2

Chapter

CONTINUOUS PROBABILITY DISTRIBUTION

INSIDE THIS CHAPTER

2.1. CONTINUOUS RANDOM VARIBLE

A random variable X which can take every value in the domain or when its range R is an interval then X is continuous random variable.

Example :

1. Age
2. Height
3. Weight
4. Temperature

2.2. PROBABILITY DENSITY FUNCTION

The probability density function of random variable X is defined as

$$f_x(x) = P(x \le X \le x + \delta x)/\delta x$$

for small interval $(x, x + \delta x)$ of length dx around the point x

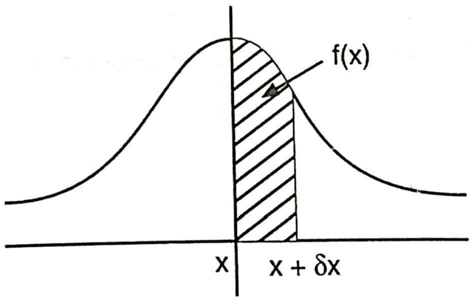

Fig. 2.1

$$P(a \le X \le b) = \int_a^b f(x)dx$$

which represent the area between the curve $y = f(x)$, x axis and the ordinate at $x = a$ and $x = b$ since total probability is unity.

i.e., $$\int_{-\infty}^{\infty} f(x)dx = 1$$

The probability density function (p.d.f) of a random variable X usually denoted by $f_x(x)$ or simply $f(x)$ has following properties.

1. $f(x) \geq 0, -\infty < x < \infty$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

2.3. CUMULATIVE DISTRIBUTION (DISTRIBUTION FUNCTION)

If X is a random variable, then $P(X \leq x)$ is called the cumulative distribution (c.d.f) or simply distribution function and it is denoted by $F(x)$.

$$\therefore \quad F(x) = P(X \leq x)$$

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

2.4. EXPECTATION OF RANDOM VARIABLE

If X is a continuous random variable, then the expectation of the random variable X as defined by

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$$

The expected value of X^2 is defined as

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$

$E(X)$ is also called mean of X

Properties

1. If X is random variable and a is constant then
   (i) $E(a) = a$
   (ii) $E(aX) = aE(X)$
   (iii) $E(X - \bar{X}) = 0$
2. If X and Y are two random variables then
$$E(X + Y) = E(X) + E(Y)$$
3. $E(XY) = E(X)\,E(Y)$ if X and Y are two independent random variable.
4. If $y = \cancel{a + bx}$ where a and b are constants then
ax+b
$$E(Y) = E(aX + b) = aE(X) + b$$

2.5. VARIANCE AND STANDARD DEVIATION OF CONTINUOUS RANDOM VARIABLE

Variance of x is defined as

$$\text{Var}(X) = V(X)$$

$$= E(X - \bar{X})^2 = E(X^2) - [E(X)]^2$$

Standard deviation of random varaible x is denoted by S.D(x) and is defined as

$$\text{S.D.}(x) = \sigma = \sqrt{V(X)} = +\sqrt{E(X)^2 - [E(X)]^2}$$

SOLVED EXAMPLES

Example 2.1. **A continuous random variables X has a probability density function defined by**

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2 & \text{if } -3 \le x - 1 \\ \frac{1}{16}(6-2x^2) & \text{if } -1 \le x < 1 \\ \frac{1}{16}(3-x)^2 & \text{if } 1 < x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

Verify that $f(x)$ is a density function and also find the mean of the random variable X.

Solution. Since $f(x)$ is density function, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-3} f(x)dx + \int_{-3}^{-1} f(x)dx + \int_{-1}^{1} f(x)dx + \int_{1}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{-3} 0.dx + \int_{-3}^{-1} \frac{1}{16}(3+x)^2 dx + \int_{-1}^{1} (6-2x^2)dx + \int_{1}^{3} \frac{1}{16}(3-x)^2 dx + \int_{3}^{\infty} 0.dx$$

$$= \frac{1}{16}\int_{-3}^{-1} (3+x)^2 dx + \int_{-1}^{1} (6-2x^2)dx + \frac{1}{16}\int_{1}^{3} (3-x)^2 dx$$

$$= \frac{1}{16}\left\{\left[\frac{(3+x)^3}{3}\right]_{-3}^{-1} + \left[6x - \frac{2x^3}{3}\right]_{-1}^{1} - \left[\frac{(3-x)^3}{3}\right]_{1}^{3}\right\}$$

$$= \frac{1}{16}\left\{\left[\frac{8}{3} - 0\right] + \left[\left(6 - \frac{2}{3}\right) - \left(-6 + \frac{2}{3}\right) - \left(0 - \frac{8}{3}\right)\right]\right\} = 1$$

$$\therefore \qquad \int_{-\infty}^{\infty} f(x)dx = 1,$$

Hence $f(x)$ is a density function.

Mean of the random variable X is

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \frac{1}{16}\int_{-3}^{-1} x(3+x)^2 dx + \frac{1}{16}\int_{1}^{-1} x(6-2x^2)dx + \frac{1}{16}\int_{1}^{3} x(3-x)^2 dx$$

$$= \frac{1}{16}\int_{-3}^{-1} x\,(9+x^2+6x)\,dx + 0 + \frac{1}{16}\int_{1}^{3} x\,(9+x^2-6x)\,dx$$

since the integrand of the second integral is odd function.

$$= \frac{1}{16}\int_{-3}^{-1}(9x+x^3+6x^2)dx + \frac{1}{16}\int_{1}^{3}(9x+x^3-6x^2)dx$$

$$= \frac{1}{16}\left\{\left[\frac{9x^2}{2}+\frac{x^4}{4}+\frac{6x^3}{3}\right]_{-3}^{-1} + \left[\frac{9x^2}{2}+\frac{x^4}{4}-\frac{6x^3}{3}\right]_{1}^{3}\right\}$$

$$= \frac{1}{16}\left\{\left[\left(\frac{9}{2}-\frac{81}{2}\right)+\left(\frac{1}{4}-\frac{81}{4}\right)+\left(\frac{-6}{3}-\frac{-162}{3}\right)\right]+\left[\left(\frac{81}{2}-\frac{9}{2}\right)+\left(\frac{81}{4}-\frac{1}{4}\right)-\left(\frac{162}{3}-\frac{1}{3}\right)\right]\right\}$$

$$= 0$$

Therefore, the mean of the random variableb X is zero.

Example 2.2. A continuous random variable X has

$$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

represents the density, find the mean and standard deviation of X.

Solution. If $f(x)$ is density function, then it satisfies

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1}\frac{1}{2}(x+1)dx + \int_{1}^{\infty} 0.dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}+x\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1^2}{2}+1\right)-\left(\frac{(-1)^2}{2}-1\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{3}{2}\right)-\left(-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\cdot\frac{4}{2} = 1$$

Hence, $$f(x) = \begin{cases} \frac{1}{2}(x+1) & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$ is a density function.

Mean of the random variable X is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_{-\infty}^{-1} 0.dx + \int_{-1}^{1} x\cdot\frac{1}{2}(x+1).dx + \int_{1}^{\infty} 0.dx$$

$$= \frac{1}{2}\int_{-1}^{1}(x^2+x)\cdot dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{3}-\frac{-1}{3}\right)+\left(\frac{1}{2}-\frac{1}{2}\right)\right]$$

$$= \frac{1}{3}$$

Therefore the mean of the random variable X is $\frac{1}{3}$

∴ The variance of the random variable X is

$$\text{Var}(X) = E(X^2) - \{E(X)\}^2$$

Now

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx$$

$$= \int_{-1}^{1} x^2 f(x)dx$$

$$= \int_{-1}^{1} x^2 \frac{1}{2}(x+1)dx$$

$$= \frac{1}{2}\int_{-1}^{1}(x^3+x^2)dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{1}{4}+\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4}+\frac{1}{3}-\frac{1}{4}+\frac{1}{3}\right]$$

$$= \frac{1}{3}$$

Now, $$Var(X) = E(X^2) - \{E(X)\}^2$$

$$= \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

$\therefore$ Standard deviation of $X = \frac{\sqrt{2}}{3}$

Example 2.3. If the probability density function

$$f(x) = \begin{cases} kx^3 & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value 'k' and find the probability between $x = \frac{1}{2}$ and $x = \frac{3}{2}$.

Solution. From the given data, $f(x) = \begin{cases} kx^3 & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$

If $f(x)$ is a density function, then it satisfies $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{0} f(x)dx + \int_{0}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$$

$$\Rightarrow \qquad k \cdot \int_0^3 x^3 dx = 1 \Rightarrow \left[\frac{x^4}{4}\right]_0^3 = 1 \Rightarrow k\left[\left(\frac{3^4}{4} - 0\right)\right] = 1$$

$$\Rightarrow \qquad \frac{81}{4}k = 1$$

$$\therefore \qquad k = \frac{4}{81}$$

Now, $$f(x) = \begin{cases} \frac{4}{81}x^3 & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

(i) $P\left(\frac{1}{2} \le x \le \frac{3}{2}\right)$

$$P\left(\frac{1}{2} \le X \le \frac{3}{2}\right) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(x)dx$$

$$= \frac{4}{81}\int_{\frac{1}{2}}^{\frac{3}{2}} x^3 dx$$

$$= \frac{4}{81}\left[\frac{x^4}{4}\right]_{\frac{1}{2}}^{\frac{3}{2}} = \frac{1}{81}[x^4]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{81}\left[\left(\frac{3}{2}\right)^4 - \left(\frac{1}{2}\right)^4\right] = \frac{1}{81}\left[\frac{80}{16}\right]$$

$$= \frac{5}{81} = 0.0617$$

Example 2.4. Is the function defined by

$$f(x) = \begin{cases} 0 & \text{if } \quad x < 2 \\ \dfrac{3+2x}{18} & \text{if } 2 \le x \le 4 \\ 0 & \text{if } \quad x > 4 \end{cases}$$

a probability density function? Find the probability that a variate having $f(x)$ as density function will fall in the interval $(2 \le X \le 3)$.

Solution. Given $f(x)$ is

$$f(x) = \begin{cases} 0 & \text{if } \quad x < 2 \\ \dfrac{3+2x}{18} & \text{if } 2 \le x \le 4 \\ 0 & \text{if } \quad x > 4 \end{cases}$$

If it is a density function, then it satisfies $\int_{-\infty}^{+\infty} f(x)dx = 1$

Now,

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{2} f(x)dx + \int_{2}^{4} f(x)dx + \int_{4}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{0} 0dx + \int_{2}^{4}\frac{3+2x}{18}dx + \int_{4}^{\infty} f(x)dx$$

$$= 0 + \frac{3}{18}[x]_2^4 + \frac{2}{18}\left[\frac{x^2}{2}\right]_2^4 + 0 = \frac{1}{6}[4-2] + \frac{1}{18}[4^2 - 2^2]$$

$$= \frac{1}{3} + \frac{2}{3} = 1$$

$$\therefore \quad \int_{-\infty}^{+\infty} f(x)dx = 1$$

Hence,
$$f(x) = \begin{cases} 0 & \text{if } x < 2 \\ \frac{3+2x}{18} & \text{if } 2 \le x \le 4 \\ 0 & \text{if } x > 4 \end{cases}$$ is a density function

Now
$$P(2 \le X \le 3) = \int_2^3 f(x)dx$$
$$= \int_2^3 \frac{3+2x}{18} dx$$
$$= \frac{3}{18}[x]_2^3 + \frac{2}{18}\left[\frac{x^2}{2}\right]_2^3$$
$$= \frac{1}{6}[3-2] + \frac{1}{18}[3^2 - 2^2]$$
$$= \frac{1}{6} + \frac{5}{18} = \frac{8}{18} = \frac{4}{9}$$
$$= 0.44$$

Example 2.5. A continuous random variable X has the probability density function

$$f(x) = \begin{cases} kx & \text{for } 0 \le x < 2 \\ 2k & \text{for } 2 \le x < 4 \\ -kx + 6k & \text{for } 4 \le x < 6 \\ 0 & \text{elsewhere} \end{cases}$$

Find k and mean of the density function.

Solution. If $f(x)$ is a density function, then it satisfies $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^2 f(x)dx + \int_2^4 f(x)dx + \int_4^6 f(x)dx + \int_6^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)dx + \int_0^2 f(x)dx + \int_2^4 f(x)dx + \int_4^6 f(x)dx + \int_6^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0.dx + \int_0^2 kxdx + \int_2^4 2kdx + \int_4^6 k(6-x)dx + \int_6^{\infty} 0.dx = 1$$

$$\Rightarrow k.\int_0^2 xdx + 2k\int_2^4 dx + k\int_4^6 (6-x)dx = 1$$

$$\Rightarrow \quad k\left[\frac{x^2}{2}\right]_0^2 + 2k[x]_2^4 + k\left[6(x)_4^6 - \left(\frac{x^2}{2}\right)_4^6\right] = 1$$

$$\Rightarrow \quad k\left[\frac{4}{2}\right] + 2k[2] + k\left[6(2) - \left(\frac{36}{2} - \frac{16}{2}\right)\right] = 1$$

$$\Rightarrow \quad 2k + 4k + k\,(12 - 10) = 1$$

$$\Rightarrow \quad 8k = 1 \Rightarrow k = \frac{1}{8}$$

Now the probability density function becomes

$$f(x) = \begin{cases} \frac{1}{8}x & \text{for } 0 \le x < 2 \\ 2.\frac{1}{8} = \frac{1}{4} & \text{for } 2 \le x < 4 \\ \frac{1}{8}(6-x) & \text{for } 4 \le x < 6 \\ 0 & \text{elsewhere} \end{cases}$$

Mean of the random variable X is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_{-\infty}^{0} 0.dx + \int_0^2 x\frac{1}{8}x \cdot dx + \int_2^4 \frac{1}{4}x \cdot dx + \int_4^6 \frac{1}{8}(6-x) \cdot xdx + \int_6^{\infty} 0 \cdot dx$$

$$= \frac{1}{8}\int_0^2 x^2 dx + \frac{1}{4}\int_2^4 xdx + \frac{1}{8}\int_4^6 (6x - x^2)dx$$

$$= \frac{1}{8}\left[\frac{x^3}{3}\right]_0^2 + \frac{1}{4}\left[\frac{x^2}{2}\right]_2^4 + \frac{1}{8}\left[6\left(\frac{x^2}{2}\right)_4^6 - \left(\frac{x^3}{3}\right)_4^6\right]$$

$$= \frac{1}{8}\left[\frac{8}{3} - 0\right] + \frac{1}{4}\left[\frac{16}{2} - \frac{4}{2}\right] + \frac{1}{8}\left[6\left(\frac{36}{2} - \frac{16}{2}\right) - \left(\frac{216}{3} - \frac{64}{3}\right)\right]$$

$$= \frac{1}{3} + \frac{1}{4}(6) - \frac{1}{8}\left[60 - \frac{152}{3}\right] = \frac{1}{3} + \frac{3}{2} + \frac{7}{6}$$

$$\Rightarrow \quad x = 3$$

Therefore the mean of random variable X is 3.

