8·1. INTRODUCTION

In this chapter we shall study some of the probability distributions that figure most prominently in statistical theory and application. We shall also study their parameters, *i.e.*, the quantities that are constants for particular distributions but that can take on different values for different members of families of distributions of the same kind. We shall introduce number of discrete probability distributions that have been successfully applied in a wide variety of decision situations. The purpose of this chapter is to show the types of situations in which these distributions can be applied.

It may be mentioned that a theoretical probability distribution gives us a law according to which different values of the random variable are distributed with specified probabilities according to some definite law which can be expressed mathematically. It is possible to formulate such laws either on the basis of given conditions (a prior considerations) or on the basis of the results (a posterior inferences) of an experiment. The present study will also enable us to fit a mathematical model or a function of the form $y = p(x)$ to the observed data.

This chapter is devoted to the study of univariate (except for the multinomial) distributions like Rectangular, Binomial, Poisson, Negative Binomial, Geometric, Hypergeometric, Multinomial and Power-series distributions. We have already defined distribution function, mathematical expectation, *m.g.f.*, characteristic function and moments. This prepares us for a study of discrete theoretical probability distributions.

8·2. DISCRETE UNIFORM DISTRIBUTION

Definition. *A r.v. X is said to have a discrete uniform distribution over the range [1, n] if its p.m.f. is expressed as follows :*

$$P(X = x) = \begin{cases} \dfrac{1}{n} & \text{for } x = 1, 2, \ldots, n \\ 0, & \text{otherwise} \end{cases} \qquad \ldots (8 \cdot 1)$$

Here n is known as the parameter of the distribution and lies in the set of all positive integers. Equation (8·1) is also called a discrete rectangular distribution.

Such distributions can be conceived in practice if under the given experimental conditions, the different values of the random variable become equally likely. Thus for a die experiment, and for an experiment with a deck of cards such distribution is appropriate.

8·2·1. Moments.

$$E(X) = \frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}, \quad E(X^2) = \frac{1}{n}\sum_{i=1}^{n} i^2 = \frac{(n+1)(2n+1)}{6}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{(n+1)(n-1)}{12}$$

The *m.g.f* of X is : $\quad M_X(t) = E(e^{tX}) = \dfrac{1}{n}\displaystyle\sum_{x=1}^{n} e^{tx} = \dfrac{e^t(1-e^{nt})}{n(1-e^t)}$

8·3. BERNOULLI DISTRIBUTION

Definition. *A r.v. X is said to have a Bernoulli distribution with parameter p if its p.m.f. is given by :*

$$P(X = x) = \begin{cases} p^x (1-p)^{1-x}, & \text{for } x = 0, 1 \\ 0 & \text{otherwise} \end{cases} \qquad \ldots (8 \cdot 2)$$

The parameter p satisfies $0 \le p \le 1$. Often $(1-p)$ is denoted as q.

A random experiment whose outcomes are of two types, success S and failure F, occurring with probabilities p and q respectively, is called a *Bernoulli trial*. If for this experiment, a *r.v.* X is defined such that it takes value 1 when S occurs and 0 if F occurs, then X follows a Bernoulli distribution.

8·3·1. Moments of Bernoulli Distribution.

The rth moment about origin is : $\ldots (8 \cdot 2a)$

$$\mu_r' = E(X^r) = 0^r . q + 1^r . p = p; \; r = 1, 2, \ldots$$

$$\mu_1' = E(X^1) = p, \; \mu_2' = E(X^2) = p, \text{ so that } \mu_2 = \text{Var}(X) = \mu_2' - \mu_1'^2 = p^2 - p = pq$$

$p - p^2$

The *m.g.f.* of Bernoulli variate is :

$$M_X(t) = e^{0 \times t} P(X=0) + e^{1 \times t} P(X=1) = q + pe^t \quad \ldots(8\cdot2b)$$

Remark. *Degenerate Random Variable*. Sometimes we may come across a variate X which is degenerate at a point 'c', say, so that $P(X = c) = 1$ and $= 0$ otherwise, *i.e.*, the whole mass of the variable is concentrated at a single point 'c'. Hence, a degenerate *r.v.* X is characterised by : Var (X) = 0.

Since $P(X = c) = 1$, Var (X) = 0.

m.g.f. of degenerate *r.v.* is :

$$M_X(t) = E(e^{tX}) = e^{tc} P(X = c) = e^{ct}.$$

8·4. BINOMIAL DISTRIBUTION

Binomial distribution was discovered by James Bernoulli (1654-1705) in the year 1700 and was first published posthumously in 1713, eight years after his death. Let a random experiment be performed repeatedly, each repitition being called a trial and let the occurrence of an event in a trial be called a success and its non-occurrence a failure. Consider a set of n independent Bernoullian trials (n being finite) in which the probability 'p' of success in any trial is constant for each trial, then $q = 1 - p$, is the probability of failure in any trial.

The probability of x successes and consequently $(n - x)$ failures in n independent trials, in a specified order (say) $SSFSFFFS...FSF$ (where S represents success and F represents failure) is given by the compound probability theorem by the expression :

$$P(SSFSFFFS...FSF) = P(S)P(S)\,P(F)\,P(S)\,P(F)\,P(F)\,P(F)\,P(S) \times \ldots \times P(F)\,P(S)\,P(F)$$

$$= p \cdot p \cdot q \cdot p \cdot q \cdot q \cdot q \cdot p \ldots q \cdot p \cdot q$$

$$= \underbrace{p \cdot p \cdot p \ldots p}_{\{x \text{ factors}\}} \cdot \underbrace{q \cdot q \cdot q \ldots\ldots q}_{\{(n-x) \text{ factors}\}} = p^x q^{n-x}$$

But x successes in n trials can occur in $\binom{n}{x}$ ways and the probability for each of these ways is same, *viz.*, $p^x q^{n-x}$. Hence the probability of x successes in n trials in *any order* is given by the *addition theorem* of probability by the expression $\binom{n}{x} p^x q^{n-x}$.

The probability distribution of the number of successes, so obtained is called the *Binomial probability distribution*, for the obvious reason that the probabilities of $0, 1, 2, \ldots, n$ successes, *viz.*, $q^n, \binom{n}{1} q^{n-1} p, \binom{n}{2} q^{n-2} p^2, \ldots, p^n$, are the successive terms of the binomial expansion $(q + p)^n$.

Definition. *A random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by :*

$$P(X = x) = p(x) = \begin{cases} \binom{n}{x} p^x q^{n-x} \,; x = 0, 1, 2, \ldots, n \,; q = 1 - p \\ 0 \qquad , \textit{otherwise} \end{cases} \quad \ldots(8\cdot3)$$

The two independent constants n and p in the distribution are known as the *parameters* of the distribution. 'n' is also sometimes, known as the degree of the binomial distribution.

Binomial distribution is a discrete distribution as X can take only the integral values, *viz.*, $0, 1, 2, \ldots, n$. Any random variable which follows binomial distribution is known as *binomial variate*.

We shall use the notation $X \sim B(n, p)$ to denote that the random variable X follows binomial distribution with parameters n and p.

The probability $p(x)$ in (8·3) is also sometimes denoted by $b(x, n, p)$.

***Remarks* 1.** The assignment of probabilities in (8·3) is permissible because

$$\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (q+p)^n = 1$$

2. Let us suppose that n trials constitute an experiment. Then, if this experiment is repeated N times, the *frequency function* of the binomial distribution is given by :

$$f(x) = Np(x) = N \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}; \; x = 0, 1, 2, \ldots, n \qquad \ldots (8\cdot3a)$$

and the expected frequencies of 0, 1, 2, ..., n successes are the successive terms of the binomial expansion, $N(q+p)^n, q+p=1$.

3. *Physical conditions for Binomial Distribution.* We get the binomial distribution under the following experimental conditions :

(*i*) Each trial results in two exhaustive and mutually disjoint outcomes, termed as success and failure.

(*ii*) The number of trials 'n' is finite.

(*iii*) The trials are independent of each other.

(*iv*) The probability of success 'p' is constant for each trial.

The trials satisfying the conditions (*i*), (*iii*) and (*iv*) are also called *Bernoulli trials.*

The problems relating to tossing of a coin or throwing of dice or drawing cards from a pack of cards with replacement lead to binomial probability distribution.

4. Binomial distribution is important not only because of its wide applicability, but because it gives rise to many other probability distributions. Tables for $p(x)$ are available for various values of n and p.

Example 8·1. *Ten coins are thrown simultaneously. Find the probability of getting at least seven heads.*

Solution. p = Probability of getting a head $= \frac{1}{2}$

q = Probability of not getting a head $= \frac{1}{2}$

The probability of getting x heads in a random throw of 10 coins is :

$$p(x) = \binom{10}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{10-x} = \binom{10}{x}\left(\frac{1}{2}\right)^{10}; \; x = 0, 1, 2, \ldots, 10$$

∴ Probability of getting at least seven heads is given by :

$$P(X \geq 7) = p(7) + p(8) + p(9) + p(10)$$

$$= \left(\frac{1}{2}\right)^{10}\left\{\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right\} = \frac{120+45+10+1}{1024} = \frac{176}{1024}.$$

Example 8·2. *A and B play a game in which their chances of winning are in the ratio 3 : 2. Find A's chance of winning at least three games out of the five games played.*

Solution. Let p be the probability that 'A' wins the game. Then we are given :

$$n = 5, \; p = \frac{3}{5} \quad \Rightarrow \quad q = 1 - p = \frac{2}{5}.$$

Hence, by binomial probability law, the probability that out of 5 games played, A wins 'x' games is given by :

$$P(X = x) = p(x) = \binom{5}{x}\left(\frac{3}{5}\right)^x\left(\frac{2}{5}\right)^{5-x} ; x = 0, 1, 2, \ldots, 5$$

The requird probability that 'A' wins at least three games is given by :

$$P(X \geq 3) = \sum_{r=3}^{5}\binom{5}{r}\frac{3^r \cdot 2^{5-r}}{5^5} = \frac{3^3}{5^5}\left[\binom{5}{3}2^2 + \binom{5}{4}\cdot 3 \times 2 + 1 \cdot 3^2 \times 1\right]$$

$$= \frac{27 \times (40 + 30 + 9)}{3125} = 0{\cdot}68$$

Example 8·3. *A coffee connoisseur claims that he can distinguish between a cup of instant coffee and a cup of percolator coffee 75% of the time. It is agreed that his claim will be accepted if he correctly identifies at least 5 of the 6 cups. Find his chances of having the claim (i) accepted, (ii) rejected, when he does have the ability he claims.*

Solution. If p denotes the probability of a correct distinction between a cup of instant coffee and a cup of percolator coffee, then we are given :

$$p = \frac{75}{100} = \frac{3}{4} \Rightarrow q = 1 - p = \frac{1}{4}, \quad \text{and} \quad n = 6$$

If the random variable X denotes the number of correct distinctions, then by the Binomial probability law, the probability of x correct identifications out of 6 cups is given by :

$$P(X = x) = p(x) = \binom{6}{x}\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{6-x} ; x = 0, 1, 2, \ldots, 6$$

(*i*) The probability of the claim being accepted is :

$$P(X \geq 5) = p(5) + p(6) = \binom{6}{5}\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)^{6-5} + \binom{6}{6}\left(\frac{3}{4}\right)^6 = \frac{1458}{4096} + \frac{729}{4096} = 0{\cdot}534.$$

(*ii*) The probability of the claim being rejected is :

$$P(X \leq 4) = 1 - P(X \geq 5) = 1 - 0{\cdot}534 = 0{\cdot}466.$$

Example 8·4. *A multiple-choice test consists of 8 questions with 3 answers to each question (of which only one is correct). A student answers each question by rolling a balanced die and checking the first answer if he gets 1 or 2, the second answer if he gets 3 or 4 and the third answer if he gets 5 or 6. To get a distinction, the student must secure at least 75% correct answers. If there is no negative marking, what is the probability that the student secures a distinction ?*

Solution. Since there are three answers to each question, out of which only one is correct, the probability of getting an answer to a question correctly is given by :

$$p = \frac{1}{3}, \quad \text{so that} \quad q = 1 - p = \frac{2}{3}$$

By Binomial probability law, the probability of getting r correct answers in a 8-question test is given by :

$$P(X = x) = p(x) = \binom{8}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{8-x} ; x = 0, 1, 2, \ldots, 8$$

Hence, the required probability of securing a distinction (*i.e.*, of getting correct answers to at least 6 out of the 8 questions) is given by :

$$p(6) + p(7) + p(8) = \binom{8}{6}\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^{8-6} + \binom{8}{7}\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right)^{8-7} + \binom{8}{8}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^{8-8}$$

$$= \frac{1}{3^6}\left[28 \times \frac{4}{9} + 8 \times \frac{1}{3} \times \frac{2}{3} + \frac{1}{9}\right] = \frac{129}{729 \times 9} = 0{\cdot}0197.$$

Example 8·5. *An irregular six-faced die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many items in 10,000 sets of 10 throws each, would you expect it to give no even number.*

Solution. Let p be the probability of getting an even number in a throw of a die. Then the probability of getting x even numbers in ten throws of a die is given by :

$$P(X = x) = \binom{10}{x} p^x q^{10-x} \; ; x, 0, 1, 2, \ldots, 10 \qquad \ldots (*)$$

We are given that : $P(X = 5) = 2P(X = 4) \Rightarrow \binom{10}{5} p^5 q^5 = 2\binom{10}{4} p^4 q^6$

$$\Rightarrow \frac{10!\,p}{5!\,5!} = 2\,\frac{10!\,q}{4!\,6!} \Rightarrow \frac{p}{5} = \frac{2q}{6} = \frac{q}{3} \Rightarrow 3p = 5(1-p) \Rightarrow p = \frac{5}{8} \text{ and } q = \frac{3}{8}.$$

Thus $\quad P(X = x) = \binom{10}{x}\left(\frac{5}{8}\right)^x\left(\frac{3}{8}\right)^{10-x}$ $\qquad$ ... [From (*)]

Hence the required number of times that in 10,000 sets of 10 throws each, we get no even number $= 10{,}000 \times P(X = 0) = 10{,}000 \times \left(\frac{3}{8}\right)^{10} = 1$ (approx.).

