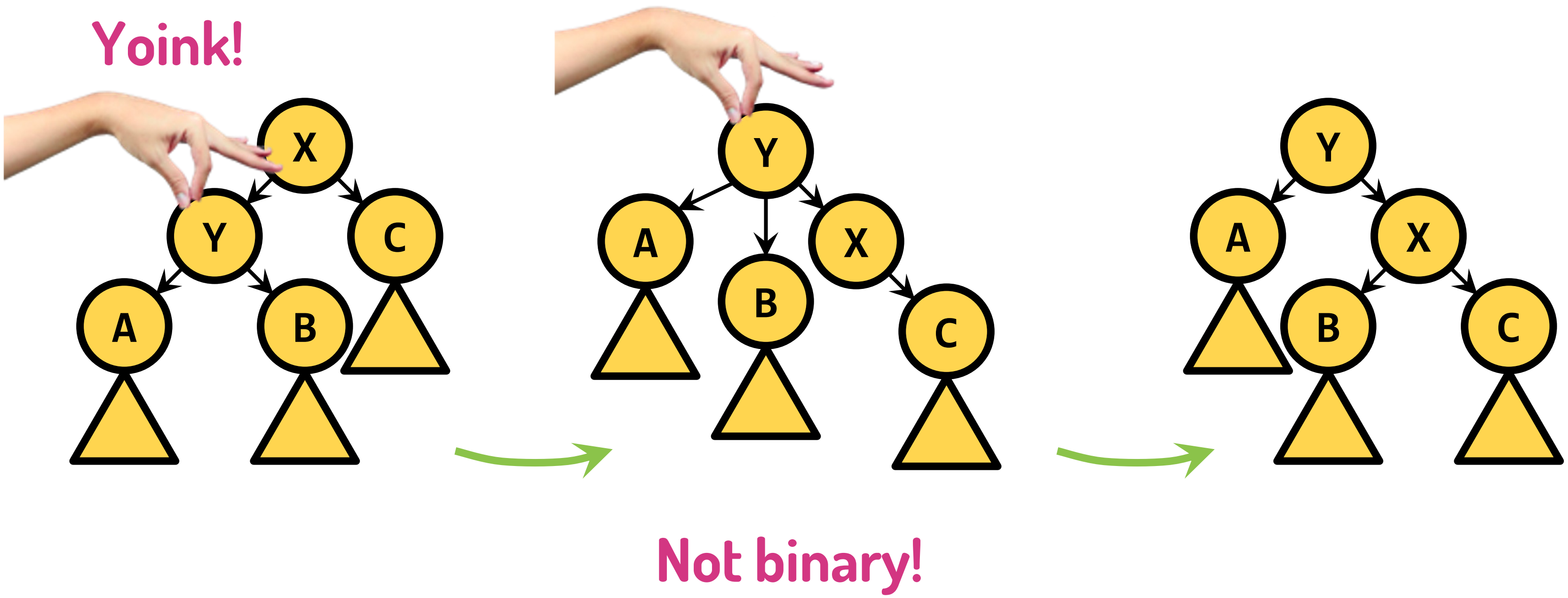
But this is time-consuming because we have to reconstruct the whole tree frequently…

Any other ideas?



Idea 1: Rotations

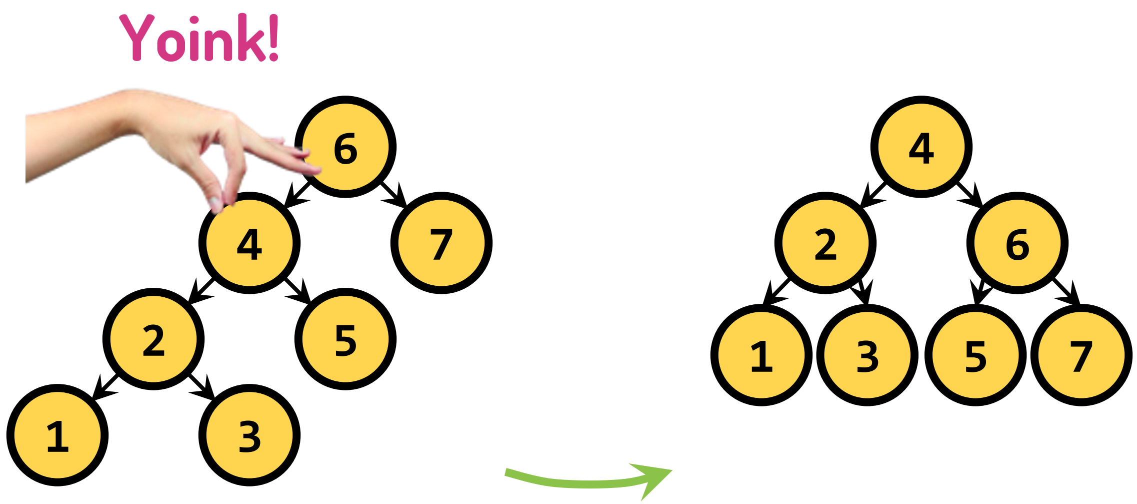
Maintain the BST property, and move some of the vertices (but not all of them) around.



Check the BST properties on Y and X

Idea 1: Rotations

Maintain the BST property, and move some of the vertices (but not all of them) around.



Explain on board

Idea 2: Proxy for Balance

Maintaining **perfect balance** is too difficult.

Checking for balance and which node to rotate is difficult

Instead, let's determine some proxy for balance.

i.e. If the tree satisfies some property, then it's "pretty balanced."

If during tree modification (construction, insert, delete, etc.) the property no long holds, we can maintain this property using rotations.

Red-Black Trees (RB-Tree)

Red-Black Trees

There exist many ways to achieve this proxy for balance, but here we'll study the **red-black tree**.

1. Every vertex is colored **red** or **black**.
2. The root vertex is a **black** vertex.
3. A NIL child is a **black** vertex.
4. The child of a **red** vertex must be a **black** vertex.
5. For all vertices v, all paths from v to its NIL descendants have the same number of **black** vertices.

We can be sure that the tree is pretty balanced as long as these proxy properties hold.

Red-Black Trees by Example

1. Every vertex is colored **red** or **black**.
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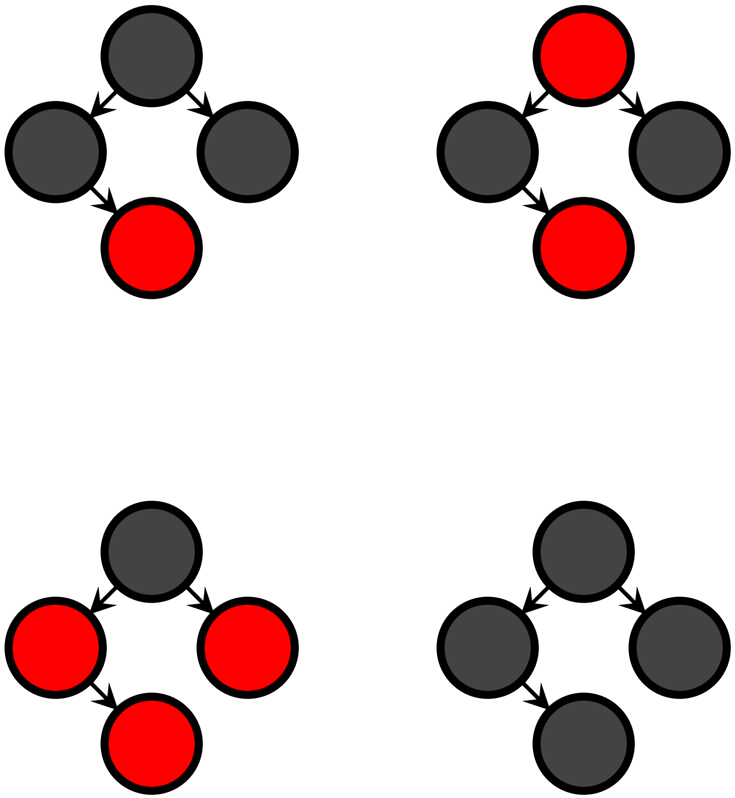