Example 2.6. If X is a continuous random variable and K is a constant then prove that: (i) $V(x + k) = V(X)$, (ii) $V(Xk) = k^2\, V(X)$.

Solution. Since $\quad V(X) = E[X^2] - [E(X)]^2$

$$Var\,[X + k] = E\,[(X + k)^2] - [E(X + k))]^2$$

$$= E[X^2 + 2kX + k^2] - [E(X) + k]^2$$

$$= E[X^2] + 2kE(X) + k^2 - \{[E(X)]^2 + 2k[E(X)] + k^2\}$$

$$= E[X^2] + 2kE(X) + k^2 - [E(X)]^2 - 2k[E(X)] - k^2$$

$$E[X^2] - [E(X)]^2 = Var[X]$$

$\therefore$ $$Var[X + k] = Var[X]$$

Now, $$Var[Xk] = E[(Xk)^2] - [E(Xk)]^2$$

$$= E[k^2 X^2] - [kE(X)]^2$$

$$= k^2 E[X^2] - k^2 [E(X)]^2$$

$$= k^2 [E(X^2) - [E(X))^2]$$

$$= k^2 Var[X]$$

$\therefore$ $$Var[Xk] = k^2 Var[X]$$

Example 2.7. If the probability density function of a random variables is given by

$$f(x) = \begin{cases} k(1-x^2) & \text{if } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value of k and the probabilities that a random variable will take on a value (*i*) between 0.1 and 0.2, (*ii*) greater than 0.5, (*iii*) Mean and (*iv*) Variance.

Solution. Since $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \quad \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(xdx + \int_{1}^{\infty} f(x)dx = 1$$

$$\Rightarrow \quad 0 + \int_{0}^{1} k(1-x^2)dx + 0 = 1$$

$$\Rightarrow \quad k\left[x - \frac{x^3}{3}\right]_0^1 = 1 \Rightarrow k\left[1 - \frac{1^3}{3}\right] = 1$$

$$\Rightarrow \quad \frac{2}{3}k = 1 \Rightarrow k = \frac{3}{2}$$

Now the probability density function of the random variables X is

$$f(x) = \begin{cases} \frac{3}{2}(1-x^2) \text{ if } 0 \le x \le 1 \\ 0 \quad \text{elsewhere} \end{cases}$$

(*i*) $$P(0.1 < X < 0.2) = \int_{0.1}^{0.2} f(x)dx = \int_{0.1}^{0.2} \frac{3}{2}(1-x^2)dx = \frac{3}{2}\int_{0.1}^{0.2}(1-x^2)dx$$

$$= \frac{3}{2}(0.0977) = 0.1466$$

(ii) $$P(X > 0.5) = \int_{0.5}^{\infty} f(x)dx = \frac{3}{2}\int_{0.5}^{1}(1-x^2)dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.5}^{1} = \frac{3}{2}\left[\left(1-\frac{(1)^3}{3}\right)-\left(0.5-\frac{(0.5)^3}{3}\right)\right]$$

$$= \frac{3}{2}(0.208) = 0.3125$$

(iii) Mean of the random variable X is

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^{0} x.0.dx + \int_{0}^{1} x\frac{3}{2}(1-x^2)dx + \int_{1}^{\infty} x.0.dx$$

$$= \frac{3}{2}\int_{0}^{1} x(1-x^2)dx = \frac{3}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{0}^{1} = \frac{3}{2}\left[\left(\frac{1}{2}-\frac{(1)^4}{4}\right)-0\right]$$

$$= \frac{3}{2}\times\frac{1}{4} = \frac{3}{8} = 0.375$$

(iv) Variance

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_{0}^{1} x^2\,\frac{3}{2}(1-x^2)dx$$

$$= \frac{3}{2}\int_{0}^{1}(x^2 - x^4)dx$$

$$= \frac{3}{2}\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_{0}^{1}$$

$$= \frac{3}{2}\left[\left(\frac{1}{3}-\frac{1}{5}\right)-0\right]$$

$$= \frac{3}{2}(0.133) = 0.2$$

$\therefore$ The variance of the random variable X is

$$V(X) = E(X^2) - \{E(X)\}^2$$
$$= 0.2 - (0.375)^2 = 0.0594$$

Example 2.8. A continuous random variable has the pdf,

$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probabilities that it will take on a value: (*i*) between 1 and 3 and (*ii*) greater then 0.5.

Solution. (*i*) $P(1 < X < 3) = \int_1^3 f(x)dx = \int_1^3 2e^{-2x}dx = 2\int_1^3 e^{-2x}dx$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_1^3 = -\left[e^{-6} - e^{-2}\right]$$

$$= e^{-2} - e^{-6}$$

$$= 0.1353 - 0.003478$$

$$= 0.1338$$

$\therefore \quad P(1 < X < 3) = 0.1338$

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)dx$$

$$= \int_{0.5}^{\infty} 2e^{-2x}dx$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty} = -[e^{-2x}]_{0.5}^{\infty}$$

$$= -(0 - e^{-1}) = e^{-1}$$

$$= 0.3687$$

(*ii*) $$P(X > 0.5) = \int_{0.5}^{\infty} f(x)dx = 1 - \int_{-\infty}^{0.5} f(x)dx$$

$$= 1 - \left[\int_{-\infty}^{0} f(x)dx + \int_{0}^{0.5} f(x)dx\right]$$

$$= 1 - \left[\int_{0}^{0.5} f(x)dx\right] \qquad \left(\because \int_{-\infty}^{0} f(x)dx = 0\right)$$

$$= 1 - \int_{0}^{0.5} 2e^{-2x}dx$$

$$= 1 - 2\int_{0}^{0.5} e^{-2x}dx$$

Example 2.9. If $f(x) = ke^{-|x|}$ is p.d.f. in $-\infty < x < \infty$ find the values of k and variance of the random variable and also find the probability between 0 and 4.

Solution. Since we know that

$$\int_{-\infty}^{+\infty} f(x)dx = 1 \Rightarrow \int_{-\infty}^{+\infty} ke^{-|x|}dx = 1$$

$$\Rightarrow \quad \int_{-\infty}^{0} ke^{-|x|}dx + \int_{0}^{+\infty} ke^{-|x|}dx = 1$$

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$$\Rightarrow \qquad \int_{-\infty}^{0} ke^{x}dx + \int_{0}^{+\infty} ke^{-x}dx = 1$$

$$\because \qquad |x| = \begin{cases} -x \text{ if } x<0 \\ x \text{ if } x>0 \end{cases}$$

$$\therefore \qquad k\left[e^{x}\right]_{-\infty}^{0} + k\left[e^{-x}\right]_{0}^{\infty} = 1 \Rightarrow k\,[1-0] - k[0-1] = 1$$

$$2k = 1 \Rightarrow k = \frac{1}{2}$$

$$\therefore \qquad f(x) = \frac{1}{2}e^{-|x|}$$

Now,

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx$$

$$= \int_{-\infty}^{+\infty} x.\frac{1}{2}e^{-|x|}dx = 0 \qquad \text{(since integrand is odd function)}$$

and

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_{-\infty}^{+\infty} x^2 \cdot \frac{1}{2}e^{-|x|}dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^2 \cdot e^{-|x|}dx = \frac{1}{2}2\int_{0}^{\infty} x^2 \cdot e^{-|x|}dx$$

(since integrand is even function)

$$= \int_{0}^{\infty} x^2 \cdot e^{-|x|}dx$$

$$= \left[x^2\frac{e^{-x}}{1} - 2x\frac{e^{-x}}{1} + 2\frac{e^{-x}}{-1}\right]_0^{\infty} = \{0-(-2)\} = 2 \quad (\because \lim_{x\to\infty} e^{-x} = 0)$$

The variance of the random variable X is

$$V(X) = E(X^2) - \{E(X)\}^2$$

$$= 2 - 0 = 2$$

Now,

$$P(0 \le X \le 4) = \int_{0}^{4} f(x)dx$$

$$= \int_{0}^{4} \frac{1}{2}e^{-|x|}dx$$

$$= \frac{1}{2}\int_{0}^{4} e^{-x}dx \qquad (\because \ |x| = x \ \text{ as } \ 0<x<4)$$

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$$= \frac{1}{2}\left[\frac{e^{-x}}{-1}\right]_0^4 = \frac{1}{2}\{e^{-4} - e^0)$$

$$= \frac{1}{2}\{1 - e^{-4}\} \approx 0.491$$

Example 2.10. A continuous random variable has the pdf $f(x) = \begin{cases} k\,x^2 e^{-x} & \text{if } x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$.

Determine the constant k**, find the mean and variance.**

Solution. Since $f(x)$ is a density function, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 1$$

$$\Rightarrow \quad \int_0^{\infty} k\,x^2 e^{-x} dx = 1 \Rightarrow k\int_0^{\infty} x^2 e^{-x} dx = 1$$

$$\Rightarrow \quad k\left[x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(\frac{e^{-x}}{1}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty} = 1$$

$$\Rightarrow \quad k[(x^2.\,0 - 2x.0 + 2.0) - (0 - 0 - 2.1)] = 1 \qquad \left(\because \lim_{x\to\infty} e^{-x} = 0\right)$$

$$\Rightarrow \quad 2k = 1 \Rightarrow k = \frac{1}{2}$$

$$\therefore \quad f(x) = \frac{1}{2}x^2 e^{-x} \text{ for } x \geq 0$$

Mean of the random variable X is

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{0} xf(x)dx + \int_0^{\infty} xf(x)dx$$

$$= 0 + \frac{1}{2}\int_0^{\infty} x^3 e^{-x} dx$$

$$= \frac{1}{2}\left[x^3\left(\frac{e^{-x}}{-1}\right) - 3x^2\left(\frac{e^{-x}}{1}\right) + 6x\left(\frac{e^{-x}}{-1}\right) - 6\left(\frac{e^{-x}}{1}\right)\right]_0^{\infty}$$

$$= \frac{1}{2}[(x^2 \cdot 0 - 3x^2 \cdot 0 + 6x \cdot 0 - 6.0) - (0 - 0 - 6.1)] = 3 \qquad \left(\because \lim_{x\to\infty} e^{-x} = 0\right)$$

Therefore the mean of the random variable X is 3.

$\therefore$ The variance of the random variable X is

$$Var(X) = E(X^2) - \{E(X)\}^2$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx$$

$$= \int_{-\infty}^{0} x^2 f(x)dx + \int_{0}^{\infty} x^2 f(x)dx = 0 + \frac{1}{2}\int_{0}^{\infty} x^4 e^{-x} dx$$

$$= \frac{1}{2}\left[x^4\left(\frac{e^{-x}}{-1}\right) - 4x^3\left(\frac{e^{-x}}{1}\right) + 12x^2\left(\frac{e^{-x}}{-1}\right) - 24x\left(\frac{e^{-x}}{1}\right) + 24\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty}$$

$$= \frac{1}{2}[(x^4 \cdot 0 - 4x^3 \cdot 0 + 12x^2 \cdot 0 - 24x \cdot 0 + 24 \cdot 0) - (0 - 0 - 0 - 24.1)]$$

$$= 12 \qquad \left(\because \ \underset{x \to 0}{Lt}\, e^{-x}\right)$$

$\therefore$ $$V(X) = E(X^2) - \{E(X)^2 = 12 - (3)^2 = 3$$

$\therefore$ The variance of the random variable X is 3.

Example 2.11. Let $F(x)$ be the distribution function of a random variable X given by

$$F(x) = \begin{cases} cx^3 & \text{when } 0 \le x \le 3 \\ 1 & \text{when } x > 3 \\ 0 & \text{elsewhere} \end{cases}$$

If $P(X = 3) = 0$, Determine: (*i*) c, (*ii*) mean and (*iii*) $P(X > 1)$.

