J.C. BOSE UNIVERSITY OF

SCIENCE & TECHNOLOGY, YMCA

Report on the Physical Significance of Gamma and Beta Distributions

STATISTICS ASSIGNMENT

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Introduction

Probability distributions are essential tools in understanding and modeling real-world phenomena. Among them, the **Gamma** and **Beta** distributions are widely used in physics, engineering, biology, and reliability analysis. While both are defined over the positive real numbers or within the [0,1] interval, their applications are distinct due to their shapes and properties. This report explores the **physical significance** of these two important distributions.

Definitions and Mathematical Formulations

1. Gamma Distribution

Definition:

> The Gamma distribution is a two-parameter family of continuous probability distributions defined for positive real numbers. It is used to model the waiting time until α events occur, where each event happens independently at a constant average rate.

Mathematical Formulation

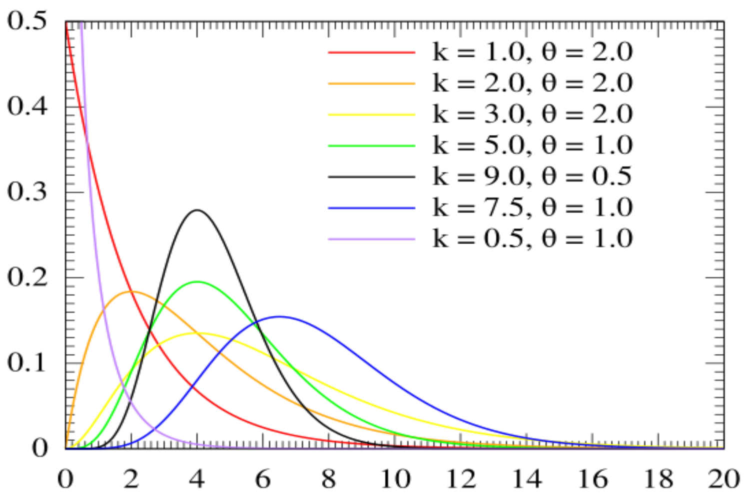
The probability density function (PDF) of the Gamma distribution is given by:

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0, \alpha, \beta > 0$$

Where:

- xx is the random variable (e.g., time, distance)
- α\alpha is the shape parameter
- β\beta is the rate parameter (inverse of scale)
- Γ(α)\Gamma(\alpha) is the Gamma function:

Γ(α)=∫0∞tα−1e−tdt\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt



The shape of the distribution varies significantly based on the value of α and β.

Key Properties

- Mean: $\mu = \frac{\alpha}{\beta}$
- Variance: $\sigma^2 = \frac{\alpha}{\beta^2}$

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2. Beta Distribution

Definition:

> The Beta distribution is a family of continuous probability distributions defined on the interval [0, 1]. It is particularly useful for modeling random variables that represent proportions or probabilities.

Mathematical Formulation

The probability density function (PDF) of the Beta distribution is:

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1, \alpha, \beta > 0$$

Where:

- α\alpha and β\beta are shape parameters
- B(α,β)B(\alpha, \beta) is the Beta function, a normalization constant:

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

The shape of the distribution can be skewed left, right, or even U-shaped depending on the values of α and β.

Key Properties

- Mean: $\mu = \frac{\alpha}{\alpha+\beta}$
- Variance: $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Historical Context of Gamma and Beta Distributions

The Gamma and Beta distributions have rich mathematical histories, deeply rooted in the development of probability theory and mathematical statistics.

Gamma Distribution: Origins and Evolution

The Gamma distribution was first introduced in the context of mathematical analysis and complex integrals. The foundation of this distribution lies in the Gamma function, denoted by $\Gamma(n)$, which was generalized by the renowned mathematician Leonhard Euler in the 18th century. The Gamma function extends the concept of factorials beyond integers, allowing it to be applied to real and complex numbers:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

Euler's work laid the groundwork for what would later be recognized as the Gamma distribution, which appears naturally in problems related to waiting times and processes involving the sum of exponential variables. It wasn't until the 20th century that the distribution took on its modern probabilistic form, thanks to contributions from statisticians and applied mathematicians like Jacques Hadamard and Pearson.