Example 8·6. *A department in a works has 10 machines which may need adjustment from time to time during the day. Three of these machines are old, each having a probability of 1/11 of needing adjustment during the day, and 7 are new, having corresponding probabilities of 1/21.*

Assuming that no machine needs adjustment twice on the same day, determine the probabilities that on a particular day

(i) just 2 old and no new machines need adjustment.

(ii) If just 2 machines need adjustment, they are of the same type.

Solution. Let p_1 = Probability that an old machine needs adjustment $= \frac{1}{11} \Rightarrow q_1 = \frac{10}{11}$.

and $\quad p_2$ = Probability that a new machine needs adjustment $= \frac{1}{21} \Rightarrow q_2 = \frac{20}{21}$.

Then $\quad P_1(x)$ = Probability that 'x' old machines need adjustment

$$= \binom{3}{x} p_1^{\,x} q_1^{\,3-x} = \binom{3}{x}\left(\frac{1}{11}\right)^x\left(\frac{10}{11}\right)^{3-x} \; ; x = 0, 1, 2, 3$$

and $\quad P_2(x)$ = Probability that 'x' new machines need adjustment

$$= \binom{7}{x} p_2^{\,x} q_2^{\,7-x} = \binom{7}{x}\left(\frac{1}{21}\right)^x\left(\frac{20}{21}\right)^{7-x} \; ; x = 0, 1, 2, \ldots, 7$$

(i) The probability that just two old machines and no new machine need adjustment is given (by the compound probability theorem) by the expression :

$$P_1(2)\cdot P_2(0) = \binom{3}{2}\left(\frac{1}{11}\right)^2\left(\frac{10}{11}\right)\left(\frac{20}{21}\right)^7 = 0{\cdot}016 \qquad \ldots (1)$$

(ii) Similarly, the probability that just 2 new machines and no old machine need adjustment is :

$$P_1(0).\, P_2(2) = \left(\frac{10}{11}\right)^3 \times \binom{7}{2}\left(\frac{1}{21}\right)^2\left(\frac{20}{21}\right)^5 = 0{\cdot}028 \qquad \ldots (2)$$

The probability that 'if just two machines need adjustment, they are of the same type' is the same as the probability that 'either just 2 old and no new or just 2 new and no old machines need adjustment'.

$\therefore$ Required probability = (1) + (2) = 0·016 + 0·028 = 0·044.

Example 8·7. *The probability of a man hitting a target is $\frac{1}{4}$:*

(i) If he fires 7 times what is the probability of his hitting the target at least twice ?

(ii) How many times must he fire so that the probability of his hitting the target at least once is greater than $\frac{2}{3}$?

Solution. p = Probability of the man hitting the target $= \frac{1}{4} \Rightarrow q = 1 - p = \frac{3}{4}$.

$p(x)$ = Probability of getting x hits in 7 shots $= \binom{7}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{7-x}$; $x = 0, 1, \ldots, 7$

(i) Probability of at least two hits

$$= 1 - \{p(0) + p(1)\} = 1 - \left\{\binom{7}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{7-0} + \binom{7}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{7-1}\right\} = \frac{4547}{8192}.$$

(ii) Probability of at least one hit in n shots $= 1 - p(0) = 1 - \left(\frac{3}{4}\right)^n$.

It is required to find n, so that $1 - \left(\frac{3}{4}\right)^n > \frac{2}{3} \Rightarrow \frac{1}{3} > \left(\frac{3}{4}\right)^n$

Taking logarithms of each side, $\log \frac{1}{3} > n \log \frac{3}{4} \Rightarrow \log 1 - \log 3 > n (\log 3 - \log 4)$

$\Rightarrow \quad 0 - 0·4771 > n\,(0·4771 - 0·6021) \quad \Rightarrow \quad 0·4771 < 0·1250\, n$

$\therefore \quad n > \dfrac{0·4771}{0·1250} = 3·8$

Since n cannot be fractional, the required number of shots is 4.

Example 8·8. *In a precision bombing attack there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance or better of completely destroying the target ?*

Solution. We have : p = Probability that the bomb strikes the target = 50% = $\frac{1}{2}$.

Let n be the number of bombs which should be dropped to ensure 99% chance or better of completely destroying the target. This implies that 'probability that out of n bombs, at least two strike the target, is greater than 0·99'.

Let X be a *r.v.* representing the number of bombs striking the target. Then

$X \sim B\left(n, p = \frac{1}{2}\right)$ with $P(X = x) = p(x) = \binom{n}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{n-x} = \binom{n}{x}\left(\frac{1}{2}\right)^n$; $x = 0, 1, 2, \ldots$

We should have : $P(X \geq 2) \geq 0·99 \quad \Rightarrow \quad [1 - p(X \leq 1)] \geq 0·099$

$\Rightarrow \quad [1 - \{p(0) + p(1)\}] \geq 0·99 \quad \Rightarrow \quad 1 - \left\{\binom{n}{0} + \binom{n}{1}\right\}\left(\frac{1}{2}\right)^n \geq 0·99$

$\Rightarrow \quad 0·01 \geq \dfrac{1+n}{2^n} \quad \Rightarrow \quad 2^n \times (0·01) \geq 1 + n \quad \Rightarrow \quad 2^n \geq 100 + 100n \quad \ldots(*)$

By trial method, we find that the inequality (*) is satisfied by $n = 11$. Hence the minimum number of bombs needed to destroy the target completely is 11.

Example 8·9. *In a binomial distribution consisting of 5 independent trials, probabilities of 1 and 2 successes are 0·4096 and 0·2048 respectively. Find the parameter 'p' of the distribution.*

Solution. Let $X \sim B(n, p)$. In usual notations, we are given :

$$n = 5,\ p(1) = 0{\cdot}4096 \quad \text{and} \quad p(2) = 0{\cdot}2048.$$

According to Binomial probability law :

$$P(X = x) = p(x) = \binom{5}{x} p^x (1-p)^{5-x},\ x = 0, 1, 2, \ldots, 5$$

Now $p(1) = \binom{5}{1} p (1-p)^4 = 0{\cdot}4096$ ... (*) and $p(2) = \binom{5}{2} p^2 (1-p)^3 = 0{\cdot}2048$ ... (**)

Dividing (*) by (**), we get

$$\frac{\binom{5}{1} p(1-p)^4}{\binom{5}{2} p^2 (1-p)^3} = \frac{0{\cdot}4096}{0{\cdot}2048} \quad \Rightarrow \quad \frac{5(1-p)}{10p} = 2 \quad \Rightarrow \quad p = \frac{1}{5} = 0{\cdot}2.$$

Example 8·10. *With the usual notations, find p for a binomial variate X, if n = 6 and 9 P(X = 4) = P(X = 2).*

Solution. For the binomial random variable X with parameters $n = 6$ and p, the probability function is :

$$P(X = r) = \binom{6}{r} p^r q^{6-r};\ r = 0, 1, 2, \ldots, 6$$

We are given : $9P(X = 4) = P(X = 2) \quad \Rightarrow \quad 9 \times \binom{6}{4} p^4 q^2 = \binom{6}{2} p^2 q^4$

$$\Rightarrow \quad 9p^2 = q^2 \quad \Rightarrow \quad 9p^2 = (1-p)^2 = 1 + p^2 - 2p$$

$$\Rightarrow \quad 8p^2 + 2p - 1 = 0 \quad \Rightarrow \quad p = \frac{-2 \pm \sqrt{4+32}}{2 \times 8} = \frac{-2 \pm 6}{16} = -\frac{1}{2}, \frac{1}{4}$$

Since probability cannot be negative, $p = -\frac{1}{2}$ is rejected. Hence $p = \frac{1}{4}$.

8·4·1. Moments of Binomial Distribution. The first four moments about origin of binomial distribution are obtained as follows :

$$\mu_1' = E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x} = np\,(q+p)^{n-1} = np$$

$$\left[\because \binom{n}{x} = \frac{n}{x}\binom{n-1}{x-1} = \frac{n}{x}\cdot\frac{n-1}{x-1}\cdot\binom{n-2}{x-2} = \frac{n}{x}\cdot\frac{n-1}{x-1}\cdot\frac{n-2}{x-2}\binom{n-3}{x-3}, \text{ and so on}\right]$$

Thus the mean of the binomial distribution is np.

$$\mu_2' = E(X^2) = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \{x(x-1) + x\} \frac{n(n-1)}{x(x-1)} \cdot \binom{n-2}{x-2} p^x q^{n-x}$$

$$= n(n-1)p^2 \left\{ \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x} \right\} + np$$

$$= n(n-1)\,p^2\,(q+p)^{n-2} + np = n\,(n-1)\,p^2 + np$$

$$\mu_3' = E(X^3) = \sum_{x=0}^{n} x^3\,p(x) = \sum_{x=0}^{n} \{x\,(x-1)\,(x-2) + 3x\,(x-1) + x\}\binom{n}{x} p^x q^{n-x}$$

$$= n\,(n-1)\,(n-2)\,p^3 \sum_{x=3}^{n} \binom{n-3}{x-3} p^{x-3} q^{n-x}$$

$$+ 3n\,(n-1)\,p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x} + np$$

$$= n\,(n-1)\,(n-2)\,p^3\,(q+p)^{n-3} + 3n\,(n-1)\,p^2\,(q+p)^{n-2} + np$$

$$= n\,(n-1)\,(n-2)\,p^3 + 3n\,(n-1)\,p^2 + np$$

Similarly

$$x^4 = x\,(x-1)\,(x-2)\,(x-3) + 6x\,(x-1)\,(x-2) + 7x\,(x-1) + x$$

Let $x^4 = Ax\,(x-1)\,(x-2)\,(x-3) + Bx\,(x-1)\,(x-2) + Cx\,(x-1) + x$

(By giving to x the values 1, 2 and 3, we find the values of arbitrary constants A, B and C.)

$$\therefore\ \mu_4' = E(X^4) = \sum_{x=0}^{n} x^4 \binom{n}{x} p^x q^{n-x}$$

$$= n\,(n-1)\,(n-2)\,(n-3)\,p^4 + 6n(n-1)\,(n-2)\,p^3 + 7n\,(n-1)\,p^2 + np$$ [On simplification]

Central Moments of Binomial Distribution :

$$\mu_2 = \mu_2' - \mu_1'^2 = n^2p^2 - np^2 + np - n^2p^2 = np\,(1-p) = npq$$

$$\mu_3 = \mu_3' - 3\mu_2'\,\mu_1' + 2\mu_1'^3$$

$$= \{n\,(n-1)\,(n-2)\,p^3 + 3n\,(n-1)\,p^2 + np\} - 3\,\{n\,(n-1)\,p^2 + np\}\,np + 2\,(np)^3$$

$$= np\,(-3np^2 + 3np + 2p^2 - 3p + 1 - 3npq)$$

$$= np\,\{3np\,(1-p) + 2p^2 - 3p + 1 - 3npq\}$$

$$= np\,(2p^2 - 3p + 1) = np\,(2p^2 - 2p + q) = npq\,(1-2p)$$

$$= npq\,\{q + p - 2p\} = npq\,(q-p)$$

$$\mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'\,\mu_1'^2 - 3\mu_1'^4 = npq\,\{1 + 3\,(n-2)\,pq\}$$ [On simplification]

Hence

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{n^2\,p^2\,q^2\,(q-p)^2}{n^3\,p^3\,q^3} = \frac{(q-p)^2}{npq} = \frac{(1-2p)^2}{npq} \quad \ldots (8\cdot5)$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq\,\{1 + 3\,(n-2)\,pq\}}{n^2\,p^2\,q^2} = \frac{1 + 3\,(n-2)\,pq}{npq} = 3 + \frac{1-6pq}{npq} \quad \ldots (8\cdot6)$$

$$\gamma_1 = \sqrt{\beta_1} = \frac{q-p}{\sqrt{npq}} = \frac{1-2p}{\sqrt{npq}}, \quad \gamma_2 = \beta_2 - 3 = \frac{1-6pq}{npq} \quad \ldots (8\cdot6a)$$

Remarks **1.** If $X \sim B(n, p)$, then mean $= np$ and variance $= npq$ ... (8·6b)

$\mu_3 = npq\,(q-p)$ and $\mu_4 = npq\,[1 + 3\,(n-2)\,pq]$ ... (8·6c)

2. Variance $= npq < np =$ Mean $\quad (\because 0 < q < 1)$

Hence, for the binomial distribution, variance is less than mean

Example 8·11. *Comment on the following :*

The mean of a binomial distribution is 3 and variance is 4.

Solution. If the given binomial distribution has parameters n and p, then we are given :

Mean $= np = 3$ ... (*) and Variance $= npq = 4$ ... (**)

Dividing (**) by (*), $q = \frac{4}{3}$, which is impossible, since probability cannot exceed unity. Hence the given statement is wrong.

Aliter. Since for a binomial distribution variance is always less than mean, the given statement is wrong.

Example 8·12. *The mean and variance of binomial distribution are 4 and $\frac{4}{3}$ respectively. Find $P(X \geq 1)$.*

Solution.