Solution. Since $$f(x) = \frac{d}{dx}F(x), \text{then}$$

$$f(x) = \begin{cases} 3cx^2 & \text{when } 0 \le x \le 3 \\ 0 & \text{when } x > 3 \\ 0 & \text{when } x > 3 \end{cases}$$

$\therefore$ $$f(x) = \begin{cases} 3cx^2 & \text{when } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Since $f(x)$ is a density function, $\int_{-\infty}^{\infty} f(x)dx = 1$ and the range of x is 0 to 3

$$\Rightarrow \qquad \int_{-\infty}^{\infty} f(x) \cdot dx + \int_{0}^{3} f(x)dx + \int_{3}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \qquad \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{3} 3cx^2 dx + \int_{3}^{\infty} 0\, dx = 1$$

$$\Rightarrow \qquad 3c\int_{0}^{3} x^2 dx = 1$$

$$\Rightarrow \qquad 3c\left[\frac{x^3}{3}\right]_0^3 = 1$$

$$\Rightarrow \quad c(27-0) = 1$$

$$\Rightarrow \quad 27c = 1 \Rightarrow c = \frac{1}{27}$$

$$f(x) = \begin{cases} 3 \cdot \dfrac{1}{27} x^2 & \text{when } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

i.e.,

$$f(x) = \begin{cases} \dfrac{1}{9} x^2 & \text{when } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Now the mean,

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{9}\int_0^3 x \cdot x^2 dx = \frac{1}{9}\int_0^3 x^3 dx$$

$$= \frac{1}{9}\left[\frac{x^4}{4}\right]_0^3$$

$$= \frac{1}{9}\left[\frac{1}{4}(81-0)\right] = \frac{9}{4}$$

$$\therefore \quad E(X) = \frac{9}{4} = 2.25$$

$$P(X > 1) = \int_1^{\infty} f(x)dx = \int_1^3 f(x)dx + \int_3^{\infty} f(x)dx = \int_1^3 f(x)dx + \int_3^{\infty} 0 \cdot dx$$

$$= \frac{1}{9}\int_1^3 x^2 dx = \frac{1}{9}\left[\frac{x^3}{3}\right]_1^3 = \frac{1}{9}\left[\frac{1}{3}(27-1)\right]$$

$$= 0.9629$$

$$\therefore \quad P(X > 1) = 0.9629$$

EXERCISE 2.1

1. A continuous random variable X has the distribution function

$$f(x) = \begin{cases} 0 & \text{if } x \le 1 \\ 4k(x-1)^4 & \text{if } 1 \le x \le 3 \\ 0 & \text{if } x > 3 \end{cases}$$

Determine: (*i*) k, (*ii*) the probability density function of x (*iii*) mean

$$\left[\textbf{Ans. } k = \frac{1}{16} \ \text{pdf} = \begin{cases} \dfrac{1}{4}(x-1)^3 & 1 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}; \ \text{Mean} = \frac{13}{5}\right]$$

8. If probability density function

$$f(x) = \begin{cases} kx^3 & \text{if } 1 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value of k and find the propability between $x = 1/2$ and $x = 3/2$. $\left[\textbf{Ans.} \frac{1}{20}, \frac{1}{6}\right]$

9. A continuous random variable X has distribution function

$$f(x) = \begin{cases} 0 & \text{if } x \le 1 \\ k(x-D)^4 & \text{if } 1 \le x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$$

Determine: (*i*) $f(x)$, (*ii*) mean at $D = 1$. $\left[\textbf{Ans.} f(x) = \begin{cases} 4k(x-D)^3 & 1 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}\right]$

2.6. NORMAL DISTRIBUTION

Normal distribution is the most popular and commonly used distribtion. It was discovered by De Moivre in 1733 after 20 years when Bernoulli gave Binomial distribution. This distribution is a limiting case of Binomial distribution when neither $p \to q$ is too small and n, the number of trials becomes infinitely large *i.e.*, $n \to \infty$. In fact any quantity whose variation depends on random cause will be distributed according to the normal distribution whereas in Binomial and Poisson distribution X assume values like 0, 1, 2,... and thus these distribution are discrete distributions. Cases when the variables can assume any value between 0 and 1 are classified under the continuous variate, for example, in case of height, weight etc.

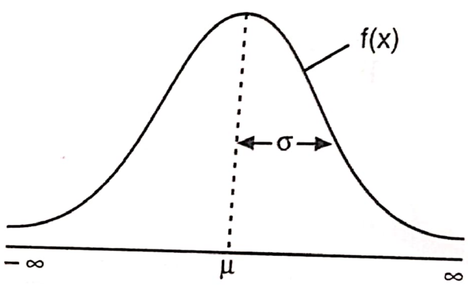

Fig. 2.2.

The continuous random varible X is said to have a normal distribution, if its probability density function is defined as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \le X \le \infty, \ \sigma > 0$$

where μ and σ are parameters of the normal distribution.

2.6.1. Mean of the Normal Distribution

The normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \le X \le \infty$$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $\frac{x-\mu}{\sigma} = z$ then $dx = \sigma dz$ and $x = \mu + \sigma z$ limits are unaltered

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2}z^2} dz \ (\sigma)$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz$$

$$= \frac{\mu}{\sqrt{2\pi}} 2\int_0^{\infty} e^{-\frac{1}{2}z^2} dz + 0$$

$\left(\because\ z e^{-\frac{1}{2}z^2} \text{ is an odd and } e^{-\frac{1}{2}z^3} \text{ is even function}\right)$

$$= \frac{2\mu}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$= \mu$$

$\therefore \qquad E(X) = \mu$

$\therefore$ The mean of the normal distribution is 'μ'.

2.6.2. Variance of the Normal Distribution

Since $\text{Var}(X) = E(X^2) - [E(X)]^2 = E(X - \mu)^2$

Now $E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

Put $\frac{x-\mu}{\sigma} = z$ then $dx = \sigma dz$ limits are unaltered

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-\frac{1}{2}z^2} \sigma dz = \frac{\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-\frac{1}{2}z^2} dz \qquad \left(\because\ z^2 e^{-\frac{1}{2}z^2} \text{ is an even function}\right)$$

Put $\frac{z^2}{2} = t$ then $zdz = dt$ and $z = \sqrt{2t}$ also limits are unaltered

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} 2te^{-t} \frac{dt}{\sqrt{2t}}$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} \sqrt{2t}\, e^{-t} dt = \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} t^{\frac{1}{2}} e^{-t} dt = \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{\frac{3}{2}-1} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \qquad \because \int_0^{\infty} e^{-t} t^{\frac{3}{2}-1} dt = \sqrt{\left(\frac{3}{2}\right)}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \sqrt{\pi} \qquad \because \sqrt{\frac{1}{2}} = \sqrt{\pi}$$

$$\because\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \sigma^2$$

Variance of normal distribution is σ^2.

2.6.3. Median of the Normal Distribution

Suppose 'M' is median of normal distribution then

$$\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2} \qquad \ldots(i)$$

Now take, $$\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2} \Rightarrow \int_{-\infty}^{\mu} f(x)\,dx + \int_{\mu}^{M} f(x)\,dx$$

$$\int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \int_{\mu}^{M} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2} \qquad \ldots(ii)$$

Consider, $\int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

Put $\frac{x-\mu}{\sigma} = z$ then $dx = \sigma dz$

Also limits are when $x = -\infty$ then $z = -\infty$

Also when $x = \mu$ then $z = 0$

$$\int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} \sigma\, dz$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{-\frac{1}{2}(z)^2} dz \qquad \text{by symmetry property}$$

$$\therefore \quad \int_{-\infty}^{\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{2}} = \frac{1}{2} \qquad \ldots(iii)$$

From (*ii*) and (*iii*)

$$\frac{1}{2} + \int_{\mu}^{M} f(x)\,dx = \frac{1}{2}$$

$\therefore \int_{\mu}^{M} f(x)\,dx = 0$. This implies that $\mu = M$ $\qquad$ ($\because$ if $\int_{a}^{b} f(x)\,dx = 0$ then $a = b$)

Hence, the median of the normal distribution is 'μ'.

2.6.4. Mode of the Normal Distribution

Mode is the value of x at which $f(x)$ has maximum value. Then $f'(x) = 0$ and $f''(x) = -$ ve at that value of 'x'.

Now the probability density function of 'x' is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty \le X \le \infty$$

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left\{-\frac{1}{2}\cdot 2\cdot\left(\frac{x-\mu}{\sigma}\right)\right\}$$

$$= -\frac{1}{\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)}(x-\mu) \qquad \text{(when } x=\mu\text{)}$$

$$= 0$$

and

$$f''(x) = (x-\mu)\left\{-\frac{1}{\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right\}\left\{-\frac{1}{2}\cdot 2\left(\frac{x-\mu}{\sigma}\right)\right\} + \left\{\frac{-1}{\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right\}$$

$$= -\frac{1}{\sigma^2\sqrt{2\pi}}$$

which is – ve at $x = \mu$.

$\therefore$ $f(x)$ has maximum value, when the value of $x = \mu$.

$\therefore$ The mode of the normal distribution is μ.

Note: For the normal distribution, the mean median, and mode are equal

i.e., $\rightarrow$ Mean = Median = Mode Imp

2.6.5. Properties of the Normal Distribution

1. The normal probability curve with mean μ and standard deviation s is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty \le X \le \infty$$

2. The curve is bell-shaped and symmetrical about the line $x = \mu$.
3. Mean, median and mode of the normal distribution of coincide the normal distribution is unimodal.
4. $f(x)$ decreases rapidly as x increases.
5. X-axis is an asymptote to the curve (the tangent to the curve at ∞ is X-axis)

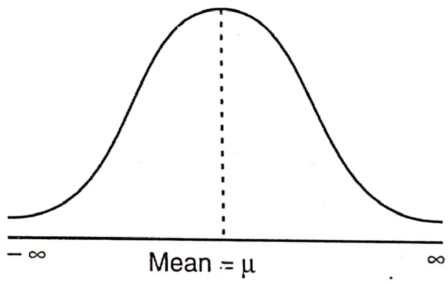

Fig. 2.2.

6. The maximum probability occurs at the point $x = \mu$ and is $\frac{1}{\sigma\sqrt{2\pi}}$.
7. Mean deviation about mean $= \frac{4}{5}\sigma$.
8. Since $f(x)$, being the probability, can never the negative, so that no portion of the curve lies below the X-axis.
9. A linear function of independent normal variates is also normal variate.
10. The point of inflextion of the curve are at $x = \mu \pm \sigma$.
    (*i*) Area of the normal curve between $(\mu - \sigma)$ and $(\mu + \sigma)$ is 0.6826
    i.e., $P(\mu - \sigma < X < \mu + \sigma) = 0.6826$

(ii) Area of the normal curve between $\mu - 2\sigma$ and $\mu + 2\sigma$ is 0.9544

i.e., $P(\mu - 2\sigma < X \mu + 2\sigma) = 0.9544$

(iii) Area of the normal curve between $\mu - 3\sigma$ and $\mu + 3\sigma$ is 0.9973

i.e., $P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$

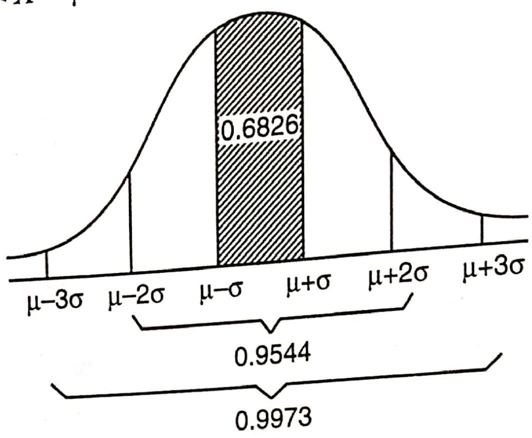

Fig. 2.3

2.6.6. Normal Distribution as a Limiting Form of Binomial Distribution

Normal distribution is another limiting form of the binomial distribution under the following conditions:

(i) The number of trials n, is indefinitely large, *i.e.*, $n \to \infty$

(ii) Neither 'p' or nor 'q' is very small.

For large n, the calculation of binomial probabilities is very difficult. In such a case we can use normal distribution and compute the required probability.

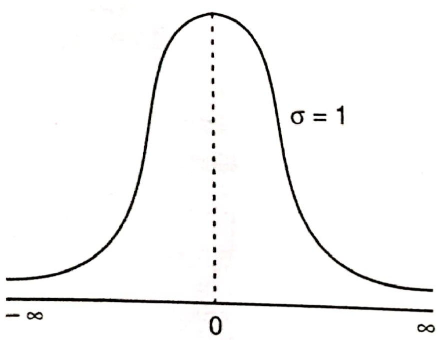

Fig. 2.4.

Suppose the number of successes ranges from x_1 to x_2 then the probability of getting x_1 to x_2 successes is given by

$$\sum_{x=x_1}^{x_2} C_x p^x q^{n-x}$$

Now consider two cases.

Case 1: $p = q = \frac{1}{2}$ and n is not large. The mean of the binomial distribution $= np$ and standard deviation $= \sqrt{npq}$.

The corresponding mean and standard deviation of the normal distribution are μ and *s* respectively.
Since we know that
Let z_1 and z_2 be the values of Z, corresponding to x_1 and z_2 of X respectively. Then

$$= P(x_1 < X < x_2)\ P = (z_1 < Z < z_2) = \int_{z_1}^{z_2} \phi(z)dz$$

And the value can be computed from the standard normal distribution.

Case 2: $p \neq q$ for large n,

We can approximate the binomial to the normal curve and calculate the corresponding probabilities.

For any success x which lies in the interval $\left(x - \frac{1}{2}, x + \frac{1}{2}\right)$ then z_1 can be calculated corresponding to the lower limit of the x_1 class and z_2 to the limit of the x_2 class.

$$\therefore \qquad z_1 = \frac{\left(x_1 - \frac{1}{2}\right) - \mu}{\sigma} = \frac{x_1 - \frac{1}{2} - np}{\sqrt{npq}}$$

and

$$z_2 = \frac{\left(x_2 + \frac{1}{2}\right) - \mu}{\sigma} = \frac{x_2 + \frac{1}{2} - np}{\sqrt{npq}}$$

$\therefore$ The required probability $= \int_{z_1}^{z_2} \phi(z)dz$,

and this value can be computed from standard normal distribution.

