

Greedy Algorithms II

Outline for Today

Greedy algorithms

Greedy graph algorithms

Minimum Spanning Trees

Prim's Algorithm

Kruskal's Algorithm

Minimum Spanning Trees

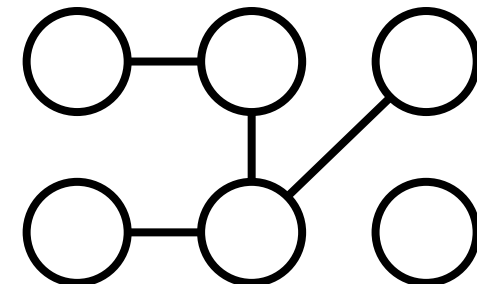
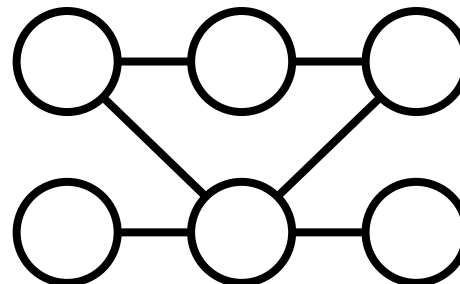
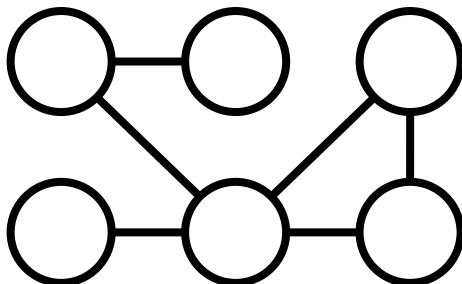
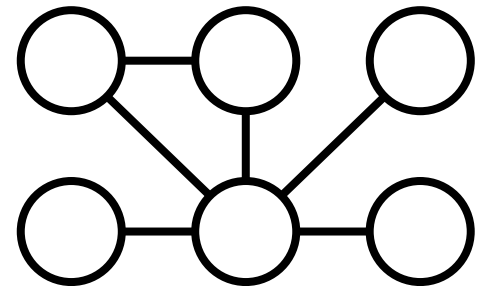
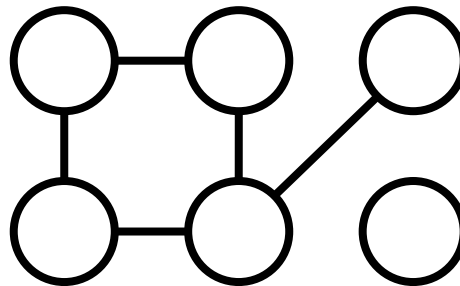
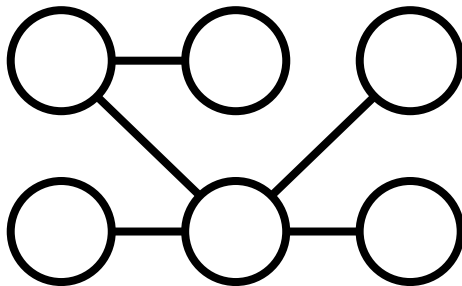
Tree

In Lecture 3, we studied trees with directed edges from parent to children vertices. In this lecture, edges will be undirected.



A **tree** is an undirected, acyclic, connected graph.

Which of these graphs **contain connected components that are trees**?



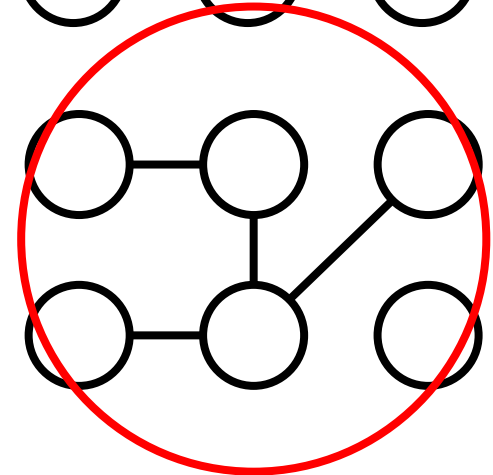
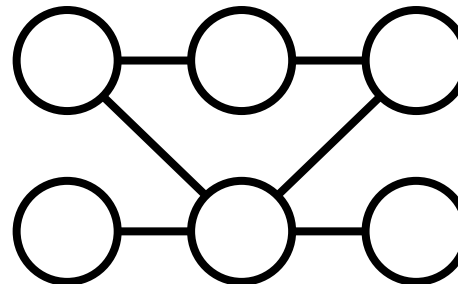
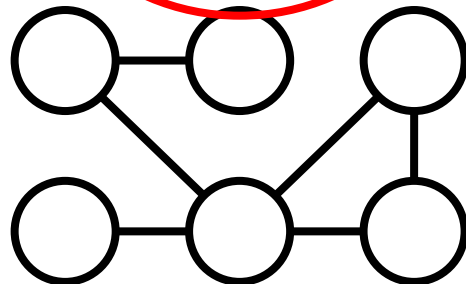
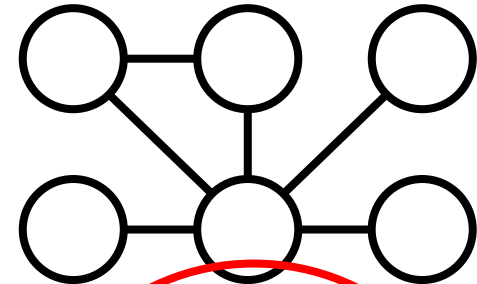
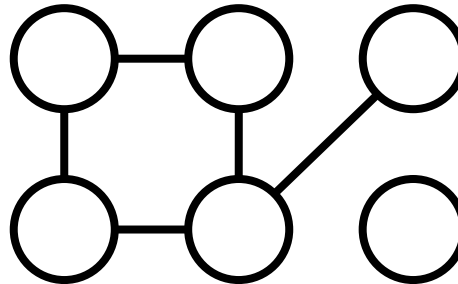
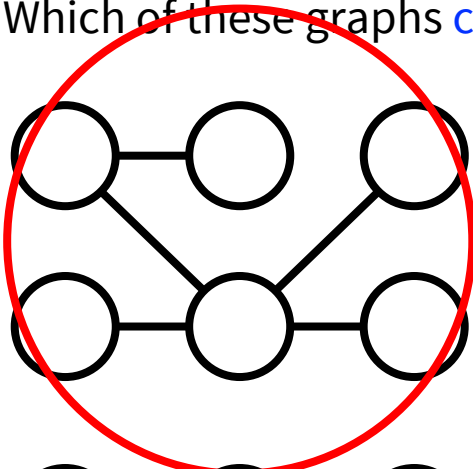
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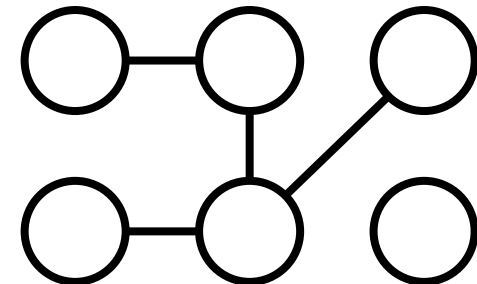
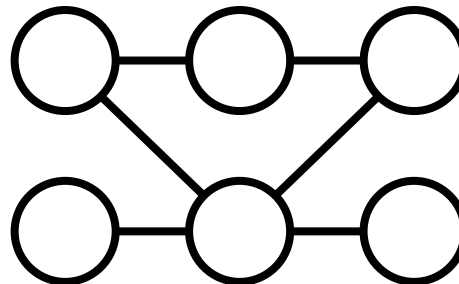
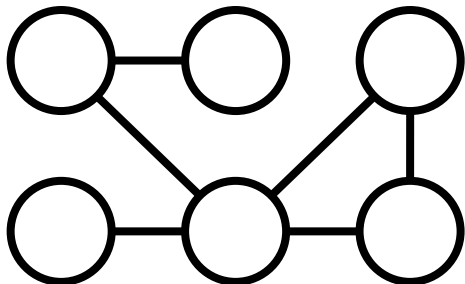
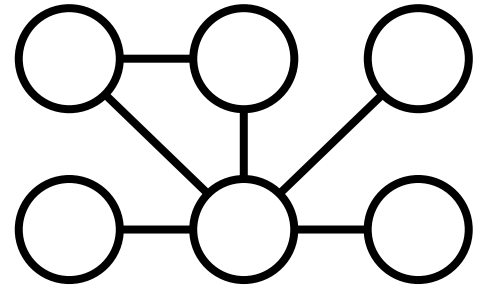
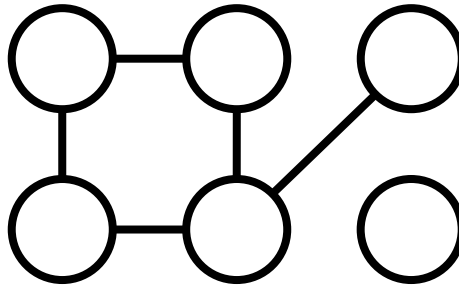
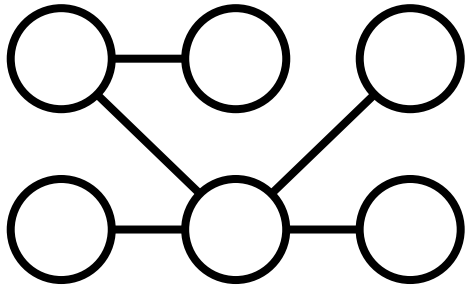
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Spanning Tree

A **spanning tree** is a tree that connects all of the vertices.

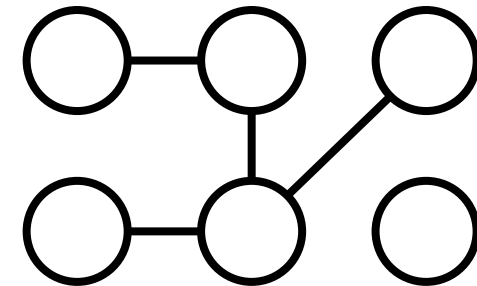
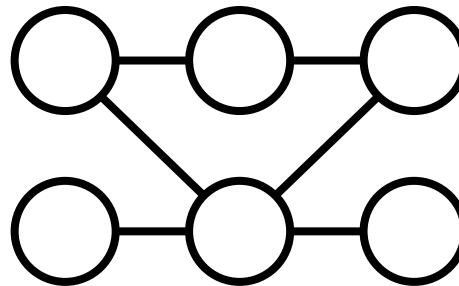
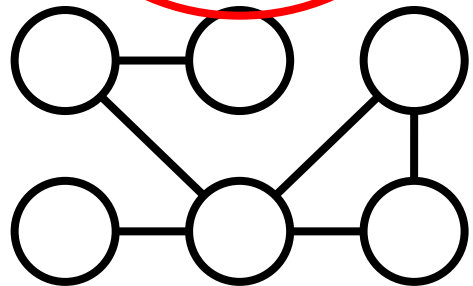
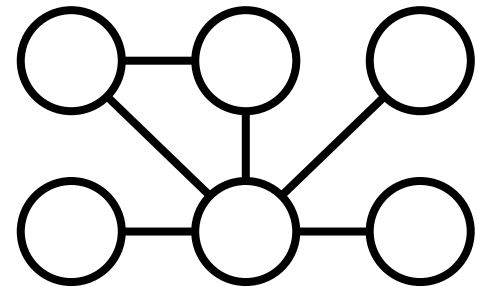
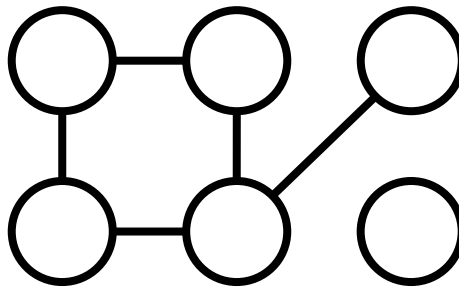
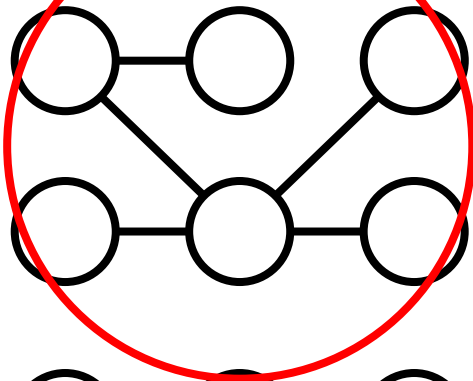
Which of these graphs are spanning trees? 🤔



Spanning Tree

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Which of these graphs are spanning trees? 🤔

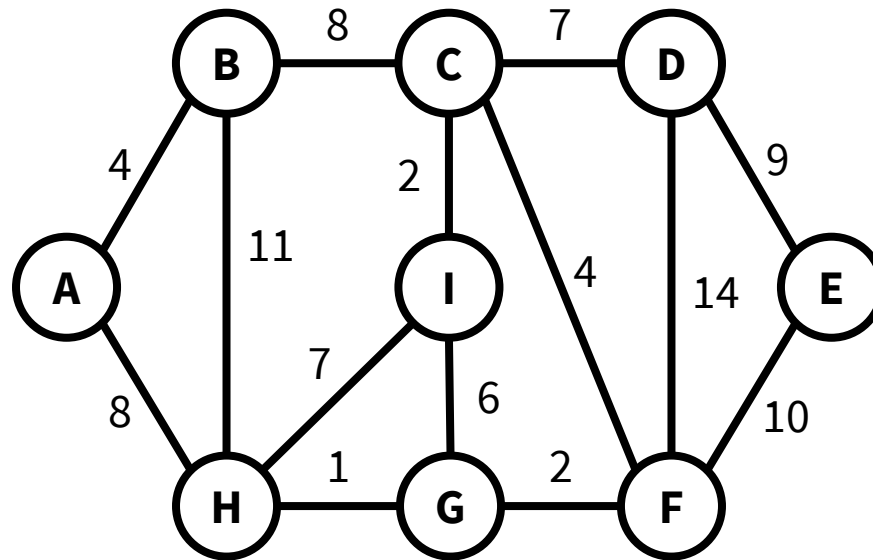


This connected component of the graph is a tree, but it doesn't include all of the vertices.

Spanning Tree

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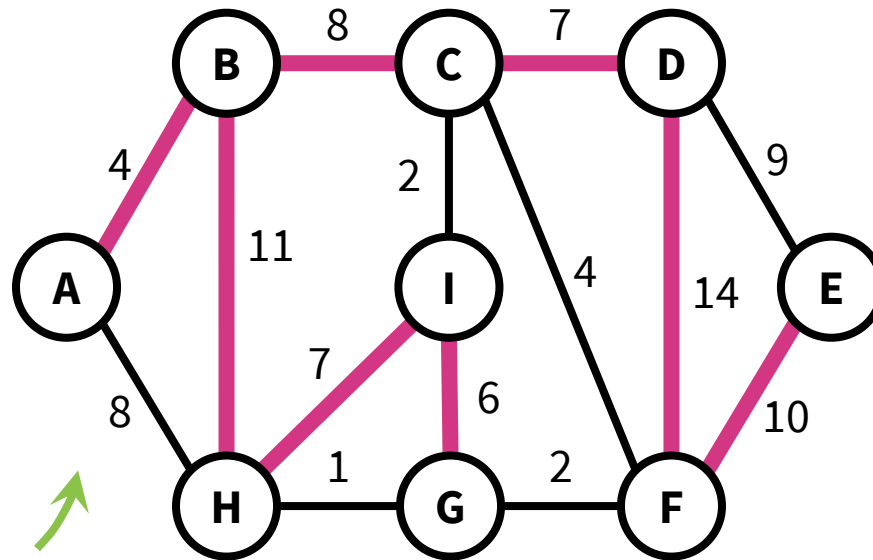
The **cost of a spanning tree** is the **sum of the weights on the edges**.