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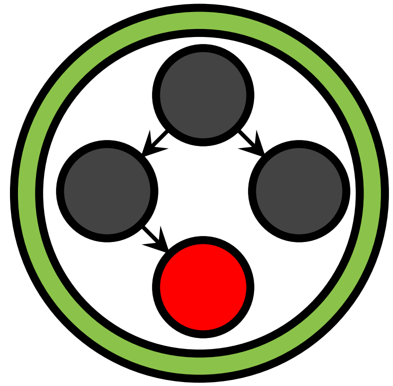
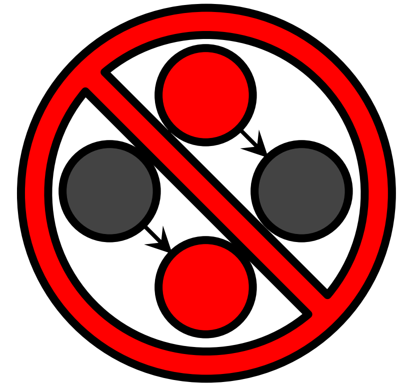
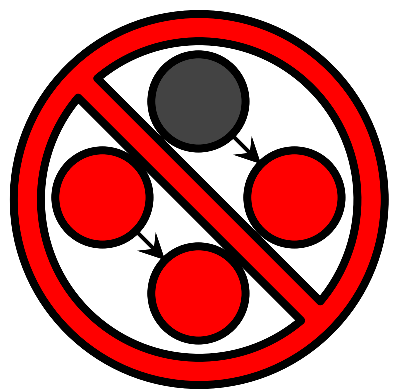
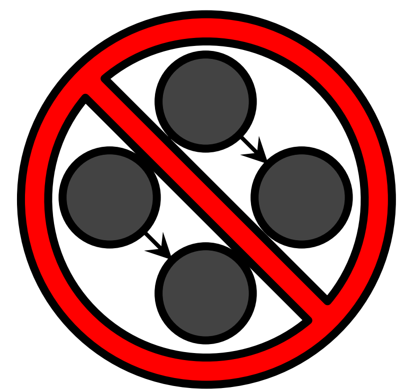




Violates 2



Violates 4



Violates 5

Red-Black Trees

Maintaining these properties maintains a "pretty balanced" BST.

Intuition:

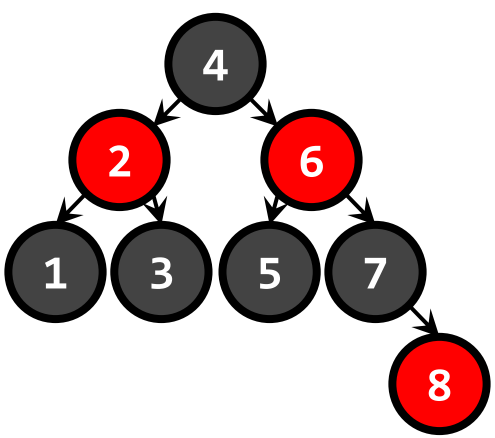
The **black** vertices are balanced.

Rule #5: For all vertices v, all paths from v to its NIL descendants have the same number of **black** vertices

The **red** vertices are "spread out."

Rule #4: The child of a **red** vertex must be a **black** vertex

We can maintain this property as we insert/delete vertices, using rotations.

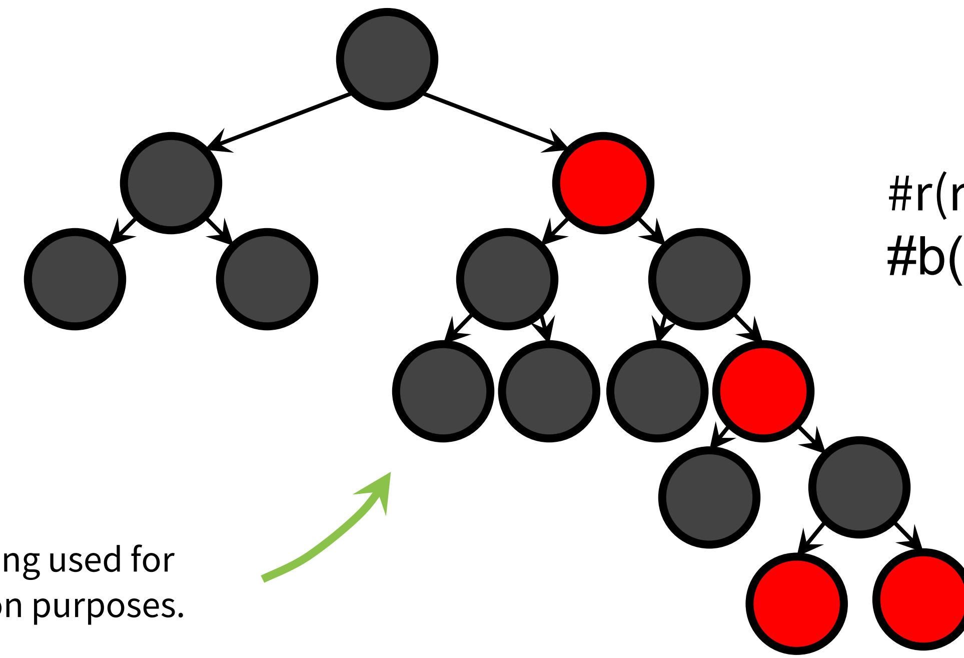


Red-Black Trees

To see why a red-black tree is "pretty balanced," consider that its height is at most O(log(n)).

Property: One path could be twice as long as the others if we pad it with red vertices, but at most twice as long as the others.

Explain: how can a path be as long as possible?



#r(root) < h/2
#b(root) ≥ h/2

A valid coloring used for demonstration purposes.

Red-Black Trees

Lemma: The number of non-NIL nodes in a subtree of x is at least $n(x) \geq 2^{b(x)} - 1$, b(x) is the number of black nodes from x to NIL descendants.

Proof:

To prove this statement, we proceed by induction.

For base case, note that a NIL node has $b(x) = 0$ and $2^0 - 1 = 0$, meaning the tree has at least 0 nodes, which is true. Same for a single black or red node.

For inductive step, let n(x) be the number of non-NIL nodes of subtree x (including x). Then:

$$\begin{aligned} n(x) &= 1 + n(x.left) + n(x.right) \\ &\geq 1 + (2^{b(x)-1} - 1) + (2^{b(x)-1} - 1) \\ &= 2^{b(x)} - 1 \end{aligned}$$

Thus, the number of non-NIL nodes of x is at least $2^{b(x)} - 1$. ■

Explain on board

Red-Black Trees

Theorem: A Red-Black Tree has height $h \leq 2\log_2(n+1) = O(\log n)$.

Proof:

By our lemma, the number of non-NIL nodes of x is at least $2^{b(x)} - 1$.