Constant of Normal Distribution

The mean of the normal distribution is $\bar{x}$

The standard deviation of the normal distribution is σ

$$\mu_2 = \sigma^2, \quad \mu_3 = 0 \quad \text{and} \quad \mu_4 = 3\sigma^4$$

β_1 or moment coefficient of skeness, $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$

β_2 or moment coefficient of kurtosis, $\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\sigma^4}{\sigma^4} = 3$

SOLVED EXAMPLES

Example 2.12. If μ = 50 and σ = 10 find : (*i*) *P* (50 ≤ *X* ≤ 80), (*ii*) *P*(60 ≤ *X* ≤ 70) (*iii*) *P*(30 ≤ *X* ≤ 40), (*iv*) *P*(40 ≤ *X* ≤ 60). Use Table : Area under the normal curve.

Solution. Standard normal variate $z = \frac{X - \mu}{\sigma} = \frac{X - 50}{10}$

(*i*) $z = \dfrac{50-50}{10} = 0$ when $X = 50$ and $z = \dfrac{80-50}{10} = 3$, when $X = 80$

Hence, $P(50 \le X \le 80) = P(0 \le z \le 3) = 0.4987$ [From table)

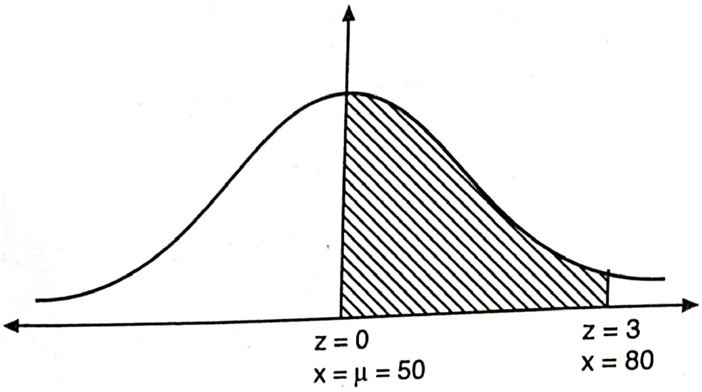

Fig. 2.5.

(*ii*) $P(60 \le X \le 70) = P(1 \le z \le 2) =$ Area from $z = 1$ to $z = 2$

$= $ (Area from $z = 0$ to $z = 2$) – (Area from $z = 0$ to $z = 1$)

$= 0.4772 - 0.3413 = 0.1359$

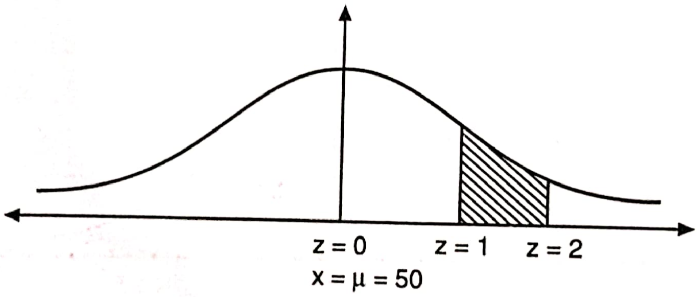

Fig. 2.6.

(*iii*) $P(30 \le X \le 4) = P(-2 \le z \le -1)$

Due to symmetry, area between $z = -1$ to $z = -2$ will be the same as between $z = + 1$ to $z = + 2$, which is the same as in (*ii*) *i.e.*, 0.1359.

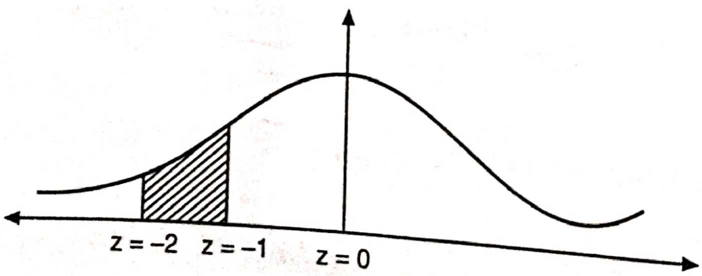

Fig. 2.7.

(*iv*) $P(40 \le X \le 60) = P(-1 \le z \le 1)$

$=$ area between $z = -1$ to $z = 1$

$=$ twice the area between $z = 0$ to $z = 1$

$= 2 \times 0.3413 = 0.6826$

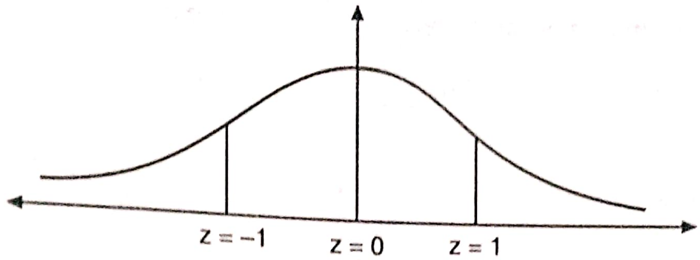

Fig. 2.8.

Example 2.13. In a normal distribution 31% of items are under 45 and 8% are over 64. Find the mean and standard deviation of the distribution.

Solution. Let μ be the mean and σ be the standard deviation of the distribution

Normal variate $z = \dfrac{x-\mu}{\sigma}$

As 31% of items are under 45 and 8% over 64

$\therefore$ At $x = 64$, we have $$z = \frac{64-\mu}{\sigma} = z_1 \text{ (say)}$$

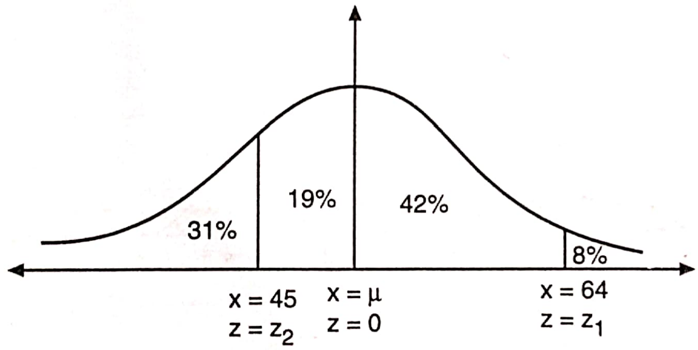

Fig. 2.9.

Area from $z = 0$ to $z = z_1$ is 42% *i.e.*, 0.42. The value of z_1 corresponding to area 0.42 from the table 1 is 1.405 $\cong$ 1.4

$$\therefore \qquad \overset{1.4}{\cancel{1.405}} = \frac{64-\mu}{\sigma} \qquad ...(i)$$

Similarly at $x = 45$, $$|z| = \left|\frac{45-\mu}{\sigma}\right| = |z_2|$$

Area between $x = 45$ (*i.e.*, $z = z_2$) to $x = \mu$ ($z = 0$) is the same numerically as between $z = 0$ to $z = z_2$ which is 19%.

Now the value of normal variate z_2 corresponding to area 0.19 is ~~0.495~~ 0.5

$$\therefore \qquad \left|\frac{45-\mu}{\sigma}\right| = |\overset{0.5}{\cancel{0.495}}|$$

$$\therefore \qquad \frac{\mu-45}{\sigma} = \cancel{0.495}\ 0.5 \qquad ...(ii)$$

$\frac{1.4}{0.5} = 2.8$

$\frac{1.4}{0.5} = 2.8$

From (*i*) and (*ii*) $\qquad \frac{64-\mu}{\mu-45} = \frac{1.4\cancel{65}}{\cancel{0}.495} = 2.8.$

$$\mu = 50 \text{ and } \sigma = 10.$$

Example 2.14. A sample of 100 dry battery cells tested to find the length of life produced the following results:

$$\mu = 12 \text{ hours}, \sigma = 3 \text{ hours}$$

Assuming the data to be normally distributed, what percentage of battery calls are expected to have life:

(*a*) more than 15 hours

(*b*) less than 6 hours

(*c*) between 10 and 14 hours.

Solution. Let X denotes the length of life of dry battery cells.

Also $\qquad z = \frac{X-\mu}{\sigma} = \frac{X-12}{3}$

(*a*) When $X = 15$, $\qquad z = \frac{15-12}{3} = 1$

$\therefore \qquad P(X > 15) = P(z > 1)$

Area of the right of $z = 1$ is $0.5 - 0.3413 = 0.1587$

$\therefore$ Percentage of battery cells having life more than 15 hours

$$= 0.1586 \times 100 = 15.87\%$$

(*b*) When $X = 6$, $\qquad z = \frac{6-12}{3} = -2$

$\therefore \qquad P(X < 6) = P(z < -2)$

$\therefore$ Percentage of battery cells have life less than 6 hours

$$= 0.0\cancel{0}28 \times 100 = 2.28\%$$

(*c*) When $X = 10$, $\qquad z = \frac{10-12}{3} = -0.67$

When $X = 14$, $\qquad z = \frac{14-12}{3} = 0.67$

$\therefore \qquad P(10 < X < 14) = P(-0.67 < z < 0.67) = 2P(0 < Z < 0.67)$

Area between $X = 10$ to $X = 14$ is twice the area of the left of $z = 0.67$

$$= 2 \times 0.2487 = 0.4974$$

$\therefore$ Percentage of battery cells having life span between 10 hours and 14 hours = 49.74%

Example 2.15. The customer accounts of certain department store have an average balance of ₹ 120 and a standard deviation of ₹ 40. Assuming that the account balances are normally distributed:

(*i*) **What proportion of the account is over ₹ 150?**

(*ii*) **What proportion of account is between ₹ 100 and ₹ 150?**

(*iii*) **What proportation of account is between ₹ 60 and ₹ 90?**

Solution. Standard normal variate $z = \frac{X - \mu}{\sigma}$

(*i*) When $X = 150$, $z = \frac{150-120}{40} = 0.75$

To find out the proportion of accounts greater than ₹ 150, we refer to right of $z = 0.75$

The area to the right of $z = 0$ is 0.5

Less the area between $z = 0$ and $z = 0.75$ is 2735.

The area of the right to $z = 0.75$ is 0.2266.

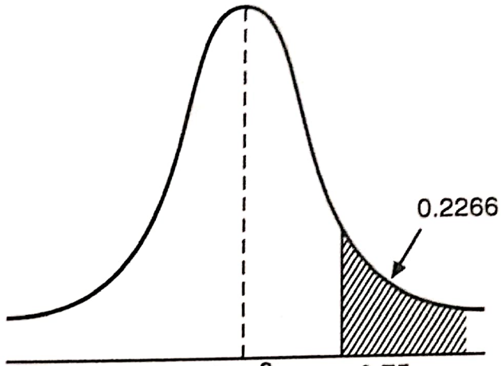

Fig. 2.10.

Therefore, 22.66% of account have balance is excess of ₹150 (shaded area)

(*ii*) When $X = 100$, $z = \frac{100-120}{40} = -0.5$

When $X = 150$, $z = \frac{150-120}{40} = 0.75$

The area between $z = -0.5$ and $z = 0$ is 0.1915.

The area between $z = 0$ and $z = 0.175$ is 0.2734

∴ The total area *i.e.*, $z = -0.5$ and $z = 0$ is 0.1915

The area between $z = 0$ and $z = 0.75$ is 0.2734.

∴ The area total area *i.e.*, $z = -0.5$ to $z = 0.75$ is $0.1915 + 0.273 = 0.4649$

∴ 46.49% of accounts have an average balance of accounts between ₹ 100 and 150.

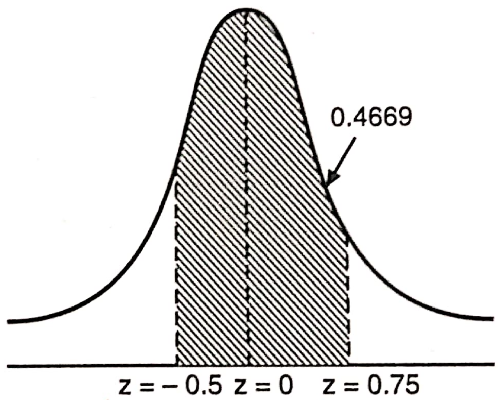

Fig. 2.11

(*iii*) When $X = 90$, $z = \frac{90-120}{40} = -0.75$

When $X = 60$, $z = \frac{60-120}{40} = -1.5$

Area between $z = 0$ to $z = -1.5$ is 0.4332

Area between $z = 0$ to $z = -0.75$ is 0.2734

∴ The area between $X = 90$ and $X = 60$

($z = -1.5$ to $z = -0.75$) is $0.4332 - 0.2734 = 0.1598$

∴ 15.98% of accounts are between ₹ 60 and ₹ 90

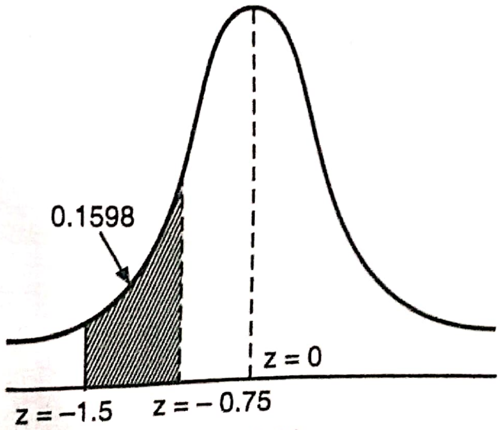

Fig. 2.12.