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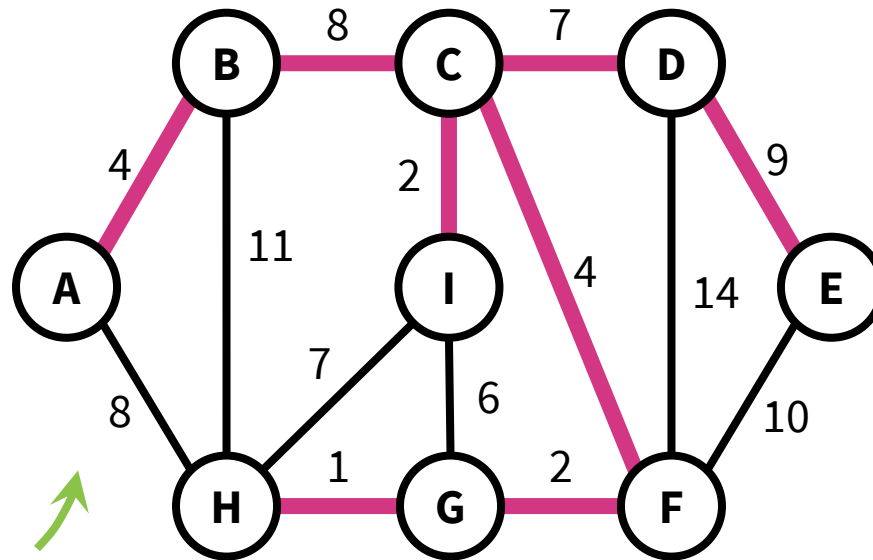


This spanning tree has
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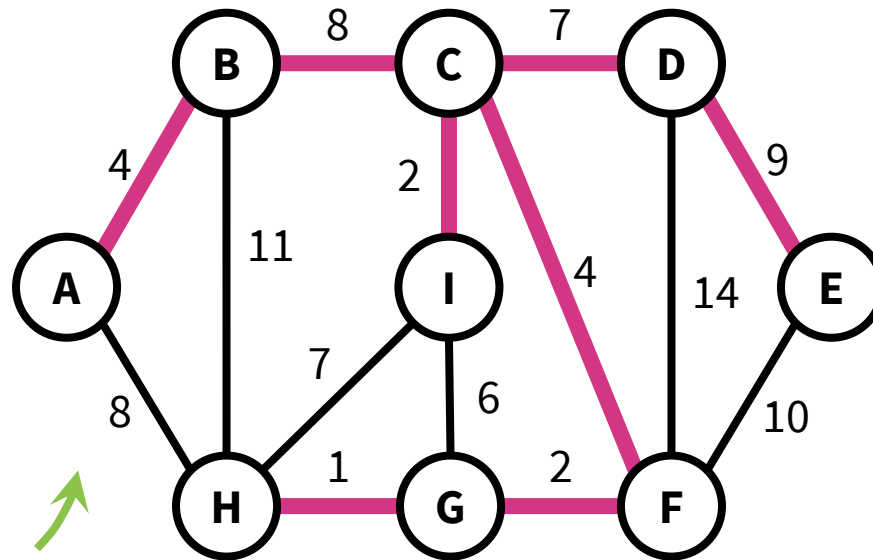


This spanning tree
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Minimum Spanning Tree

minimum **of minimal cost**
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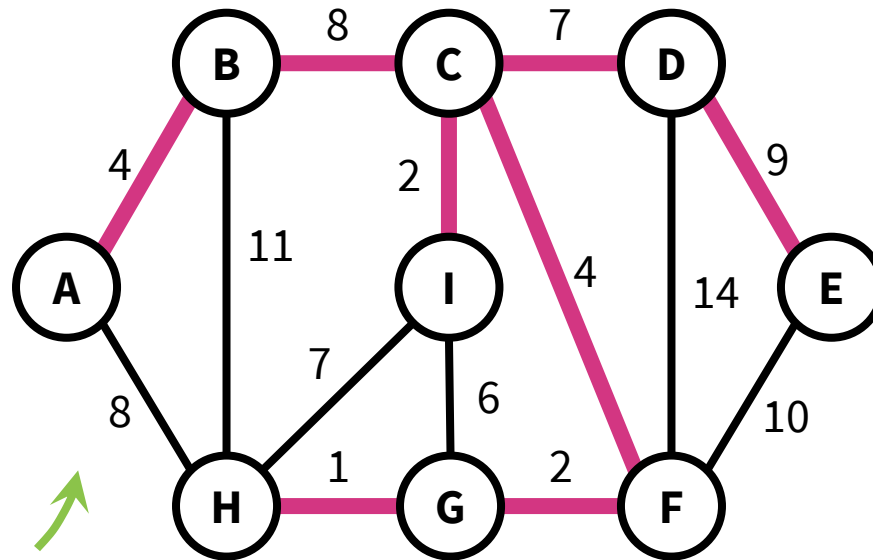


This spanning tree
has a cost of 37.
This is a minimum
spanning tree.

Minimum Spanning Tree

Finding the MST is very useful in many problems.

E.g., Find the MST to connect all telephones/computers with shortest total cable length.



Edge weight is the distance between two telephones/computes.

Minimum Spanning Tree

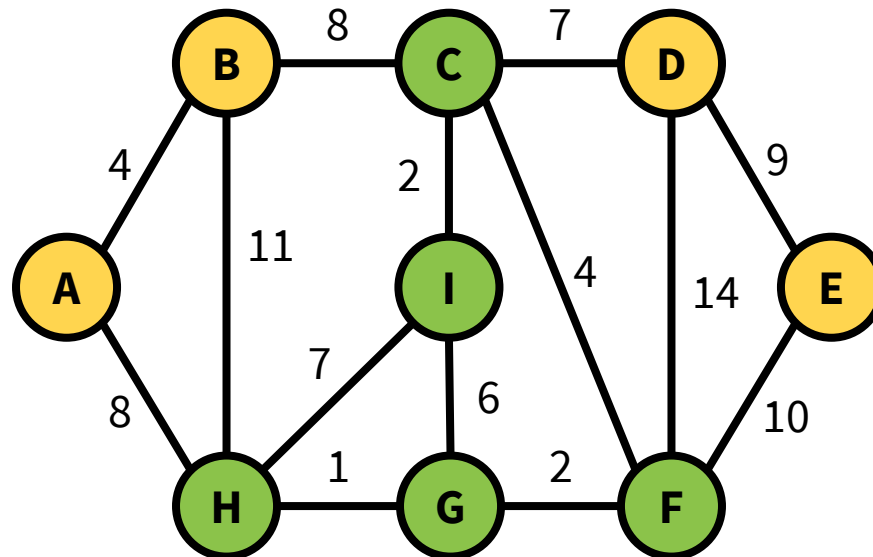
How might we find a MST?

Today, we'll see two **greedy algorithms** that find a MST.

Review: Graph Cut

Recall from Lecture 7, a **cut** is a **partition of the vertices** into **two nonempty parts**.

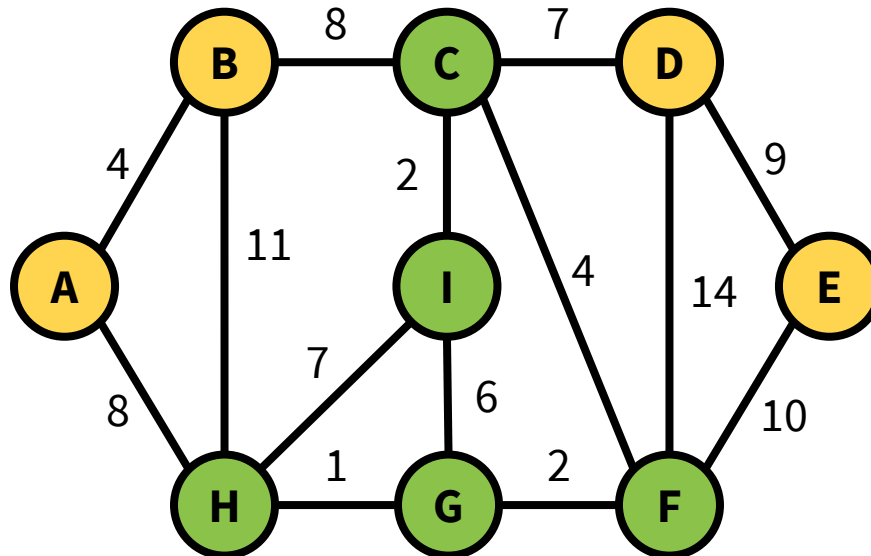
e.g. This is the cut “{A, B, D, E} and {C, I, F, G, H}”.



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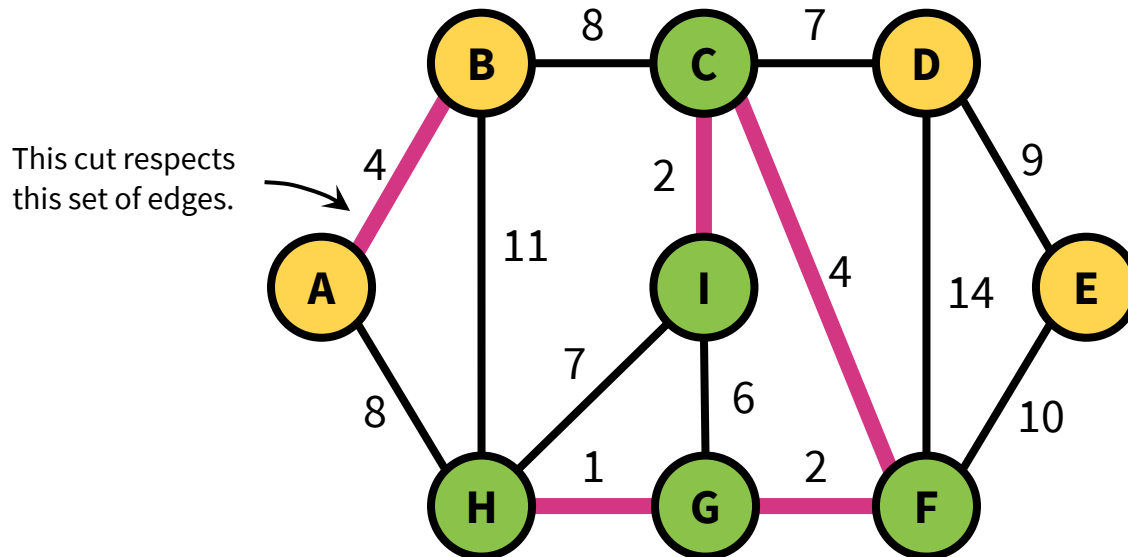


A cut **respects** a set of edges if **no edges in the set cross the cut**.

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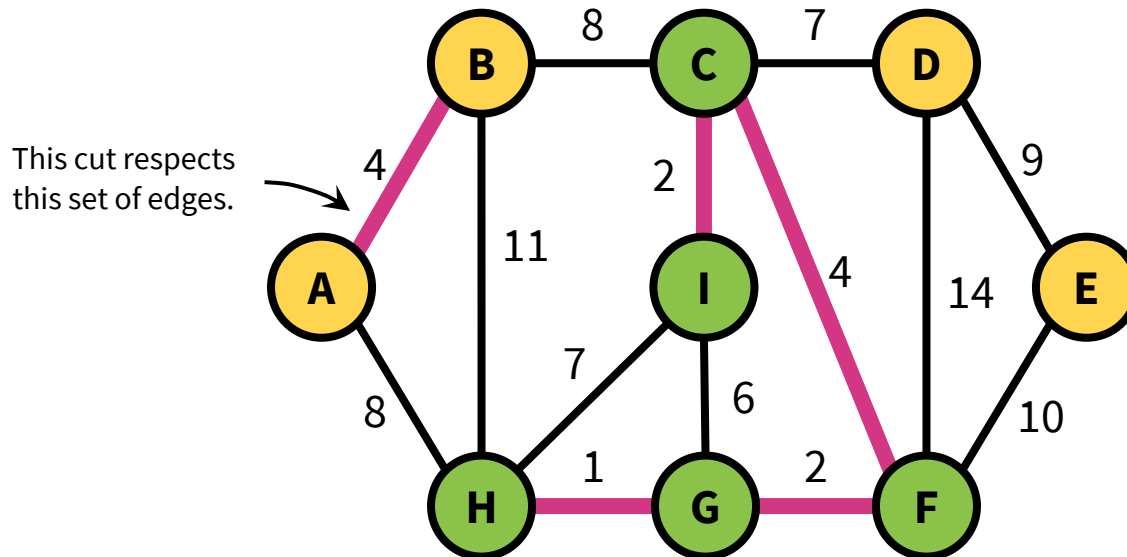


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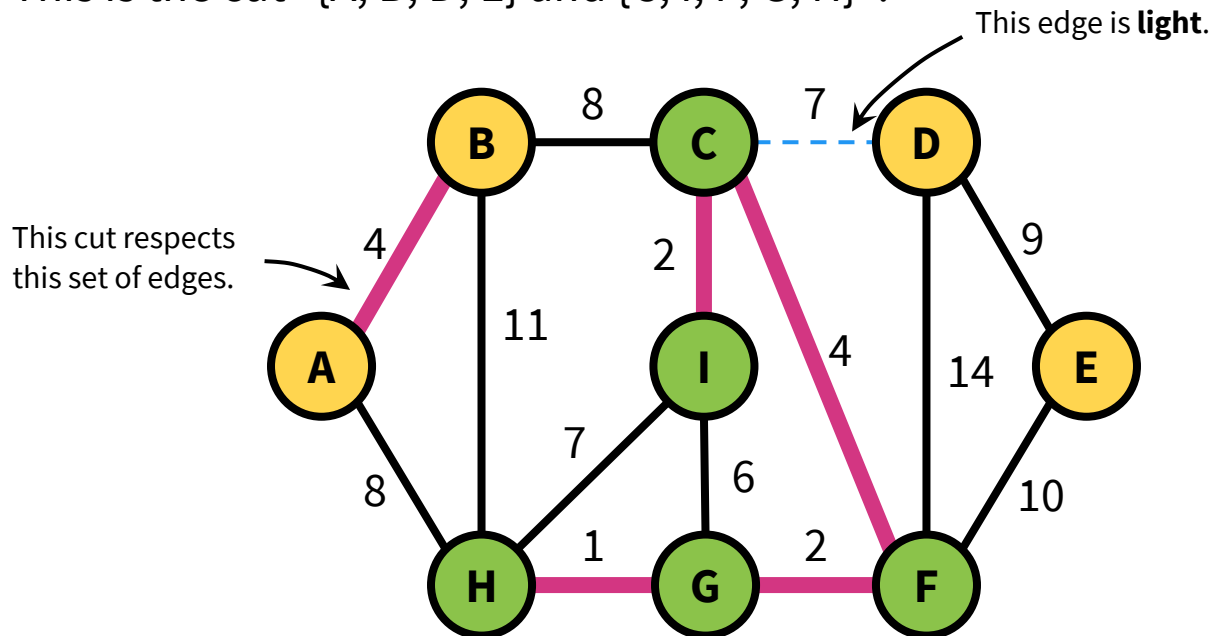
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An edge is **light** if it has the **smallest weight of any edge crossing the cut**. 17

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Lemma

Consider a cut that respects a set of edges **A**.

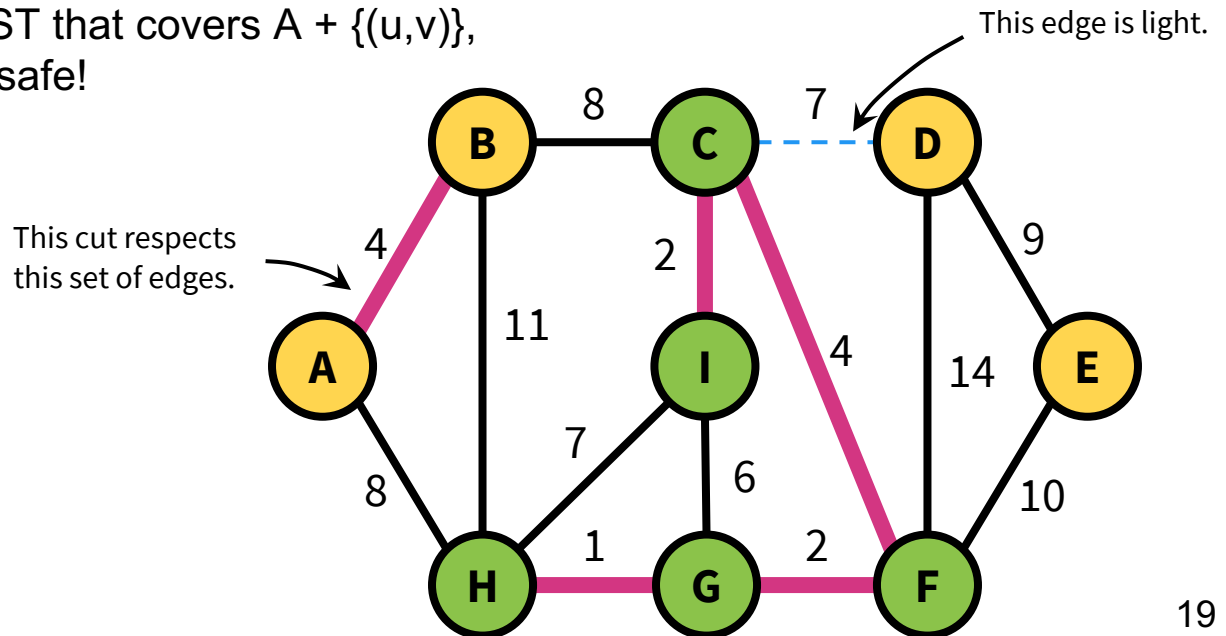
Suppose there exists an MST containing **A**.

Let (u, v) be a light edge.

Then there exists an MST containing $A \cup \{(u, v)\}$.

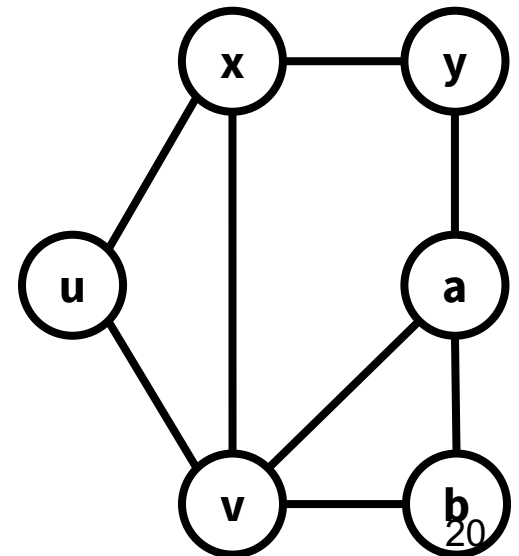
← This is precisely the sort of statement we need for a greedy algorithm: **If we haven't ruled out the possibility of success so far, then adding a light edge won't rule it out.**

If we can find a MST that covers A ,
then we can still find a MST that covers $A + \{(u, v)\}$,
so adding (u, v) would be safe!