Notice that on any root to NIL path there are no two consecutive red vertices (otherwise the tree violates rule 4);

So the number of black vertices is at least the number of red vertices.
Thus, b(root) is at least half of the height, i.e., $b(root) \geq h/2$

Let n be the number of vertices in the tree
then $n \geq 2^{b(root)} - 1 \geq 2^{h/2} - 1$, and hence $h \leq 2\log_2(n+1)$. ■

`insert` in Red-Black Trees

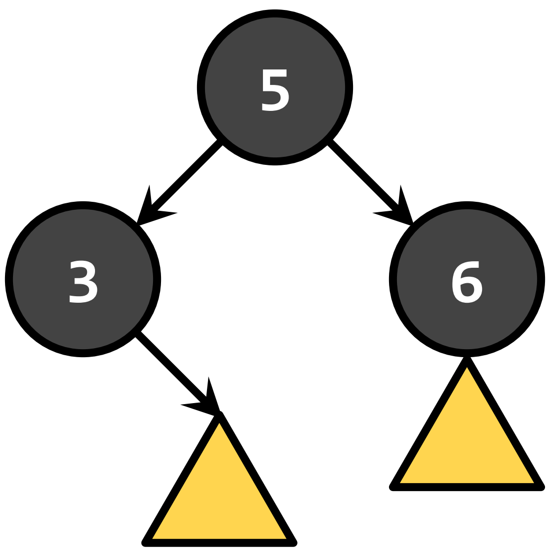
```
algorithm rb_insert(root, key_to_insert):
  x = search(root, key_to_insert)
  v = new red vertex with key_to_insert
  if key_to_insert > x.key:
    x.right = v
    fix things up, if necessary
  if key_to_insert < x.key:
    x.left = v
    fix things up, if necessary
  if key_to_insert == x.key:
    return
```

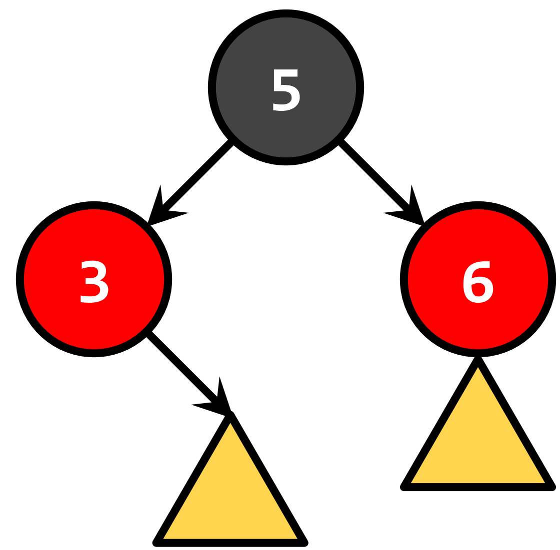
What does that mean?

`insert` in Red-Black Trees

What does "if necessary" mean?

Suppose we want to `insert(1)`.





`insert` in Red-Black Trees

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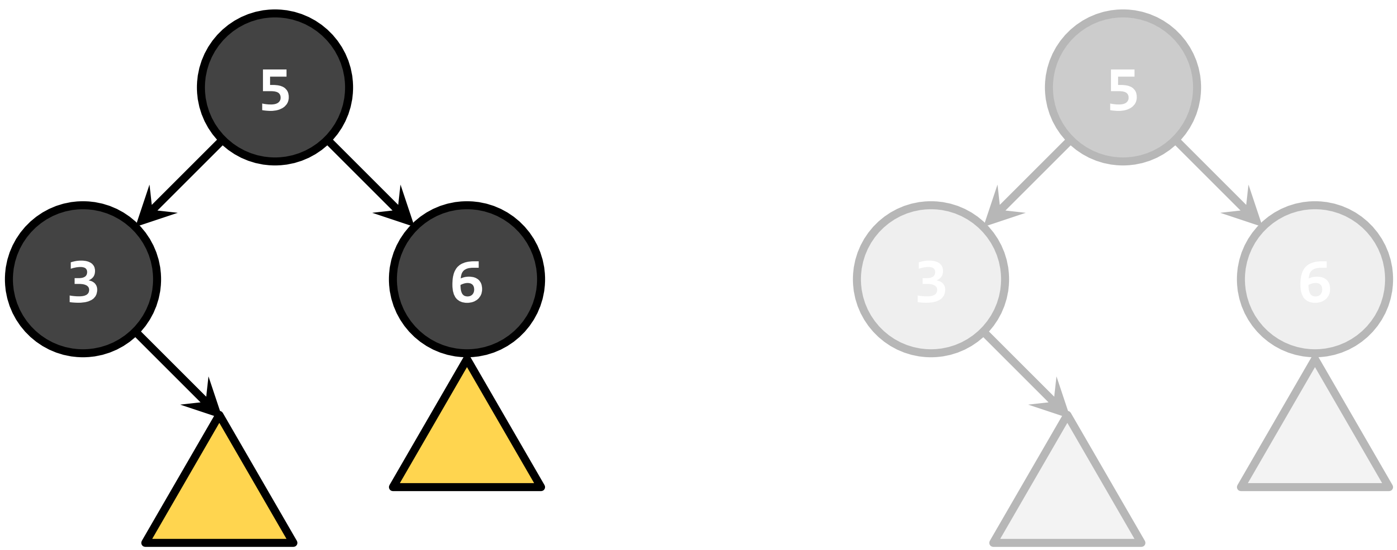

`insert` in Red-Black Trees

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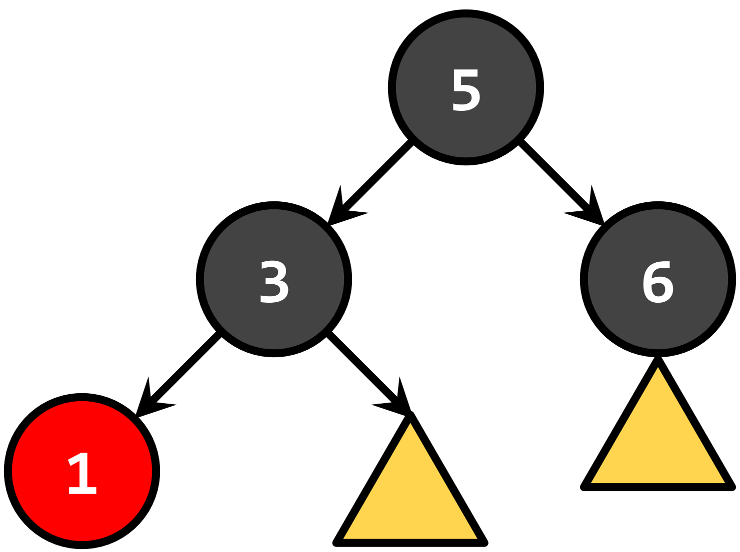