The Gamma distribution became especially prominent in queueing theory, reliability engineering, and Bayesian statistics. In the Bayesian framework, it serves as a conjugate prior for the exponential and Poisson distributions, making it a vital tool in inference problems.

Beta Distribution: Emergence and Development

The Beta distribution, like the Gamma, also derives from an integral expression — the Beta function, denoted as $B(\alpha, \beta)$, which is defined as:

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$$

This function, too, was studied by Euler in the 18th century, but the Beta distribution itself emerged in the early 1900s as part of Karl Pearson's efforts to classify a wide range of empirical distributions. Pearson introduced the Beta distribution as part of his Pearson distribution family, which aimed to model various types of skewed data observed in real-life phenomena.

One of the key reasons for the Beta distribution's importance is its flexibility in modeling random variables bounded between 0 and 1. It found applications in modeling probabilities, proportions, and Bayesian posterior distributions, especially when the prior beliefs are themselves represented as continuous distributions.

Integration into Statistical Theory

Both the Gamma and Beta distributions gained further prominence as mathematical tools when Bayesian statistics rose to popularity in the mid-20th century. They were particularly admired for their conjugacy properties, which greatly simplified analytical computations in Bayesian inference.

Furthermore, with the advent of modern computing, these distributions are now easily implemented and analyzed using software like R, Python (SciPy), and MATLAB, reinforcing their importance in both theoretical research and applied fields such as finance, medicine, machine learning, and engineering.

Probability Mass Function (PMF) and Probability Density Function (PDF)

In probability theory, the Probability Mass Function (PMF) is used for discrete random variables, while the Probability Density Function (PDF) is used for continuous random variables. Since Gamma and Beta distributions are continuous, we describe their PDFs, not PMFs.

Gamma Distribution – PDF and Derivation

The Gamma distribution models the sum of multiple independent exponentially distributed random variables. It is commonly used to represent waiting times, reliability, and stochastic processes.

Definition:

Let X be a continuous random variable that follows a Gamma distribution with shape parameter α>0\alpha > 0α>0 and scale parameter θ>0\theta > 0θ>0. Its PDF is given by:

$$f(x;\alpha,\theta) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \quad x > 0$$

Alternatively, using rate parameter $\beta = \frac{1}{\theta}$:

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0$$

Proof (Sketch of Derivation):

The Gamma function is defined as:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

To make this a probability distribution, we normalize the function so that the total area under the curve is 1. Thus:

$$\int_0^\infty f(x)dx = 1 \Rightarrow \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx = 1$$

Since $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$, this confirms the function is a valid PDF.

4.2 Beta Distribution – PDF and Derivation

The Beta distribution is used for modeling probabilities and proportions in the interval [0, 1]. It is defined by two shape parameters: α>0\alpha > 0α>0 and β>0\beta > 0β>0.

Definition:

Let X∈[0,1]X \in [0,1]X∈[0,1] be a continuous random variable following the Beta distribution. Its PDF is:

$$f(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0 < x < 1$$

Where $B(\alpha, \beta)$ is the **Beta function**, defined as:

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Proof (Sketch of Derivation):

To ensure $f(x)$ is a valid PDF:

$$\int_0^1 f(x; \alpha, \beta) dx = \int_0^1 \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} dx = 1$$

By definition of the Beta function, the normalization constant $\frac{1}{B(\alpha,\beta)}$ ensures the integral over [0, 1] equals 1, validating it as a proper probability density function.

Cumulative Distribution Function (CDF)

The Cumulative Distribution Function (CDF) of a continuous random variable provides the probability that the variable takes a value less than or equal to a specific value. It is defined as:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

For distributions like Gamma and Beta, which are defined over specific intervals, the limits of integration and evaluation of the CDF follow their domain.