Let $X \sim B(n, p)$. Then we are given : Mean $= np = 4$ ... (*) and Var $(X) = npq = \frac{4}{3}$.

Dividing, we get $q = \frac{1}{3} \Rightarrow p = \frac{2}{3}$. Substituting in (*), we obtain $n = \frac{4}{p} = \frac{4 \times 3}{2} = 6.$

$$\therefore \quad P(X \geq 1) = 1 - P(X = 0) = 1 - q^n = 1 - \left(\frac{1}{3}\right)^6 = 1 - \frac{1}{729} = 0{\cdot}99863.$$

Example 8·13. *If $X \sim B(n, p)$, show that :*

$$E\left(\frac{X}{n} - p\right)^2 = \frac{pq}{n}; \qquad Cov\left(\frac{X}{n}, \frac{n-X}{n}\right) = -\frac{pq}{n}$$

Solution. Since $X \sim B(n, p)$, $E(X) = np$ and $\text{Var}(X) = npq$

$$\therefore \quad E\left(\frac{X}{n}\right) = \frac{1}{n}E(X) = p; \qquad \text{Var}\left(\frac{X}{n}\right) = \frac{1}{n^2}.\text{Var}(X) = \frac{pq}{n} \quad \ldots (*)$$

(i) $$E\left(\frac{X}{n} - p\right)^2 = E\left[\frac{X}{n} - E\left(\frac{X}{n}\right)\right]^2 = \text{Var}\left(\frac{X}{n}\right) = \frac{pq}{n} \qquad \text{[From (*)]}$$

(ii) $$\text{Cov}\left(\frac{X}{n}, \frac{n-X}{n}\right) = E\left[\left\{\frac{X}{n} - E\left(\frac{X}{n}\right)\right\}\left\{\frac{n-X}{n} - E\left(\frac{n-X}{n}\right)\right\}\right]$$

$$= E\left[\left(\frac{X}{n} - p\right)\left\{\left(1 - \frac{X}{n}\right) - (1-p)\right\}\right] = E\left[\left(\frac{X}{n} - p\right)\left\{-\left(\frac{X}{n} - p\right)\right\}\right]$$

$$= -E\left(\frac{X}{n} - p\right)^2 = -\text{Var}\left(\frac{X}{n}\right) = -\frac{pq}{n}.$$

8·4·2. Recurrence Relation for the moments of Binomial Distribution (*Renovsky Formula*).

By *def.*, $\mu_r = E\{X - E(X)\}^r = \sum_{x=0}^{n} (x - np)^r \binom{n}{x} p^x q^{n-x}$

Differentiating *w.r. to p*, we get

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} \binom{n}{x} \left[-nr(x-np)^{r-1} p^x q^{n-x} + (x-np)^r \left\{xp^{x-1} q^{n-x} - (n-x)p^x q^{n-x-1}\right\}\right]$$

$$= -nr \sum_{x=0}^{n} \binom{n}{x} (x-np)^{r-1} p^x q^{n-x} + \sum_{x=0}^{n} \binom{n}{x} (x-np)^r p^x q^{n-x} \left(\frac{x}{p} - \frac{n-x}{q}\right)$$

$$= -nr \sum_{x=0}^{n} (x-np)^{r-1} p(x) + \sum_{x=0}^{n} (x-np)^r p(x) \frac{(x-np)}{pq}$$

$$= -nr \sum_{x=0}^{n} (x - np)^{r-1}\; p(x) + \frac{1}{pq} \sum_{x=0}^{n} (x - np)^{r+1}\, p(x)$$

$$= -nr\,\mu_{r-1} + \frac{1}{pq}\mu_{r+1}$$

$$\Rightarrow \quad \mu_{r+1} = pq\left(nr\;\mu_{r-1} + \frac{d\,\mu_r}{dp}\right) \qquad \text{(Renovsky Formula)} \qquad \ldots (8{\cdot}7)$$

Putting $r = 1, 2$ and 3 successively in (8·7), we get

$$\mu_2 = pq\left(n\,\mu_0 + \frac{d\,\mu_1}{dp}\right) = npq \qquad (\because \mu_0 = 1 \text{ and } \mu_1 = 0)$$

$$\mu_3 = pq\left[2n\,\mu_1 + \frac{d\mu_2}{dp}\right] = pq\,.\,\frac{d(npq)}{dp} = npq\,\frac{d}{dp}\{p\,(1-p)\}$$

$$= npq\,\frac{d}{dp}(p - p^2) = npq\,(1-2p) = npq\,(q-p)$$

and
$$\mu_4 = pq\left[3n\mu_2 + \frac{d\mu_3}{dp}\right] = pq\left[3n\,.\,npq + \frac{d}{dp}\left\{npq\,(q-p)\right\}\right]$$

$$= pq\left[3n^2\,pq + n\frac{d}{dp}\left\{p\,(1-p)\,(1-2p)\right\}\right]$$

$$= pq\left[3n^2\,pq + n\;\frac{d}{dp}(p - 3p^2 + 2p^3)\right]$$

$$= pq\,[3n^2\,pq + n\;\;(1 - 6p + 6p^2)] = pq\,[3n^2\,pq + n\,(1 - 6pq)]$$

$$= npq\,[3npq + 1 - 6pq] = npq\,[1 + 3pq\,(n-2)].$$

Example 8·14. *Show that the rth moment μ_r' about the origin of the binomial distribution of degree n is given by :*

$$\mu_r' = \left(p\,\frac{\partial}{\partial p}\right)^r (q+p)^n \qquad \ldots (*)$$

Solution. We shall prove this result by using the principle of mathematical induction. We have

$$(q+p)^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} \quad \Rightarrow \quad \frac{\partial}{\partial p}(q+p)^n = \sum_{x=0}^{n} \binom{n}{x} q^{n-x}\,xp^{x-1}$$

$$\therefore \quad p\,\frac{\partial}{\partial p}(q+p)^n = p\sum_{x=0}^{n} \binom{n}{x} q^{n-x}\,xp^{x-1} = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}\,x = \mu_1' \qquad \ldots (i)$$

Thus, the result (*) is true for $r = 1$.

Let-us now assume that the result (*) is true for $r = k$, say, so that

$$\left(p\,\frac{\partial}{\partial p}\right)^k (q+p)^n = \mu_k' = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}\,x^k \qquad \ldots (ii)$$

Differentiating (*ii*) partially *w.r.* to p and then multiplying both sides by p, we get

$$p\left(\frac{\partial}{\partial p}\right)\left[\left(p\,\frac{\partial}{\partial p}\right)^k (q+p)^n\right] = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}\,x^{k+1} = E\,(X^{k+1})$$

$$\Rightarrow \qquad \left(p\,\frac{\partial}{\partial p}\right)^{k+1} (q+p)^n = \mu'_{k+1}$$

Hence if the result (*) is true for $r = k$, it is also true for $r = k + 1$. But we have proved in (i) that (*) is true for $r = 1$. Hence it is true for $1 + 1 = 2$; $2 + 1 = 3$; and so on. Hence by the principle of mathematical induction, (*) is true for all positive integral values of r.

8·4·3. Factorial Moments of Binomial Distribution. The rth factorial moment of the Binomial distribution is :

$$\mu_{(r)}' = E\{X^{(r)}\} = \sum_{x=0}^{n} x^{(r)}\, p(x) = \sum_{x=0}^{n} x^{(r)} \frac{n\,!}{x\,!\,(n-x)\,!}\, p^x q^{n-x}$$

$$= n^{(r)} p^r \sum_{x=r}^{n} \frac{(n-r)\,!}{(x-r)\,!\,(n-x)\,!}\, p^{x-r} q^{n-x} = n^{(r)}\, p^r (q+p)^{n-r} = n^{(r)} p^r \qquad \ldots (8\cdot8)$$

$$\therefore \quad \mu_{(1)}' = E\{X^{(1)}\} = np = \text{Mean}, \qquad \mu_{(2)}' = E\{X^{(2)}\} = n^{(2)} p^2 = n(n-1)p^2$$

$$\mu_{(3)}' = E\{X^{(3)}\} = n^{(3)} p^3 = n(n-1)(n-2)p^3$$

Now $\mu_{(2)} = \mu_{(2)}' - \mu_{(1)}'^2 + \mu_{(1)}' = n^2p^2 - np^2 - n^2p^2 + np = npq$

$$\mu_{(3)} = \mu_{(3)}' - 3\mu_{(2)}'\mu_{(1)}' + 2\mu_{(1)}'^3 - 2\mu_{(1)}'$$

$$= n(n-1)(n-2)p^3 - 3n(n-1)p^2 \cdot np + 2n^3p^3 - 2np = -2npq(1+p)$$

[On simplification]

8·4·4. Mean Deviation about Mean of Binomial Distribution. The mean deviation η about the mean np of the binomial distribution is given by :

$$\eta = \sum_{x=0}^{n} |x - np|\, p(x) = \sum_{x=0}^{n} |x-np| \binom{n}{x} p^x q^{n-x}, \qquad (x \text{ being an integer})$$

$$= \sum_{x=0}^{np} -(x-np)\binom{n}{x} p^x q^{n-x} + \sum_{x=np}^{n} (x-np)\binom{n}{x} p^x q^{n-x}$$

$$= 2\sum_{x=np}^{n} (x-np)\binom{n}{x} p^x q^{n-x}$$

$$\left[\because \sum_{x=0}^{n} x\binom{n}{x} p^x q^{n-x} = np \quad \Rightarrow \quad \sum_{x=0}^{n} (x-np)\binom{n}{x} p^x q^{n-x} = 0\right]$$

$$= 2\sum_{\mu}^{n} (x-np)\binom{n}{x} p^x q^{n-x}, \text{ where } \mu \text{ is greatest integer contained in } np+1.$$

$$= 2\sum_{\mu}^{n} \left[\{xq - (n-x)p\}\binom{n}{x} p^x q^{n-x}\right]$$

$$= 2\sum_{\mu}^{n} \left[\frac{n\,!}{(x-1)\,!\,(n-x)\,!}\, p^x q^{n-x+1} - \frac{n\,!}{x\,!\,(n-x-1)\,!}\, p^{x+1} q^{n-x}\right]$$

$$= 2\sum_{x=\mu}^{n} (t_{x-1} - t_x), \qquad \text{where } t_x = \frac{n\,!}{x\,!\,(n-x-1)\,!}\, p^{x+1} q^{n-x}$$

$$= 2(t_{\mu-1} - t_n) = 2\,t_{\mu-1}$$

This is obtained by summing over x and using $t_n = 0$.

$$\therefore \quad \eta = 2t_{\mu-1} = 2\frac{n\,!}{(\mu-1)\,!\,(n-\mu)\,!} \cdot p^{\mu} q^{n-\mu+1} = 2npq \binom{n-1}{\mu-1} p^{\mu-1} q^{n-\mu}. \qquad \ldots (8\cdot9)$$

8·4·5. Mode of Binomial Distribution. We have

$$\frac{p(x)}{p(x-1)} = \binom{n}{x} p^x q^{n-x} \Big/ \binom{n}{x-1} p^{x-1} q^{n-x+1}$$

$$= \frac{n!}{(n-x)!\,x!} p^x q^{n-x} \Big/ \frac{n!}{(x-1)!\,(n-x+1)!} p^{x-1} q^{n-x+1}$$

$$= \frac{(n-x+1)p}{xq} = \frac{xq+(n-x+1)p-xq}{xq} = 1 + \frac{(n+1)p-x(p+q)}{xq}$$

$$= 1 + \frac{(n+1)p-x}{xq} \qquad \dots (8\cdot10)$$

Mode is the value of x for which $p(x)$ is maximum.

We discuss the following two cases :

Case I. *When $(n+1)p$ is not an integer*. Let $(n+1)p = m+f$, where m is an integer and f is fractional such that $0 < f < 1$. Substituting in (8·10), we get

$$\frac{p(x)}{p(x-1)} = 1 + \frac{(m+f)-x}{xq} \qquad \dots (*)$$

From (*), it is obvious that

$$\frac{p(x)}{p(x-1)} > 1 \text{ for } x = 0, 1, 2, \dots m \quad \text{and} \quad \frac{p(x)}{p(x-1)} < 1 \text{ for } x = m+1, m+2, \dots, n$$

$$\Rightarrow \quad \frac{p(1)}{p(0)} > 1, \frac{p(2)}{p(1)} > 1, \dots, \frac{p(m)}{p(m-1)} > 1, \text{ and } \frac{p(m+1)}{p(m)} < 1, \frac{p(m+2)}{p(m+1)} < 1, \dots, \frac{p(n)}{p(n-1)} < 1$$

$$\therefore\ p(0) < p(1) < p(2) < \dots < p(m-1) < p(m) > p(m+1) > p(m+2) > \dots > p(n).$$

$\Rightarrow$ $p(x)$ is maximum at $x = m$.

Thus, in this case there exists unique modal value for binomial distribution and it is m, the integral part of $(n+1)p$.

Case II. *When $(n+1)p$ is an integer.* Let $(n+1)p = m$ *(an integer)*.

Substituting in (8·10), we get $\quad \dfrac{p(x)}{p(x-1)} = 1 + \dfrac{m-x}{xq} \qquad \dots (**)$

From (**), it is obvious that : $\quad \dfrac{p(x)}{p(x-1)} \begin{cases} > 1 \text{ for } x = 1, 2, \dots, m-1 \\ = 1 \text{ for } x = m \\ < 1 \text{ for } x = m+1, m+2, \dots, n \end{cases}$

Now proceeding as in case 1, we have

$$p(0) < p(1) < \dots < p(m-1) = p(m) > p(m+1) > p(m+2) > \dots > p(n)$$

Thus, in this case the binomial distribution is *bimodal* and the two modal values are m and $m-1$.