Proof of Lemma

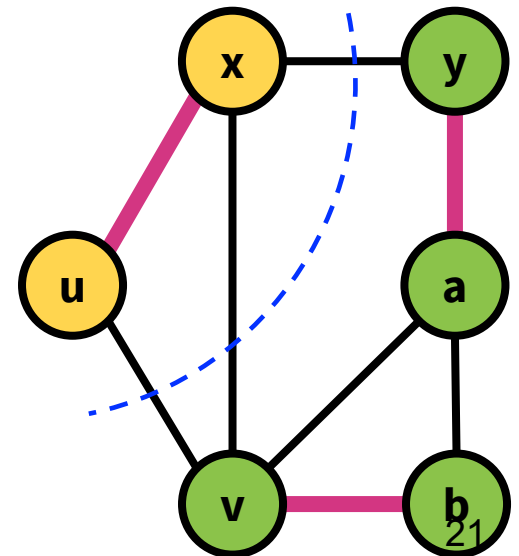
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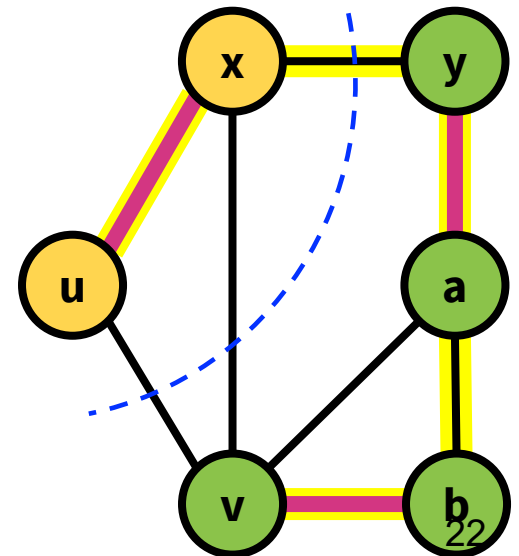
A cut that respects a set of edges **A**



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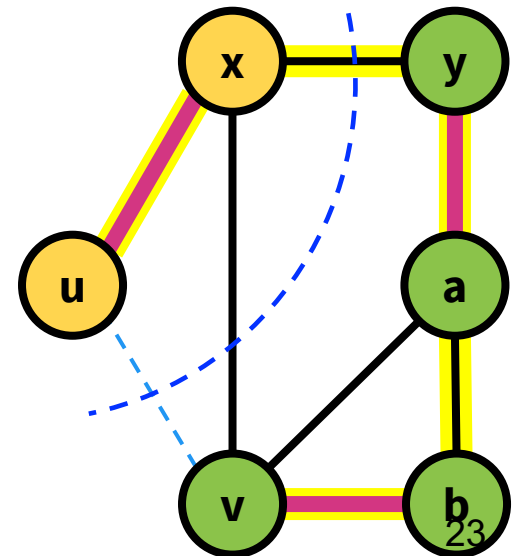
A **cut** that respects a set of edges **A**, if there's an MST **T** containing **A**,



Proof of Lemma

Consider a graph with ...

A **cut** that respects a set of edges **A**, if there's an MST **T** containing **A**,
and if there is a light edge (u, v) not in **T**.

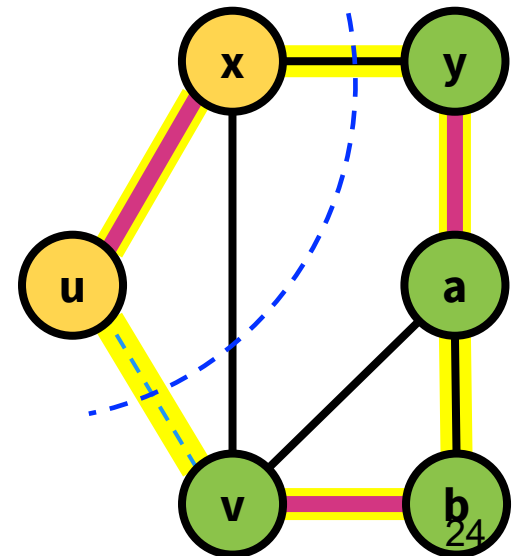


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Adding (u, v) to **T** will make a cycle. Because it has more than $n-1$ edges.



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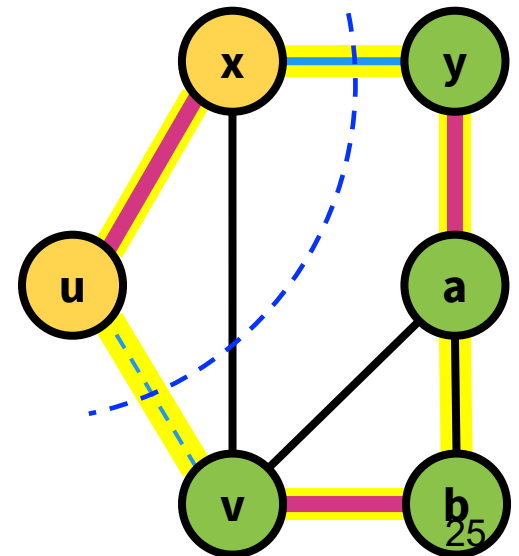
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There must be another edge in this cycle crossing this cut.

Let's call this edge (x, y) .



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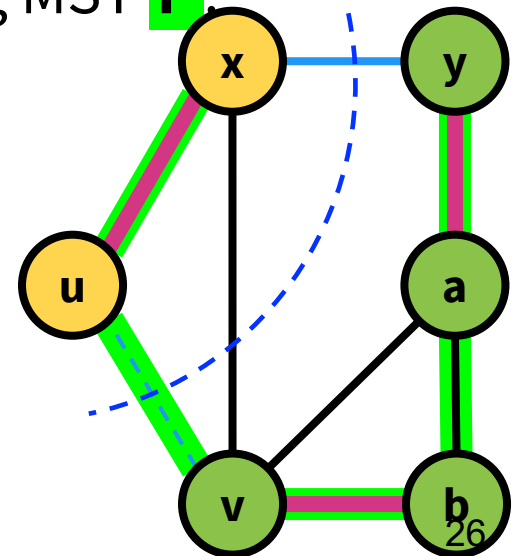
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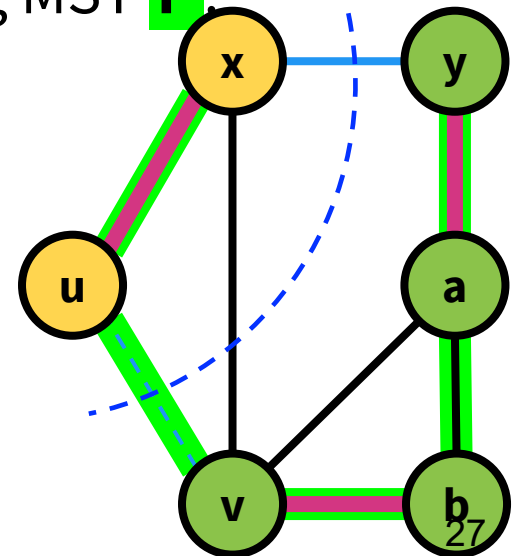
Exchange (u, v) for (x, y) in T ; call the resulting MST T' .

Claim: T' is still an MST.

Since we deleted (x, y) , T' is still a tree.

Since (u, v) is light, T' has cost at most that of T .

Thus, there exists a MST containing
 $A \cup \{(u, v)\}$.



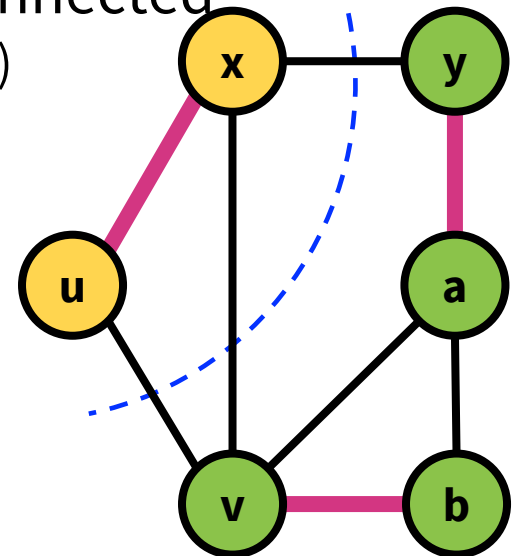
Proof of Lemma

Another (more intuitive) way to understand the lemma

If the nodes are split into two groups by a **cut**, and we have already included the edges in **A** for our MST

Then our MST must include at least one of the edges crossing the **cut**, since otherwise the MST will be disconnected (unable to connect the yellow nodes with green nodes)

Which crossing **cut** edges to select now?



Proof of Lemma

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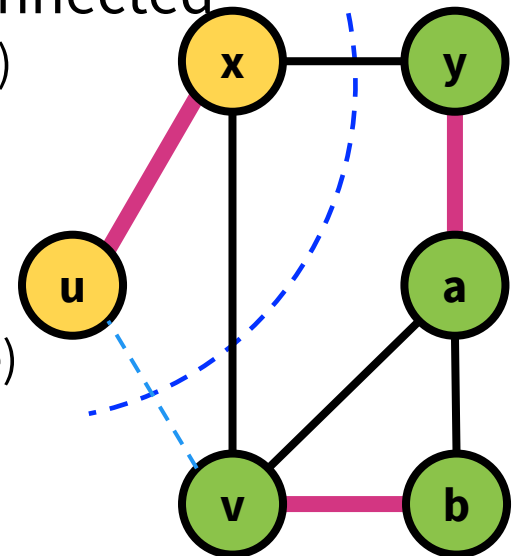
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Which crossing **cut** edges to select now?

The lightest edge!

(greedy strategy since we want minimum spanning tree)



Prim's Algorithm

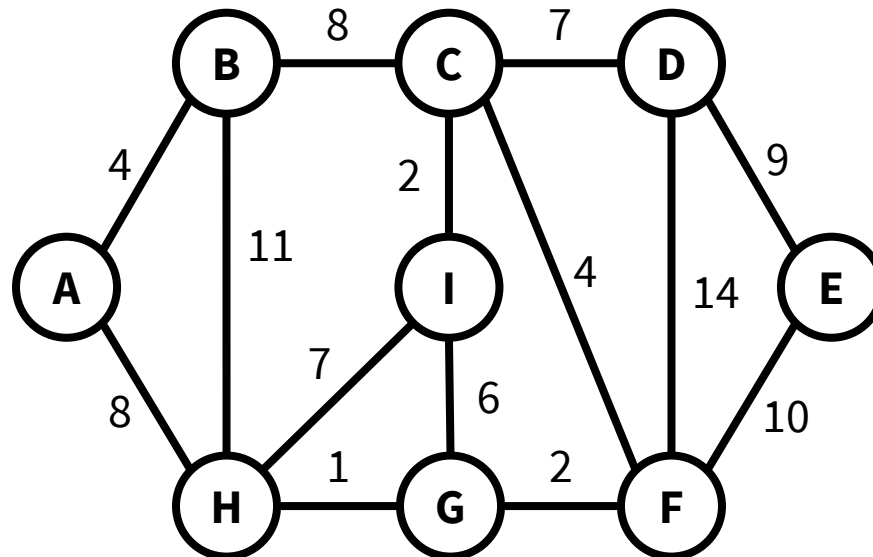
Any Ideas?

Recall our lemma:

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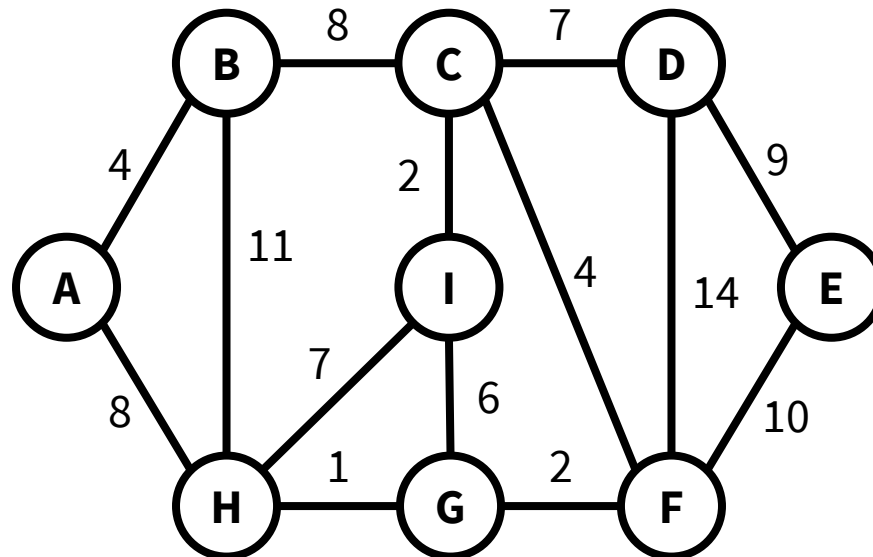
Lemma: There exists an MST containing $\mathbf{A} \cup \{(u, v)\}$.

Any ideas about what to greedily choose?



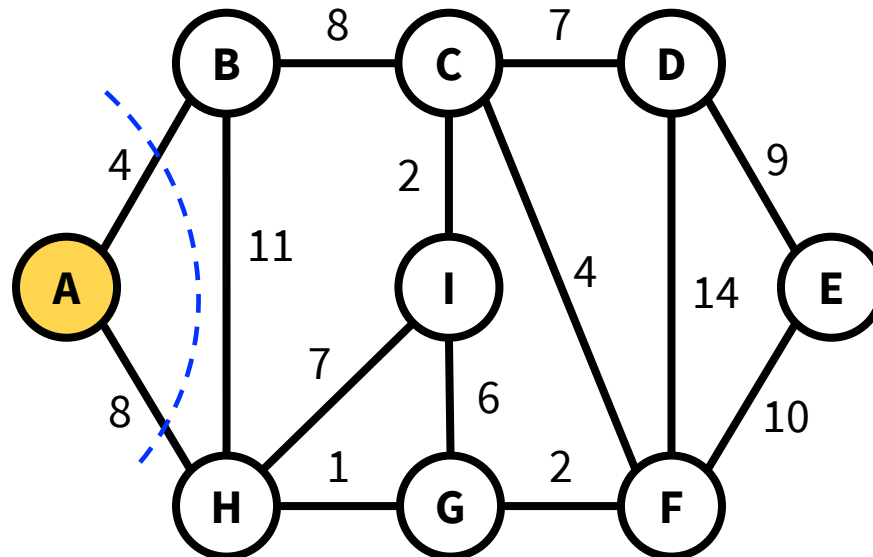
Prim's Algorithm

Main idea: Extend a single tree of visited vertices by greedily adding the closest vertex.