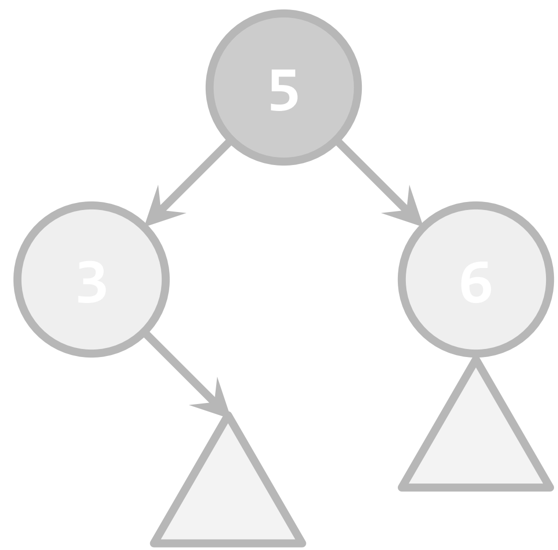

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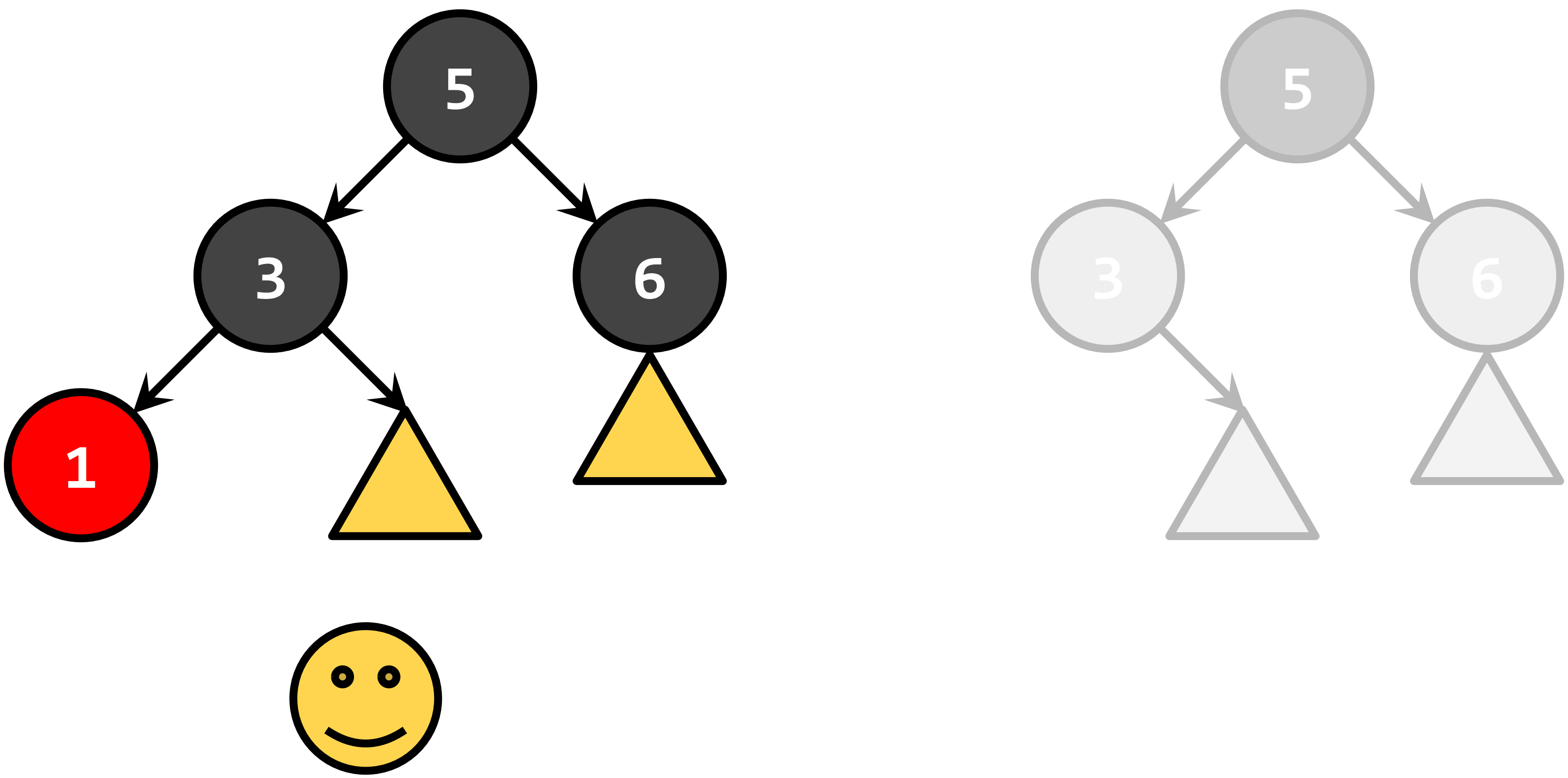

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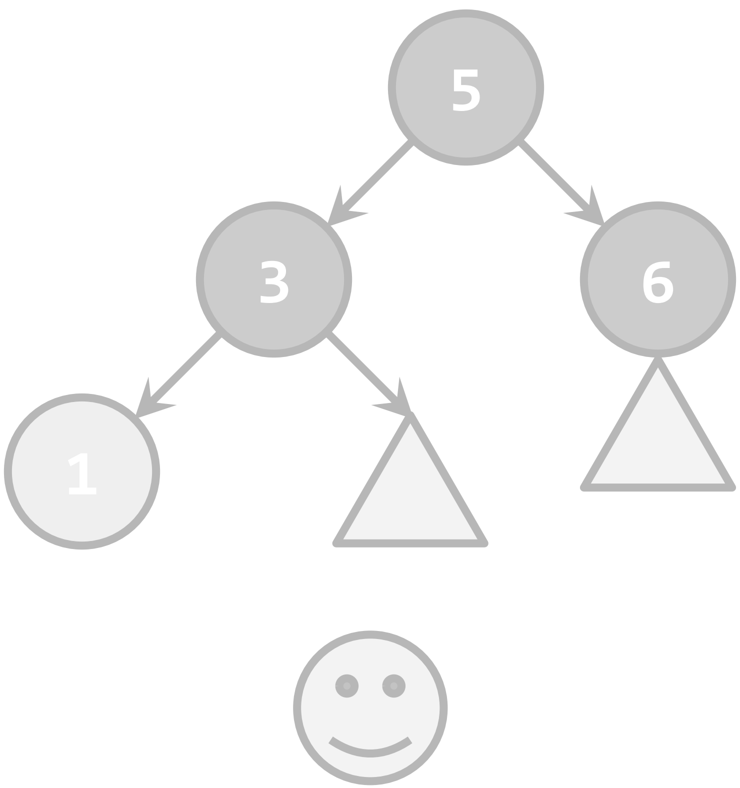