Example 8·15. *Determine the binomial distribution for which the mean is 4 and variance 3 and find its mode.*

Solution. Let $X \sim B(n, p)$, then we are given that

$$E(X) = np = 4 \quad \dots (*) \quad \text{and} \quad \text{Var}(X) = npq = 3 \quad \dots (**)$$

Dividing (**) by (*), we get $\quad q = \frac{3}{4} \Rightarrow p = 1 - q = \frac{1}{4}$

Hence from (*), we obtain $\quad n = \frac{4}{p} = 16$

Thus the given binomial distribution has parameters $n = 16$ and $p = \frac{1}{4}$.

Mode. We have $(n+1)p = 4{\cdot}25$, which is not an integer. Hence the unique mode of the binomial distribution is 4, the integral part of $(n+1)p$.

Example 8·16. *Show that for $p = 0{\cdot}5$, the binomial distribution has a maximum probability at $X = \frac{1}{2}n$, if n is even, and at $X = \frac{1}{2}(n-1)$ as well as $X = \frac{1}{2}(n+1)$, if n is odd.*

Solution. Here we have to find the mode of the binomial distribution.

(*i*) Let n be even $= 2m$, (say), $m = 1, 2, \ldots$

$\therefore$ If $p = 0{\cdot}5$, then $(n+1)p = (2m+1) \times \frac{1}{2} = m + 0{\cdot}5$. Hence in this case, the distribution is unimodal, the unique mode being at $X = m = \dfrac{n}{2}$.

(*ii*) Let n be odd $= (2m+1)$, say. Then

$$(n+1)p = (2m+2) \times \frac{1}{2} = m + 1 \text{ (Integer)} = \frac{n-1}{2} + 1 = \frac{n+1}{2}$$

Since $(n+1)p$ is an integer, the distribution is bimodal, the two modes being $\frac{1}{2}(n+1)$ and $\frac{1}{2}(n+1) - 1 = \frac{1}{2}(n-1)$.

8·4·6. Moment Generating Function of Binomial Distribution. Let $X \sim B(n, p)$, then :

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x} = (q + pe^t)^n \qquad \ldots (8{\cdot}11)$$

m.g.f. **about Mean of Binomial Distribution :**

$$E\{e^{t(X-np)}\} = e^{-tnp} \,.\, E(e^{tX}) = e^{-tnp} . M_X(t) = e^{-tnp}(q + pe^t)^n = (qe^{-pt} + pe^{tq})^n$$

$$= \left[q\left\{1 - pt + \frac{p^2t^2}{2!} - \frac{p^3t^3}{3!} + \frac{p^4t^4}{4!} - \ldots\right\} + p\left\{1 + tq + \frac{t^2q^2}{2!} + \frac{t^3q^3}{3!} - \ldots\right\}\right]^n$$

$$= \left[(q+p) + \frac{t^2}{2!}pq(q+p) + \frac{t^3}{3!}pq(q^2 - p^2) + \frac{t^4}{4!}pq(q^3 + p^3) + \ldots\right]^n$$

$$= \left[1 + \left\{\frac{t^2}{2!}\cdot pq + \frac{t^3}{3!}\cdot pq(q-p) + \frac{t^4}{4!}qp(1 - 3pq) + \ldots\right\}\right]^n$$

$$= \left[1 + \binom{n}{1}\left\{\frac{t^2}{2!}\cdot pq + \frac{t^3}{3!}pq(q-p) + \frac{t^4}{4!}pq(1-3pq) + \ldots\right\}\right.$$

$$\left. + \binom{n}{2}\left\{\frac{t^2}{2!}pq + \frac{t^2}{3!}pq(q-p) + \ldots\right\}^2 + \ldots\right]$$

Now $\qquad \mu_2 = \text{Coefficient of } \dfrac{t^2}{2!} = npq, \qquad \mu_3 = \text{Coefficient of } \dfrac{t^3}{3!} = npq(q-p)$

$$\mu_4 = \text{Coefficient of } \frac{t^4}{4!} = npq(1-3pq) + 3n(n-1)p^2q^2$$

$$= 3n^2p^2q^2 + npq(1 - 6pq).$$

Example 8·17. *X is binomially distributed with parameters n and p. What is distribution of $Y = n - X$?*

Solution. $X \sim B(n, p)$, represents the number of successes in n independent trials with constant probability p of success for each trial.

$\therefore Y = n - X$, represents the number of failures in n independent trials with constant probability 'q' of failure of each trial. Hence $Y = (n - X) \sim B(n, q)$

Aliter. Since $X \sim B(n, p)$, $\quad M_X(t) = E(e^{tX}) = (q + pe^t)^n$

$$\therefore \quad M_Y(t) = E(e^{tY}) = E[e^{t(n-X)}] = e^{nt}.\ E(e^{-tX}) = e^{nt} M_X(-t)$$

$$= e^{nt}.\ (q + pe^{-t})^n = [e^t (q + pe^{-t})]^n = (p + qe^t)^n$$

Hence, by uniqueness theorem of *m.g.f.*, $Y = (n - X) \sim B(n, q)$.

Example 8·18. *The m.g.f. of a r.v. X is* $\left(\frac{2}{3} + \frac{1}{3}e^t\right)^9$. *Show that*

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = \sum_{x=1}^{5} \binom{9}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$$

Solution. Since $M_X(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^9 = (q + pe^t)^n$, by uniqueness theorem of *m.g.f.*

$X \sim B\left(n = 9, p = \frac{1}{3}\right)$. Hence $E(X) = \mu_X = np = 3$; $\qquad \sigma_X^2 = npq = 9 \times \frac{1}{3} \times \frac{2}{3} = 2$

$$\mu \pm 2\sigma = 3 \pm 2 \times \sqrt{2} = 3 \pm 2 \times 1{\cdot}4 = (0{\cdot}2, 5{\cdot}8)$$

$$\therefore \quad P(\mu - 2\sigma < X < \mu + 2\sigma) = P(0{\cdot}2 < X < 5{\cdot}8) = P(1 \le X \le 5)$$

$$= \sum_{x=1}^{5} p(x) = \sum_{x=1}^{5} \binom{n}{x} p^x q^{n-x} = \sum_{x=1}^{5} \binom{9}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$$

8·4·7. Additive Property of Binomial Distribution.

Let $X \sim B(n_1, p_1)$ and $Y \sim B(n_2, p_2)$ be independent random variables. Then

$$M_X(t) = (q_1 + p_1 e^t)^{n_1}, \qquad M_Y(t) = (q_2 + p_2 e^t)^{n_2} \qquad \dots (*)$$

What is the distribution of $X + Y$?

We have $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$ $\qquad$ [$\because$ X and Y are independent]

$$= (q_1 + p_1 e^t)^{n_1}.\ (q_2 + p_2 e^t)^{n_2} \qquad \dots (**)$$

Since (**) cannot be expressed in the form $(q + p\,e^t)^n$, from uniqueness theorem of *m.g.f's* it follows that $X + Y$ is not a binomial variate. Hence, *in general the sum of two independent binomial variates is not a binomial variate. In other words, binomial distribution does not possess the additive or reproductive property.*

However, if we take $p_1 = p_2 = p$, (say), then from (**) $\qquad M_{X+Y}(t) = (q + pe^t)^{n_1 + n_2}$, which is the *m.g.f.* of a binomial variate with parameters $(n_1 + n_2, p)$. Hence by uniqueness theorem of *m.g.f.'s* $X + Y \sim B(n_1 + n_2, p)$. *Thus the binomial distribution possesses the additive or reproductive property if* $p_1 = p_2$.

Generalisation. *If* X_i, $(i = 1, 2, \dots, k)$ *are independent binomial variates with parameters* (n_i, p), $(i = 1, 2, \dots k)$, *then their sum* $\sum_{i=1}^{k} X_i \sim B\left(\sum_{i=1}^{k} n_i, p\right)$.

The proof is left as an exercise to the reader.

Example 8·19. *If the independent random variables X, Y are binomially distributed respectively with* $n = 3$, $p = \frac{1}{3}$, *and* $n = 5$, $p = \frac{1}{3}$, *write down the probability that* $X + Y \ge 1$.

Solution. We are given : $X \sim B(3, \frac{1}{3})$ and $Y \sim B(5, \frac{1}{5})$.

Since X and Y are independent binomial random variables, with $p_1 = p_2 = \frac{1}{3}$, by the additive property of binomial distribution, we get

$$X + Y \sim B(3 + 5, \tfrac{1}{3}), \quad i.e., \quad X + Y \sim B(8, \tfrac{1}{3})$$

$$\therefore \quad P(X + Y = r) = \binom{8}{r} \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{8-r} \qquad \dots (*)$$

Hence $\quad P(X + Y \geq 1) = 1 - P(X + Y < 1) = 1 - P(X + Y = 0) = 1 - \left(\frac{2}{3}\right)^8$. [From (*)]

8·4·8. Characteristic Function of Binomial Distribution.

$$\varphi_X(t) = E(e^{itX}) = \sum_{x=0}^{n} e^{itx} p(x) = \sum_{x=0}^{n} e^{itx} \binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} \binom{n}{x} (pe^{it})^x q^{n-x}$$

$$= (q + pe^{it})^n \qquad \dots (8·13)$$

8·4·9. Cumulants of the Binomial Distribution. Cumulant generating function is given by :

$$K_X(t) = \log M_X(t) = \log (q + pe^t)^n = n \log (q + pe^t)$$

$$= n \log\left[q + p\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right)\right]$$

$$= n \log\left[1 + p\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right)\right]$$

$$= n\left[p\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right) - \frac{p^2}{2}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right)^2\right.$$

$$\left. + \frac{p^3}{3}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right)^3 - \frac{p^4}{4}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right)^4 + \dots\right]$$

Mean $= \kappa_1 =$ Coefficient of t in $K_X(t) = np$

$\mu_2 = \kappa_2 =$ Coefficient of $\frac{t^2}{2!}$ in $K_X(t) = n(p - p^2) = np(1 - p) = npq$

The coefficient of t^3 in $K_X(t)$

$$= n\left[\frac{p}{3!} - \frac{p^2}{2!} . 2 . \frac{1}{2!} + \frac{p^3}{3}\right] = \frac{np}{3!}(1 - 3p + 2p^2)$$

$\therefore \quad \kappa_3 =$ Coefficient of $\frac{t^3}{3!}$ in $K_X(t) = np(1 - 3p + 2p^2)$

$= np(1 - p)(1 - 2p) = npq(1 - p - p) = npq(q - p)$

$\Rightarrow \quad \mu_3 = \kappa_3 = npq(q - p)$

The coefficient of t^4 in $K_X(t)$

$$= n\left[\frac{p}{4!} - \frac{p^2}{2!}\left(\frac{2}{3!} + \frac{1}{4}\right) + \frac{p^3}{3} \frac{3}{2!} - \frac{p^4}{4}\right] = \frac{np}{4!}(1 - 7p + 12p^2 - 6p^3)$$

$\therefore \quad \kappa_4 =$ Coefficient of $\frac{t^4}{4!}$ in $K_X(t) = np(1 - p)(1 - 6p + 6p^2)$

$= npq[1 - 6p(1 - p)] = npq(1 - 6pq)$

$$\therefore \quad \mu_4 = \kappa_4 + 3\kappa_2^2 = npq(1-6pq) + 3n^2p^2q^2 = npq(1-6pq+3npq)$$
$$= npq[1 + 3pq(n-2)].$$

8·4·10. Recurrence Relation for Cumulants of Binomial Distribution. By def.,

$$\kappa_r = \left[\frac{d^r}{dt^r}\log M_X(t)\right]_{t=0} = n\left[\frac{d^r}{dt^r}\log(q+pe^t)\right]$$

$$\therefore \quad \frac{d\kappa_r}{dp} = n\left[\frac{d^r}{dt^r}\cdot\frac{d}{dp}\log(q+pe^t)\right]_{t=0} = n\left[\frac{d^r}{dt^r}\cdot\frac{(-1+e^t)}{q+pe^t}\right]_{t=0}$$

$$\kappa_{r+1} = n\left[\frac{d^{r+1}}{dt^{r+1}}\log(q+pe^t)\right]_{t=0} = n\left[\frac{d^r}{dt^r}\cdot\frac{d}{dt}\log(q+pe^t)\right]_{t=0} = n\left[\frac{d^r}{dt^r}\left(\frac{pe^t}{q+pe^t}\right)\right]_{t=0}$$

$$= n\left[\frac{d^r}{dt^r}\left(1-\frac{q}{q+pe^t}\right)\right]_{t=0} = -nq\left[\frac{d^r}{dt^r}\left(\frac{1}{q+pe^t}\right)\right]_{t=0}$$

Hence

$$\kappa_{r+1} - pq\frac{d\kappa_r}{dp} = -nq\left[\frac{d^r}{dt^r}\left(\frac{1}{q+pe^t}\right)\right]_{t=0} - npq\left[\frac{d^r}{dt^r}\left(\frac{e^t-1}{q+pe^t}\right)\right]_{t=0} = -nq\left[\frac{d^r}{dt^r}\left\{\frac{1+pe^t-p}{q+pe^t}\right\}\right]_{t=0}$$

$$= -nq\left[\frac{d^r}{dt^r}\left\{\frac{q+pe^t}{q+pe^t}\right\}\right]_{t=0} = -nq\left[\frac{d^r}{dt^r}(1)\right]_{t=0} = 0$$

$$\therefore \quad \kappa_{r+1} = pq\frac{d\kappa_r}{dp} \qquad \ldots (8·14)$$

In particular,

$$\kappa_2 = pq\cdot\frac{d\kappa_1}{dp} = pq\cdot\frac{d}{dp}(np) = npq \qquad (\because \kappa_1 = \text{mean} = np)$$

$$\kappa_3 = pq\cdot\frac{d\kappa_2}{dp} = pq\cdot\frac{d(npq)}{dp} = npq(q-p)$$

$$\kappa_4 = pq\cdot\frac{d\kappa_3}{dp} = pq\cdot\frac{d}{dp}\{npq(q-p)\} = npq\frac{d}{dp}\{p(1-p)(1-2p)\}$$
$$= npq\cdot\frac{d}{dp}(p-3p^2+2p^3) = npq(1-6p+6p^2) = npq[1-6p(1-p)]$$
$$= npq(1-6pq).$$

8·4·11. Probability Generating Function of Binomial Distribution.

$$P(s) = \sum_{k=0}^{n} P(X=k)s^k = \sum_{k=0}^{n}\binom{n}{k}(ps)^k q^{n-k} = (ps+q)^n \qquad \ldots (8·14a)$$

The fact that this generating function is nth power of $(q+ps)$ shows that $p(x) = \{b(x; n, p)\}$ is the distribution of the sum $S_n = X_1 + X_2 + \ldots + X_n$, of n independent random variables with the common generating function $(q+ps)$. Each variable X_i assumes the value 0 with probability q and 1 with probability p.