Gamma Distribution – CDF

Let $X \sim \mathrm{Gamma}(\alpha, \theta)$, with shape parameter $\alpha > 0$ and scale parameter $\theta > 0$. The PDF is:

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \quad x > 0$$

The **CDF** is defined as:

$$F(x; \alpha, \theta) = P(X \leq x) = \int_{0}^{x} \frac{1}{\Gamma(\alpha)\theta^{\alpha}} t^{\alpha-1} e^{-t/\theta} dt$$

This integral does not generally have a **closed-form solution** for arbitrary α, but it is expressed using the **Lower Incomplete Gamma Function**, $\gamma(\alpha, x)$:

$$F(x; \alpha, \theta) = \frac{1}{\Gamma(\alpha)} \gamma\left(\alpha, \frac{x}{\theta}\right)$$

Where the lower incomplete Gamma function is:

$$\gamma(\alpha, x) = \int_{0}^{x} t^{\alpha-1} e^{-t} dt$$

Thus, the CDF can be written compactly as:

$$F(x) = \frac{\gamma(\alpha, x/\theta)}{\Gamma(\alpha)}$$

Key Points:

- The Gamma CDF is monotonically increasing.
- It approaches 1 as x→∞x \to \inftyx→∞.
- It is used to determine waiting time probabilities and reliability in engineering contexts.

Beta Distribution – CDF

Let $X \sim \mathrm{Beta}(\alpha, \beta)$, with $\alpha, \beta > 0$, and PDF:

$$f(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1$$

The **CDF** is given by:

$$F(x; \alpha, \beta) = P(X \leq x) = \int_0^x \frac{1}{B(\alpha, \beta)} t^{\alpha-1}(1-t)^{\beta-1} dt$$

This is known as the **Regularized Incomplete Beta Function**, denoted as:

$$F(x; \alpha, \beta) = I_x(\alpha, \beta)$$

Where:

$$I_x(\alpha, \beta) = \frac{1}{B(\alpha, \beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt$$

The Beta function B(α,β)B(\alpha, \beta)B(α,β) is related to the Gamma function:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Key Points:

- The Beta CDF is bounded in [0, 1].
- It represents the probability of proportions, making it useful in Bayesian analysis.
- The CDF is flexible in shape depending on the values of α\alphaα and β\betaβ (U-shaped, skewed, uniform, etc.).

Moment Generating Function (MGF)

The Moment Generating Function (MGF) of a random variable X is a function that encodes all of the moments (mean, variance, etc.) of a distribution. It is defined as:

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

- If it exists, the MGF uniquely determines the probability distribution.
- The nthn^{th}nth moment of a random variable can be found by differentiating the MGF nnn times and evaluating at t=0t = 0t=0:

$$\mu_n' = \mathbb{E}[X^n] = M_X^{(n)}(0)$$

MGF of Gamma Distribution

Let $X \sim \mathrm{Gamma}(\alpha, \theta)$, where:

- $\alpha > 0$ is the **shape** parameter,
- $\theta > 0$ is the **scale** parameter.

The **PDF** is:

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \quad x > 0$$

MGF Derivation:

The MGF is:

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^\infty e^{tx} f(x)\, dx$$

Substitute the PDF:

$$M_X(t) = \int_0^\infty e^{tx} \cdot \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta} dx$$

Combine the exponential terms:

$$M_X(t) = \frac{1}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x\left(\frac{1}{\theta}-t\right)} dx$$

Let:

$$\lambda = \frac{1}{\theta} - t$$

For the MGF to converge, we require:

$$t < \frac{1}{\theta}$$

Now, this integral becomes:

$$M_X(t) = \frac{1}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{1}{\Gamma(\alpha)\theta^\alpha} \cdot \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

$$M_X(t) = \left(\frac{1}{1-\theta t}\right)^\alpha \quad \text{for } t < \frac{1}{\theta}$$

