Gamma and Beta Distribution Report

Contents

Introduction

Probability distributions play a central role in statistics and data analysis. Among the wide variety of distributions available, Gamma and Beta distributions are two continuous probability distributions widely used across various disciplines including statistics, engineering, economics, and machine learning. They are primarily used to model waiting times and proportions, respectively. This report provides a detailed overview of the Gamma and Beta distributions, highlighting their definitions, mathematical properties, historical development, and practical applications.

Definition and Formulation

The Gamma distribution is a two-parameter family of continuous probability distributions. It is defined for $x > 0$ and is used to model the time until an event occurs a certain number of times.

- Shape parameter (k or α)

- Scale parameter (θ) or rate parameter ($\beta = 1/\theta$)

PDF: $f(x; \alpha, \theta) = (x^{\alpha-1}e^{-x/\theta}) / (\theta^\alpha \Gamma(\alpha))$ for $x > 0$.

The Beta distribution is a family of continuous distributions defined on the interval [0, 1], parameterized by two positive shape parameters α and β.

PDF: $f(x; \alpha, \beta) = (x^{\alpha-1}(1-x)^{\beta-1}) / B(\alpha, \beta)$ for $0 < x < 1$.

Historical Context

The Gamma distribution was introduced by Carl Friedrich Gauss and developed by others in the 19th century for modeling waiting times. The Beta distribution has roots in Euler's work and became prominent in Bayesian statistics for modeling prior distributions, especially in the early 20th century.

Probability Mass Function

Gamma and Beta distributions are continuous distributions and thus do not have a probability mass function (PMF). Instead, their probability density function (PDF) serves a similar role in defining the distribution of probabilities over continuous intervals.

Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF) of the Gamma distribution is:

$F(x; \alpha, \theta) = \int_0^x (t^{\alpha-1}e^{-t/\theta}) / (\theta^\alpha \Gamma(\alpha))\, dt.$

For the Beta distribution:

$F(x; \alpha, \beta) = \int_0^x (t^{\alpha-1}(1-t)^{\beta-1}) / B(\alpha, \beta)\, dt.$

These are computed using numerical integration methods.

Moment Generating Function (MGF)

Gamma Distribution:

MGF: $M(t) = (1 - \theta t)^{-\alpha}$ for $t < 1/\theta$.

Beta Distribution:

The MGF does not have a closed-form but can be expressed using the moment series expansion.

Properties

Gamma Distribution:

- Mean: $E[X] = \alpha\theta$

- Variance: $Var(X) = \alpha\theta^2$

Beta Distribution:

- Mean: $E[X] = \alpha / (\alpha + \beta)$

- Variance: $Var(X) = (\alpha\beta) / ((\alpha + \beta)^2(\alpha + \beta + 1))$

- Support: [0, 1]

Parameter Estimation

Parameters for both distributions can be estimated using:

- Method of Moments: Using sample moments.

- Maximum Likelihood Estimation (MLE): Maximizing the likelihood function.

- Bayesian Methods: Using prior knowledge in parameter inference.

Applications

Gamma Distribution:

- Queuing theory

- Reliability analysis

- Environmental modeling

Beta Distribution:

- Bayesian inference

- Modeling proportions

- Quality control and project scheduling

Graphical Representation and Analysis

Gamma distribution varies in shape depending on the α parameter. For small α, it's skewed; for large α, it resembles a normal distribution.

Beta distribution shows diverse shapes based on α and β. For example:

- Uniform when $\alpha = \beta = 1$

- U-shaped when $\alpha, \beta < 1$

- Bell-shaped when $\alpha, \beta > 1$

Conclusion

Gamma and Beta distributions are versatile tools for modeling continuous phenomena. Their diverse applications across various fields highlight their importance. A solid understanding of their properties and behavior helps in effective statistical modeling and inference.

References

1. Casella, G., & Berger, R. L. (2002). Statistical Inference.

2. DeGroot, M. H., & Schervish, M. J. (2012). Probability and Statistics.

3. Ross, S. M. (2014). Introduction to Probability Models.

4. Wikipedia: Gamma and Beta distributions.

5. Online resources such as Khan Academy and Statlect.