Thus $$\{b(k; n, p)\} = \{b(k; 1, p)\}^n \qquad \ldots (8·14b)$$

Let X and Y be two independent $r.v.$'s having $b(k; m, p)$ and $b(k; n, p)$ as their distributions, then

$$P_X(s) = (q+ps)^m \quad \text{and} \quad P_Y(s) = (q+ps)^n$$

$$P_{X+Y}(s) = P_X(s).\ P_Y(s) = (q+ps)^m (q+ps)^n = (q+ps)^{m+n}$$

$\therefore \quad \{b\,(k;m,p)\} * \{b\,(k\,;n,p)\} = \{b\,(k\ ;m+n,p)\}$ ... (8·14c)

$\Rightarrow \quad \mu_{(1)}' = [n\,(q+ps)^{n-1}\,p]_{s=1} = np$

Also $\mu_{(2)}' = [n\,(n-1)\,(q\ +ps)^{n-2}\,p^2]_{s=1} = n\,(n-1)\,p^2$, and so on.

$\mu_{(r)}' = [n\,(n\ -1)\,(n-2)\ldots(n-r+1)\,(q+ps)^{n-r}\,p^r]_{s=1} = n\,(n-1)\,(n-2)\ldots(n-r+1)p^r.$

Example 8·20. *how that :* $E\left(\frac{20}{X+a}\right) = \int_0^1 t^{a-1}\,G(t)\,dt,\ a>0$ ... (*)

where $G(t)$ is the probability generating function of X. Find it when $X \sim B\,(n,p)$, and $a=1$.

Solution. R.H.S. $= \int_0^1 t^{a-1}.\ G(t)\,dt = \int_0^1 t^{a-1}\,E(t^X)\,dt$

$$= \int_0^1 \left\{t^{a-1}\left(\sum_x p(x)t^x\right)\right\} dt = \sum_x \left[p(x)\int_0^1 t^{x+a-1}dt\right] = \sum_x p(x)\,.\,\frac{1}{(x+a)} = E\left(\frac{1}{X+a}\right) \quad \ldots (**)$$

If $X \sim B\,(n,p)$, then $\quad G(t) = \sum_{x=0}^{n} t^x\,p(x) = (q+pt)^n$

Hence taking $a=1$ in (*) and using (**), we get

$$E\left[\frac{1}{(X+1)}\right] = \int_0^1 (q+pt)^n\,dt = \left|\frac{(q+pt)^{n+1}}{(n+1)p}\right|_0^1 = \frac{1-q^{n+1}}{(n+1)p}.$$

8·4·12. Recurrence Relation for the Probabilities of Binomial Distribution (*Fitting of Binomial Distribution*).

We have $\dfrac{p\,(x+1)}{p(x)} = \dfrac{\binom{n}{x+1}\,p^{x+1}\,q^{n-x-1}}{\binom{n}{x}\,p^x\,q^{n-x}} = \dfrac{n-x}{x+1}\cdot\dfrac{p}{q}$ (On simplification)

$$\therefore \quad p\,(x+1) = \left\{\frac{n-x}{x+1}\cdot\frac{p}{q}\right\}p\,(x), \quad \ldots (8·15)$$

which is the required recurrence formula for the probabilities of binomial distribution.

This formula provides us a very convenient method of graduating the given data by a binomial distribution. The only probability we need to calculate is $p(0)$ which is given by $p\,(0) = q^n$, where q is estimated from the given data by equating the mean $\bar{x}$ of the distribution to np, the mean of the binomial distribution. Thus $\hat{p} = \bar{x}\,/n.$

The remaining probabilities, *viz.*, $p(1), p(2), \ldots$ can now be easily obtained from (8·15) as explained below :

$$p(1) = [p\,(x+1)]_{x=0} = \left(\frac{n-x}{x+1}\cdot\frac{p}{q}\right)_{x=0} \times p\,(0),\quad p(2) = [p\,(x+1)]_{x=1} = \left(\frac{n-x}{x+1}\cdot\frac{p}{q}\right)_{x=1} \times p\,(1)$$

$$p(3) = [p\,(x+1)]_{x=2} = \left(\frac{n-x}{x+1}\cdot\frac{p}{q}\right)_{x=2} \times p\,(2), \text{ and so on.} \quad \ldots (8·15a)$$

Example 8·21. *The following data due to Weldon shows the results of throwing 12 fair dice 4,096 times; a throw of 4, 5, or 6 being called success.*

Success	Frequency	Success	Frequency
0	—	7	847
1	7	8	536
2	60	9	257
3	198	10	71
4	430	11	11
5	731	12	—
6	948		

Fit a binomial distribution and find the expected frequencies.

Solution. In the usual notations, we are given : $n = 12, N = 4,096$

p = Probability of success, *i.e.*, throw of 4, 5 or 6 $= \frac{1}{2} \Rightarrow q = 1 - p = \frac{1}{2}$.

Thus, by the Binomial probability law, the probability of x successes in a throw of 12 dice is given by :

$$P(X = x) = p(x) = \binom{12}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{12-x} = \binom{12}{x}\left(\frac{1}{2}\right)^{12} = \frac{1}{4096}\cdot\binom{12}{x}$$

Hence in 4,096 throws of 12 dice, the frequencies of x successes are given by :

$$f(x) = N.\, p(x) = 4096.\frac{1}{4096}\binom{12}{x} = \binom{12}{x}.$$

Thus the expected frequencies are as tabulated below :

COMPUTATION OF EXPECTED FREQUENCIES

Success (x)	Expected Frequency $f(x) = N\,p(x)$	Success (x)	Expected Frequency $f(x) = N\,p(x)$
0	$\binom{12}{0} = 1$	7	$\binom{12}{7} = 792$
1	$\binom{12}{1} = 12$	8	$\binom{12}{8} = 495$
2	$\binom{12}{2} = 66$	9	$\binom{12}{9} = 220$
3	$\binom{12}{3} = 220$	10	$\binom{12}{10} = 66$
4	$\binom{12}{4} = 495$	11	$\binom{12}{11} = 12$
5	$\binom{12}{5} = 792$	12	$\binom{12}{12} = 1$
6	$\binom{12}{6} = 924$	Total	4,096

Example 8·22. *Seven coins are tossed and number of heads noted. The experiment is repeated 128 times and the following distribution is obtained :*

No. of heads	0	1	2	3	4	5	6	7	Total
Frequencies	7	6	19	35	30	23	7	1	128

Fit a binomial distribution assuming, (i) The coin is unbaised. (ii) The nature of the coin is not known. (iii) Probability of a head for four coins is 0·5 and for the remaining three coins is 0·45.

Solution. In fitting binomial distribution, first of all the mean and variance of the data are equated to np and npq respectively. Then the expected frequencies are calculated from these values of n and p. Here $n = 7$ and $N = 128$.

Case I. *When the coin is unbiased :* $p = q = \frac{1}{2}$, $\quad \frac{p}{q} = 1$

$$p(0) = q^n = \left(\frac{1}{2}\right)^7 = \frac{1}{128} \quad \text{so that} \quad f(0) = Nq^n = 128\left(\frac{1}{2}\right)^7 = 1$$

Using the recurrence formula (8·15), the various probabilities, *viz.*, $p(1), p(2), \ldots$ can be easily calculated as shown below :

COMPUTATION OF EXPECTED BINOMIAL FREQUENCIES

x	f	fx	$\frac{n-x}{x+1}$	$\frac{n-x}{x+1}\cdot\frac{p}{q}$	Expected frequency $f(x) = Np(x)$
0	7	0	7	7	$f(0) = Np(0) = 1$
1	6	6	3	3	$f(1) = 1 \times 7 = 7$
2	19	38	$\frac{5}{3}$	$\frac{5}{3}$	$f(2) = 7 \times 3 = 21$
3	35	105	1	1	$f(3) = 21 \times \frac{5}{3} = 35$
4	30	120	$\frac{3}{5}$	$\frac{3}{5}$	$f(4) = 35 \times 1 = 35$
5	23	115	$\frac{1}{3}$	$\frac{1}{3}$	$f(5) = 35 \times \frac{3}{5} = 21$
6	7	42	$\frac{1}{7}$	$\frac{1}{7}$	$f(6) = 21 \times \frac{1}{3} = 7$
7	1	7			$f(7) = 7 \times \frac{1}{7} = 1$
Total	128	433			

Case II. *When the nature of the coin is not known,* then

$$\text{Mean} = np = \bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i = \frac{433}{128} = 3{\cdot}3828\,; \qquad n = 7$$

$$\therefore \quad p = \frac{3{\cdot}3828}{7} = 0{\cdot}48326 \quad \text{and} \quad q = 1 - p = 0{\cdot}51674, \quad \Rightarrow \quad \frac{p}{q} = 0{\cdot}93521$$

$$f(0) = Nq^7 = 128\,(0{\cdot}5167)^7 = 1{\cdot}2593 \text{ (using logarithms)}$$

COMPUTATION OF EXPECTED BINOMIAL FREQUENCIES

x	$\frac{n-x}{x+1}$	$\frac{n-x}{x+1}\cdot\frac{p}{q}$	Expected frequency $f(x) = N\,p(x)$		
0	7	6·54647	$f(0) = Np(0)$	$= 1{\cdot}2593$	$\simeq 1$
1	3	2·80563	$f(1) = 1{\cdot}2593 \times 6{\cdot}54647$	$= 8{\cdot}2438$	$\simeq 8$
2	$\frac{5}{3}$	1·55868	$f(2) = 2{\cdot}80563 \times 8{\cdot}2438$	$= 23{\cdot}129$	$\simeq 23$
3	1	0·93521	$f(3) = 1{\cdot}55868 \times 23{\cdot}129$	$= 36{\cdot}05$	$\simeq 34$
4	$\frac{3}{5}$	0·56113	$f(4) = 0{\cdot}93521 \times 36{\cdot}05$	$= 33{\cdot}715$	$\simeq 34$
5	$\frac{1}{3}$	0·31174	$f(5) = 0{\cdot}56113 \times 33{\cdot}715$	$= 18{\cdot}918$	$\simeq 19$
6	$\frac{1}{7}$	0·13360	$f(6) = 0{\cdot}31174 \times 18{\cdot}918$	$= 5{\cdot}897$	$\simeq 6$
7			$f(7) = 0{\cdot}13360 \times 5{\cdot}897$	$= 0{\cdot}788$	$\simeq 1$

(*iii*) The probability generating functions (*p.g.f.*), say $P_X(s)$ for the 4 coins and $P_Y(S)$ for the remaining 3 coins are given by : ... [*c.f.* 8·14 (*a*)]

$$P_X(s) = (0{\cdot}50 + 0{\cdot}50s)^4, \qquad P_Y(s) = (0{\cdot}55 + 0{\cdot}45s)^3$$

Since all the throws are independent, the *p.g.f.* $P_{X+Y}(s)$ for the whole experiment is given by: ... (*c.f.* 8·14 (*b*)]

$$\begin{aligned} P_{X+Y}(s) &= P_X(s)\, P_Y(s) \\ &= (0{\cdot}50 + 0{\cdot}50\, s)^4 \times (0{\cdot}55 + 0{\cdot}45\, s)^3 \\ &= (0{\cdot}0625 + 0{\cdot}25s + 0{\cdot}375s^2 + 0{\cdot}25s^3 + 0{\cdot}0625s^4) \\ &\quad \times (0{\cdot}166375 + 0{\cdot}408375s + 0{\cdot}334125\, s^2 + 0{\cdot}091125s^3) \end{aligned}$$

$f(x) = N \times$ Coefficient of s^x in $P_{X+Y}(s)$

$\therefore$ $f(0) = 128 \times {\cdot}0625 \times {\cdot}16637 = 1{\cdot}13310.$

$f(1) = 128\,({\cdot}25 + {\cdot}166375 + {\cdot}408375 \times {\cdot}0625) = 8{\cdot}5910$

$f(2) = 128\,({\cdot}28396) = 36{\cdot}3470$ $\qquad f(5) = 128\,({\cdot}14602) = 18{\cdot}6934$

$f(3) = 128\,({\cdot}184117) = 23{\cdot}5669$ $\qquad f(6) = 128\,({\cdot}04366) = 5{\cdot}5889$

$f(4) = 128\,({\cdot}26057) = 33{\cdot}3529$ $\qquad f(7) = 128\,({\cdot}005695) = {\cdot}72896$

These frequencies, rounded to the nearest integer, keeping in mind that total frequeny is $N = 128$, are given below :

x	0	1	2	3	4	5	6	7	Total
f	1	9	36	23	33	19	6	1	$N = 128$

Example 8·23. *Let X and Y be independent binomial variates, each with parameters n and p. Find P (X − Y = k).*

Solution. Since each of the variables X and Y takes the values $0, 1, 2, \ldots, n$, $Z = X - Y$ takes on the values $-n, -(n-1), \ldots, -1, 0, 1, \ldots, n$

$$P(Z = k) = P(X - Y = k) = \sum_{r=0}^{n} P(X = k + r \cap Y = r)$$

$$= \sum_{r=0}^{n} P(X = k + r).\, P(Y = r) \qquad (\because X \text{ and } Y \text{ are independent}).$$

$$= \sum_{r=0}^{n} \binom{n}{k+r} p^{k+r} q^{n-k-r} \binom{n}{r} p^r q^{n-r} = \sum_{r=0}^{n} \binom{n}{k+r}\binom{n}{r} p^{2r+k} q^{2n-2r-k}, \quad \ldots (*)$$

where $k = -n, -(n-1), \ldots -2, -1, 0, 1, 2, \ldots, n$; and $q = 1 - p$.