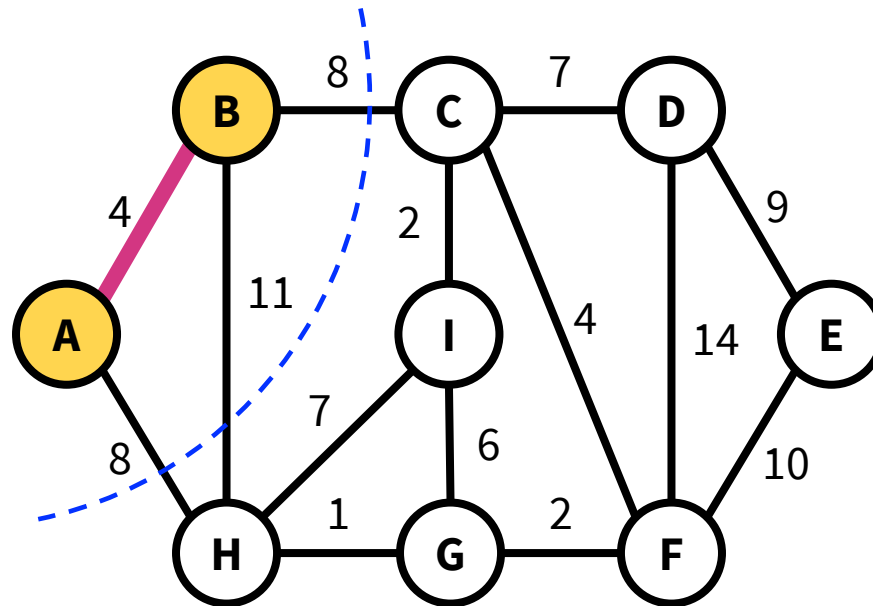
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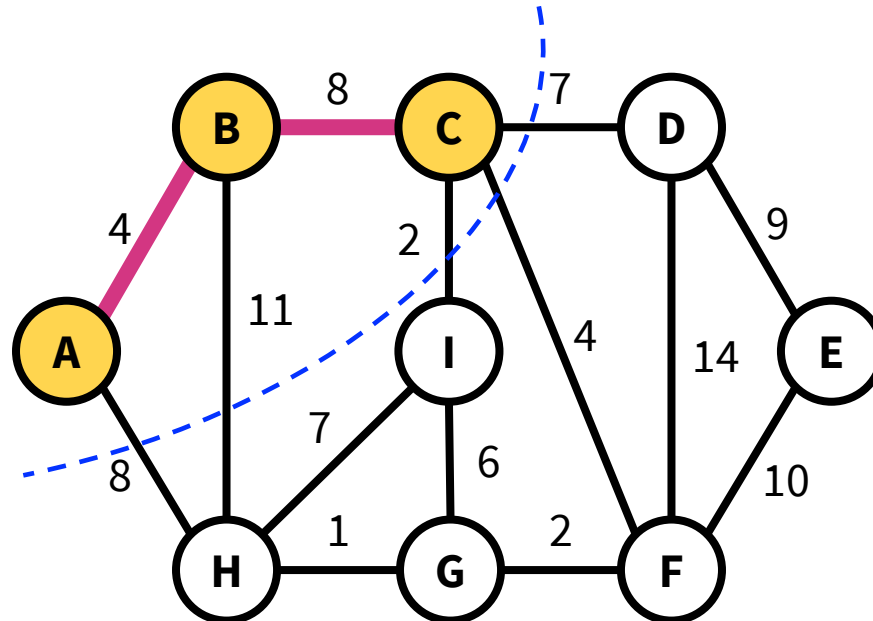
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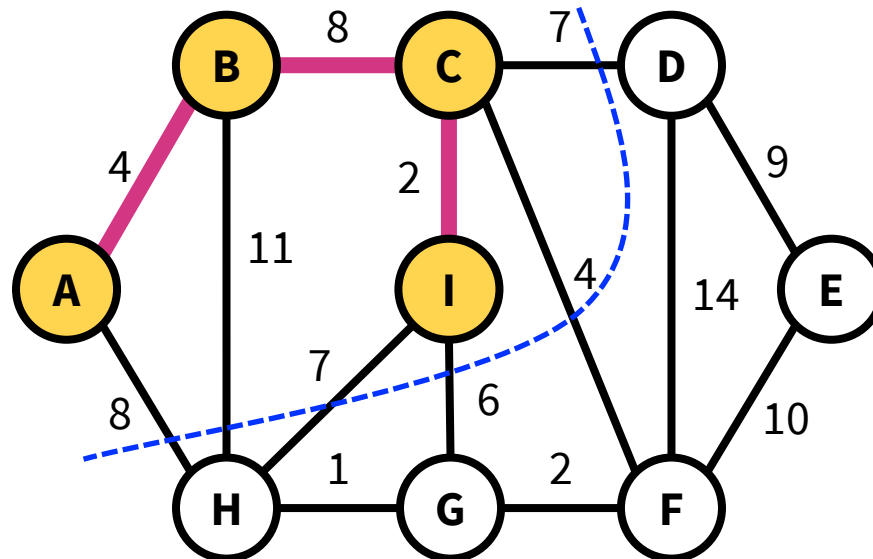
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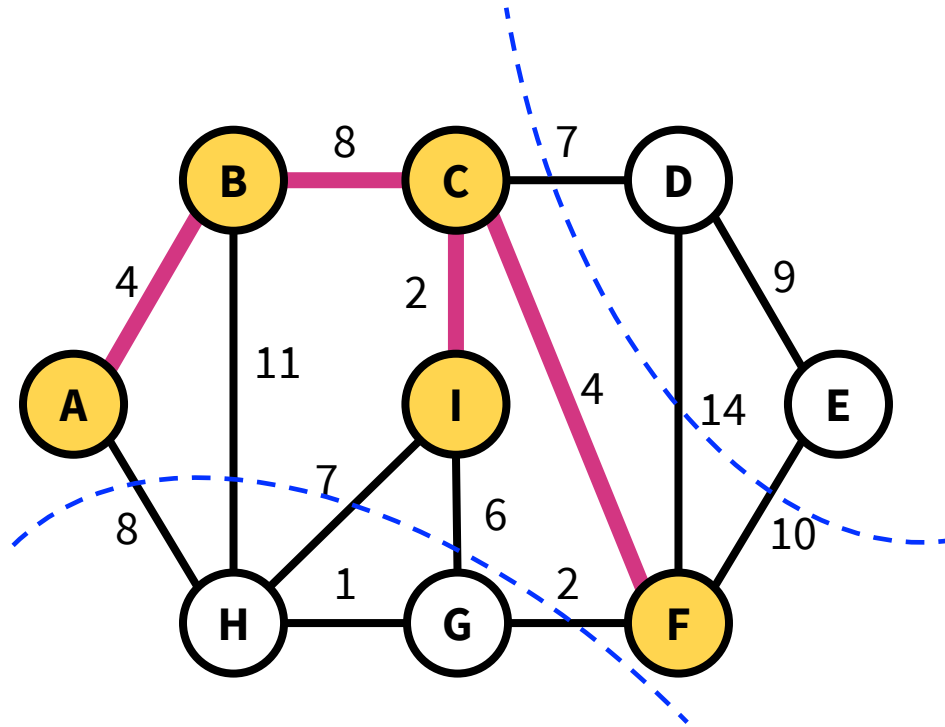
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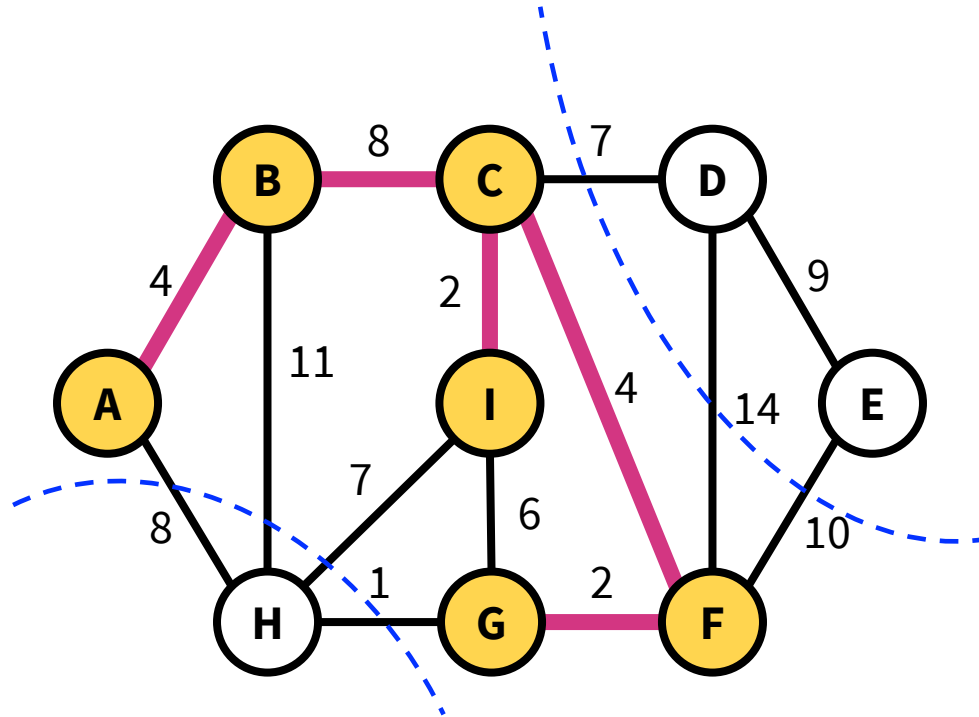
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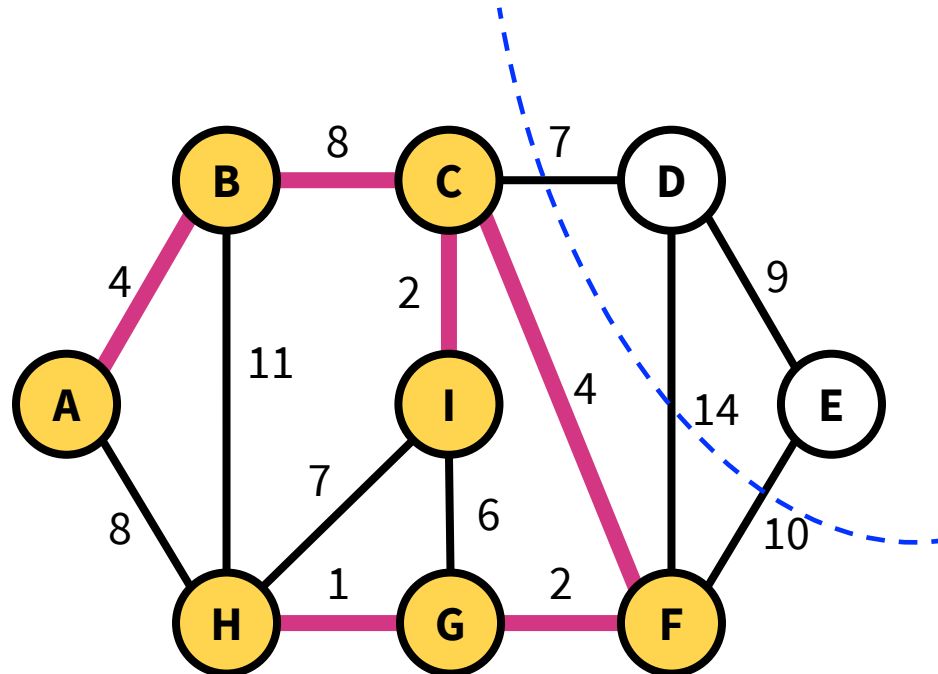
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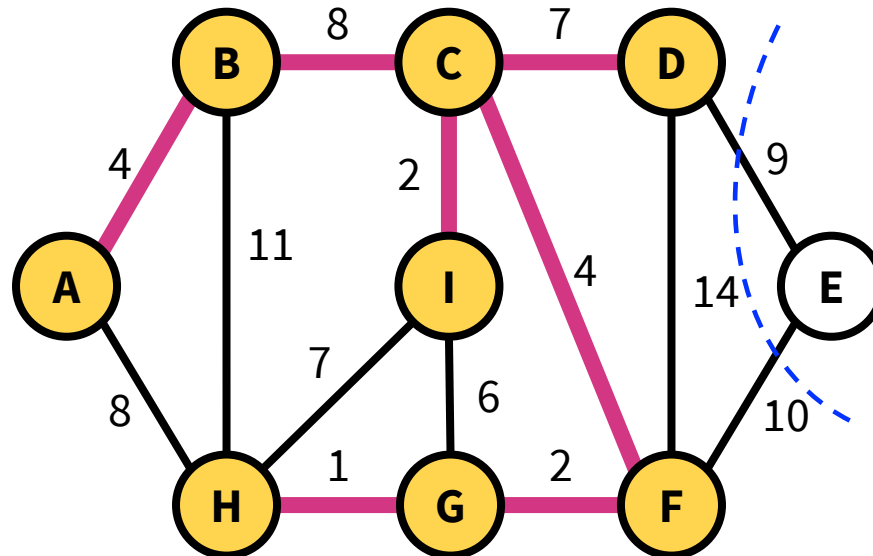
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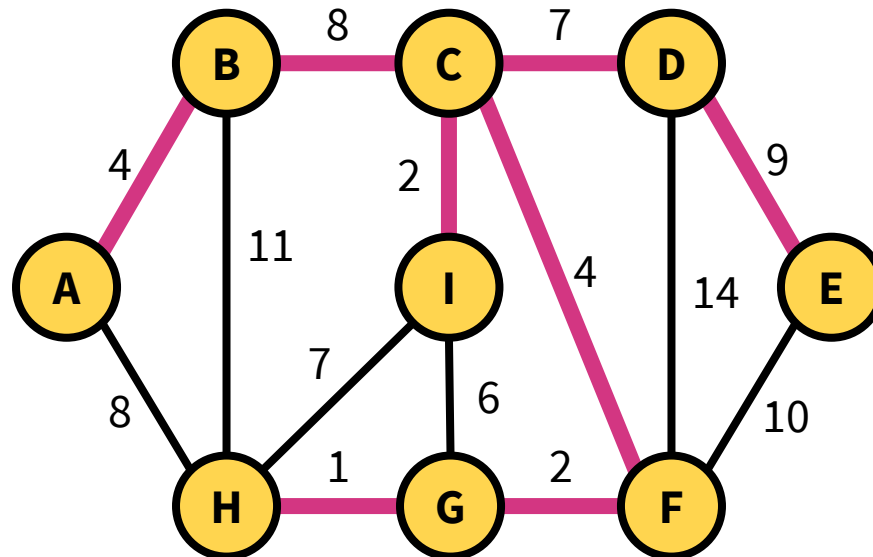
Prim's Algorithm

Main idea: Extend a single tree of visited vertices by greedily adding the closest vertex.



Prim's Algorithm

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Prim's Algorithm

algorithm slow_prim(G):

s = random vertex in G

MST = {}

visited_vertices = {s}

while |visited_vertices| < |V|:

(x, v) = lightest_edge(G, visited_vertices)

MST.add((x, v))

visited_vertices.add(v)

return MST

aka while we
haven't visited all
of the vertices

Finds the lightest
edge (x, v) in E
such that x is in
visited_vertices
and v is not.

Runtime: $O(|V| \cdot |E|)$

For each of the |V|
iterations of the
while loop, might
need to iterate
through all edges.

Proving Feasibility

Theorem: prim finds a feasible spanning tree.

Proof:

To prove this statement, we prove the loop invariant: MST contains edges of a spanning tree of the vertices in `visited_vertices`.

At the start of the first iteration, MST contains no edges, which corresponds to a spanning tree of one vertex.

Now, we prove the inductive step. Suppose that the invariant holds at the start of iteration i , so the edges in MST are (1) acyclic and (2) connect all vertices in `visited_vertices`. Then prim adds an edge (x, v) to MST and vertex v to `visited_vertices`. By construction, v has not been visited yet, so the edges in MST must still be acyclic. Furthermore, v connects to x , which connects to the rest of the vertices in `visited_vertices`; therefore, the edges in MST must still connect all vertices in `visited_vertices`, completing the induction.

At the termination of the loop, `visited_vertices` contains all of the vertices, so MST contains a spanning tree over the entire graph. ■

Proving Optimality

Recall our lemma:

Consider a cut that respects a set of edges A , such that there's an MST T containing A , and a light edge (u, v) not in T .

Lemma: There exists an MST containing $A \cup \{(u, v)\}$.

Theorem: `slow_prim` returns a minimum spanning tree.

Proof:

At the start of the first iteration of the while loop, there exists a minimum spanning tree with the edges in MST. This trivially holds since we initialize MST to the empty set.

Consider the cut of visited vertices and unvisited vertices; MST is respected by this cut. By our lemma, there exists a minimum spanning tree containing $MST \cup \{(x, v)\}$.

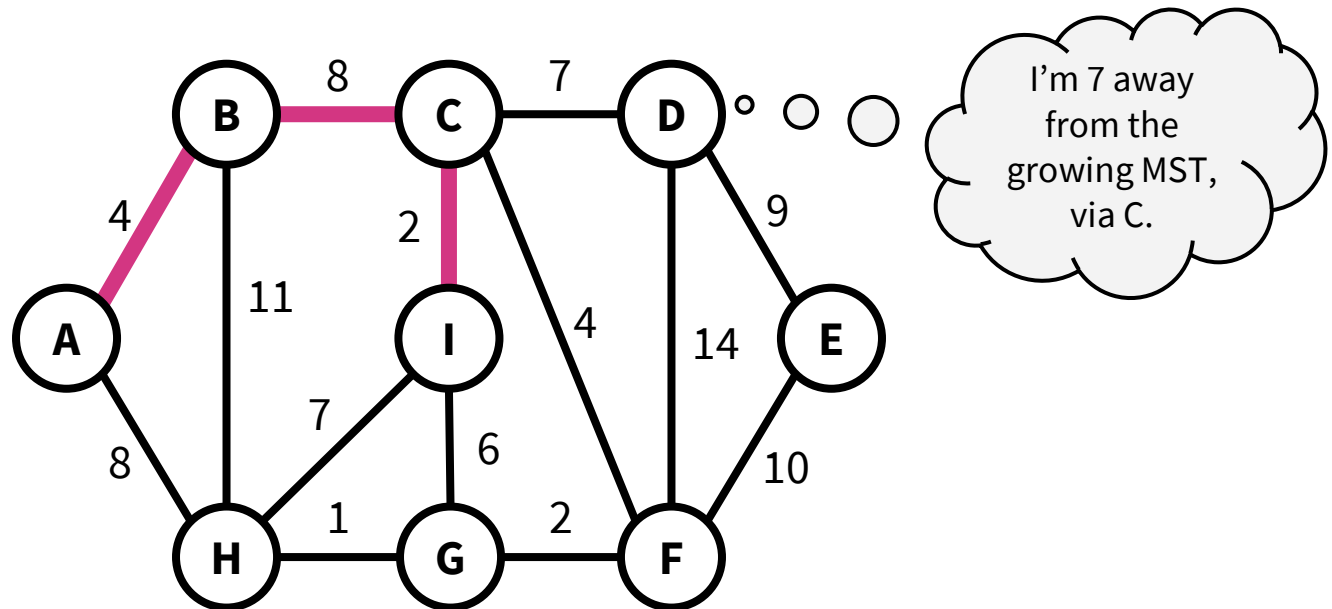
Recall, we proved our lemma with an exchange argument!

After adding the $(n-1)^{\text{st}}$ edge, we have a spanning tree; because each time we add an edge we always add the light edge, therefore, MST contains a minimum spanning tree. ■

Prim's Algorithm

We called the algorithm `slow_prim`. There's a more efficient implementation.