Example 2.16. The average height of soliders of a country is given as 68.22 inches with variance 10.8 sq. inch. How many soldiers out of 1000 would you expect to be over 72 inches tall? Given that the area under the normal curve between $z = 0$ to $z = 0.35$ is 0.1368 and between $z = 0$ and $z = 1.15$ is 0.3746.

Solution. Standard normal variate

$$z = \frac{X - \mu}{\sigma} = \frac{72 - 68.22}{\sqrt{10.8}} = 1.15$$

$P(X \geq 72) = P(z \geq 1.15) =$ Total area to the right to ordinate $z = 0) -$ (area between $z = 0$ to $z = 1.15$)

$$= 0.5 - 0.3746 = 0.1254$$

$\therefore$ Total number of soldiers k above 72″ tall $= 1000 \times 0.1254 = 125$

Example 2.17. In an intelligence test administered to 1000 students the average score was 42 and standard deviation 24. Find: (*a*) the number of students exceeding a score of 50, (*b*) the number of students lying between 30 and 54, (*c*) the value of score exceeded by the top 100 students.

Solution. (*a*) Given $\mu = 42, x = 50, \sigma = 24$

$$\therefore \quad z = \frac{x - \mu}{\sigma} = \frac{50 - 42}{24} = 0.333$$

Area of the right to ordinate at 0.333 is $0.5 - 0.1304 = 0.3696$

$\therefore$ The expected number of children exceeding a score of 50

$$= 0.3696 \times 1000 = 369.6 \text{ or } 370$$

(*b*) Standard normal variate for score 30

$$z = \frac{x - \mu}{\sigma} = \frac{30 - 42}{24} = -0.5$$

Standard normal variate for score 54

$$z = \frac{x - \mu}{\sigma} = \frac{54 - 42}{24} = 0.5$$

Area from $z = 0$ to $z = 0.5$ is 0.1915

Area from $z = -0.5$ to $z = 0$ is 0.1915

$\therefore$ Area from $z = -0.5$ to $z = 0.5 = 0.1915 + 0.1915 = 0.3830$

Thus the number of children having score between 30 and 54 $= 0.383 \times 1000 = 383$

(*c*) Probability of getting top 100 students $= \dfrac{100}{1000} = 0.1$

Standard normal variate having 0.1 area of the right $= 1.281$

Standard normal variate for score x

$$z = \frac{x - \mu}{\sigma}$$

$$1.281 = \frac{x - 42}{24}$$

$$1.28 \times 24 = x - 42$$
$$x = (1.28 \times 24) + 42 = 72.72 \text{ or } 73$$

Example 2.18. The distribution of a random variable is given by

$$f(x) = Ce^{-\frac{1}{50}(9x^2 - 30x)}, -\infty \le x \le \infty$$

Find the constant C, the mean and the variance of the random variable. Find also the upper 5% value of the random variable.

Solution. $P(-\infty \le x \le \infty) = 1$ = unit area under the curve $y = f(x)$ and x-axis

$$\therefore \quad 1 = \int_{-\infty}^{\infty} f(x)dx$$

$$1 = Ce^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{\frac{(x-\mu)^2}{2\sigma^2}} dx$$

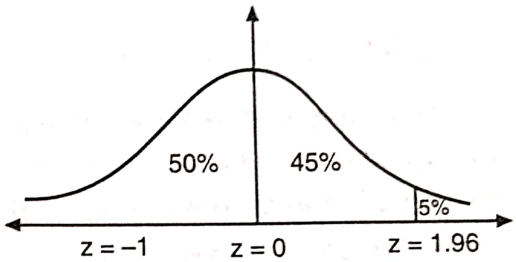

Fig. 2.13.

Comparing with normal distribution, we have

We have $$\mu = \frac{5}{3} = 1.667$$

$$2\sigma^2 = \frac{50}{9} \text{ or } \sigma^2 = \frac{25}{9} = 2.778$$

$$\sigma = \frac{5}{3}, \; C = \frac{1}{\sqrt{e}\sqrt{2\pi\sigma}} = \frac{1}{\sqrt{2\pi \times 1.667}\sqrt{e}} = 0.145$$

Standard normal variate $t = \dfrac{x-\mu}{\sigma} = \dfrac{x - \frac{5}{3}}{\frac{5}{3}}$ corresponding to 95% area, we have $1.96 = \dfrac{x - \frac{5}{3}}{\frac{5}{3}}$

$$\therefore \quad 1.96 \times \frac{5}{3} + \frac{5}{3} = x \text{ or } x = 4.933$$

Example 2.19. Assuming that the diameters of 1000 brass plugs taken consecutively from a machine form a normal distribution with mean 0.7515 cm and standard deviation 0.002 cm. Find the number of plugs likely to be rejected if the approved diameter is 0.752 ± 0.004 cm.

Solution. $\mu = 0.7515, \sigma = 0.002$

Limits of the diameter of non-defective plugs are

$$0.752 - 0.004 = 0.748 \text{ and } 0.752 + 0.004 = 0.756 \text{ cm}$$

At $x = 0.748$ m the standard normal variate

$$z_1 = \frac{x-\mu}{\sigma} = \frac{0.7480 - 0.7515}{0.002} = -1.75$$

and at $$x = 0.756, z_2 = \frac{0.756 - 0.7515}{0.002} = 2.25$$

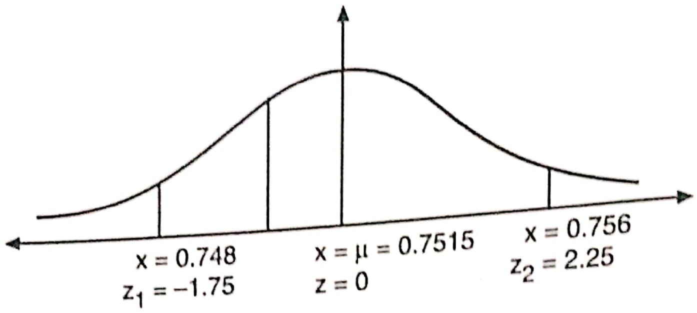

Fig. 2.14.

Area between z_1 and z_2 is numerical equal to the sum of the area between 0 to z_1 and z_2. From the table, when $z = 1.75$ area is equal to 0.4599 and when $z = 2.25$ area is equal to 0.4878.

Hence, the required are for non-defective items = 0.4599 + 0.4878 = 0.9477.

Non-defective plugs are 100 × 0.9477 = 948

Hence, defective plugs are 1000 − 948 = 52

∴ 52 pulgs are likely to be rejected.

Example 2.20. A sales tax officer reported that the average sales of 500 business establishment in a year is ₹ 36,000/and standard deviation of ₹ 10,000/-. Assuming that the sales in these business are normally distributed, find: (*i*) the number of businesses the sales of which are more than ₹ 40,000/-, (*ii*) the percentage of businesses the sales of which are likely to range between ₹ 30,000 and ₹ 40,000.

Solution. From the given data

$$N = 500,\ \mu = 36{,}000 \quad \text{and} \quad \sigma = 10{,}000$$

The standard normal variate $z = \dfrac{X-\mu}{\sigma} \Rightarrow z = \dfrac{X - 36{,}000}{10{,}000}$

(*i*) $P(X > 40{,}000)$:

When $x = 40{,}000$ then $z = \dfrac{40{,}000 - 36{,}000}{10{,}000}$

$$= 0.4 \text{ (say } z_1)$$

$$P(X > 40{,}000) = P(z > 0.4)$$
$$= 1 - P(z \le 0.4)$$
$$= 1 - (F(0.4)) = 1 - 0.6554$$
$$= 0.3446$$

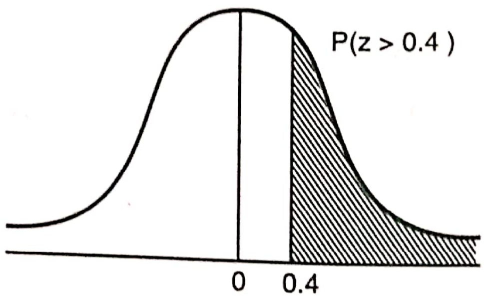

Fig. 2.15.

∴ The number of businesses the sales which are more than ₹ 40,000/- are

$$= N.P(X > 40{,}000)$$
$$= 500 \times 0.3446 = 172.3$$
$$= 172$$

(*ii*) $P(30{,}000 < X < 40{,}000)$:

When $x = 30{,}000$, then $z = \dfrac{30{,}000 - 36{,}000}{10{,}000} = -0.6$ (say z_1)

When $x = 40,000$ then $z = \dfrac{40,000 - 36,000}{10,000} = 0.4$ (say z_2)

(iii) $P(30,000 < X < 40,000) = P(-0.6 < z < 0.4)$

$$= F(0.4) - F(-0.6)$$
$$= 0.6554 - 0.2743$$
$$= 0.3811$$

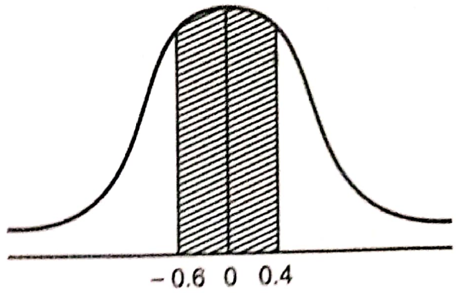

Fig. 2.16.

∴ The number of business the sales which are likely to range between 30,000 and 40,000 are

$$= N.P(30,000 < X < 40,000)$$
$$= 500 \times 0.3811 = 190.55 = 191$$

Out of 500 businesses 191 having sales between 30,000 and 40,000. For 100 businesses 38.2.

∴ The required percenatge is 38.2%.

Example 2.21. Suppose 10 percent of probability for a normal distribution $N(\mu, \sigma^2)$ is below 25 and 5 percent above 90, what are the value of μ and σ.

Solution. $P(x \leq 35) = \dfrac{10}{100} = 0.1$ ... (i)

$P(x > 90) = \dfrac{5}{100} = 0.05$ ... (ii)

The standard normal variate $z = \dfrac{x - \mu}{\sigma}$

When $x = 35$ then $z = \dfrac{35 - \mu}{\sigma}$

When $x =$ then $z = \dfrac{90 - \mu}{\sigma}$

From eqn. (i) $P(x \leq 35) = 0.1$

$$\Rightarrow \quad P\left(z \leq \frac{35 - \mu}{\sigma}\right) = 0.1$$

$$\Rightarrow \quad \left(\frac{35 - \mu}{\sigma}\right) = -1.28 \text{ from table}$$

$$\Rightarrow \quad 35 - \mu = 1.28\,\sigma$$

$$\Rightarrow \quad \mu - 1.28\,\sigma = 35 \qquad \text{...(iii)}$$

From eqn. (ii) $P(x > 90) = 0.$

$$P\left(z > \frac{90 - \mu}{\sigma}\right) = 0.05$$

$$\Rightarrow \quad \left(\frac{90 - \mu}{\sigma}\right) = -1.645$$

$$\Rightarrow \quad 90 - \mu = -1.645$$

$$\Rightarrow \qquad \mu - 1.645\,\sigma = 90 \qquad \ldots (iv)$$

From eqns. (*iii*) and (*iv*)

$$\mu = 157.87$$
$$\sigma = 150.68$$

EXERCISE 2.2

1. Students of a class were given an aptitude test. Their marks were found to be normally distributed with mean 60 and standard 5. What percentage of students scored more than 60 marks? [**Ans.** 50]
2. Find the value of z in each of the cases
   (*i*) area between $z = 0$ and $z = 0.3770$.
   (*ii*) area of the left of z is 0.8621. [**Ans.** (*i*) ±1.6, (ii) 1.09]
3. In a sample of 1000 cases, the mean of a certain test is 14 and standard deviation is 2.5. Assuming the distribution to be normal, find:
   (*i*) how many students score between 12 and 15?
   (*ii*) how many score above 18?
   (*iii*) how many score below 8?
   (*iv*) how many score 16? [**Hint :** (*iv*) Take marks between 15.5 and 16.5]
   [**Ans.** (*i*) 444, (*ii*) 55, (*iii*) 8, (*iv*) 116]
4. Find the area under the normal curven in each of the following case :
   (*i*) $z = 0$ and $z = 1.2$
   (*ii*) $z = -0.68$ and $z = 0$
   (*iii*) $z = -0.46$ and $z = 2.21$
   (*iv*) $z = 0.81$ and $z = 1.94$ [**Ans.** (*i*) 0.3849, (*ii*) 0.2518, (*iii*) 0.6637, (*iv*) 0.1828)
   [**Hints:** Use standard curve table]
5. Students of a class were given a mechanical aptitude test. These marks were found to be normally distributed with mean 60 and standard deviation 5. What percent of students scored: (*i*) more than 60 marks, (*ii*) less than 56 marks, (*iii*) between 45 and 65 marks.
   (**Ans.** (*i*) 50%, (ii) 21.21%, (*iii*) 84%)
6. The mean height of 500 students's is 151 cm and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many students's heights lie between 120 and 155 cm.
   [**Ans.** 294]
7. Fit a normal curve

$$P(x) = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} ; -\infty \le x \le \infty$$

Given the following data :

x	8.60	8.59	8.57	8.56	8.55	8.54	8.53	8.52
f	2	3	4	10	8	4	1	1

$$\left[\textbf{Ans. } P(x) = \frac{42}{0.176\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-8.563}{0.176}\right)^2}\right]$$

2.7. EXPONENTIAL DISTRIBUTION

A random variable X is said to have an exponential distribution with parameter $\lambda > 0$; if its probability density function is given by:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The cumulative distribution function $F(x)$ is given by $F(x) = P(X \leq x) = \int_{-\infty}^{x} \lambda e^{-\lambda x}\, dx$

$$\Rightarrow \qquad F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \\ 1, & x = \infty \end{cases}$$

Remark: Sometimes exponential distribution is defined by the p.d.f.

$$f(x) = \frac{1}{\beta} e^{-\frac{1}{\beta}x}, \quad x > 0$$
$$0, \quad \text{otherwise}$$

Graph of Exponential Probability Density Function

For $\qquad f(x) = \lambda e^{-\lambda x}$

x	0	1	2	....	∞
$f(x)$	λ	$\lambda e^{-\lambda}$	$\lambda e^{-2\lambda}$	....	0

when $\qquad \lambda = 1, \quad f(x) = 1, e^{-1}, e^{-2}, \ldots\ldots 0$

$\qquad \lambda = 2, \quad f(x) = 2, 2e^{-2}, 2e^{-4}, \ldots\ldots 0$

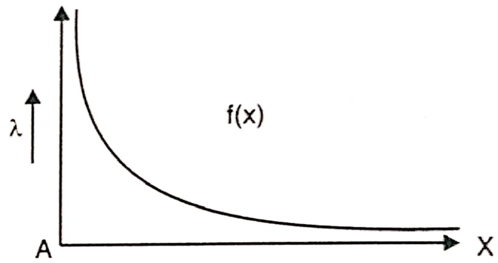

Fig. 2.17.