In particular, we have $P(Z = 0) = \sum_{r=0}^{n} \binom{n}{r}^2 \cdot p^{2r} q^{2n-2r}$

$P(Z = -n) = \sum_{r=0}^{n} \binom{n}{-n+r}\binom{n}{r} p^{2r-n} q^{3n-2r} = p^n q^n$, because we get the result when $r = n$ and for other values of $r < n$, $\binom{n}{-n+r}$ is not defined and hence taken as 0.

Example 8·24. *Find the m.g.f. of standard binomial variate* $(X - np)/\sqrt{npq}$ *and obtain its limiting form as* $n \to \infty$. *Also interpret the result.*

Solution. We know that if $X \sim B(n, p)$, then $M_X(t) = (q + pe^t)^n$.

The *m.g.f.* of standard binomial variate $Z = \dfrac{X - np}{\sqrt{npq}} = \dfrac{X - \mu}{\sigma}$, (say),

where $\mu = np$ and $\sigma^2 = npq$, is given by :

$$
\begin{aligned}
M_Z(t) &= e^{-\mu t/\sigma} M_X(t/\sigma)\\
&= e^{-npt/\sqrt{npq}} \cdot \left(q + p\, e^{t/\sqrt{npq}}\right)^n = \left[q e^{-pt/\sqrt{npq}} + p\, e^{qt/\sqrt{npq}}\right]^n\\
&= \left[q\left\{1 - \frac{pt}{\sqrt{npq}} + \frac{p^2 t^2}{2npq} + 0'(n^{-3/2})\right\} + p\left\{1 + \frac{qt}{\sqrt{npq}} + \frac{q^2 t^2}{2npq} + 0''(n^{-3/2})\right\}\right]^n
\end{aligned}
$$

where $0'(n^{-3/2})$ and $0''(n^{-3/2})$ involve terms containing $n^{3/2}$ and higher powers of n in the denominator. Thus

$$
M_Z(t) = \left[(q + p) + \frac{t^2\, pq}{2npq}(p + q) + 0\,(n^{-3/2})\right]^n = \left[1 + \frac{t^2}{2n} + 0\,(n^{-3/2})\right]^n,
$$

where $0(n^{-3/2})$ involves terms with $n^{3/2}$ and higher powers of n in the denominator.

$$
\begin{aligned}
\therefore \quad \log M_Z(t) &= n \log\left[1 + \frac{t^2}{2n} + 0\,(n^{-3/2})\right]\\
&= n\left[\left\{\frac{t^2}{2n} + 0\,(n^{-3/2})\right\} - \frac{1}{2}\left\{\frac{t^2}{2n} + 0\,(n^{-3/2})\right\}^2 + \dots\right]\\
&= \frac{t^2}{2} + 0'''\,(n^{-1/2}),
\end{aligned}
$$

where $0'''(n^{-1/2})$ involve terms with $n^{1/2}$ and higher powers of n in the denominator. Proceeding to the limit as $n \to \infty$, we get

$$
\lim_{n\to\infty} \log M_Z(t) = \frac{t^2}{2} \quad \Rightarrow \quad \lim_{n\to\infty} M_Z(t) = \exp\,(t^2/2) \qquad \dots (**)
$$

Interpretation. (**) is the *m.g.f* of standard normal variate [c.f. Remark to § 9·2·5]. Hence by uniqueness theorem of moment generating functions, standard binomial variate tends to standard normal variate as $n \to \infty$. In other words, binomial distribution tends to normal distribution as $n \to \infty$.

Example 8·25. *A drunk person performs a random walk over positions 0, ± 1, ±2, ..., as follows. He starts at 0. He takes successive one unit steps, going to the right with probability p and to the left with probability (1–p). His steps are independent. Let X denote his position after n steps. Find the distribution of $\frac{1}{2}$ (X + n) and find E (X).*

Solution. With the ith step of the drunk and, let us associate a variable X_i defined as follows :

$$
X_i = \begin{cases} 1, & \text{if he takes the step to the right}\\ -1, & \text{if he takes the step to the left} \end{cases}
$$

Then $X = X_1 + X_2 + \dots + X_n$, gives the position of the drunkard after n steps.

Define $Y_i = \dfrac{1}{2}(X_i + 1)$. Then $Y_i = \begin{cases} \frac{1}{2}(1+1) = 1, & \text{with probability } p\\ \frac{1}{2}(-1+1) = 0, & \text{with probability } 1 - p = q, \text{ (say)}. \end{cases}$

Since the n steps of drunkard are independent, Y_i's, $(i = 1, 2, \dots n)$ are *i.i.d.* Bernoulli variates with parameter p. Hence

$$\sum_{i=1}^{n} Y_i \sim B(n, p) \Rightarrow \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} \left(\frac{X_i + 1}{2}\right) = \frac{1}{2}\left(\sum_{i=1}^{n} X_i + n\right) = \frac{X+n}{2} \sim B(n, p)$$

where $X = \sum_{i=1}^{n} X_i$, is the position of the drunkard after n steps.

Since $\frac{1}{2}(X+n) \sim B(n, p)$, $\quad E\left\{\frac{1}{2}(X+n)\right\} = np \Rightarrow E(X+n) = 2np$

$$\Rightarrow \quad E(X) + n = 2np \quad \Rightarrow \quad E(X) = n(2p-1).$$

Example 8·26. *Suppose that the r.v. X is uniformly distributed on (0, 1), i.e.,*

$$f_X(x) = 1\,;\, 0 \le x \le 1 \qquad \ldots (*)$$

Assume that the conditional distribution $Y \mid X = x$ *has a binomial distribution with parameters n and* $p = x$, *i.e.,*

$$P(Y = y \mid X = x) = \binom{n}{y} x^y (1-x)^{n-y};\ y = 0, 1, 2, \ldots, n \qquad \ldots (**)$$

Find (a) E(Y), and (b) the distribution of Y.

Solution. (*a*) We are given that the conditional distribution of $Y \mid X = x \sim B(n, x)$

so that $\quad E(Y \mid X = x) = nx. \qquad \ldots (***)$

$\therefore \quad E(Y) = E[E(Y \mid X)] = E(nX) = nE(X). \quad$ [From (*)] $\qquad \ldots (****)$

Now $\quad E(X) = \int_0^1 x f(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}.$

Substituting in (****), we get : $\quad E(Y) = \frac{n}{2}.$

Hence $\quad E(Y) = \frac{1}{2} n$

(*b*) We have $\quad f_{XY}(x\, y) = f_X(x) . f_{Y \mid X}(y \mid x)$

Since X has (continuous) uniform distribution on (0, 1), marginal distribution of Y is given by :

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_0^1 f_{Y \mid X}(y \mid x) . f_X(x)\, dx$$

$$= \int_0^1 {}^nC_y\, x^y (1-x)^{n-y} . 1\, dx = {}^nC_y \int_0^1 x^y (1-x)^{n-y}\, dx \qquad \text{[From (*) and (**)]}$$

$$= {}^nC_y\, \beta(y+1, n-y+1) = \frac{n!}{y!\,(n-y)!} \frac{\Gamma(y+1)\,\Gamma(n-y+1)}{\Gamma(n+2)}$$

$$= \frac{n!}{y!\,(n-y)!} \times \frac{y!\,(n-y)!}{(n+1)!} = \frac{1}{n+1};\ y = 0, 1, 2, \ldots, n$$

Since Y takes the values 0, 1, 2, ..., n each with equal probability $\frac{1}{n+1}$, Y has discrete uniform distribution on $[0, n]$.

Remark. We could find $E(Y)$ on using the distribution of Y in (*b*).

$$E(Y) = \sum_{y=0}^{n} y f_Y(y) = \frac{1}{n+1} \sum_{y=0}^{n} y = \frac{1}{n+1}(0 + 1 + 2 + \ldots + n) = \frac{n}{2}.$$

Example 8·26. *If a fair coin is tossed an even number 2n times, show that the probability of obtaining more heads than tails is :* $\frac{1}{2}\left\{1 - \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right\}$.

Solution. Let X = No. of heads ; $\quad Y$ = No. of tails; $\quad$ No. of trials $= 2n$

8·5. POISSON DISTRIBUTION

Poisson distribution was discovered by the French mathematician and physicist Simeon Denis Poisson (1781-1840) who published it in 1837. Poisson distribution is a limiting case of the binomial distribution under the following conditions :

(i) n, the number of trials is indefinitely large , *i.e.*, $n \to \infty$.

(ii) p, the constant probability of success for each trial is indefinitely small, *i.e.*, $p \to 0$.

(iii) $np = \lambda$, (say) is finite.

Thus $p = \lambda/n$, $\quad q = 1 - \lambda/n$, where λ is a positive real number.

The probability of x successes in a series of n independent trials is :

$$b(x\,;n,p) = \binom{n}{x} p^x q^{n-x};\ x = 0, 1, 2, \ldots, n \qquad \ldots(*)$$

We want the limiting form of (*) under the above conditions. Hence

$$\lim_{n\to\infty} b(x;n,p) = \lim_{n\to\infty} \frac{n\,!}{x\,!(n-x)\,!}\left(\frac{\lambda}{n}\right)^x \cdot \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

Using Stirling's approximation for $n\,!$ as $n \to \infty$, *viz.*,

$$\lim_{n\to\infty} n\,! \approx \sqrt{2\pi}\, e^{-n}\, n^{n+(1/2)}\text{, we get}$$

$$\lim_{n\to\infty} b(x;n,p) = \lim_{n\to\infty}\left\{\frac{\sqrt{2\pi}\, e^{-n} \,.\, n^{n+(1/2)}}{x\,!\,\sqrt{2\pi}\, e^{-(n-x)} . (n-x)^{n-x+(1/2)}}\right\}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{e^x . x\,!}\cdot \lim_{n\to\infty}\frac{n^{n-x+(1/2)}}{(n-x)^{n-x+(1/2)}}\cdot\left(1-\frac{\lambda}{n}\right)^{n-x} = \frac{\lambda^x}{e^x\, x\,!}\cdot\lim_{n\to\infty}\frac{\left(1-\frac{\lambda}{n}\right)^{n-x}}{\left(1-\frac{x}{n}\right)^{n-x+(1/2)}}$$

$$= \frac{\lambda^x}{e^x\, x\,!}\cdot\frac{\lim\limits_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n . \lim\limits_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-x}}{\lim\limits_{n\to\infty}\left(1-\frac{x}{n}\right)^n \lim\limits_{n\to\infty}\left(1-\frac{x}{n}\right)^{-x+(1/2)}}$$

But we know that

$$\left.\begin{aligned}&\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda},\\ &\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^\alpha = 1,\ \alpha \text{ is not a function of } n\end{aligned}\right\} \qquad \ldots(**)$$

$$\therefore \quad \lim_{n\to\infty} b(x;n,p) = \frac{\lambda^x}{e^x\, x\,!}\times\frac{e^{-\lambda}.\,1}{e^{-x}.\,1} = \frac{e^{-\lambda}.\,\lambda^x}{x\,!}\,;\ x = 0, 1, 2, \ldots, \infty, \qquad \text{[Using (**)]}$$

which is the required probability function of the Poisson distribution. 'λ' is known as the *parameter* of Poisson distribution.

Aliter. Poisson distribution can also be derived without using Stirling's approximation as follows :

$$b(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x}\left(\frac{p}{1-p}\right)^x (1-p)^n$$

$$= \frac{n(n-1)(n-2)\ldots(n-x+1)}{x!} \cdot \frac{\left(\frac{\lambda}{n}\right)^x}{\left(1-\frac{\lambda}{n}\right)^x}\left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)}{x!\left(1-\frac{\lambda}{n}\right)^x}\lambda^x\left(1-\frac{\lambda}{n}\right)^n$$

$$\therefore \quad \lim_{n\to\infty} b(x;n,p) = \frac{e^{-\lambda}\lambda^x}{x!};\ x = 0, 1, 2, \ldots \qquad \text{[From (**)]}$$

Definition. *A random variable X is said to follow a Poisson distribution if it assumes only non-negative values and its probability mass function is given by :*

$$p(x,\lambda) = P(X=x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!};\ x = 0, 1, 2, \ldots;\ \lambda > 0 \\ 0,\ \textit{otherwise} \end{cases} \qquad \ldots (8{\cdot}15)$$

Here λ is known as the parameter of the distribution. We shall use the notation $X \sim P(\lambda)$, to denote that X is a Poisson variate with parameter λ.