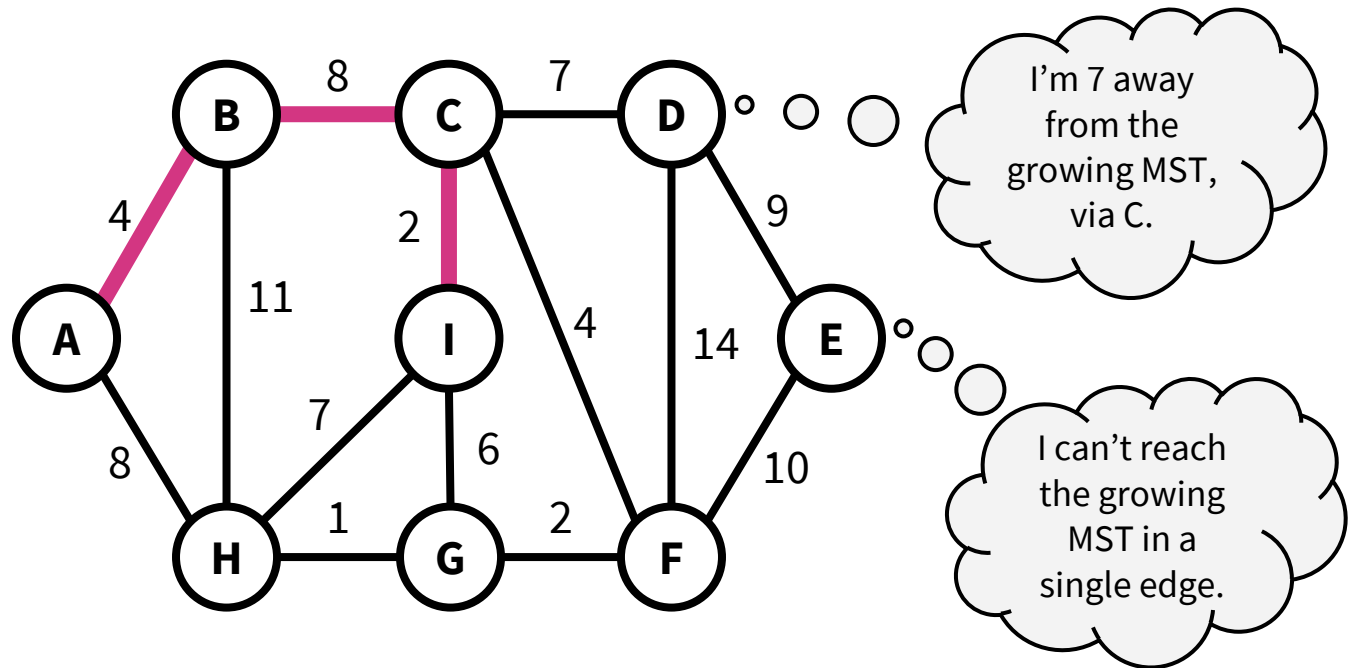
Main idea: vertices maintain information about the distance from itself to the growing spanning tree (if one edge away) and how to get there.



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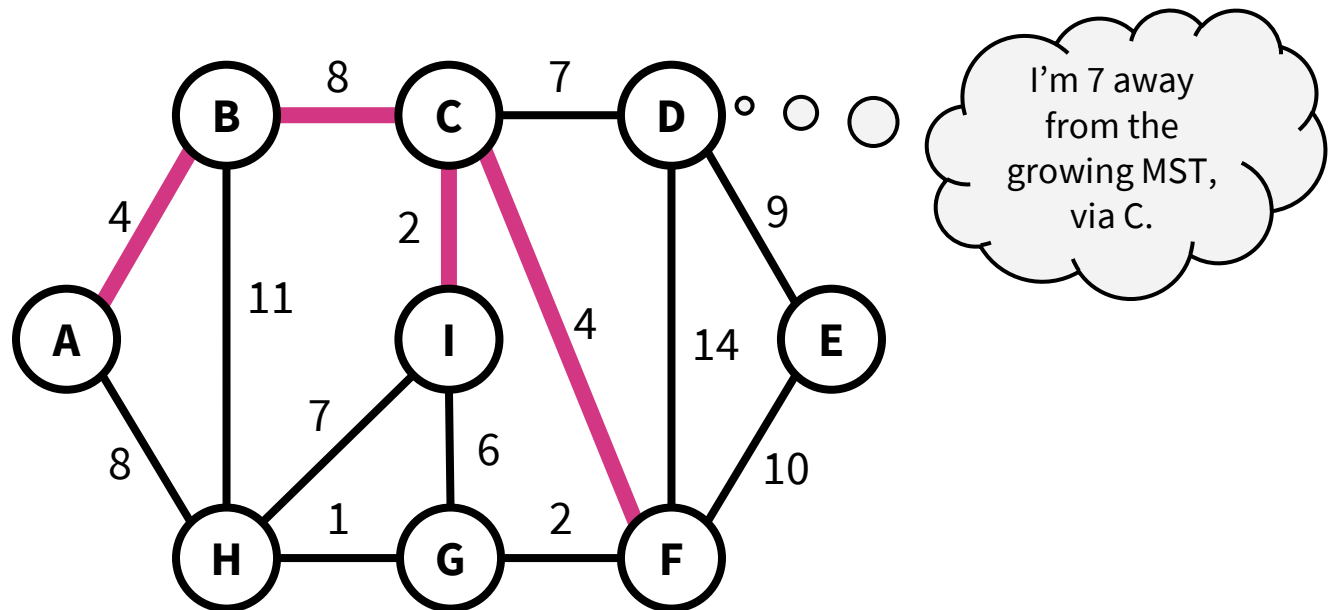
Main idea: vertices maintain information about the distance from itself to the growing spanning tree (if one edge away) and how to get there.



Prim's Algorithm

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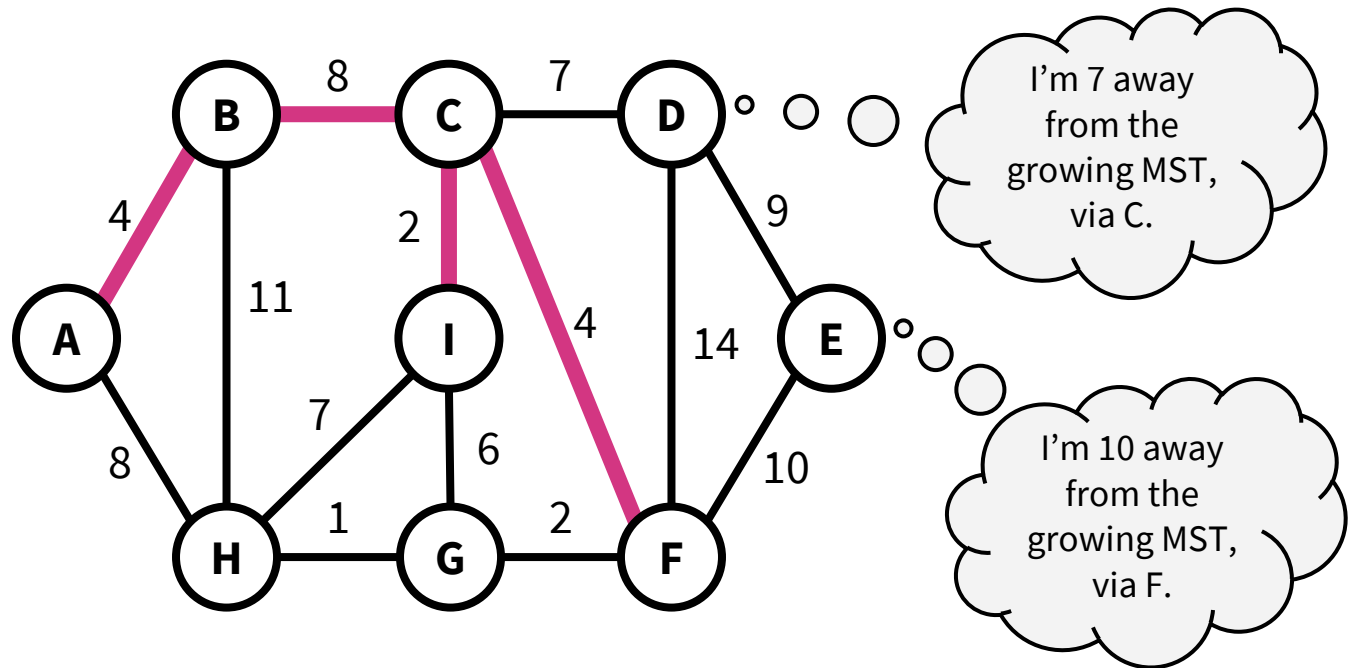
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Prim's Algorithm

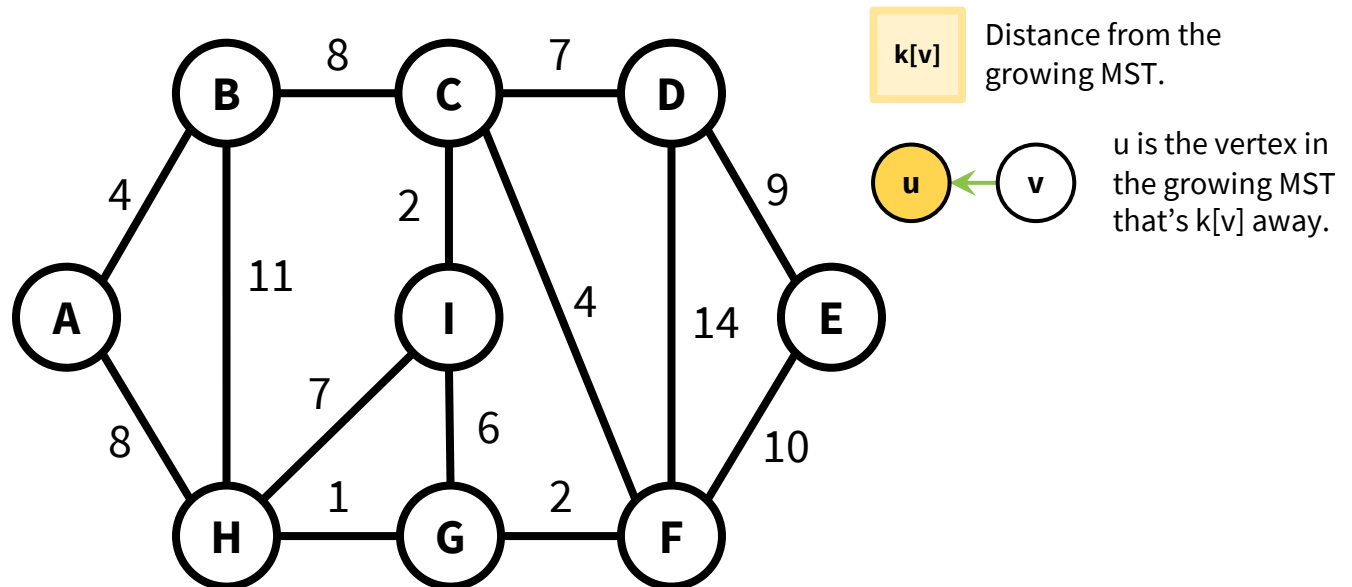
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Prim's Algorithm

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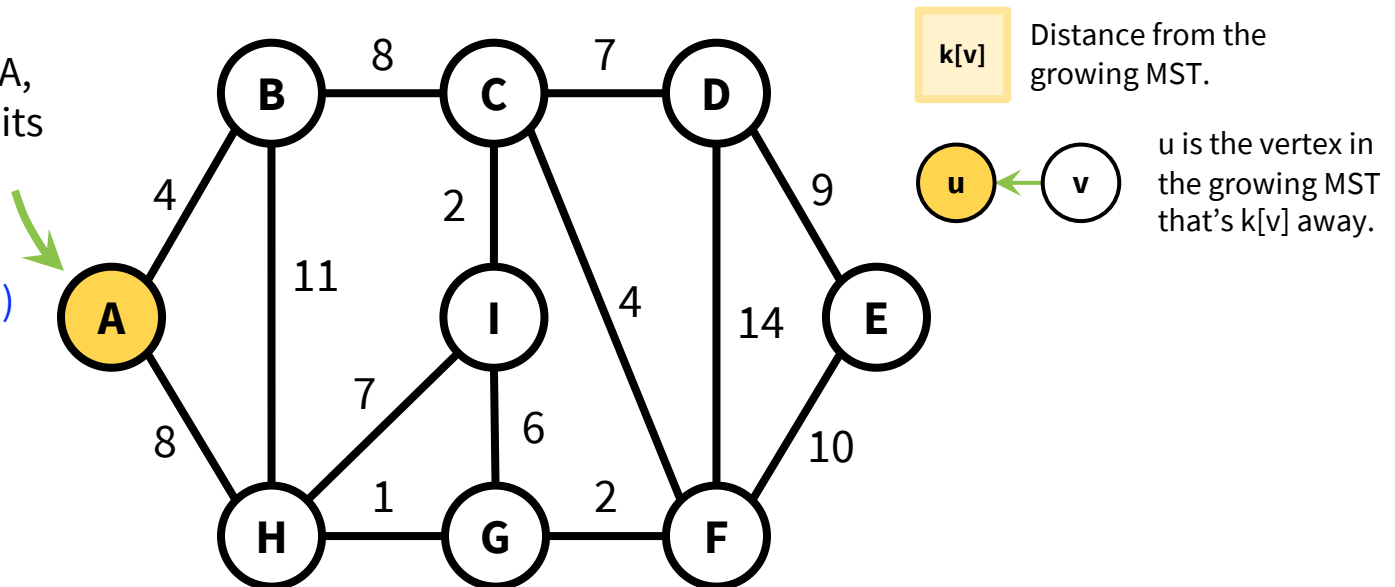


Prim's Algorithm

Main idea: vertices maintain information about the distance from the growing spanning tree and how to get there.

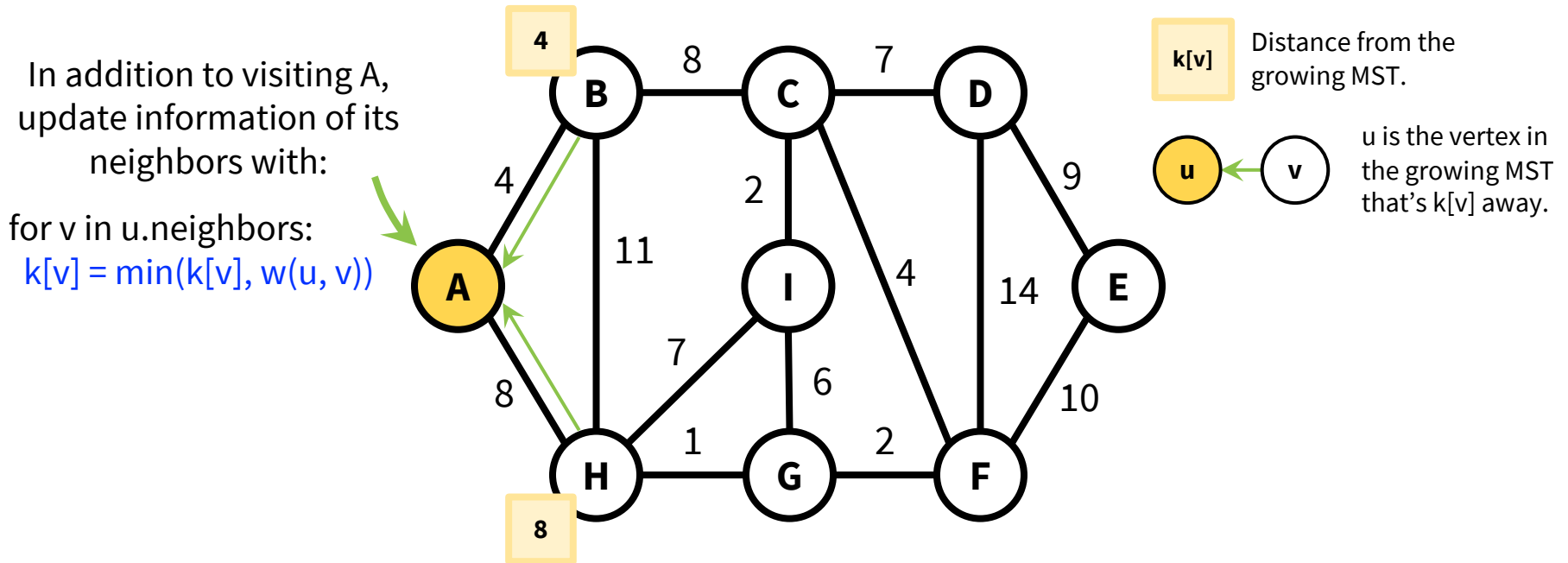
In addition to visiting A,
update information of its
neighbors with:

for v in u .neighbors:
 $k[v] = \min(k[v], w(u, v))$



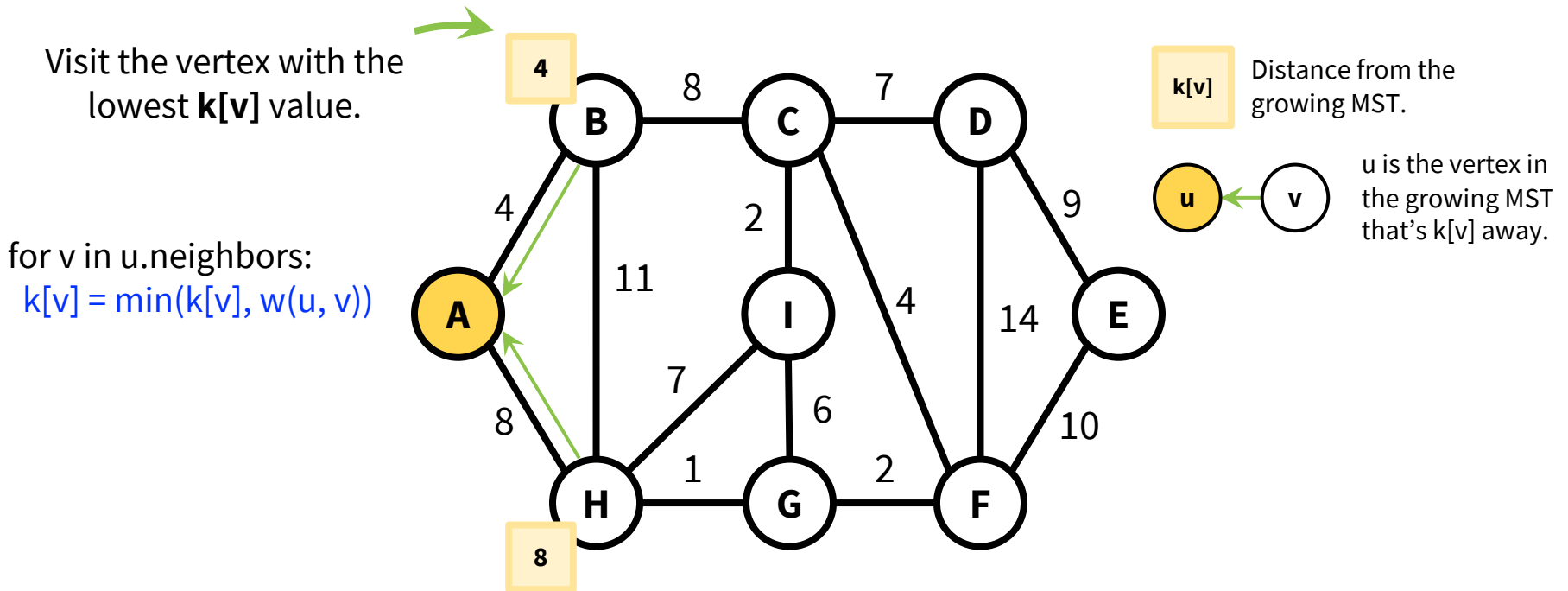
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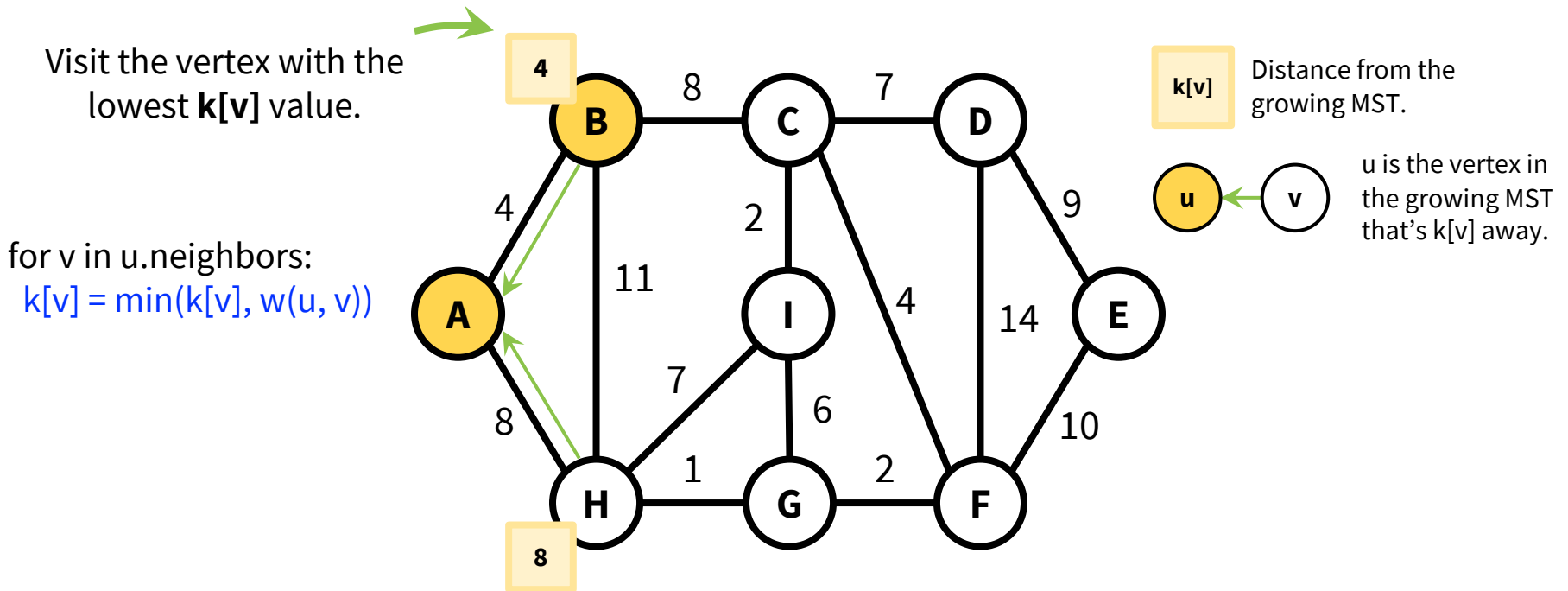
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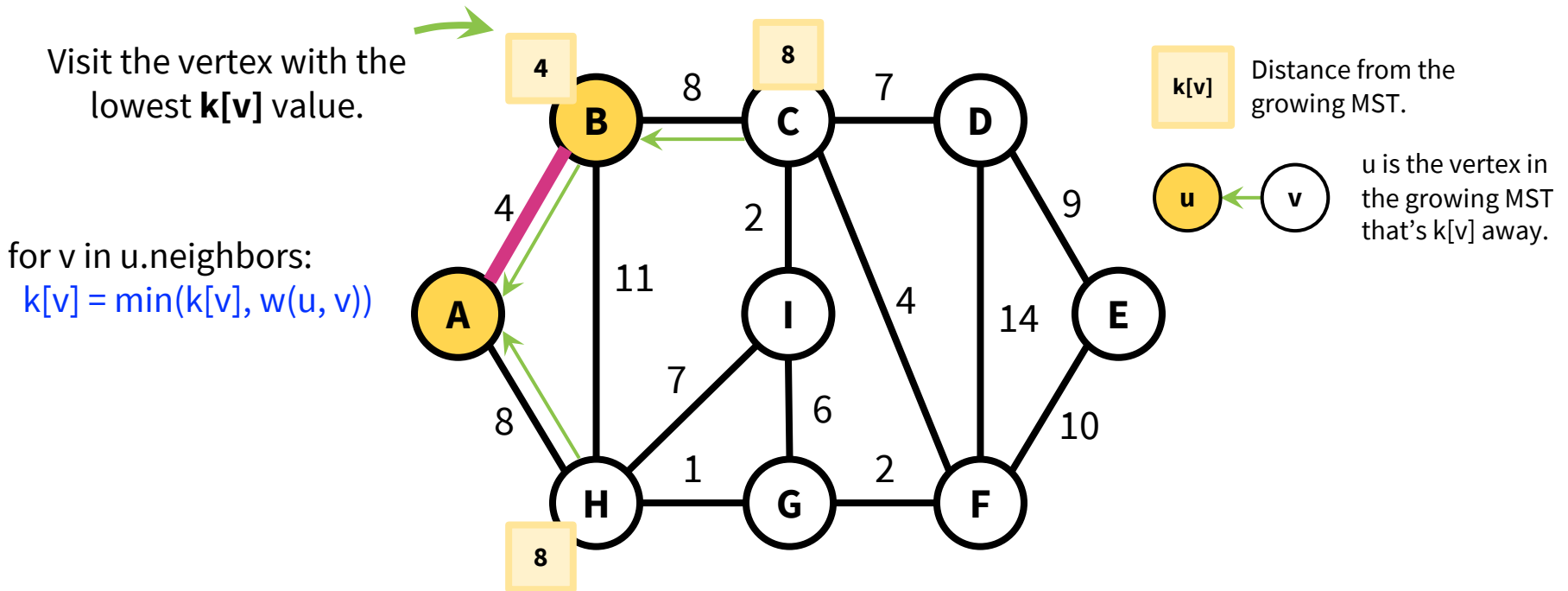
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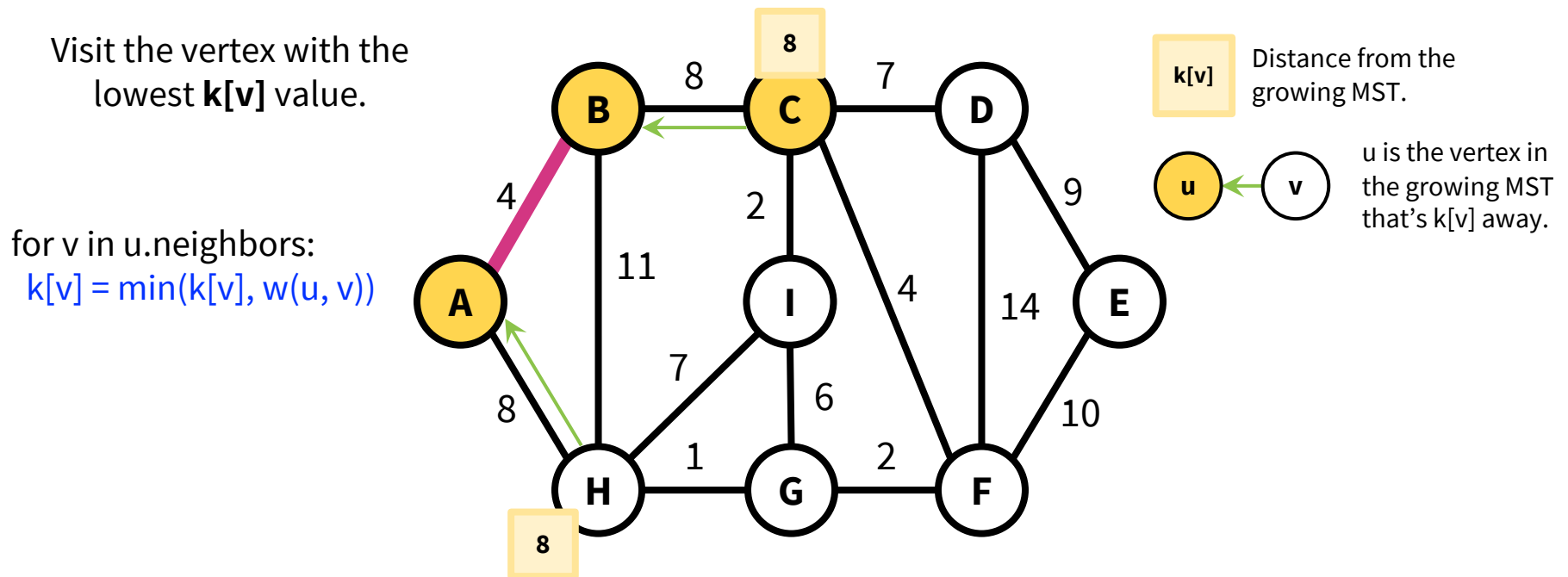
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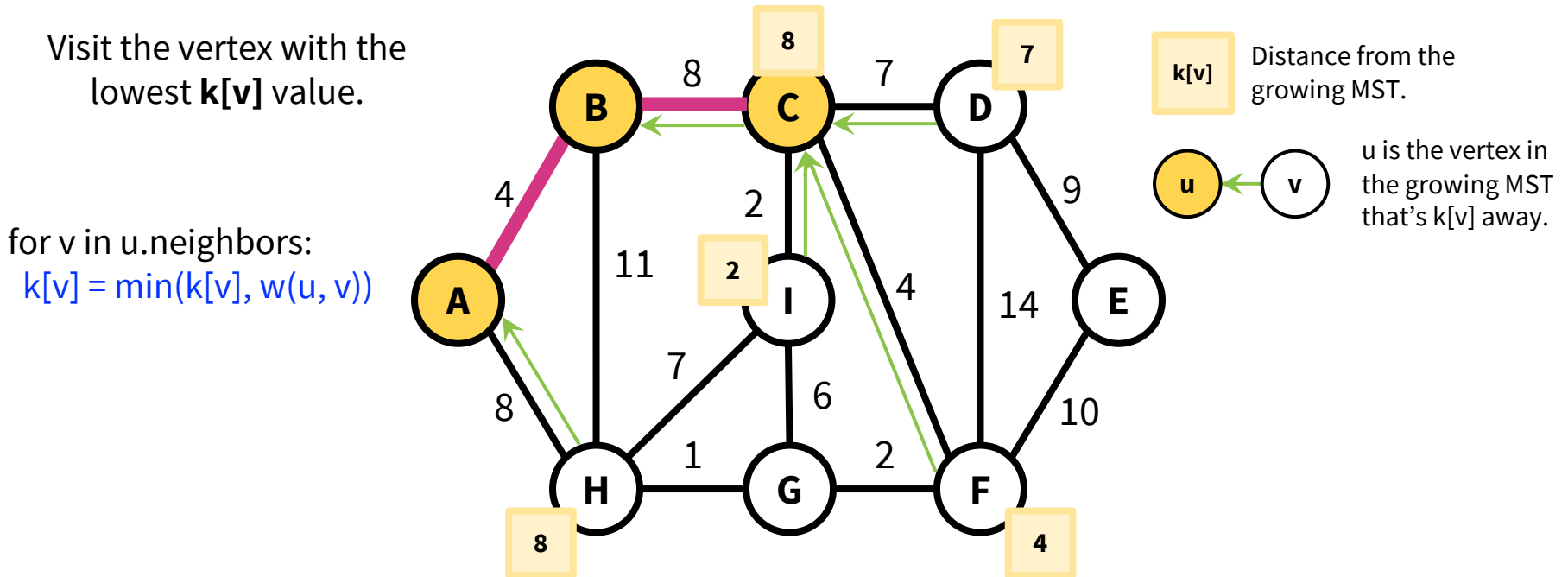
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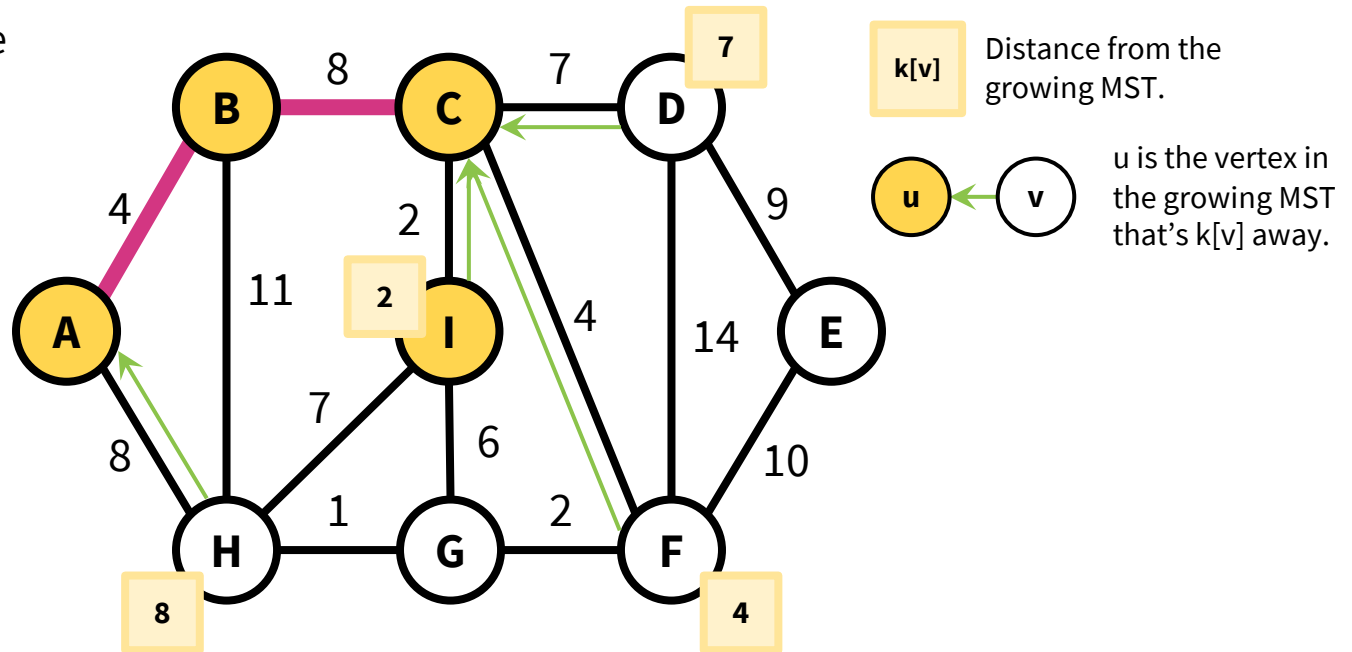


Prim's Algorithm

Main idea: vertices maintain information about the distance from the growing spanning tree and how to get there.