Graph of Distribution Function

For $\qquad F(x) = 1 - e^{-\lambda x}$

x	0	1	2	....	∞
$F(x)$	0	$1 - e^{-\lambda}$	$1 - e^{-2\lambda}$	....	1

when $\qquad \lambda = 1, \quad F(x) = 0, 1 - e^{-1}, 1 - e^{-2}, \ldots\ldots 1$

$\qquad \lambda = 2, \quad F(x) = 0, 1 - e^{-2}, 1 - c^{-4}, \ldots\ldots 1$

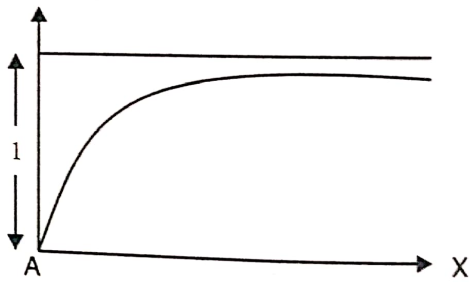

Fig. 2.18.

2.8. CONSTANTS OF EXPONENTIAL DISTRIBUTION

1. Moments About Origin

$$\mu_1{}^1 \text{ or Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\int_0^{\infty} x\lambda e^{-\lambda x}\, dx = \lambda \int_0^{\infty} x e^{-\lambda x}\, dx$$

$$= \lambda \cdot \frac{\boxed{(2)}}{\lambda^2} \qquad \left[\because \int_0^{\infty} x^{n-1} e^{-ax}\, dx = \frac{\boxed{(n)}}{a^n}\right]$$

$$= \frac{1}{\lambda}.$$

$$\mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x}\,dx = \lambda \int_0^{\infty} x^2 e^{-\lambda x}\,dx$$

$$= \lambda \,.\, \frac{\overline{|(3)}}{\lambda^3} = \frac{2}{\lambda^2}$$

Similarly,

$$\mu_3' = E(X^3) = \frac{6}{\lambda^3}$$

and

$$\mu_4' = E(X^4) = \frac{24}{\lambda^4}.$$

2. Moments About Mean

$$\mu_1 = 0 \quad \text{(always)}$$

$$\mu_2 = \mu_2' - \mu_1'^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2} = \text{variance } (\sigma^2)$$

$\therefore$ Standard deviation,

$$\sigma = \sqrt{\mu_2} = \sqrt{\text{var }(X)} = \frac{1}{\lambda}.$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= \frac{6}{\lambda^3} - \frac{3\times 2}{\lambda^2}\times\frac{1}{\lambda} + 2\left(\frac{1}{\lambda}\right)^3$$

$$= \frac{6}{\lambda^3} - \frac{6}{\lambda^3} + \frac{2}{\lambda^3} = \frac{2}{\lambda^3}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$= \frac{24}{\lambda^4} - 4.\frac{6}{\lambda^3}.\frac{1}{\lambda} + 6\times\frac{2}{\lambda^2}\times\left(\frac{1}{\lambda}\right)^2 - 3\left(\frac{1}{\lambda}\right)^4$$

$$= \frac{24}{\lambda^4} - \frac{24}{\lambda^4} + \frac{12}{\lambda^4} - \frac{3}{\lambda^4} = \frac{9}{\lambda^4}$$

3. Beta and Gamma Coefficients

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\left(\dfrac{2}{\lambda^3}\right)^2}{\left(\dfrac{1}{\lambda^2}\right)^3} = \frac{\left(\dfrac{4}{\lambda^6}\right)}{\left(\dfrac{1}{\lambda^6}\right)} = 4$$

$$\gamma_1 = \sqrt{\beta_1} = 2$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\dfrac{9}{\lambda^4}}{\left(\dfrac{1}{\lambda^2}\right)^2} = 9$$

$$\gamma_2 = \beta_2 - 3 = 9 - 3 = 6$$

4. Mean Deviation About Mean

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Then $$E(X) = \text{Mean} = \frac{1}{\lambda}$$

Hence, mean deviation about mean,

$$\text{M.D.} = E|X - E(x)|$$

$$= \int_0^\infty \left| x - \frac{1}{\lambda} \right| \lambda e^{-\lambda x}\, dx$$

$$= \int_0^\infty |\lambda x - 1| e^{-\lambda x}\, dx$$

$$= \frac{1}{\lambda} \int_0^\infty |t - 1| e^{-t}\, dt \quad \text{where } t = \lambda x$$

$$= \frac{1}{\lambda}\left[\int_0^1 (1-t)e^{-t}\, dt + \int_1^\infty (t-1)e^{-t}\, dt\right]$$

$$= \frac{1}{\lambda}[e^{-1} + e^{-1}] = \frac{2}{\lambda}e^{-1}.$$

5. Moment Generating Function

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$M_x(t) = E(e^{tx}) = \int_0^\infty \lambda e^{-\lambda x} . e^{tx}\, dx$$

$$= \int_0^\infty \lambda e^{-(\lambda - t)x}\, dx$$

$$= \frac{\lambda}{\lambda - t}(e^{-(\lambda - t)x})_0^\infty = \frac{\lambda}{\lambda - t} \quad ...(i)$$

$$= \left(\frac{\lambda - t}{\lambda}\right)^{-1} = \left(1 - \frac{t}{\lambda}\right)^{-1} \quad ...(ii)$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \frac{t^3}{\lambda^3} + \frac{t^4}{\lambda^4} + ...$$

$\therefore$ $$\mu_r' = \text{coefficient of } \frac{t^r}{r!} \text{ in } M_x(t) = \frac{r!}{\lambda^r}.$$

In particular, $$\mu_1' = \frac{1}{\lambda}, \mu_2' = \frac{2!}{\lambda^2}, \mu_3' = \frac{3!}{\lambda^3}, \mu_4' = \frac{4!}{\lambda^4} \text{ and so on.}$$

6. Cumulant Generating Function

$$f(x) = \lambda e^{-\lambda x}, x \geq 0$$

$$M_x(t) = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$\Rightarrow \quad K_x(t) = \log M_x(t) = -\log\left(1 - \frac{t}{\lambda}\right)$$

$$= \frac{t}{\lambda} + \frac{1}{2}\left(\frac{t^2}{\lambda^2}\right) + \frac{1}{3}\left(\frac{t^3}{\lambda^3}\right) + \ldots + \frac{1}{r}\left(\frac{t^r}{\lambda^r}\right) + \ldots$$

$$\therefore \quad K_1 = \frac{1}{\lambda}, K_2 = \frac{1}{\lambda^2}, K_3 = \frac{2}{\lambda^3}, \ldots, K_r = \frac{(r-1)!}{\lambda^r}.$$

SOLVED EXAMPLES

Example 2.22. A power supply unit for a computer component is assumed to follow an exponential distribution with a mean life of 1200 hours. What is the probability that the component will

$P(X \le x) = 1 - e^{-x/\lambda}$

$P(x \geq X) = e^{-x/\lambda}$

(*i*) **fail in the first 300 hours?**

(*ii*) **survive more than 1500 hours?**

(*iii*) **last between 1200 hours and 1500 hours?**

Solution. Given that $\lambda = 1200$

(*i*) The probability that the component will fail in the first 300 hours is

$$P(X \le 300) = 1 - e^{-\frac{300}{1200}}$$

$$= 1 - e^{-0.25}$$

$$= 1 - 0.7788 = 0.2212$$

(*ii*) The probability that the component will servive more than 1500 hours is

$$P(X \ge 1500) = e^{-1500/1200}$$

$$= e^{-1.25} = 0.2865$$

(*iii*) The probability that the component will lost between 1200 hours and 1500 hours is given as $P(1200 \le X \le 1500)$

$$\Rightarrow \quad P[\text{inside}] = 1 - P[\text{outside}]$$

$$= 1 - [P(X \le 1200) + P(X \ge 1500)]$$

Now, $$P(X \le 1200) = 1 - e^{-\frac{1200}{1200}} = 0.6321$$

and $$P(X \ge 1500) = e^{-\frac{1500}{1200}} = 0.2865$$

$$P(\text{Inside}) = 1 - [0.6321 + 0.2865] = 0.0813$$

Hence, $$P(1200 \le X \le 1500) = 0.0813.$$

Example 2.23. A random variable has an exponential distribution with probability density function given by

$$f(x) = \begin{cases} 3e^{-3x}, & \text{for } x > 0 \\ 0, & \text{for } x \le 0. \end{cases}$$

What is the probability, that X is not less than 4? Find mean and standard deviation. Show that

$$\text{Coefficient of variation} = \left(\frac{\text{mean}}{\text{S.D.}}\right) = 1.$$

Solution. $P(X \nless 4) = P(X \ge 4) = \int_4^\infty 3e^{-3x}\, dx = [-e^{-3x}]_4^\infty = e^{-12}.$

$$\text{Mean} = E(X) = \int_0^\infty x3e^{-3x}\, dx = 3\int_0^\infty xe^{-3x}\, dx$$

$$= 3.\frac{\Gamma(2)}{3^2} = \frac{1}{3}.$$

$$\text{Variance} = E[X - E(X)]^2$$

$$= E(X^2) - [E(X)]^2$$

$$= \int_0^\infty x^2 . 3e^{-3x}\, dx - \left(\frac{1}{3}\right)^2$$

$$= 3.\frac{\Gamma(3)}{3^3} - \frac{1}{9} = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

$\therefore$ Standard deviation, S.D. $= \sqrt{\frac{1}{9}} = \frac{1}{3}$

Hence, $\frac{\text{Mean}}{\text{S.D.}} = \frac{1/3}{1/3} = \frac{1}{3} \times \frac{3}{1} = 1.$

Example 2.24. The sales tax, X, of a shopkeeper has an exponential distribution with *p.d.f.*

$$f(x) = \frac{1}{4}e^{-x/4}, \quad x \ge 0$$
$$= 0, \quad x < 0$$

If sales tax is levied at the rate of 5% what is the probability that his sales exceed ₹ 10,000?

Solution. Sales tax on ₹ 10,000 = 10,000 × $\frac{5}{100}$ = ₹ 500

$\therefore$ If sales exceed ₹ 10,000 tax will exceed ₹ 500. Hence, the required probability is

$$P(X > 500) = \int_{500}^\infty \frac{1}{4}e^{-x/4}\, dx$$

$$= \left[\frac{1}{4}\frac{e^{-x/4}}{\left(-\frac{1}{4}\right)}\right]_{500}^\infty = e^{-\frac{500}{4}} = e^{-125}$$

Example 2.25. If families are selected at random in a certain thickly populated area and their annual income in excess of ₹ 4000 is treated as a random variable having an exponential distribution

$$f(x) = \frac{1}{2000} e^{-\frac{x}{2000}}$$

for $x > 0$, what is the probability that 3 out of 4 families selected in the area have income in excess of ₹ 5000?

Solution. ₹ 5000 – ₹ 4000 = ₹ 1000

$$\therefore \qquad P(X > 1000) = \int_{1000}^{\infty} \frac{1}{2000} e^{-\frac{x}{2000}} dx = \left[e^{-\frac{x}{2000}} \right]_{1000}^{\infty}$$

$$= e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

$$\therefore \qquad P(X < 1000) = 1 - \frac{1}{\sqrt{e}}.$$

Hence, required probability $= {}^4C_3 \left(\frac{1}{\sqrt{e}}\right)^3 \left(1 - \frac{1}{\sqrt{e}}\right) = \frac{4(\sqrt{e} - 1)}{e^2}$

Example 2.26. The income tax of a man is exponentially distributed with the probability density function given by

$$f(x) = \frac{1}{3} e^{-\frac{1}{3}x}, \text{ for } x > 0$$

$$= 0, \qquad \text{for } x < 0.$$

What is the probability that his income will exceed ₹ 17,000 assuming that the income tax is levied at the rate of 15% on the income above ₹ 15,000?

Solution. If the income exceeds ₹ 17,000 then income tax will exceed by 15% of (17000 – 15000), *i.e.*, exceeds by $\frac{15}{100} \times 2000$ *i.e.*, exceeds by ₹ 300.

Hence, the required probability

$$= P(X > 300) = \int_{300}^{\infty} \frac{1}{3} e^{-\frac{1}{3}x} dx$$

$$= \frac{1}{3} \left[-3e^{-\frac{1}{3}x} \right]_{300}^{\infty} = e^{-\frac{300}{3}} = e^{-100}.$$

Example 2.27. If X has a uniform distribution in [10, 1] with probability density function

$$f_x(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the distribution (probability density function) of $-2 \log x$.