Remarks 1. It should be noted that $\sum_{x=0}^{\infty} P(X=x) = e^{-\lambda}\sum_{x=0}^{\infty}\lambda^x/x! = e^{-\lambda}e^{\lambda} = 1$

2. The corresponding distribution function is :

$$F(x) = P(X \le x) = \sum_{r=0}^{x} p(r) = e^{-\lambda}\sum_{r=0}^{x}\lambda^r/r!;\ x = 0, 1, 2, \ldots$$

3. Poisson distribution occurs when there are events which do not occur as outcomes of a definite number of trials (unlike that in binomial distribution) of an experiment but which occur at random points of time and space wherein our interest lies only in the number of occurrences of the event, not in its non-occurrences.

4. Following are some instances where Poisson distribution may be successfully employed :

(i) Number of deaths from a disease (not in the form of an epidemic) such as heart attack or cancer or due to snake bite.

(ii) Number of suicides reported in a particular city.

(iii) The number of defective material in a packing manufactured by a good concern.

(iv) Number of faulty blades in a packet of 100.

(v) Number of air accidents in some unit of time.

(vi) Number of printing mistakes at each page of the book.

(vii) Number of telephone calls received at a particular telephone exchange in some unit of time or connections to wrong numbers in a telephone exchange.

(viii) Number of cars passing a crossing per minute during the busy hours of a day.

(ix) The number of fragments received by a surface area '*A*' from a fragment atom bomb.

(x) The emission of radioactive (alpha)particles.

8·5·1. The Poisson Process. The Poisson distribution may also be obtained independently (*i.e.*, without considering it as a limiting form of the binomial distribution) as follows :

Let X_t be the number of telephone calls received in time interval 't' on a telephone switch board. Consider the following experimental conditions :

(1) The probability of getting a call in small time interval $(t, t + dt)$ is $\lambda\, dt$, where λ is a positive constant and dt denotes a small increment in time 't'.

(2) The probability of getting more than one call in this time interval is very small, *i.e.*, is of the order of $(dt)^2$, *i.e.*, $0\,[(dt)^2]$ such that $\lim\limits_{dt \to 0} \dfrac{0\,(dt)^2}{dt} = 0.$

(3) The probability of any particular call in the time interval $(t, t + dt)$ is independent of the actual time t and also of all previous calls.

Under these conditions it can be shown that the probability of getting x calls in time 't', say, $P_X(t)$ is given by : $\quad P_X(t) = \dfrac{e^{-\lambda t}(\lambda t)^x}{x\,!}\,; x = 0, 1, 2, \ldots \infty$

which is a Poisson distribution with parameter $\lambda\, t$.

Proof. Let $P_x(t) = P$ {*of getting x calls in a time interval of length 't'*}.

Also P {of at least one call during $(t, t + dt)$} $= \lambda\, dt + 0\,[(dt)^2]$

and P {of more than one call during $(t, t + dt)$} $= 0\,[(dt)^2]$.

The event of getting exactly x calls in time $t + dt$ can materialise in the following two mutually exclusive ways :

(*i*) x calls in $(0, t)$, and none during $(t, t + dt)$ and the probability of this event is:

$$P_x(t)\,[1 - \{(\lambda\, dt + 0\,(dt)^2\}].$$

(*ii*) exactly $(x - 1)$ calls during $(0, t)$ and one call in $(t, t + dt)$ and the probability of this event is $\quad P_{x-1}(t)\,(\lambda\, dt)$.

Hence by the addition theorem of probability, we get

$$\begin{aligned} P_x(t + dt) &= P_x(t)\left\{1 - \lambda\, dt - 0\,(dt)^2\right\} + P_{x-1}(t)\,\lambda\, dt. \\ &= P_x(t)\,(1 - \lambda\, dt) + P_{x-1}(t)\,\lambda\, dt - 0\,(dt)^2\, P_x(t) \qquad \ldots (1) \end{aligned}$$

$$\Rightarrow \quad \frac{P_x(t + dt) - P_x(t)}{dt} = -\lambda\, P_x(t) + \lambda P_{x-1}(t) - \frac{0\,(dt)^2}{dt}\, P_x(t)$$

Proceeding to the limit as $dt \to 0$, we get

$$\lim_{dt \to 0} \frac{P_x(t + dt) - P_x(t)}{dt} = -\lambda\, P_x(t) + \lambda P_{x-1}(t)$$

$$\Rightarrow \quad P_x'(t) = -\lambda\, P_x(t) + \lambda\, P_{x-1}(t),\ x \geq 1 \qquad \ldots (2)$$

where (′) denotes differentiation *w.r.* to 't'.

For $x = 0$, $P_{x-1}(t) = P_{-1}(t) = P$ {(-1) calls in time 't'} $= 0$

Hence from (1), we get $\quad P_0(t + dt) = P_0(t)\left\{1 - \lambda\, dt\right\} - 0\,(dt)^2$

which on taking the limit $dt \to 0$, gives , $\quad P_0'(t) = -\lambda\, P_0(t) \quad \Rightarrow \quad \dfrac{P_0'(t)}{P_0(t)} = -\lambda.$

Integrating *w.r. to* 't', $\quad \log P_0(t) = -\lambda t + C,$

where C is an arbitrary constant to be determined from the condition $P_0(0) = 1$.

Hence $\quad C = \log 1 = 0 \quad \therefore \quad \log P_0(t) = -\lambda t \quad \Rightarrow \quad P_0(t) = e^{-\lambda t}$

Substituting this value of $P_0(t)$ in (2), we get, with $x = 1$,

$$P_1'(t) = -\lambda P_1(t) + \lambda e^{-\lambda t} \quad \Rightarrow \quad P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}$$

This is an ordinary linear differential equation whose integrating factor is $e^{\lambda t}$.

Hence its solution is :
$$e^{\lambda t} P_1(t) = \lambda \int e^{\lambda t} e^{-\lambda t} dt + C_1 = \lambda t + C_1,$$

where C_1 is an arbitrary constant to be determined from $P_1(0) = 0$, which gives $C_1 = 0$.

$$\therefore \quad P_1(t) = e^{-\lambda t} \lambda t$$

Again substituting this in (2) with $x = 2$, $\quad P'_2(t) + \lambda P_2(t) = \lambda e^{-\lambda t} \lambda t.$

Integrating factor of this equation is $e^{\lambda t}$ and its solution is :

$$P_2(t)\, e^{\lambda t} = \lambda^2 \int t\, e^{-\lambda t} e^{\lambda t} dt + C_2 = \frac{\lambda^2 t^2}{2} + C_2$$

where C_2 is an arbitrary constant to be determined from $P_2(0) = 0$, which gives $C_2 = 0$.

Hence $\quad P_2(t) = e^{-\lambda t} \dfrac{(\lambda t)^2}{2}$. $\quad$ Proceeding similarly step by step, we shall get

$$P_x(t) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}; \; x = 0, 1, 2, \ldots, \infty \qquad \ldots (8\cdot 15a)$$

which is the *p.m.f.* of Poisson distribution with parameter λt.

8·5·2. Moments of the Poisson Distribution.

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x\, p(x, \lambda) = \sum_{x=0}^{\infty} x . \frac{e^{-\lambda} \lambda^x}{x!} = \lambda e^{-\lambda} \left\{ \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right\}$$

$$= \lambda e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots \right) = \lambda e^{-\lambda} . e^{\lambda} = \lambda$$

$\mu_1' = \lambda$

Hence *the mean of the Poisson distribution is* λ.

$$\mu_2' = E(X^2) = \sum_{x=0}^{\infty} x^2 p(x, \lambda) = \sum_{x=0}^{\infty} \{x(x-1) + x\} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!} = \lambda^2 e^{-\lambda} \left[\sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} \right] + \lambda$$

$\mu_2' = \lambda^2 + \lambda$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda = \lambda^2 + \lambda$$

$$\mu_3' = E(X^3) = \sum_{x=0}^{\infty} x^3 p(x, \lambda) = \sum_{x=0}^{\infty} \{x(x-1)(x-2) + 3x(x-1) + x\} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2) \frac{e^{-\lambda} \lambda^x}{x!} + 3 \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \lambda^3 \left\{ \sum_{x=3}^{\infty} \frac{\lambda^{x-3}}{(x-3)!} \right\} + 3e^{-\lambda} \lambda^2 \left\{ \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} \right\} + \lambda$$

$\mu_3' = \lambda^3 + 3\lambda^2 + \lambda$

$$= e^{-\lambda} \lambda^3 e^{\lambda} + 3e^{-\lambda} \lambda^2 e^{\lambda} + \lambda = \lambda^3 + 3\lambda^2 + \lambda$$

$$\mu_4' = E(X^4) = \sum_{x=0}^{\infty} x^4 . p(x, \lambda)$$

$$= \sum_{x=0}^{\infty} \{x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x\} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\lambda^4 \left\{ \sum_{x=4}^{\infty} \frac{\lambda^{x-4}}{(x-4)!} \right\} + 6e^{-\lambda}\lambda^3 \left\{ \sum_{x=3}^{\infty} \frac{\lambda^{x-3}}{(x-3)!} \right\} + 7e^{-\lambda}\lambda^2 \left\{ \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} \right\} + \lambda$$

$$= \lambda^4 (e^{-\lambda} e^{\lambda}) + 6\lambda^3 (e^{-\lambda} e^{\lambda}) + 7\lambda^2 (e^{-\lambda} e^{\lambda}) + \lambda$$

$$= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

The four central moments are now obtained as follows :

$$\mu_2 = \mu_2' - \mu_1'^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda,$$

Thus the mean and the variance of the Poisson distribution are each equal to λ.

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = (\lambda^3 + 3\lambda^2 + \lambda) - 3\lambda(\lambda^2 + \lambda) + 2\lambda^3 = \lambda,$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$
$$= (\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda) - 4\lambda(\lambda^3 + 3\lambda^2 + \lambda) + 6\lambda^2(\lambda^2 + \lambda) - 3\lambda^4 = 3\lambda^2 + \lambda$$

Co-efficients of skewness and kurtosis are given by :

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda} \quad \text{and} \quad \gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}$$

Also $$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{\lambda} \quad \text{and} \quad \gamma_2 = \beta_2 - 3 = \frac{1}{\lambda} \qquad \ldots (8\cdot16)$$

Hence the Poisson distribution is always a skewed distribution.

Proceeding to the limit as $\lambda \to \infty$, $\beta_1 = 0$ and $\beta_2 = 3$.

8·5·3. Mode of the Poisson Distribution

$$\frac{p(x)}{p(x-1)} = \frac{\dfrac{e^{-\lambda}\lambda^x}{x!}}{\dfrac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}} = \frac{\lambda}{x} \qquad \ldots (8\cdot17)$$

We discuss the following cases :

Case I. *When λ is not an integer.* Let us suppose that S is the integral part of λ, so that $\lambda = S + f, 0 < f < 1$. Hence from (8·17), we get :

$$\frac{p(x)}{p(x-1)} = \frac{S+f}{x} = \begin{cases} > 1 & , \text{if } x = 0, 1, \ldots, S \\ < 1 & , \text{if } x = S+1, S+2, \ldots \end{cases}$$

$$\frac{p(1)}{p(0)} > 1, \frac{p(2)}{p(1)} > 1, \ldots, \frac{p(S-1)}{p(S-2)} > 1, \frac{p(S)}{p(S-1)} > 1,$$

and $$\frac{p(S+1)}{p(S)} < 1, \frac{p(S+2)}{p(S+1)} < 1, \ldots$$

Combining the above expressions into a single expression, we get

$$p(0) < p(1) < p(2) < \ldots < p(S-2) < p(S-1) < p(S) > p(S+1) > p(S+2) > \ldots,$$

which shows that $p(S)$ is the maximum value. Hence, in this case, the distribution is unimodal and the integral part of λ is the unique modal value.

Case II. When $\lambda = k$ *(say) is an integer*. Here, as in case I, we have

$$\frac{p(1)}{p(0)} > 1, \quad \frac{p(2)}{p(1)} > 1, \ldots, \frac{p(k-1)}{p(k-2)} > 1$$

and $$\frac{p(k)}{p(k-1)} = 1, \quad \frac{p(k+1)}{pk} < 1, \frac{p(k+2)}{p(k+1)} < 1, \ldots$$

$\therefore$ $$p(0) < p(1) < p(2) < \ldots < p(k-2) < p(k-1) = p(k) > p(k+1) > p(k+2) \ldots$$

In this case we have two maximum values, *viz.*, $p(k-1)$ and $p(k)$ and thus the distribution is bimodal and two modes are at $(k-1)$ and k, *i.e.*, at $(\lambda - 1)$ and λ, (since $k = \lambda$).