MGF of Gamma Summary:

$$M_X(t) = \left(\frac{1}{1-\theta t}\right)^\alpha, \quad t < \frac{1}{\theta}$$

- **First moment (mean):** $\mu = \alpha\theta$
- **Second moment:** $\mathbb{E}[X^2] = \alpha(\alpha+1)\theta^2$
- **Variance:** $\sigma^2 = \alpha\theta^2$

MGF of Beta Distribution

Let $X \sim \mathrm{Beta}(\alpha, \beta)$, with $\alpha, \beta > 0$

The **PDF** is:

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1$$

MGF Derivation:

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^1 e^{tx} \cdot \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} dx$$

This integral does not simplify to a closed form in terms of elementary functions. However, it is expressed as a confluent hypergeometric function or evaluated numerically.

Therefore, the MGF of the Beta distribution exists for all real ttt, but has no closed form in general.

Moments of Beta Distribution:

- **Mean:**

$$\mathbb{E}[X] = \frac{\alpha}{\alpha + \beta}$$

- **Variance:**

$$\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Properties of Gamma and Beta Distributions

The Gamma distribution is a two-parameter family of continuous probability distributions. It is defined by:

- Shape parameter α>0\alpha > 0α>0
- Scale parameter θ>0\theta > 0θ>0

1. Support

$$x \in (0, \infty)$$

The distribution is defined only for positive values, making it ideal for modeling waiting times, lifespans, and accumulated events.

2. Probability Density Function (PDF)

$$f(x; \alpha, \theta) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \quad x > 0$$

Where $\Gamma(\alpha)$ is the gamma function:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

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3. Cumulative Distribution Function (CDF)

$$F(x; \alpha, \theta) = \frac{1}{\Gamma(\alpha)} \int_0^{x/\theta} t^{\alpha-1} e^{-t} dt = \gamma(\alpha, x/\theta)$$

Where γ(α,x/θ)\gamma(\alpha, x/\theta)γ(α,x/θ) is the lower incomplete gamma function.

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4. Mean and Variance

$$\text{Mean (Expected value)} = \mathbb{E}[X] = \alpha\theta$$

$$\text{Variance} = \text{Var}(X) = \alpha\theta^2$$

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5. Skewness and Kurtosis

$$\text{Skewness} = \frac{2}{\sqrt{\alpha}}, \quad \text{Kurtosis (Excess)} = \frac{6}{\alpha}$$

As α\alphaα increases, the distribution becomes more symmetric.

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6. Additivity Property

If $X_1 \sim \text{Gamma}(\alpha_1, \theta)$ and $X_2 \sim \text{Gamma}(\alpha_2, \theta)$, then:

$$X_1 + X_2 \sim \text{Gamma}(\alpha_1 + \alpha_2, \theta)$$

This is useful in modeling total waiting times or sums of exponential variables.

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7. Memorylessness (Special Case)

Although the gamma distribution itself is not memoryless, when α=1\alpha = 1α=1, it reduces to the exponential distribution, which is memoryless.

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8. Applications

- Modeling waiting time until α\alphaα events happen (e.g., Poisson process)
- Rainfall, insurance claims, queuing systems
- Bayesian statistics (as a conjugate prior for rate parameters)

Properties of the Beta Distribution

The Beta distribution is a family of continuous probability distributions on the interval (0,1)(0, 1)(0,1), defined by:

- Shape parameters: α>0\alpha > 0α>0, β>0\beta > 0β>0

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1. Support

x∈(0,1)x \in (0, 1)x∈(0,1)

This makes it ideal for modeling proportions, probabilities, and fractions.

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2. Probability Density Function (PDF)

$$f(x;\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

Where $B(\alpha, \beta)$ is the **Beta function**:

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

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3. Cumulative Distribution Function (CDF)

$$F(x;\alpha,\beta) = I_x(\alpha,\beta)$$

Where $I_x(\alpha, \beta)$ is the **regularized incomplete beta function**.