Solution. Suppose $y = -2 \log X$. Now the distribution function G of y will be defined as follows:

$$G_y(y) = P(Y \le y) = P(-2 \log X \le y)$$

$$= P\left(\log X \ge -\frac{1}{2}y\right) = P(X \ge e^{-y/2}) = 1 - P(X \le e^{-y/2})$$

$$= 1 - \int_0^{e^{-y/2}} f(x)\,dx = 1 - \int_0^{e^{-y/2}} 1\,dx = 1 - e^{-y/2}$$

$\therefore$ $$g_y(y) = (d/dy)G(y) = \frac{1}{2}e^{-y/2}$$

Example 2.28. A random variable X has p.d.f.

$$f(x) = \frac{1}{\sigma}e^{-x/\sigma}, \; 0 \le x \le \infty.$$

Find the rth cumulant.

Solution. Moment generating function of X is

$$M_x(t) = E(e^{tX}) = \int_0^\infty e^{tx}.\frac{1}{\sigma}e^{-x/\sigma}\,dx = \frac{1}{\sigma}\int_0^\infty e^{(t-1/\sigma)x}\,dx$$

$$= -\frac{1}{\sigma}.\frac{1}{t-1/\sigma} = \frac{1}{t\sigma - 1} = (1 - t\sigma)^{-1}$$

$\therefore$ $$K_x(t) = \log(1 - t\sigma)^{-1} = t\sigma + \frac{t^2\sigma^2}{2} + ... + \frac{t^r\sigma^r}{r} + ...$$

$\therefore$ rth cumulant = coefficient of $\frac{t^r}{r!}$ in $K_x(t) = (r-1)!\,\sigma^r$.

EXERCISE 2.3

1. Define exponential distribution. Give its distribution function.
2. Find mean and variance for the exponential distribution.

$$\left[\textbf{Ans. } \text{Mean} = \frac{1}{\lambda}, \text{ Variance} = \frac{1}{\lambda^2}\right]$$

3. What is exponential distribution? Obtain μ_2, μ_3, μ_4, β_1 and β_2.
4. Obtain moment generating function for the exponential distribution

$$f(x) = \begin{cases} \theta e^{-\theta x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

Hence, find mean and variance.

5. For the exponential distribution defined by $y = Ce^{-Cx}$, where $C > 0$ and $0 \le x \le \infty$, prove that $\mu_2 = \frac{1}{C^2}$.
6. For the exponential distribution $dF = e^{-x}\,dx$, $x > 0$, find μ_2, μ_3 and μ_4. Also find mean deviation about a. Find for what value of a, this deviation is minimum. State the theorem in this connection?

$$\left[\textbf{Ans. } \mu_2 = 1, \mu_3 = 2, \mu_4 = 9; e^{-a} = \frac{1}{2} \Rightarrow a = \text{Median}\right]$$

7. Derive the following forgetful property of the exponential distribution

$$P[X > (s + t)/X > s] = P(X > t), \text{ for any } s, t > 0.$$

8. Let p.d.f. of X be $g(x) = e^{-x}, \quad x \geq 0$
and p.d.f. of y be $h(y) = e^{-y}, \quad y \geq 0$
Show that p.d.f. of $Z = X + Y$ is
$f(Z) = Ze^{-Z}, \quad Z \geq 0$
if X and Y are independent random variables.

2.9. GAMMA DISTRIBUTION

A continuous random variable X is said to be follow gamma distribution with parameter α and λ if its p.d.f. is

$$f(x) = \begin{cases} \dfrac{\alpha^{\lambda} e^{-\alpha x} x^{\lambda-1}}{\lceil(\lambda)}, & x \geq 0, \alpha, \lambda > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Remarks:

1. The function $f(x)$ represents a probability density function since

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{\infty} f(x)\, dx$$

$$= \int_{0}^{\infty} f(x)\, dx \qquad (\because \; x \geq 0)$$

$$= \int_{0}^{\infty} \frac{\alpha^{\lambda} e^{-\alpha x}}{\lceil(\lambda)} x^{\lambda-1}\, dx$$

$$= \frac{\alpha^{\lambda}}{\lceil(\lambda)} \int_{0}^{\infty} e^{-\alpha x} . x^{\lambda-1}\, dx$$

$$= \frac{\alpha^{\lambda}}{\lceil(\lambda)} . \frac{\lceil(\lambda)}{\alpha^{\lambda}} = 1 \quad \left[\because \int_{0}^{\infty} e^{-ax} x^{n-1}\, dx = \frac{\lceil(\lambda)}{a^{n}} \text{(gamma function)}\right]$$

2. **Mean of Gamma Distribution**

As we know,

$$\text{Mean} = \mu_1' = E(X) = \overline{x}$$

i.e.,

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{0} x\, f(x)\, dx + \int_{0}^{\infty} x\, f(x)\, dx$$

$$= 0 + \int_{0}^{\infty} x\, f(x)\, dx \qquad (\because \; x \geq 0)$$

$$\int_{0}^{\infty} x\, f(x)\, dx = \int_{0}^{\infty} \frac{x\, \alpha^{\lambda} e^{-\alpha x}}{\lceil(\lambda)} . x^{\lambda-1}\, dx$$

$$= \frac{\alpha^{\lambda}}{\lceil(\lambda)} \int_{0}^{\infty} e^{-\alpha x} . x^{\lambda-1+1}\, dx$$

$$= \frac{\alpha^\lambda}{\overline{|(\lambda)}} \int_0^\infty e^{-\alpha x} \cdot x^{(\lambda+1)-1}\, dx$$

$$= \frac{\alpha^\lambda}{\overline{|(\lambda)}} \cdot \frac{\Gamma(\lambda+1)}{\alpha^{\lambda+1}} \quad \left[\because \int_0^\infty e^{-ax} x^{n-1}\, dx = \frac{\overline{|(\lambda)}}{a^n} \text{(gamma function)}\right]$$

$$= \frac{\cancel{\alpha^\lambda} \cdot \Gamma(\lambda+1)}{\overline{|(\lambda)}\, \cancel{\alpha^\lambda} \cdot \alpha}$$

$$= \frac{\Gamma(\lambda+1)}{\overline{|(\lambda)} \cdot \alpha} = \frac{\lambda\, \cancel{\Gamma\lambda}}{\cancel{\Gamma\lambda} \cdot \alpha} = \frac{\lambda}{\alpha} \qquad (\because \quad \Gamma(\lambda+1) = \lambda \overline{|(\lambda)})$$

$$\therefore \qquad \boxed{E(X) = \frac{\lambda}{\alpha}.}$$

3. Variance of Gamma Distribution

$$\text{Var}(X) = \int_0^\infty x^2 f(x)\, dx - [E(x)]^2$$

So,

$$\text{Var}(X) = \frac{\alpha^\lambda}{\overline{|(\lambda)}} \int_0^\infty x^{(\lambda+1)} e^{-\alpha x}\, dx - \left(\frac{\lambda}{\alpha}\right)^2$$

$$= \frac{\alpha^\lambda}{\overline{|(\lambda)}} \int_0^\infty \left(\frac{t}{\alpha}\right)^{\lambda+1} e^{-t} \frac{dt}{\alpha} - \frac{\lambda^2}{\alpha^2} \qquad [\because \text{ substituting } t = \alpha x]$$

$$= \frac{\cancel{\alpha^\lambda}}{\alpha^{\lambda+2}\Gamma(\lambda)} \int_0^\infty t^{\lambda+1} e^{-t}\, dt - \frac{\lambda^2}{\alpha^2}$$

$$= \frac{\Gamma(\lambda+2)}{\alpha^2\Gamma(\lambda)} - \frac{\lambda^2}{\alpha^2} \qquad (\because \text{ by definition of gamma function})$$

$$= \frac{\Gamma(\lambda+2) - \lambda^2\Gamma\lambda}{\alpha^2\Gamma(\lambda)}$$

$$= \frac{\lambda(\lambda+1)\,\Gamma(\lambda) - \lambda^2\overline{|(\lambda)}}{\alpha^2\Gamma(\lambda)}$$

$$= \frac{\lambda\Gamma(\lambda)\,(\lambda+1-\lambda)}{\alpha^2\Gamma(\lambda)} = \frac{\lambda}{\alpha^2}$$

$$\therefore \qquad \boxed{\text{Var}(X) = \frac{\lambda}{\alpha^2}}$$

4. M.G.F. of Gamma Distribution

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)\, dx$$

$$= \frac{\alpha^\lambda}{\Gamma(\lambda)} \int_0^\infty e^{-\alpha x} x^{\lambda-1} . e^{tx}\, dx$$

$$= \frac{a^\lambda}{\Gamma(\lambda)} \int_0^\infty e^{-(\alpha-t)x} x^{\lambda-1}\, dx$$

Put $y = (\alpha - t)x$

$$dy = (\alpha - t)\, dt$$

$$\Rightarrow \quad \frac{1}{\alpha - t} dy = dt$$

$$\therefore \quad M_X(t) = \frac{\alpha^\lambda}{\Gamma(\lambda)} \int_0^\infty \left(\frac{y}{\alpha - t}\right)^{\lambda-1} . e^{-y} . \frac{1}{\alpha - t} dt$$

$$= \frac{\alpha^\lambda}{(\alpha - t)^\lambda} \cdot \frac{1}{\Gamma(\lambda)} \int_0^\infty y^{\lambda-1} e^{-y}\, dy$$

$$= \frac{\alpha^\lambda}{(\alpha - t)^\lambda} \cdot \frac{1}{\Gamma(\lambda)} \cdot \Gamma(\lambda) \qquad (\because \text{ by definition of gamma function})$$

$$= \left(\frac{\alpha}{\alpha - t}\right)^\lambda ; t < \lambda \quad \text{or} \quad \left(1 - \frac{t}{\alpha}\right)^{-\lambda} ; \ t < \lambda$$

5. Cumulative Distribution Function of Gamma Distribution

The cdf of $X \sim$ gamma (α, λ) is

$$F(x) = \int_0^x \frac{\alpha^\lambda e^{-\alpha x}\, x^{\lambda-1}}{\Gamma(\lambda)} dx$$

$$= \frac{\alpha^\lambda}{\Gamma(\lambda)} \int_0^x x^{\lambda-1} e^{-\alpha x} dx \qquad \text{Put} \quad \alpha x = y \ \Rightarrow \ dx = \frac{dy}{\alpha}$$

$$= \frac{\alpha^\lambda}{\Gamma(\lambda)} \int_0^{\alpha x} \left(\frac{y}{\alpha}\right)^{\lambda-1} e^{-y} . \frac{1}{\alpha} dy$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^{\alpha x} y^{\lambda-1} e^{-y}\, dy$$

This is called incomplete gamma function.

6. Special case of Gamma Distribution

Consider the gamma density function,

$$f(x) = \frac{\alpha^\lambda e^{-\alpha x} x^{\lambda-1}}{\Gamma(\lambda)}; \quad x \geq 0 \text{ and } \alpha, \ \lambda > 0$$

If $\lambda = 1$ then $$f(x) = \frac{\alpha . e^{-\alpha x} . x^0}{\overline{|1}}$$

$$= \alpha e^{-\alpha x} \quad x \geq 0$$

and this is the density function of exponential distribution.

7. Limiting form of Gamma Distribution as $\lambda \to \infty$

We know that if X follows gamma distribution, then $E(X) = \frac{\lambda}{\alpha}$ and $\text{var}(X) = \frac{\lambda}{\alpha^2}$. Then standard gamma variate is given by

$$Z = \frac{X - \mu}{\sigma} = \frac{X - \frac{\lambda}{\alpha}}{\frac{\sqrt{\lambda}}{\alpha}}$$

$$= \frac{X}{\left(\frac{\sqrt{\lambda}}{\alpha}\right)} - \sqrt{\lambda}$$

$\therefore$ $$M_z(t) = E[e^{zt}]$$

$$= E\left[e^{t(X/(\sqrt{\lambda}/\alpha) - \sqrt{\lambda})}\right]$$

$$= e^{-t\sqrt{\lambda}} E[e^{t(X/(\sqrt{\lambda}/\alpha))}$$

$$= e^{-t\sqrt{\lambda}} M_X(t/(\sqrt{\lambda}/\alpha))$$

$$= e^{-t\sqrt{\lambda}}\left(1 - \frac{t}{\sqrt{\lambda}}\right)^{-\lambda} \quad t < \sqrt{\lambda}$$

$\Rightarrow$ $$K_z(t) = -\sqrt{\lambda}t - \lambda \log\left(1 - \frac{t}{\sqrt{\lambda}}\right)$$

$$= -\sqrt{\lambda}t + \lambda\left(\frac{t}{\sqrt{\lambda}} + \frac{t^2}{2\lambda} + \frac{t^3}{3\lambda^{3/2}} + \ldots\right)$$

$$= -\sqrt{\lambda}t + \sqrt{\lambda}t + \frac{t^2}{2} + 0\,(\lambda^{-1/2})$$

where $0\,(\lambda^{-1/2})$ are terms containing $\frac{1}{2}$ and higher powers of λ in the denominator

$\therefore$ $$\lim_{\lambda \to \infty} K_z(t) = \frac{t^2}{2} \Rightarrow \lim_{\lambda \to \infty} M_z(t) = e^{t^2/2}$$

which is the m.g.f. of a standard normal variate.

In other words, gamma distribution tends to normal distribution for large valves of parameter λ.