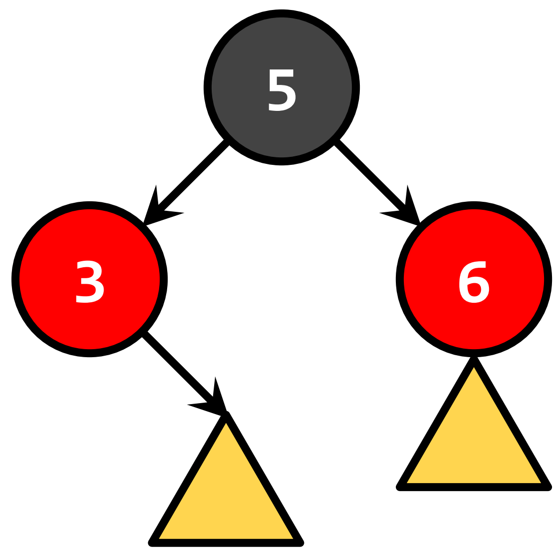

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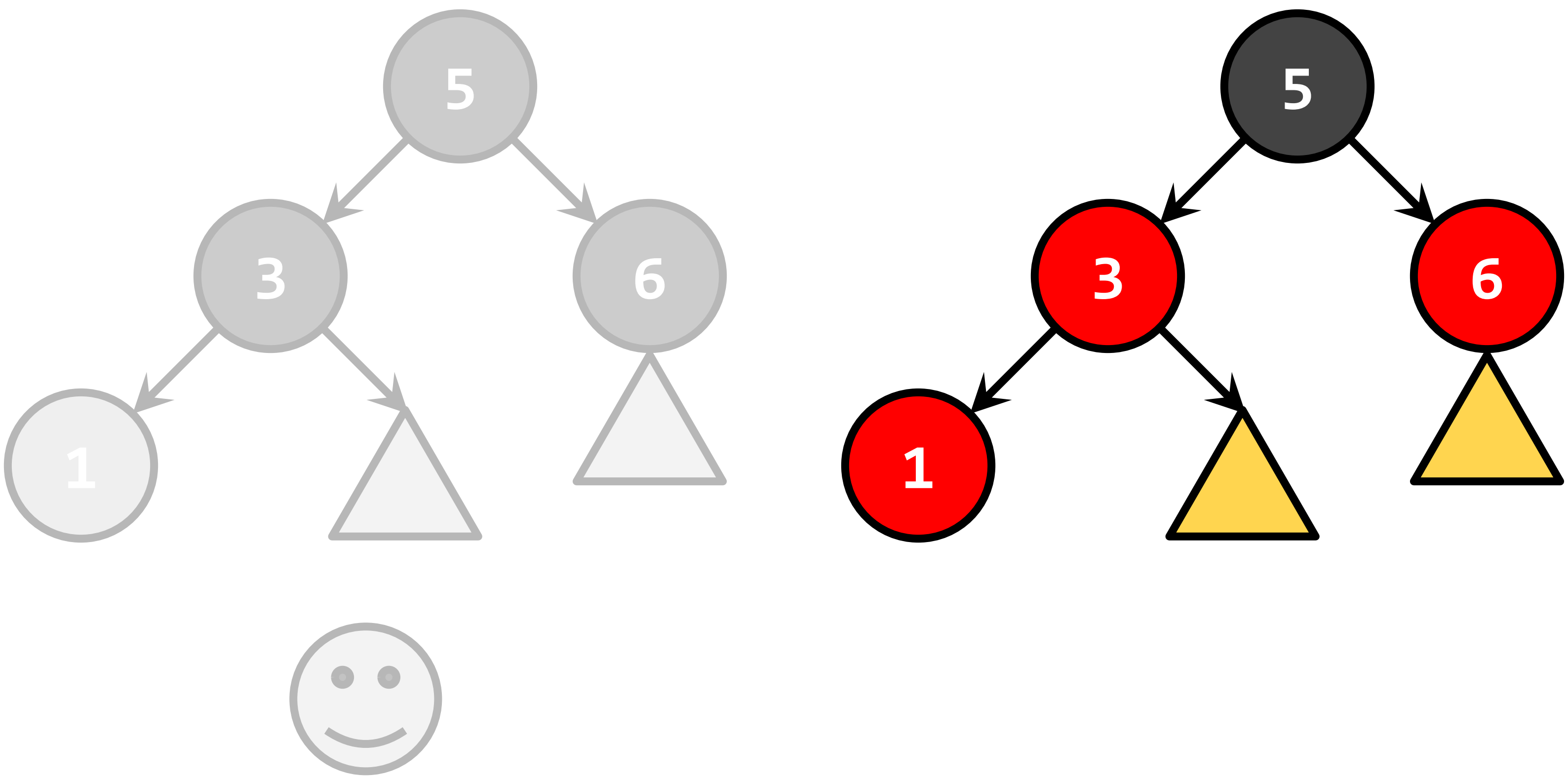

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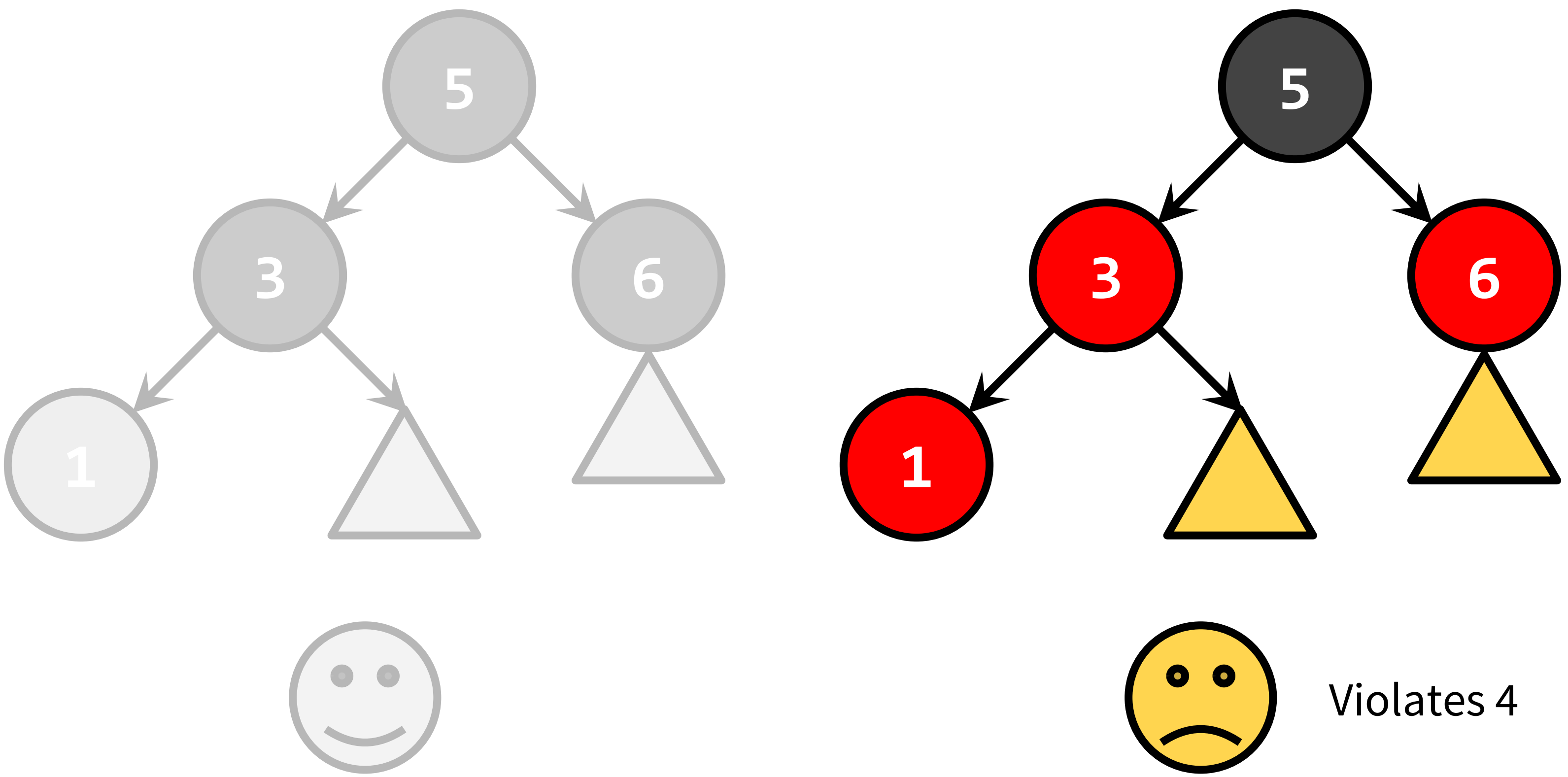