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4. Mean and Variance

$$\text{Mean} = \mathbb{E}[X] = \frac{\alpha}{\alpha+\beta}$$

$$\text{Variance} = \text{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

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5. Mode

If $\alpha > 1$ and $\beta > 1$:

$$\text{Mode} = \frac{\alpha - 1}{\alpha + \beta - 2}$$

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6. Symmetry

- If $\alpha = \beta$, the distribution is **symmetric** around 0.5
- If $\alpha > \beta$, it is **right-skewed**
- If $\alpha < \beta$, it is **left-skewed**

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7. Conjugacy (Bayesian Property)

The Beta distribution is the conjugate prior for the binomial and Bernoulli likelihoods in Bayesian inference.

If the prior is Beta(α,β)\text{Beta}(\alpha, \beta)Beta(α,β) and we observe xxx successes in nnn trials, the posterior becomes:

$$\text{Beta}(\alpha + x, \beta + n - x)$$

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8. Flexibility of Shape

The Beta distribution is highly flexible. Depending on the values of α\alphaα and β\betaβ, it can be:

- Uniform (α=β=1\alpha = \beta = 1α=β=1)
- U-shaped
- Bell-shaped
- J-shaped or reverse J-shaped

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9. Applications

- Modeling probabilities, success rates, proportions
- Bayesian estimation of probabilities
- Modeling task completion ratios, time allocation (in project management)

Parameter Estimation Techniques

Parameter estimation is a critical step in statistical modeling. It involves using observed data to estimate the parameters of a probability distribution. For Gamma and Beta distributions, the common techniques used are:

- Method of Moments (MoM)
- Maximum Likelihood Estimation (MLE)

Each method has its own advantages depending on the context and nature of data.

A. Gamma Distribution

Let X1,X2,…,XnX_1, X_2, \ldots, X_nX1,X2,…,Xn be independent and identically distributed (i.i.d.) random variables from a Gamma distribution with shape parameter α\alphaα and scale parameter θ\thetaθ.

1. Method of Moments (MoM)

Let:

- $\mu = \mathbb{E}[X] = \alpha\theta$
- $\sigma^2 = \mathrm{Var}(X) = \alpha\theta^2$

From the sample, compute:

- Sample mean: $\bar{X} = \frac{1}{n}\sum X_i$
- Sample variance: $S^2 = \frac{1}{n-1}\sum (X_i - \bar{X})^2$

Using:

$$\bar{X} = \alpha\theta \quad \text{and} \quad S^2 = \alpha\theta^2$$

Solving the equations:

- $\hat{\alpha} = \frac{\bar{X}^2}{S^2}$
- $\hat{\theta} = \frac{S^2}{\bar{X}}$

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2. Maximum Likelihood Estimation (MLE)

The likelihood function for the Gamma distribution:

$$L(\alpha, \theta) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x_i^{\alpha-1} e^{-x_i/\theta}$$

The log-likelihood:

$$\log L = n[\alpha \log(1/\theta) - \log \Gamma(\alpha)] + (\alpha - 1) \sum \log x_i - \frac{1}{\theta} \sum x_i$$

To estimate α\alphaα and θ\thetaθ, we solve:

$$\frac{\partial \log L}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial \log L}{\partial \theta} = 0$$

This leads to a transcendental equation involving the digamma function ψ(α)\psi(\alpha)ψ(α). Iterative numerical methods such as Newton-Raphson are used for solving α\alphaα. Once α\alphaα is known, estimate θ\thetaθ as:

$$\hat{\theta} = \frac{\bar{X}}{\hat{\alpha}}$$

B. Beta Distribution

Let X1,X2,…,XnX_1, X_2, \ldots, X_nX1,X2,…,Xn be i.i.d. random variables from a Beta distribution with parameters α\alphaα and β\betaβ.