Visit the vertex with the lowest $k[v]$ value.

for v in u .neighbors:
 $k[v] = \min(k[v], w(u, v))$

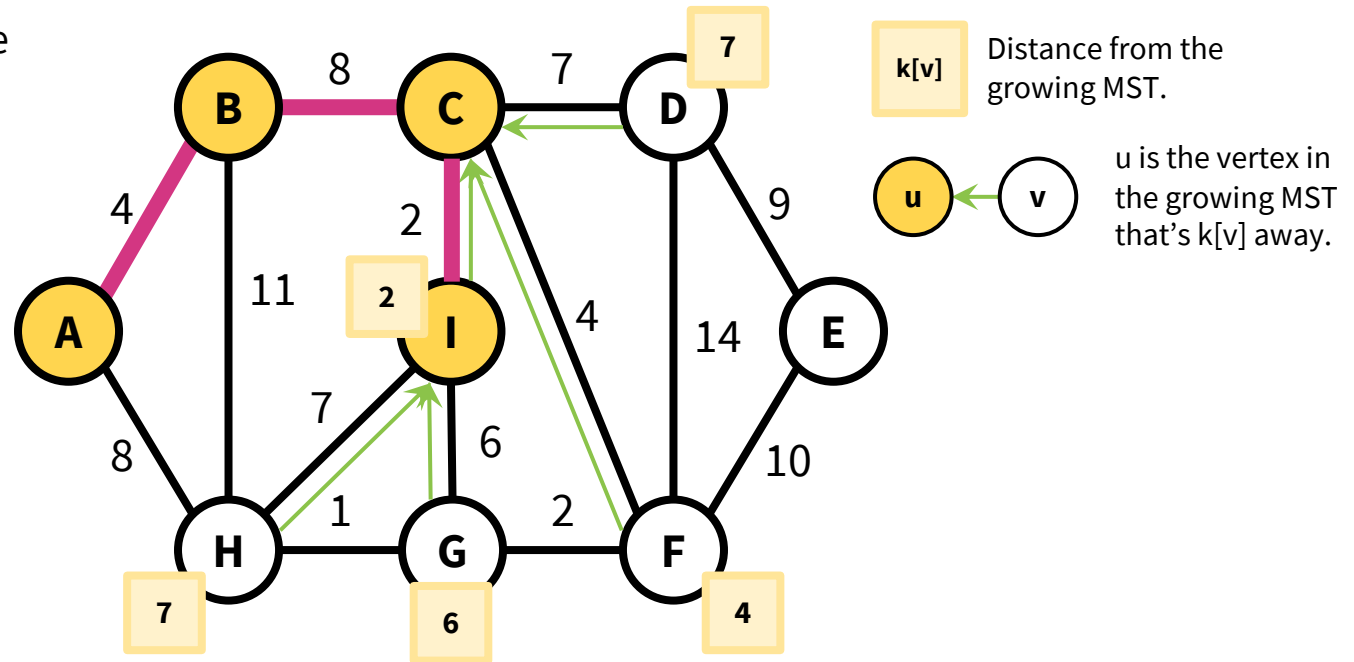


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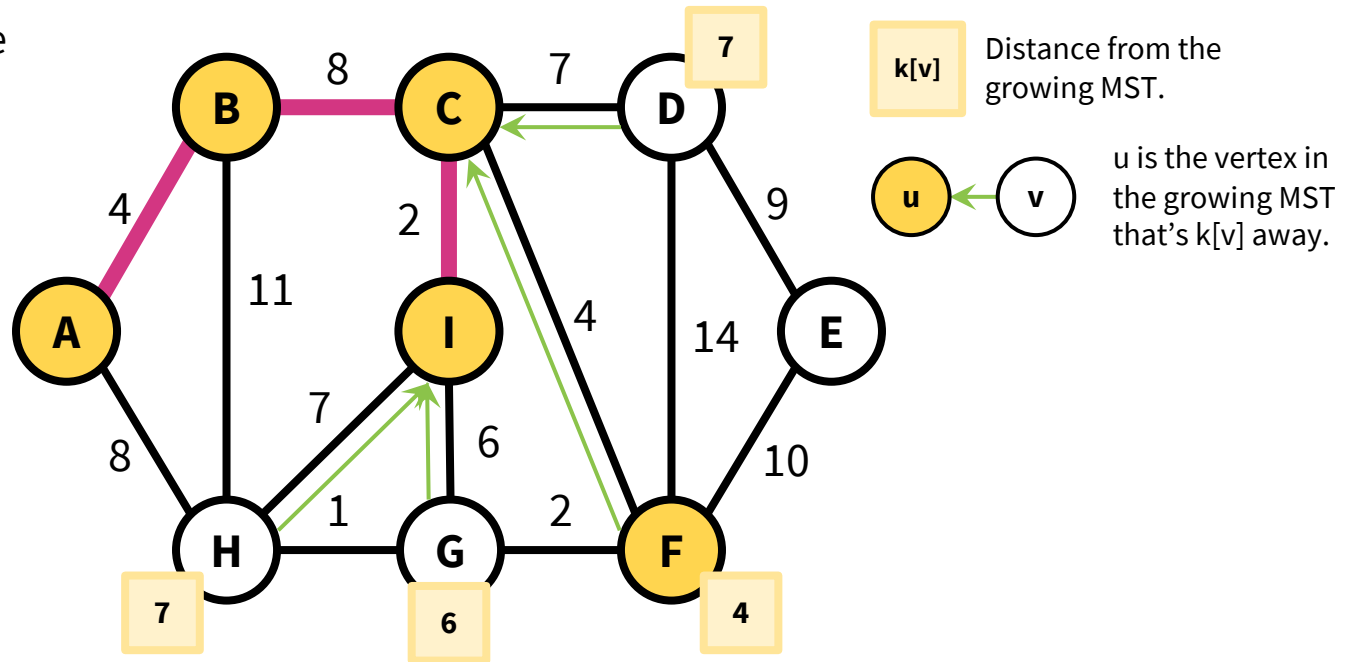


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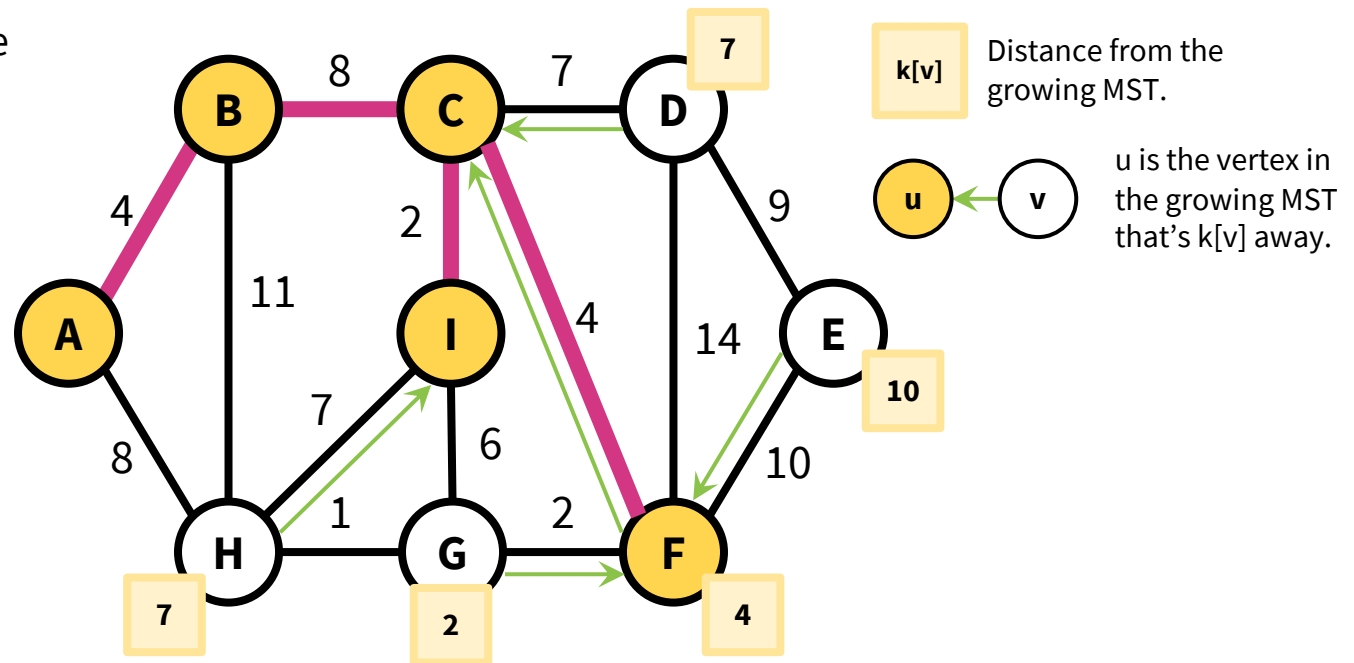


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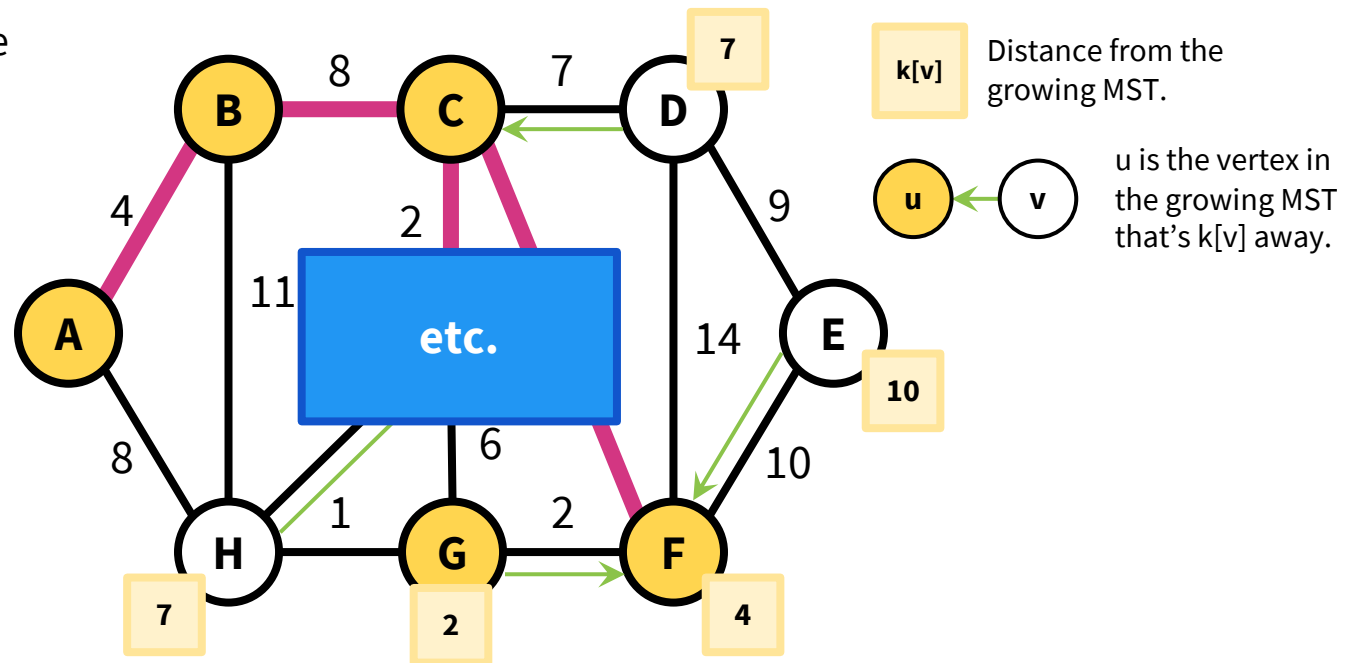


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Prim's Algorithm

```
algorithm prim(G):  
  s = random vertex in G  
  MST = {}  
  visited_vertices = {s}  
  update_info(G, s)  
  while |visited_vertices| < |V|:  
    (x, v) = lightest_edge(G, visited_vertices)  
    MST.add((x, v))  
    visited_vertices.add(v)  
    update_info(G, v)  
  return MST
```

Updates information about distance from the growing MST.

Runtime:

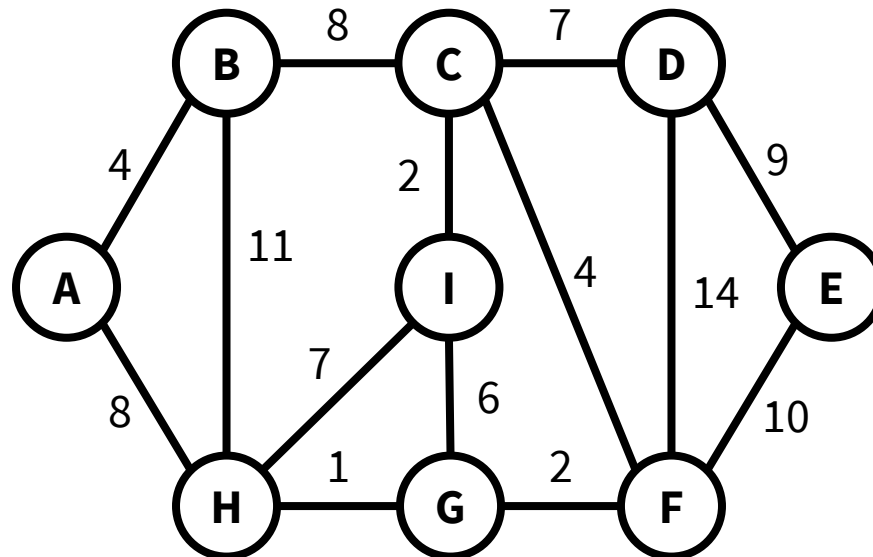
$$O(|V| \log(|V|) + |E|)$$

$|V|$ is the number of while loops, $\log|V|$ comes from picking up the lightest edge using a priority queue implemented by RB-tree. Comes from updating k values of nodes in $\text{update_info}(G, v)$ subroutine. Eventually, each edge will be visited for one and only once.

Kruskal's Algorithm

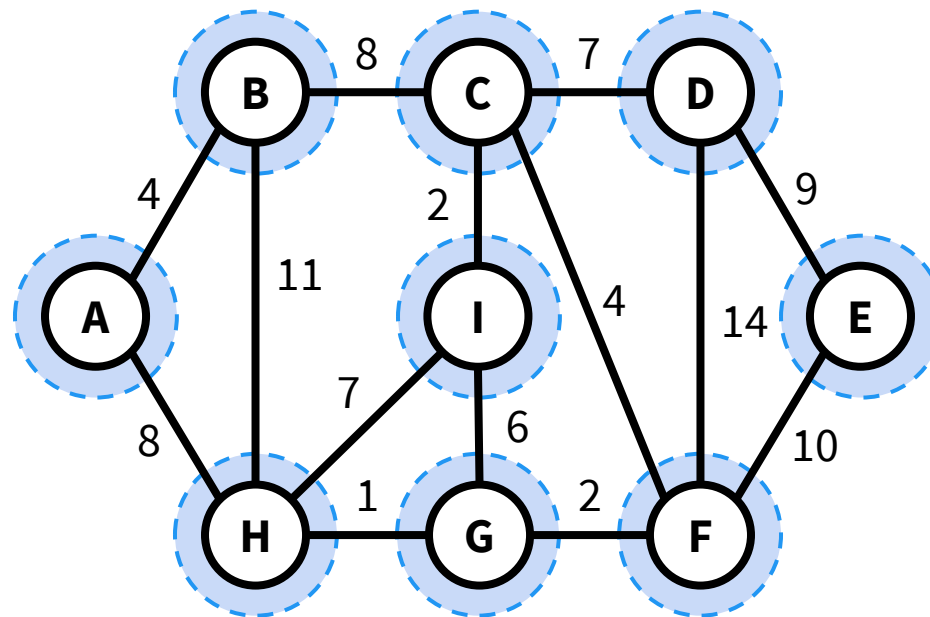
Kruskal's Algorithm

Main idea: Maintain a forest of trees of visited vertices by greedily adding the cheapest edge.