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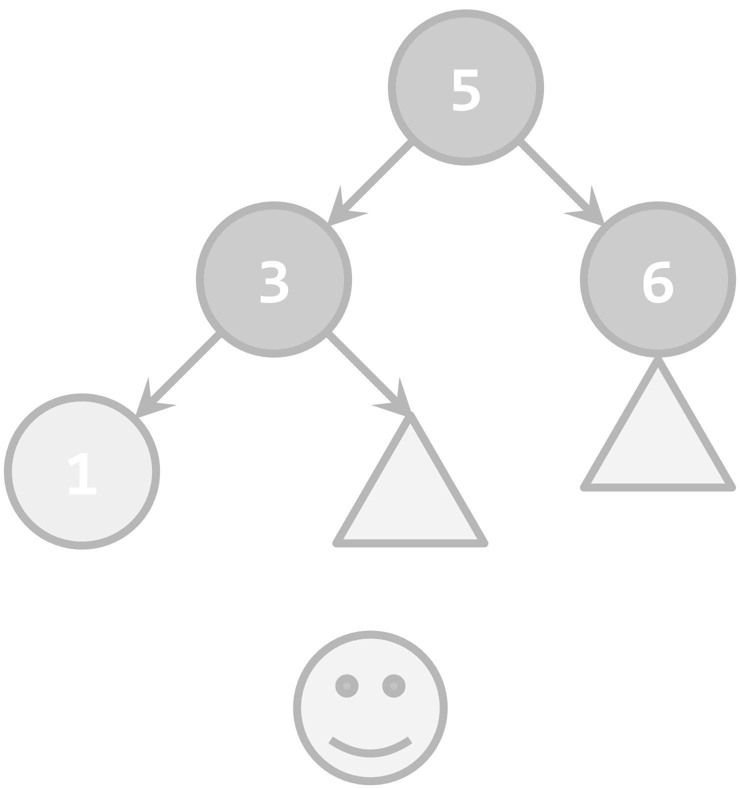
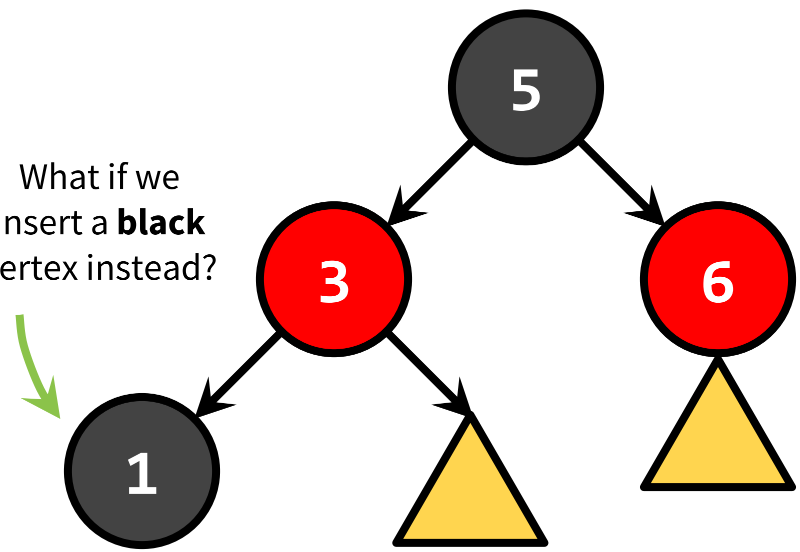


What if we
insert a **black**
vertex instead?



`insert` in Red-Black Trees

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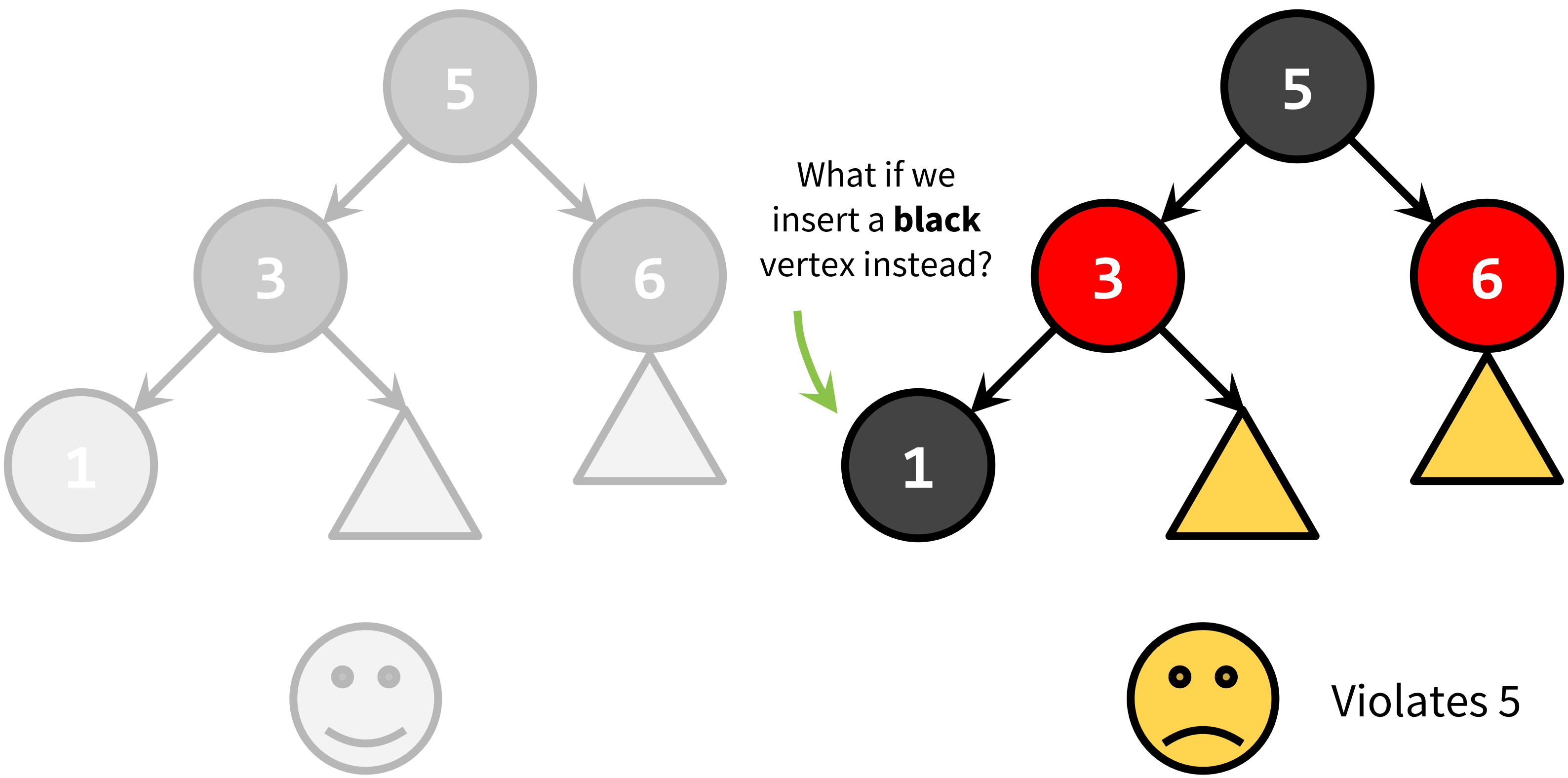


`insert` in Red-Black Trees

What does "if necessary" mean?

So it seems we're happy if the parent of the inserted vertex is **black**.

Check the rules:

1. Every vertex is colored **red** or **black**.
2. The root vertex is a **black** vertex.
3. A NIL child is a **black** vertex.
4. The child of a **red** vertex must be a **black** vertex.
5. For all vertices v, all paths from v to its NIL descendants have the same number of **black** vertices.

But there's an issue if the parent of the inserted vertex is **red**.