1. Method of Moments (MoM)

Let:

- $\mu = \frac{\alpha}{\alpha+\beta}$
- $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

From the sample:

- Sample mean: X¯\bar{X}X¯
- Sample variance: S2S^2S2

Now define:

$$v = \frac{\bar{X}(1-\bar{X})}{S^2} - 1$$

Then:

$$\hat{\alpha} = \bar{X} \cdot v \quad , \quad \hat{\beta} = (1-\bar{X}) \cdot v$$

This method works well for initial approximations and small samples.

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2. Maximum Likelihood Estimation (MLE)

The likelihood function for the Beta distribution is:

$$L(\alpha, \beta) = \prod_{i=1}^{n} \frac{x_i^{\alpha-1}(1-x_i)^{\beta-1}}{B(\alpha, \beta)}$$

Log-likelihood:

$$\log L = (\alpha - 1)\sum \log x_i + (\beta - 1)\sum \log(1 - x_i) - n \log B(\alpha, \beta)$$

The derivatives:

$$\frac{\partial \log L}{\partial \alpha} = \sum \log x_i - n\psi(\alpha) + n\psi(\alpha + \beta)$$
$$\frac{\partial \log L}{\partial \beta} = \sum \log(1 - x_i) - n\psi(\beta) + n\psi(\alpha + \beta)$$

Where ψ(·)\psi(\cdot)ψ(·) is the digamma function. These equations are solved using iterative methods (e.g., Newton-Raphson).

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3. Numerical Estimation Techniques

For both distributions, especially when MLE becomes complex due to non-linear equations, estimation is done using numerical techniques:

- Newton-Raphson
- Method of Scoring
- Expectation-Maximization (EM) Algorithm

Statistical software such as R, Python (SciPy), or MATLAB can be used to obtain precise parameter estimates.

Illustrative Example (Gamma MoM):

Let's say the sample data:

$X = \{2.3, 1.8, 3.2, 2.9, 3.0\}$

- $\bar{X} = 2.64$, $S^2 = 0.285$

Estimate parameters:

$$\hat{\alpha} = \frac{(2.64)^2}{0.285} \approx 24.47, \quad \hat{\theta} = \frac{0.285}{2.64} \approx 0.108$$

Key Takeaways

- MoM is simple but can be inaccurate with skewed distributions or small sample sizes.
- MLE is statistically efficient but computationally intensive.
- Choosing the method depends on data quality, sample size, and computational resources.

Notations and Symbols Used

In statistical theory and probability distribution analysis, consistent and clear notation is essential for expressing mathematical formulations, derivations, and theoretical concepts. Below is a comprehensive list of commonly used symbols and notations in the context of Gamma and Beta Distributions, along with their descriptions.

A. Common Mathematical Symbols

Symbol	Name	Description
X	Random Variable	A variable whose possible values are numerical outcomes of a random phenomenon.
x	Realization	A specific observed value of the random variable X.
$f(x)$	Probability Density Function (PDF)	Function describing the likelihood of X assuming a particular value x.
$F(x)$	Cumulative Distribution Function (CDF)	Probability that $X \leq x$.
$\mathbb{E}[X]$	Expected Value	The mean or average of a random variable.
$\mathrm{Var}(X)$	Variance	Measure of dispersion or spread of the distribution.
$\Gamma(\cdot)$	Gamma Function	A function that generalizes factorial to real and complex numbers.
$B(\alpha, \beta)$	Beta Function	Normalizing constant for the Beta distribution.
$\psi(\cdot)$	Digamma Function	Derivative of the logarithm of the gamma function.
μ	Mean	Average of the distribution.
σ^2	Variance	Spread of the distribution.
α	Shape Parameter (Gamma/Beta)	Controls the shape of the distribution.
β	Shape Parameter (Beta)	Often used as the second shape parameter in Beta distribution.
θ	Scale Parameter (Gamma)	Controls the spread of the Gamma distribution.
n	Sample Size	Number of observations in the dataset.
$\bar{X}$	Sample Mean	Average of the sample data.