8. Additive Property of Gamma Distribution

If X_1 and X_2 be independent random variables following gamma distribution with parameters λ_1 and λ_2 respectively. Then, the moment generating function of the sum of two gamma distribution will have parameter $\lambda_1 + \lambda_2$.

This result may be extended to the case of any number of independent gamma variables, *i.e.*, if $X_1, X_2, ..., X_k$ are independent gamma variates with parameters $\lambda_1, \lambda_2, ..., \lambda_K$ respectively then $X_1 + X_2 + ... + X_K$ is also a gamma variate with parameter $\lambda_1 + \lambda_2 + ... + \lambda_K$.

Proof: Since X_i is a gamma variate with parameter λ_i,

$$M_{X_i}(t) = \left(1-\frac{t}{\alpha}\right)^{-\lambda_i}$$

The m.g.f. of sum $X_1 + X_2 + ... + X_K$ is given by

$$M_{X_1+X_2+...+X_K}(t) = M_{X_1}(t) . M_{X_2}(t) . M_{X_3}(t), ..., M_{X_K}(t)$$

(since $X_1, X_2, ..., X_K$ are independent)

$$= \left(1-\frac{t}{\alpha}\right)^{-\lambda_1} . \left(1-\frac{t}{\alpha}\right)^{-\lambda_2} \left(1-\frac{t}{\alpha}\right)^{-\lambda_3} ... \left(1-\frac{t}{\alpha}\right)^{-\lambda_K}$$

$$= \left(1-\frac{t}{\alpha}\right)^{-(\lambda_1+\lambda_2+...+\lambda_K)}$$

which is the m.g.f. of a gamma variate with parameter $\lambda_1 + \lambda_2 + ... + \lambda_K$.

SOLVED EXAMPLES

Example 2.29. The daily consumption of milk in a city, in excess of 20,000 litres, is approximately distributed as a Gamma variate with parameters $a = \frac{1}{10,000}$ and $\lambda = 2$. The city has a daily stock of 30,000 litres. What is the probability that the stock is insufficient on a particular day?

Solution. If the *r.v.* X denotes the daily consumption of milk (in litres) in a city, then the *r.v.* $Y = X - 20,000$ has a gamma distribution with p.d.f.;

$$g(y) = \frac{1}{(10,000)^2\,\Gamma(2)} y^{2-1} e^{-y/10,000} = y\frac{e^{-y/10,000}}{(10,000)^2};\ 0 < y < \infty$$

Since the daily stock of the city is 30,000 litres, the required probability 'p' that the stock is insufficient on a particular day is given by:

$$p = P(X > 30,000) = P(Y > 10,000)$$

$$= \int_{10,000}^{\infty} g(y)\,dy = \int_{10,000}^{\infty} \frac{y\,e^{-y/10,000}}{(10,000)^2}\,dy$$

$$= \int_1^\infty z e^{-z}\, dz \qquad \text{[Taking } z = y/10{,}000\text{]}$$

Integrating by parts, $p = \left| -ze^{-z} \right|_1^\infty + \int_1^\infty e^{-z}\, dz = e^{-1} - \left| e^{-z} \right|_1^\infty = e^{-1} + e^{-1} = \frac{2}{e}$.

Remark. Since $\lambda = 2$, the integration is easily done. However, for general values of a and λ, the integral is evaluated by using tables of Incomplete Gamma Integral of the form:

$\int_0^\infty \frac{e^{-x} x^{n-1}}{\Gamma(n)} dx$, which have been tabulated for different values of α and n.

Example 2.30. If $X \sim N(\mu, \sigma^2)$, obtain the p.d.f. of $U = \frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2$.

Solution. Since $X \sim N(\mu, \sigma^2) \Rightarrow z = \frac{x-\mu}{\sigma} \sim N(0, 1)$ with p.d.f.:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right); \quad -\infty < z < \infty. \qquad ...(i)$$

The distribution function $G(\cdot)$ of $U = \frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2$ is given by:

$$G_U(u) = P(U \le u) = P\left[\frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2 \le u\right]$$

$$= P(Z^2 \le 2u) = P(-\sqrt{2u} \le Z \le \sqrt{2u}), \text{ where } Z \sim N(0, 1)$$

$$= P(Z \le \sqrt{2u}\,) - P(Z \le \sqrt{2u}\,)$$

$$= \Phi(\sqrt{2u}\,) - \Phi(-\sqrt{2u}\,) \qquad ...(ii)$$

where $\Phi(\cdot)$ is the distribution function of standard normal variate (SNV)Z.
Differentiating w.r.to u, the p.d.f. $g(\cdot)$ of U is given by:

$$g(u) = \phi(\sqrt{2u}).\frac{d}{du}(\sqrt{2u}) - \phi(-\sqrt{2u}).\frac{d}{du}(-\sqrt{2u})$$

[$\phi(\cdot)$ is p.d.f. of SNV Z]

$$= \frac{1}{\sqrt{2u}}\left[\phi(\sqrt{2u}) + \phi(-\sqrt{2u})\right]$$

$$= \frac{1}{\sqrt{2u}}.2\phi(\sqrt{2u}) \qquad [\because\ \phi(u) \text{ is even function of } u]$$

$$= \sqrt{\frac{2}{u}} \cdot \frac{1}{\sqrt{2\pi}} \exp.\left(-\frac{1}{2} . 2u\right) \qquad \text{[From } (i)]$$

$$= \frac{1}{\Gamma(1/2)} e^{-u} . u^{(1/2)-1} \quad u \geq 0$$

$$\left[\because \sqrt{\pi} = \Gamma(1/2) \text{ and } u = \frac{1}{2}\left(\frac{x-u}{\sigma}\right)^2 \geq 0\right]$$

which is the p.d.f. of gamma distribution with parameter $\frac{1}{2}$.

Hence, $$U = \frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2 \text{ is a } \gamma\left(\frac{1}{2}\right) \text{ variate.}$$

Example 2.31. Show that the mean value of positive square root of a $\gamma(\mu)$ variate is $\Gamma\left(\mu+\frac{1}{2}\right) / \Gamma(\mu)$. Hence, prove that the mean deviation of a normal variate from its mean is $\sqrt{2/\pi}$, where σ is the standard deviation of the distribution.

Solution. Let X be a $\gamma(\mu)$ variate. Then $f(x) = \frac{e^{-x} x^{\mu-1}}{\Gamma(\mu)}$; $\mu > 0, 0 < x < \infty$

$$\therefore \quad E(\sqrt{X}) = \int_0^\infty x^{1/2} f(x)\, dx = \frac{1}{\Gamma(\mu)} \int_0^\infty e^{-x} x^{\mu+(1/2)-1}\, dx = \frac{\Gamma\left(\mu+\frac{1}{2}\right)}{\Gamma(\mu)} \qquad ...(i)$$

If $X \sim N(\mu, \sigma^2)$, then $U = \frac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2$ is a $\gamma\left(\frac{1}{2}\right)$ variate. (refer example 2.30)

$$\therefore \quad |X-\mu| = \sqrt{2}\sigma U^{1/2}, \text{ where } U \text{ is a } \gamma\left(\frac{1}{2}\right) \text{ variate.}$$

Hence, mean deviation of X about mean is given by:

$$E|X-\mu| = E(\sqrt{2}\,\sigma\, U^{1/2}) = \sqrt{2}\,\sigma\, E(U^{1/2})$$

$$= \sqrt{2}\sigma \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}{r\left(\frac{1}{2}\right)} = \frac{\sqrt{2}\sigma}{\sqrt{\pi}} = \sigma\sqrt{2/\pi}. \quad \left[\text{Using } (i) \text{ with } \mu = \frac{1}{2}\right]$$

Example 2.32. If X and Y are independent Gamma variates with parameters μ and respectively, show that the variables $U = X + Y, Z = \dfrac{X}{X+Y}$ are independent and that U is a $\gamma(\mu + \nu)$ variate and Z is a $\beta_1(\mu, \nu)$ variate.

Solution. Since X is a $\gamma(\mu)$ variate and Y is a $\gamma(\nu)$ variate, we have

$$f_1(x)dx = \frac{1}{\Gamma(\mu)} e^{-x} x^{\mu-1} dx;\ 0 < x < \infty,\ \mu > 0$$

$$f_2(y)\, dy = \frac{1}{\Gamma(\nu)} e^{-y} y^{\nu-1} dy\ ;\ 0 < y < \infty,\ \nu > 0$$

Since X and Y are independent distributed, their joint probability differential is given by the compound probability theorem as shown below:

$$dF(x, y) = f_1(x) f_2(y)\, dx\, dy = \frac{1}{\Gamma\mu\, \Gamma\nu} e^{-(x+y)} x^{\mu-1} y^{\nu-1}\, dxdy$$

Now $u = x + y,\ z = \dfrac{x}{x+y}$ so that $x = uz,\ y = u - x = u(1-z)$

Jacobian of transformation J is given by:

$$J = \frac{\partial(x, y)}{\partial(u, z)} = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta y}{\delta u} \\ \frac{\delta x}{\delta z} & \frac{\delta y}{\delta z} \end{vmatrix} = \begin{vmatrix} z & 1-z \\ u & -u \end{vmatrix} = -u$$

As X and Y range from 0 to ∞, u ranges from 0 to ∞ and z from 0 to 1 $\left(\because \dfrac{x}{x+y} \le 1\right)$

Hence, the joint distribution of U and Z is given by:

$$dG(u, z) = g(u, z)\, du\, dz = \frac{1}{\Gamma(\mu)\,\Gamma(\nu)} . e^{-u} (uz)^{\mu-1} [u(1-z)]^{\nu-1}\, |J|\, dudz$$

$$= \frac{1}{\Gamma(\mu)\,\Gamma(\nu)} . e^{-u} u^{\mu+\nu-1} z^{\mu-1} (1-z)^{\nu-1}\, dudz$$

$$= \left\{\frac{e^{-u} u^{\mu+\nu-1}}{\Gamma(\mu+\nu)} du\right\} \left\{\frac{1}{B(\mu, \nu)} z^{\mu-1} (1-z)^{\nu-1}\, dz\right\}$$

$$= [g_1(u)\, du]\, [g_2(z)\, dz],\quad \text{(say)},$$

where

$$g_1(u) = \frac{1}{\Gamma(\mu+\nu)} e^{-u} u^{\mu+\nu-1}, 0 < u < \infty \qquad ...(i)$$

and

$$g_2(z) = \frac{1}{B(\mu, \nu)} z^{\mu-1} (1-z)^{\nu-1},\ 0 < z < 1 \qquad ...(ii)$$

From (i) and (ii), we conclude that U and Z are independently distributed, U is a $\gamma\,(\mu + \nu)$ variate and Z as a $\beta_1(\mu, \nu)$ variate.

Example 2.33. If X and Y are independent Gamma variates with parameters μ and ν respectively, show that $U = X + Y$, $Z = \dfrac{X}{Y}$ are independent and that U is a $\gamma(\mu + \nu)$ variate and Z is a β_2 (μ, ν) variate.

Solution. As in example 2.32, we have

$$dF(x, y) = \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-(x+y)} x^{\mu-1} y^{\nu-1}\, dxdy,\ 0 < (x, y) < \infty$$

Since $u = x + y$ and $z = \dfrac{x}{y}$, $1 + z = 1 + \dfrac{x}{y} = \dfrac{u}{y} \Rightarrow y = \dfrac{u}{1+z}$ and $x = \dfrac{uz}{1+z} = u\left(1 - \dfrac{1}{1+z}\right)$

$$J = \frac{\partial(x, y)}{\partial(u, z)} = \frac{-u}{(1+z)^2}$$

As x and y range from 0 to ∞, both u and z range from 0 to ∞. Hence, the joint probability differential of random variables U and Z becomes:

$$dG(u, z) = \frac{1}{\Gamma(\mu)\Gamma(\nu)} e^{-u} \left(\frac{uz}{1+z}\right)^{\mu-1} \left(\frac{u}{1+z}\right)^{\nu-1} |J|\, dudz$$

$$= \left[\frac{e^{-u}\, u^{\mu+\nu-1}}{\Gamma(\mu+\nu)} du\right]\left[\frac{1}{B(\mu, \nu)} \cdot \frac{z^{\mu-1}}{(1+z)^{\mu+\nu}} dz\right];$$

$$0 < u < \infty,\ 0 < z < \infty$$

Hence, U and Z are independently distributed, U as a $\gamma(\mu + \nu)$ variate and Z as a β_2 (μ, ν) variate.

Remark: The above two examples lead to the following important results.

If X is a $\gamma(\mu)$ variate and Y is an independent $\gamma(\nu)$ variate, then

(*i*) $X + Y$ is a $\gamma(\mu + \nu)$ variate, *i.e.*, the sum of two independent Gamma variates is also a Gamma variate.

(*ii*) $\dfrac{X}{Y}$ is a β_2 (μ, ν) variate *i.e.*, the ratio of two independent Gamma variates is β_2 – variate.

(*iii*) $X/(X + Y)$ is a $\beta_1(\mu, \nu)$ variate.

EXERCISE 2.4

1. The time spent on a computer (X) is a gamma distribution with mean 20 and variance 80 min^2.
   (*i*) What are the value of λ and α?
   (*ii*) What is $P(X < 24)$?
   (*iii*) Find $P(20 < X < 40)$ **[Ans.** (*i*) 5 and 4, (*ii*) 0.715, (*iii*) 0.411]
2. In a certain city, the daily consumption of electric power in millions of kilowatt-hours can be treated as a random variable having a gamma distribution with $\lambda = 3$, $\alpha = \dfrac{1}{2}$. If the power plant of this city has a daily consumption of 12 million kilowatt-hours, what is the probability that this power supply will be inadequate on any given day? (**Ans.** 0.58003)