`insert` in Red-Black Trees

```
algorithm rb_insert(root, key_to_insert):
  x = search(root, key_to_insert)
  v = new red vertex with key_to_insert
  if key_to_insert > x.key:
    x.right = v
    recolor(v)
  if key_to_insert < x.key:
    x.left = v
    recolor(v)
  if key_to_insert == x.key:
    return
```

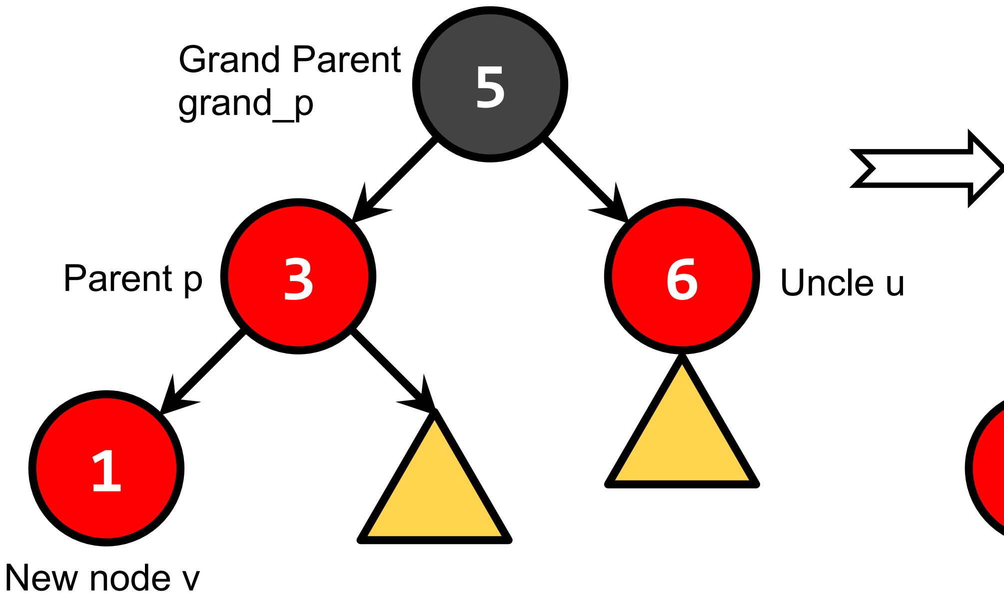
Runtime: `O(log n)`

`insert` in Red-Black Trees

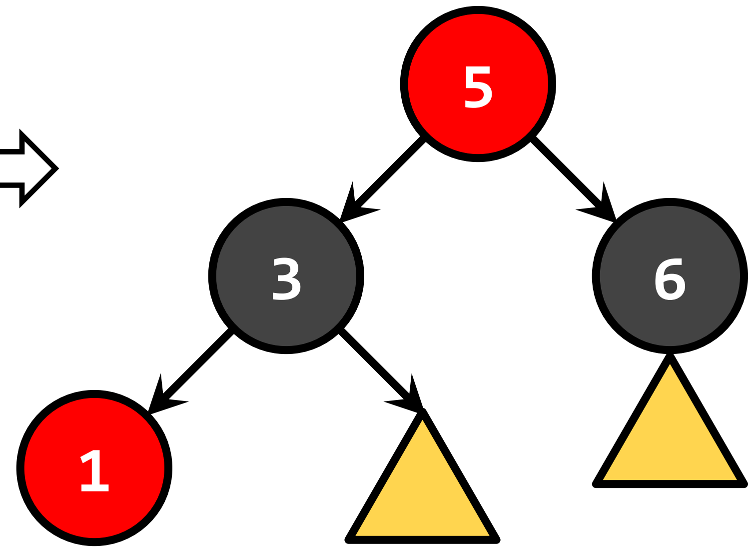
```
algorithm recolor(v):
  if v == root:
    v.color = black
    return
  p = parent(v)
  if p.color == black:
    return
  grand_p = p.parent
  uncle = grand_p.right
  if uncle.color == red:
    p.color = black
    uncle.color = black
    grand_p.color = red # maintain number of black vertices
    recolor(grand_p) # fix up color recursively
  else:  # uncle.color == black
    p.color = black
    grand_p.color = red
    right_rotate(grand_p)  # yoink
```