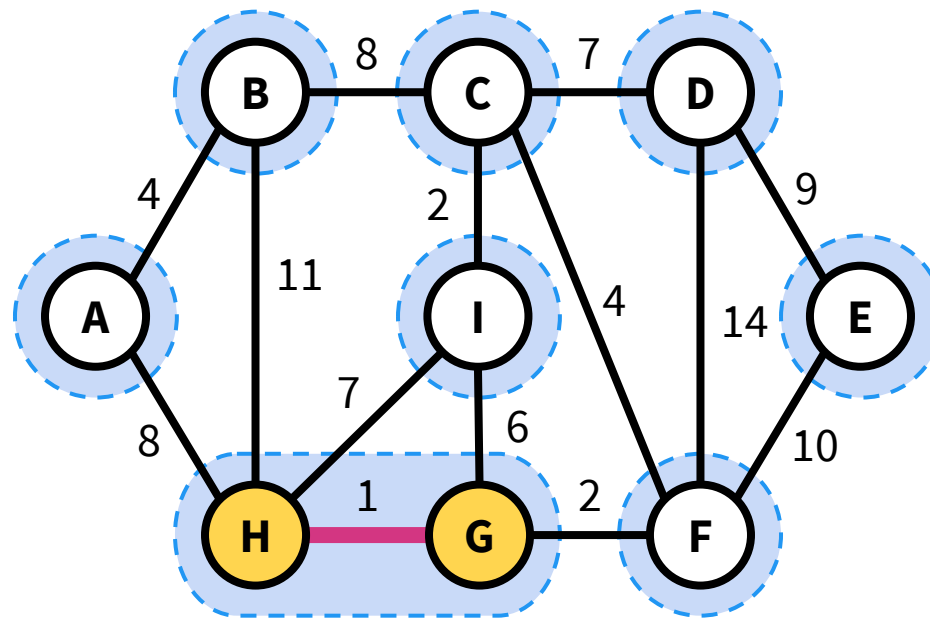
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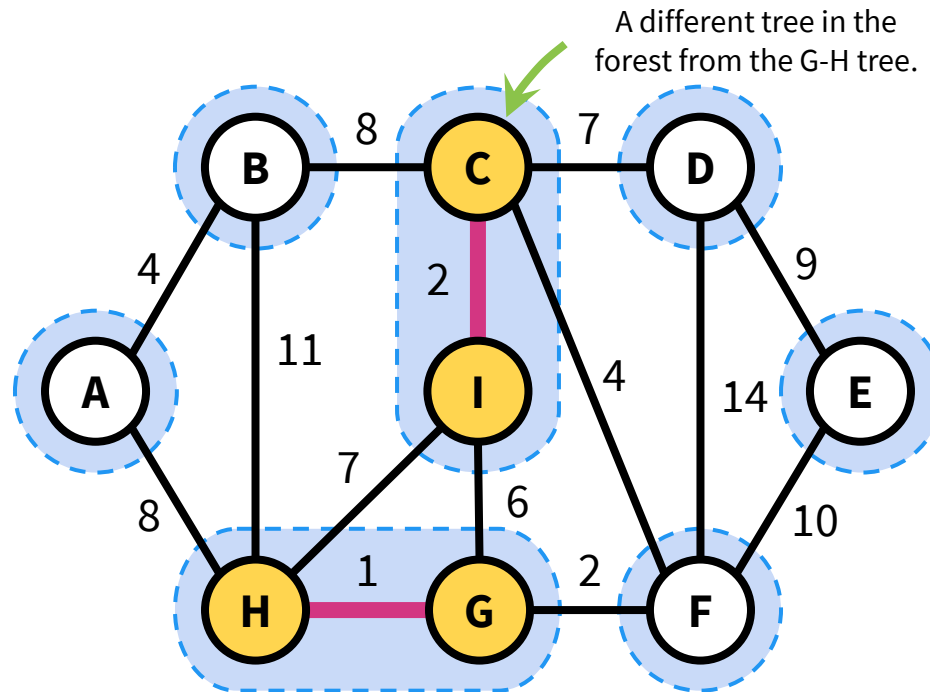
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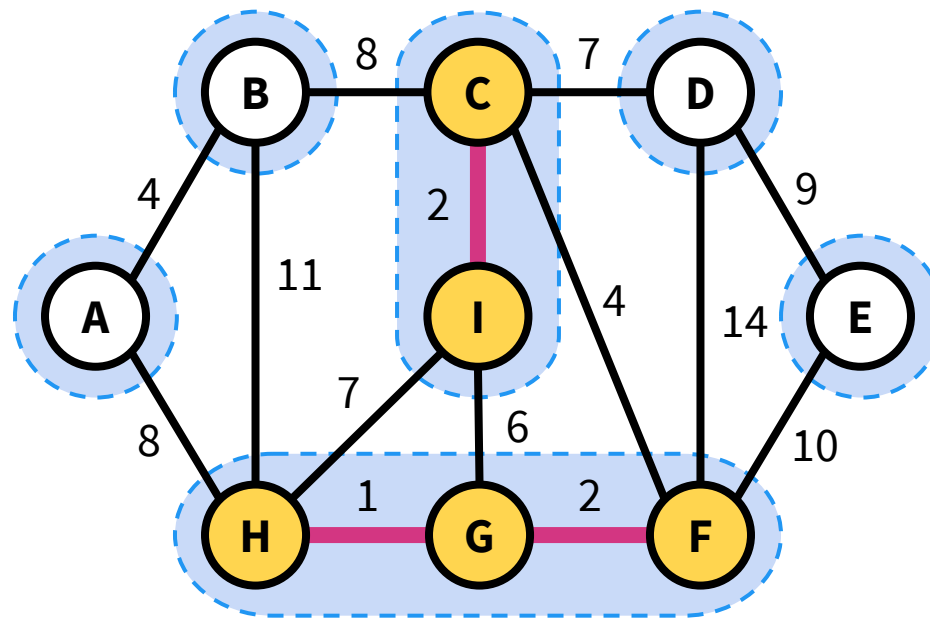
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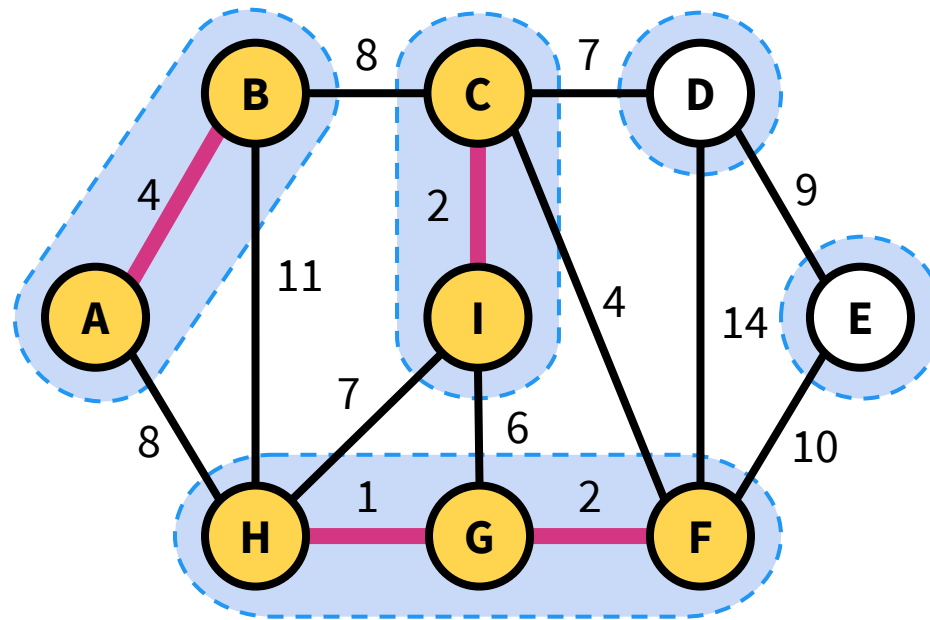
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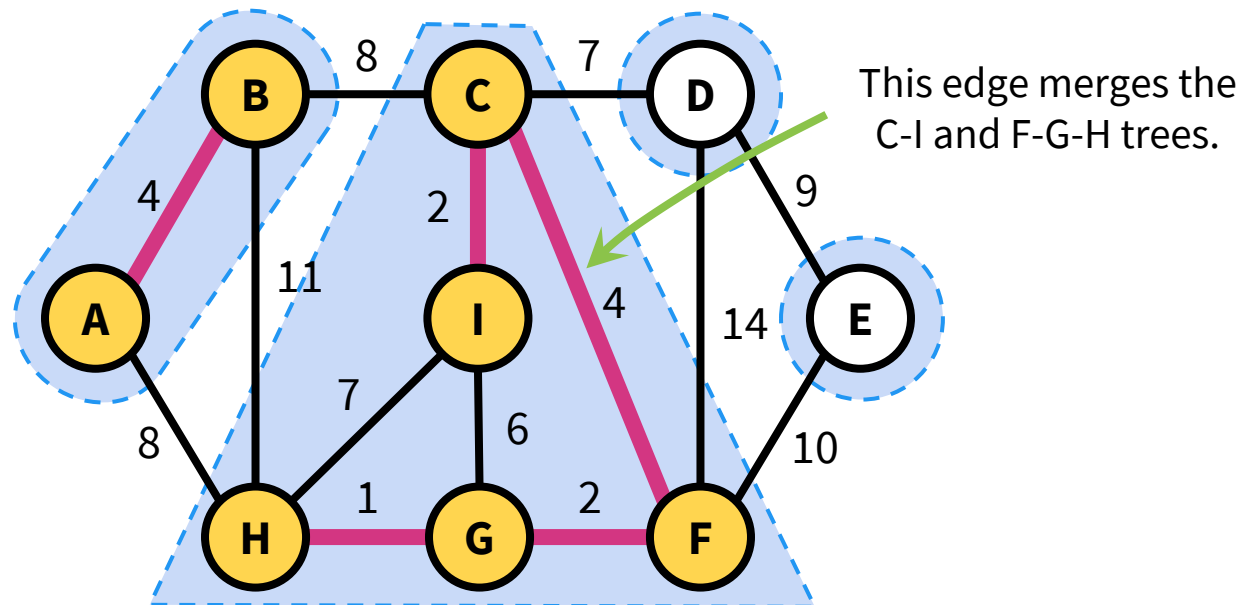
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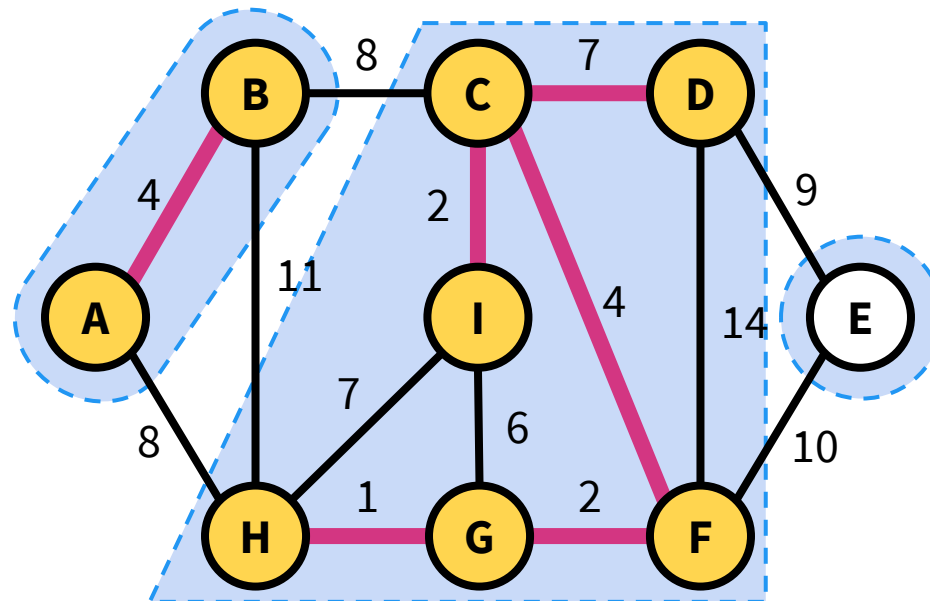
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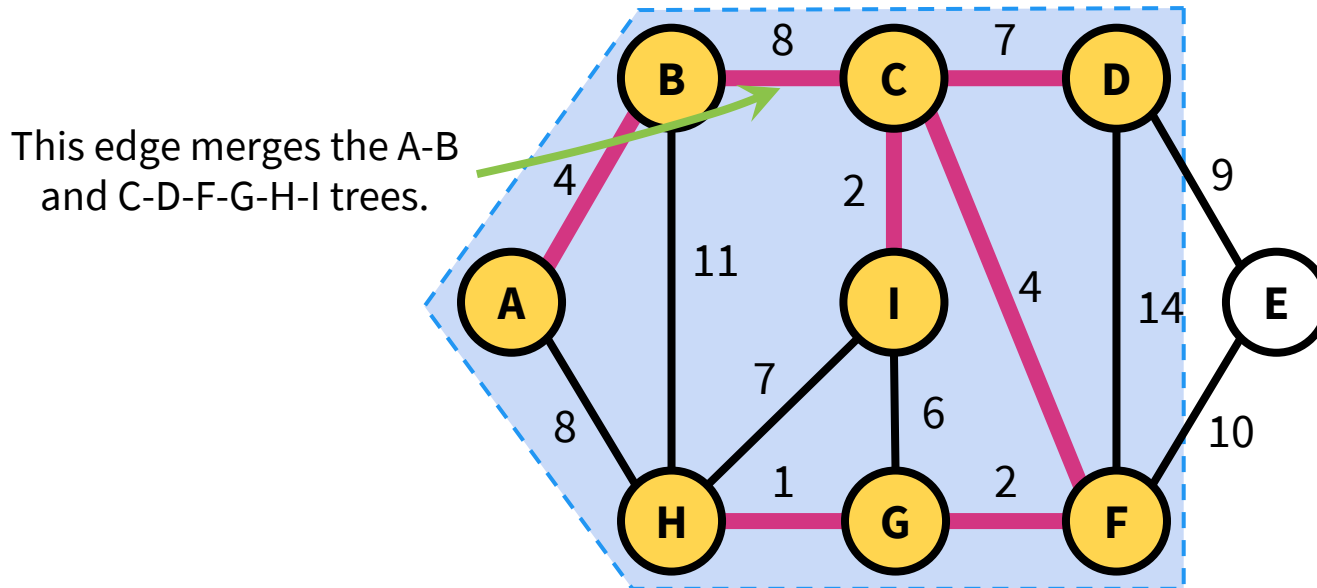
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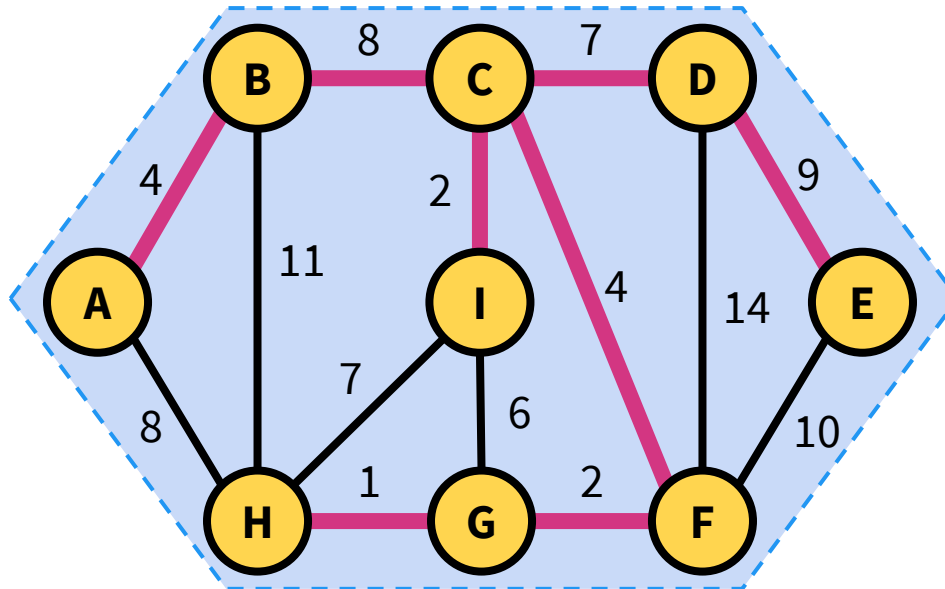
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Kruskal's Algorithm

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Kruskal's Algorithm

kruskal uses union-find data structure, which supports ...

make_set(u): create a set $\{u\}$ in $\mathbf{O(1)}$

find(u): returns the set containing u in $\mathbf{O(1)}$

union(u, v): merges the sets containing u and v in $\mathbf{O(1)}$



Technically, these operations all run in amortized-time $\alpha(|V|)$; $\alpha(n) \leq 4$, provided $n < \#$ of atoms in the universe. We will discuss amortized analysis in greater detail later.

Kruskal's Algorithm

```
algorithm kruskal(G):  
    E_sorted = sort the edges in E by non-decreasing weight  
    MST = {}  
    for v in V:  
        make_set(v) # put each vertex in its own tree  
    for (u, v) in E_sorted:  
        if find(u) != find(v): # u and v in different trees  
            MST.add((u, v))  
            union(u, v) # merge u's tree with v's tree  
    return MST
```

Runtime:

$O(|E| \log(|E|))$

Using comparison-based sort.

$O(|E|)$

Using radix sort

Kruskal's Algorithm

Recall our lemma:

Consider a cut that respects a set of edges A , such that there's an MST T containing A , and a light edge (u, v) not in T .

Lemma: There exists an MST containing $A \cup \{(u, v)\}$.

Theorem: `kruskal` returns a minimum spanning tree.

Proof:

At the start of the first iteration of the while loop, there exists a minimum spanning tree with the edges in MST. This trivially holds since we initialize MST to the empty set.

`kruskal` finds an edge (u, v) that merges two trees T_1 and T_2 . Consider the cut $\{T_1, V - T_1\}$; MST respects this cut. By our lemma, there exists a minimum spanning tree containing $MST \cup \{(u, v)\}$.

Recall, we proved our lemma with an exchange argument!

After adding the $(n-1)^{\text{st}}$ edge, we have a valid spanning tree; and we know there exists a MST containing this spanning tree, therefore, this is exactly a minimum spanning tree. ■

Prim's and Kruskal's

	Description	Runtime	Use-cases
Prim's	Grows a tree	$O(V \log(V)+ E)$ with red-black tree	Better on dense graphs
Kruskal's	Grows a forest	$O(E \log(E))$ with union-find $O(E)$ with union-find and radix sort	Better on sparse graphs and if the edge weights can be radix sorted.


Beyond Prim's and Kruskal's

Karger-Klein-Tarjan (1995): Las Vegas randomized algorithm

$O(|E|)$ expected, $O(\min\{|E|\log(|V|), |V|^2\})$ worst-case

Chazelle (2000): $O(|E|\alpha(|V|))$ deterministic algorithm

Inverse
Ackermann
function




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Acknowledgement: Part of the materials are adapted from Virginia Williams and David Eng's lectures on algorithms. We appreciate their contributions.