Conclusion: Comparative Analysis of Gamma and Beta Distributions

The Gamma and Beta distributions are two of the most significant continuous probability distributions in the field of statistics and applied mathematics. While they share some similarities in their formulation and mathematical underpinnings, they serve fundamentally different purposes and are used in different contexts. This report has delved into their definitions, formulations, historical development, probability functions, moment generating functions, properties, parameter estimation methods, and applications—revealing both their unique strengths and complementary roles.

Domain and Support

- Gamma Distribution: Defined on the interval (0,∞)(0, \infty)(0,∞), making it suitable for modeling positive continuous variables such as waiting times, life spans, or energy levels.
- Beta Distribution: Defined on the interval (0,1)(0, 1)(0,1), making it ideal for modeling proportions and probabilities, such as success rates and confidence scores.

Parameterization and Shape

- Both distributions use shape parameters α\alphaα (and β\betaβ for Beta), but their interpretations vary:
    - In Beta, α\alphaα and β\betaβ control the skewness and concentration of probability within [0,1][0,1][0,1].
    - In Gamma, α\alphaα (shape) and θ\thetaθ (scale) influence the rate of decay and peak of the distribution.
- Gamma becomes exponential when α=1\alpha = 1α=1, and Beta becomes uniform when α=β=1\alpha = \beta = 1α=β=1.

Probability Functions

- The PDF of both distributions includes a gamma function in their denominator for normalization, but the shapes of the functions vary drastically depending on parameters.
- CDFs in both distributions do not generally have closed-form expressions, but they are computable via numerical methods or incomplete gamma/beta functions.

- MGFs exist for the Gamma distribution (for t<1/θt < 1/\thetat<1/θ), but the Beta distribution does not have a closed-form MGF, often relying on series expansion.

Properties and Moments

- **Gamma** distribution has a linear relationship between mean and variance: $\mu = \alpha\theta$, $\sigma^2 = \alpha\theta^2$.
- **Beta** distribution has bounded variance and mean within the unit interval: $\mu = \frac{\alpha}{\alpha+\beta}$, $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.
- The properties of both distributions make them flexible but purpose-specific.

Estimation Techniques

- Both employ Maximum Likelihood Estimation (MLE) and Method of Moments (MoM) for parameter estimation, though the procedures differ due to the structure of their respective likelihood functions.
- Gamma's estimation often involves digamma functions; Beta may require numerical root-finding techniques.

Applications

- Gamma Distribution is widely used in:
  - Queuing models
  - Reliability engineering
  - Meteorology (e.g., rainfall modeling)
  - Bayesian priors for Poisson rates
- Beta Distribution is used in:
  - Bayesian inference (priors for binomial proportions)
  - Proportion modeling
  - A/B testing
  - Machine learning (conjugate priors)

Symbolism and Notation

- Both distributions share some mathematical symbols such as Γ(·)\Gamma(\cdot)Γ(·) (Gamma function), but their application in formulas and behavior differs.
- Proper understanding of notation like f(x)f(x)f(x), μ\muμ, σ2\sigma^2σ2, and normalization constants is essential for working with either distribution.

Graphical Representation

- Beta distributions are bounded between 0 and 1 and show great versatility in shapes (U-shaped, bell-shaped, J-shaped).
- Gamma distributions are right-skewed and become more symmetric with increasing α\alphaα.
- Visual tools and graphs highlight how parameter values influence the behavior of both distributions.

Final Thoughts

In conclusion, the Gamma and Beta distributions are not competitors but rather complementary tools in the statistician's toolkit. Where Gamma shines in modeling time and skewed continuous data, Beta excels in modeling uncertainty in probabilities and proportions. Their deep mathematical structure, including reliance on the Gamma function, ties them together, but their use cases and characteristics clearly differentiate them.

A solid understanding of these distributions allows practitioners to choose the right model for the right scenario, improving the quality and interpretability of statistical modeling, machine learning, and real-world decision-making processes.

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