Runtime: `O(log n)`

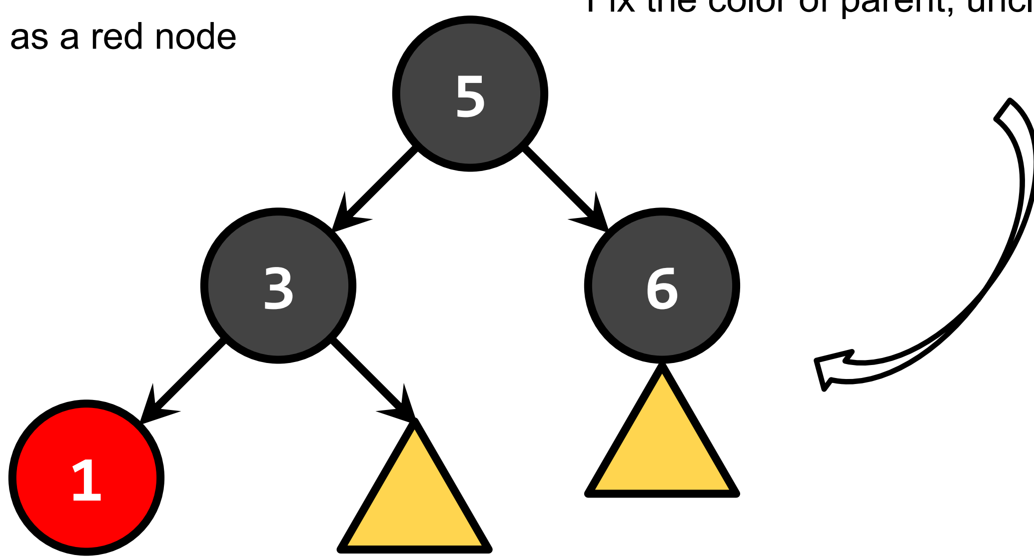
`insert` in Red-Black Trees



Insert new node v as a red node

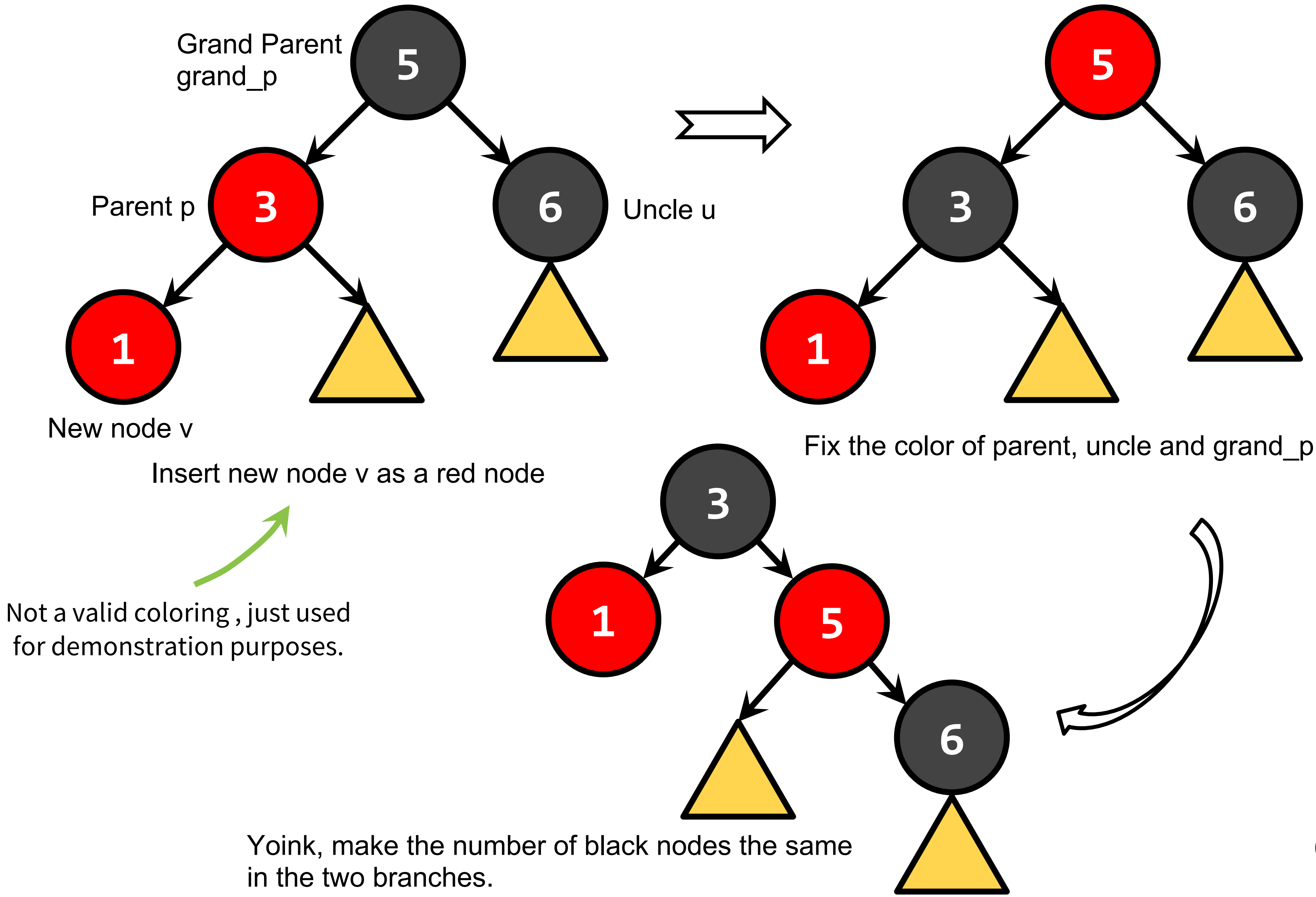


Fix the color of parent, uncle and grand_p



Fix the color of grand_p recursively

`insert` in Red-Black Trees



Red-Black Trees

Since we maintain the red-black property in `O(log n)`, then insert, delete, and search all require `O(log n)`-time.

	Search	Insertion	Deletion
Linked list	O(n)	O(n)	O(n)
Arrays	O(log n)	O(n)	O(n)
BST (unbalanced)	O(n)	O(n)	O(n)
BST (balanced)	O(log n)	O(log n)	O(log n)
RBT (always balanced)	O(log n)	O(log n)	O(log n)

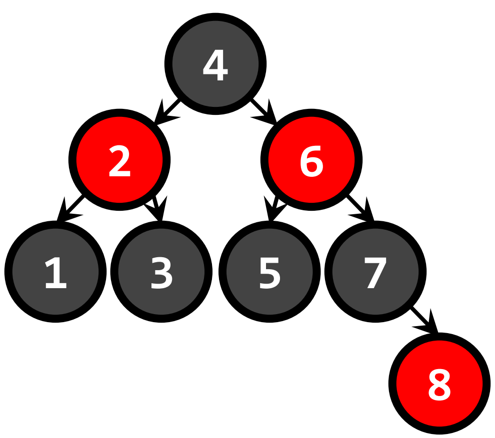
Red-Black Trees

Why is RB-Tree important at all?

In many languages (such as C++ and Java), RB-Trees are used as the foundations for sets and dictionaries.

Set = {1, 3, 4, 5, 8, …}

Operations: insert a value into the set
delete a value from the set
check if a value exists in the set (i.e., search)



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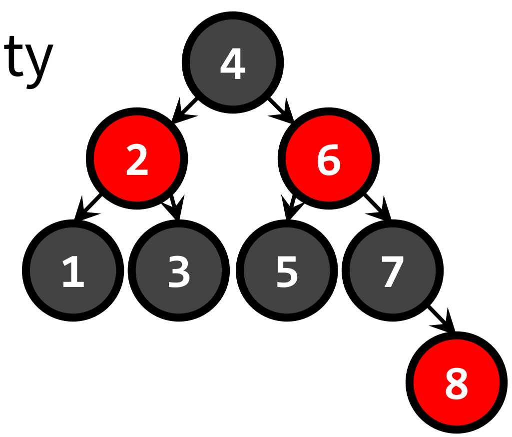

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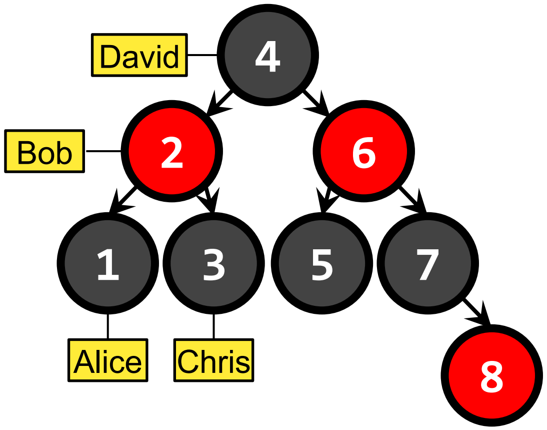


Dict = {1: “Alice”,
2: “Bob”,
3: “Chris”,
4: “David”,
5: …}

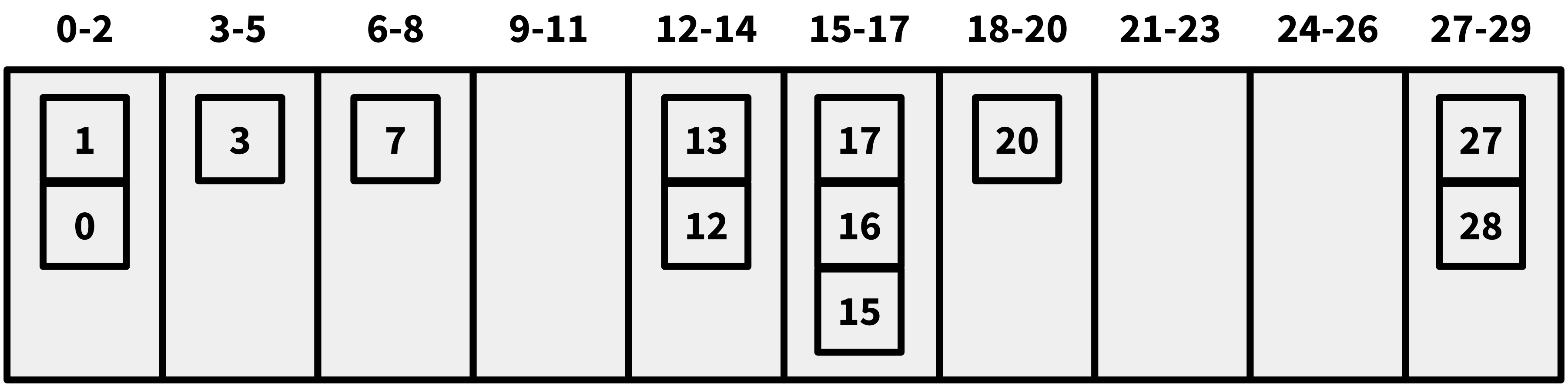
Operations: insert a key-value pair into the dict
delete a key-value pair from the dict
retrieve the value of a given key from the dict (i.e., search)

All operations in O(log n) complexity

Red-Black Trees

Why is RB-Tree important at all?

In some “newer” languages (such as Python), sets and dictionaries are implemented as hash-tables, which we will introduce later.



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Acknowledgement: Part of the materials are adapted from Mary Wootter, Virginia Williams and David Eng's lectures on algorithms. We appreciate their contributions.