8·5·4. Recurrence Relation for Moments of the Poisson Distribution. By *def.*,

$$\mu_r = E\{X - E(X)\}^r = \sum_{x=0}^{\infty} (x-\lambda)^r p(x, \lambda) = \sum_{x=0}^{\infty} (x-\lambda)^r \frac{e^{-\lambda}\lambda^x}{x!}$$

Differentiating *w.r.to* λ, we get

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} r(x-\lambda)^{r-1}(-1)\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{(x-\lambda)^r}{x!}\{x\lambda^{x-1}e^{-\lambda} - \lambda^x e^{-\lambda}\}$$

$$= -r\sum_{x=0}^{\infty}(x-\lambda)^{r-1}.\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty}\frac{(x-\lambda)^r}{x!}\{\lambda^{x-1}e^{-\lambda}(x-\lambda)\}$$

$$= -r\sum_{x=0}^{\infty}(x-\lambda)^{r-1}.\frac{e^{-\lambda}\lambda^x}{x!} + \frac{1}{\lambda}\sum_{x=0}^{\infty}(x-\lambda)^{r+1}.\frac{e^{-\lambda}\lambda^x}{x!}$$

$\therefore$ $$= -r\,\mu_{r-1} + \frac{1}{\lambda}\mu_{r+1}$$

$\Rightarrow$ $$\mu_{r+1} = r\lambda\,\mu_{r-1} + \lambda\frac{d\mu_r}{d\lambda} \qquad \ldots (8{\cdot}18)$$

Putting $r = 1, 2$ and 3 successively, we get

$$\mu_2 = \lambda\mu_0 + \lambda\frac{d\mu_1}{d\lambda} = \lambda, \quad \mu_3 = 2\lambda\,\mu_1 + \lambda\frac{d\mu_2}{d\lambda} = \lambda, \quad \mu_4 = 3\lambda\,\mu_2 + \lambda\frac{d\mu_3}{d\lambda} = 3\lambda^2 + \lambda.$$

8·5·5. Moment Generating Function of the Poisson Distribution.

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}.\frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty}\frac{e^{-\lambda}(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left\{1 + \lambda e^t + \frac{(\lambda e^t)^2}{2!} + \ldots\right) = e^{-\lambda}.e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)} \qquad \ldots (8{\cdot}19)$$

8·5·6. Characteristic Function of the Poisson Distribution.

$$\phi_X(t) = \sum_{x=0}^{\infty} e^{itx}.p(x,\lambda) = \sum_{x=0}^{\infty} e^{itx}\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^{it})^x}{x!} = e^{-\lambda}e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)} \qquad \ldots (8{\cdot}20)$$

8·5·7. Cumulants of the Poisson Distribution.

$$K_X(t) = \log M_X(t) = \log[e^{\lambda(e^t-1)}] = \lambda(e^t - 1)$$

$$= \lambda\left[\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^r}{r!} + \ldots\right) - 1\right] = \lambda\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^r}{r!} + \ldots\right]$$

κ_r = rth cumulant = Coefficient of $\frac{t^r}{r!}$ in $K_X(t) = \lambda$

$$\Rightarrow \qquad \kappa_r = \lambda\ ;\ r = 1, 2, 3, \ldots \qquad \ldots (8\cdot 20\ldots)$$

Hence, *all cumulants of the Poisson distribution are equal, each being equal to* λ. In particular, we have

$$\text{Mean} = \kappa_1 = \lambda,\ \mu_2 = \kappa_2 = \lambda,\ \mu_3 = \kappa_3 = \lambda \text{ and } \mu_4 = \kappa_4 + 3\kappa_2^2 = \lambda + 3\lambda^2$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\lambda + 3\lambda^2}{\lambda^2} = \frac{1}{\lambda} + 3$$

Remark. If m is the mean and σ is the s.d. of Poisson distribution with parameter λ, then

$$m\,\sigma\,\gamma_1\,\gamma_2 = \lambda\,.\,\sqrt{\lambda}\,.\,\sqrt{\beta_1}\,(\beta_2 - 3) = \lambda.\,\sqrt{\lambda}\,.\,\frac{1}{\sqrt{\lambda}}\,.\,\frac{1}{\lambda} = 1.$$

8·5·8. Additive or Reproductive Property of Independent Poisson Variates. *Sum of independent Poisson variates is also a Poisson variate. More elaborately, if* X_i, ($i = 1, 2, \ldots, n$) *are independent Poisson variates with parameters* λ_i; $i = 1, 2, \ldots, n$ *respectively, then* $\sum_{i=1}^{n} X_i$ *is also a Poisson variate with parameter* $\sum_{i=1}^{n} \lambda_i$.

Proof. $$M_{X_i}(t) = e^{\lambda_i (e^t - 1)};\ i = 1, 2, \ldots, n \qquad [\because X_i \sim P(\lambda_i)]$$

$$M_{X_1 + X_2 + \ldots + X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t), \quad [\text{Since } X_i\ ;\ i = 1, 2, \ldots, n \text{ are independent}]$$

$$= e^{\lambda_1(e^t-1)}\, e^{\lambda_2(e^t-1)} \ldots e^{\lambda_n(e^t-1)} = e^{(\lambda_1+\lambda_2+\ldots+\lambda_n)\,(e^t-1)}$$

which is the $m.g.f.$ of a Poisson variate with parameter $\lambda_1 + \lambda_2 + \ldots + \lambda_n$. Hence, by uniqueness theorem of $m.g.f.$'s, $\sum_{i=1}^{n} X_i$ is also a Poisson variate with parameter $\sum_{i=1}^{n} \lambda_i$.

Remarks 1. In fact, the converse of the above result is also true, *i.e., if* $X_1, X_2, \ldots, X_n$ *are independent and* $\sum_{i=1}^{n} X_i$ *has a Poisson distribution, then each of the random variables* $X_1, X_2, \ldots, X_n$ *has a Poisson distribution.*

Let X_1 and X_2 be independent $r.\,v.$'s so that $X_1 \sim P(\lambda_1)$ and $X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$. Then we want to prove that $X_2 \sim P(\lambda_2)$.

Proof. Since X_1 and X_2 are independent, we have

$$M_{X_1+X_2}(t) = M_{X_1}(t)\, M_{X_2}(t) \quad \Rightarrow \quad e^{(\lambda_1+\lambda_2)(e^t-1)} = e^{\lambda_1(e^t-1)} M_{X_2}(t)$$

or $$M_{X_2}(t) = e^{\lambda_2(e^t-1)} \quad \Rightarrow \quad X_2 \sim P(\lambda_2), \text{ by uniqueness theorem of } m.g.f.$$

2. The difference of two independent Poisson variates is not a Poisson variate.

$$M_{X_1-X_2}(t) = M_{X_1+(-X_2)}(t) = M_{X_1}(t).\, M_{(-X_2)}(t), \text{ (since } X_1 \text{ and } X_2 \text{ are independent.)}$$

$$\therefore \quad M_{X_1-X_2}(t) = M_{X_1}(t)\, M_{X_2}(-t) = e^{\lambda_1(e^{-t}-1)}.\, e^{\lambda_2(e^{-t}-1)} \qquad [\because M_{cX}(t) = M_X(ct)]$$

which cannot be put in the form $e^{\lambda(e^t-1)}$. Hence $(X_1 - X_2)$ is not a Poisson variate.

Moreover, the difference $(X_1 - X_2)$ cannot be a Poisson variate is evident from the fact that it may have positive as well as negative values, while a Poisson variate is always non-negative.

8.5.9. Probability Generating Function of Poisson Distribution.

$$\text{P.G.F. of } X = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \cdot s^k = \sum_{k=0}^{\infty} e^{-\lambda} \frac{(\lambda s)^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{\lambda(s-1)} \quad \dots (8\cdot21)$$

Example 8·32. *A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantees that not more than 10 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality ?*

Solution. In the usual notations, we are given : $n = 100$,

and p = Probability of a defective pin = 5% = 0·05

(Since 'p' is small, we may use Poisson distribution.)

$\therefore$ λ = Mean number of defective pins = $np = 100 \times 0{\cdot}05 = 5$

Let the random variable X denote the number of defective pins in a box of 100. Then by Poisson probability law, the probability of x defective pins in a box is :

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-5}5^x}{x!};\ x = 0, 1, 2, \dots$$

Probability that a box will fail to meet the guaranteed quality is :

$$P(X > 10) = 1 - P(X \le 10) = 1 - \sum_{x=0}^{10} \frac{e^{-5}5^x}{x!} = 1 - e^{-5} \sum_{x=0}^{10} \frac{5^x}{x!}$$

Example 8·33. *A car hire firm has two cars, which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1·5. Calculate the proportion of days on which (i) neither car is used, and (ii) the proportion of days on which some demand is refused.*

Solution. Here the random variable X, which denotes the number of demands for a car on any day follows Poisson distribution with mean $\lambda = 1{\cdot}5$. The proportion of days on which there are x demands for a car is given by :

$$P(X = x) = \frac{e^{-1\cdot5}(1{\cdot}5)^x}{x!};\ x = 0, 1, 2, \dots$$

(i) Proportion of days on which neither car is used is given by :

$$P(X = 0) = e^{-1\cdot5} = \left\{1 - 1{\cdot}5 + \frac{(1{\cdot}5)^2}{2!} - \frac{(1{\cdot}5)^3}{3!} + \frac{(1{\cdot}5)^4}{4!} - \dots\right\} = 0{\cdot}2231$$

(ii) Proportion of days on which some demand is refused is :

$$P(X > 2) = 1 - P(X \le 2) = 1 - \{P(X = 0) + P(X = 1) + P(X = 2)\}$$

$$= 1 - e^{-1\cdot5}\left\{1 + 1{\cdot}5 + \frac{(1{\cdot}5)^2}{2!}\right\} = 1 - 0{\cdot}2231 \times 3{\cdot}625 = 0{\cdot}19126.$$

Example 8·34. *An insurance company insures 4,000 people against loss of both eyes in a car accident. Based on previous data, the rates were computed on the assumption that on the average 10 persons in 1,00,000 will have car accident each year that result in this type of injury. What is the probability that more than 3 of the insured will collect on their policy in a given year ?*

Solution. In usual notations, we are given : n = 4,000, and

p = Probability of loss of both eyes in a car accident $= \frac{10}{1{,}00{,}000} = 0{\cdot}0001.$

Since p is very small and n is large, we may approximate the given distribution by Poisson distribution. Thus the parameter λ of the Poisson distribution is :

$$\lambda = np = 4{,}000 \times 0{\cdot}0001 = 0{\cdot}4$$

Let the random variable X denote number of car accidents in the batch of 4,000 people. Then by Poisson probability law :

$$P(X = x) = \frac{e^{-\lambda} \cdot \lambda^x}{x\,!} = \frac{e^{-0\cdot4}\,(0{\cdot}4)^x}{x\,!}\,;\ x = 0, 1, 2, \ldots \qquad \ldots (*)$$

Hence the required probability that more than 3 of the insured will collect on their policy is given by :

$$P\,(X > 3) = 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]$$

$$= 1 - e^{-0\cdot4}\left\{(0{\cdot}4)^0 + (0{\cdot}4) + \frac{(0{\cdot}4)^2}{2\,!} + \frac{(0{\cdot}4)^3}{3\,!}\right\}$$

$$= 1 - 0{\cdot}6703\,(1 + 0{\cdot}4 + 0{\cdot}08 + 0{\cdot}0107) = 1 - 0{\cdot}6703 \times 1{\cdot}4907 = 0{\cdot}0008.$$

Example 8·35. *A manufacturer, who produces medicine bottles, finds that 0·1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 100 boxes from the producer of bottles. Using Poisson distribution, find how many boxes will contain :*

(i) no defective, and (ii) at least two defectives. [*Given* $e^{-0\cdot5} = 0{\cdot}6065$]

Solution. In the usual notations, we are given :

$N = 100$, $n = 500$, p= Probability of a defective bottle $= 0{\cdot}001$, and $\lambda = np = 500 \times 0{\cdot}001 = 0{\cdot}5$.

Let the random variable X denote the number of defective bottles in a box of 500. Then by Poisson probability law, the probability of x defective bottles in a box is given by :

$$P(X = x) = \frac{e^{-0\cdot5}\,(0{\cdot}5)^x}{x\,!} = \frac{0{\cdot}6065 \times (0{\cdot}5)^x}{x\,!}\,;\ x = 0, 1, 2, \ldots$$

Hence in a consignment of 100 boxes, the frequency (number) of boxes containing x defective bottles is :

$$f(x) = N.P\,(X = x) = \frac{100 \times 0{\cdot}6065 \times (0{\cdot}5)^x}{x\,!}$$

(*i*) Number of boxes containing no defective bottle

$$= 100 \times P\,(X = 0) = 100 \times 0{\cdot}6065 \simeq 61$$

(*ii*) Number of boxes containing at least two defective bottles

$$= 100[P(X \geq 2)] = 100\,\{1 - P\,(X = 0) - P(X = 1)\}$$

$$= 100(1 - 0{\cdot}6065 - 0{\cdot}6065 \times 0{\cdot}5) = 100 \times 0{\cdot}09025 \simeq 9.$$

Example 8·36. *Six coins are tossed 6,400 times. Using the Poisson distribution, find the approximate probability of getting six heads r times.*

Solution. The probability of obtaining six heads in one throw of six coins (*a single* trial), is $p = (1/2)^6$, assuming that head and tail are equally probable.

$$\therefore \qquad \lambda = np = 6{,}400 \times (1/2)^6 = 100$$

Hence, using Poisson probability law, the required probability of getting 6 heads r times is given by :

$$P(X = r) = \frac{e^{-\lambda}.\lambda^r}{r!} = \frac{e^{-100}.(100)^r}{r!} ; r = 0, 1, 2, \ldots$$

Example 8·37. *In a book of 520 pages, 390 typo-graphical errors occur. Assuming Poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.*

Solution. The average number of typographical errors per page in the book is given by $\lambda = (390/520) = 0{\cdot}75$

Hence using Poisson probability law, the probability of x errors per page is given by :

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.75}(0{\cdot}75)^x}{x!} ; x = 0, 1, 2, \ldots$$

The required probability that a random sample of 5 pages will contain no error is given by :

$$[P(X = 0)]^5 = (e^{-0.75})^5 = e^{-3.75}.$$

Example 8·38. *In a Poisson frequency distribution, frequency corresponding to 3 successes is 2/3 times frequency corresponding to 4 successes. Find the mean and standard deviation of the distribution.*

Solution. Let X be a random variable following Poisson distribution with parameter λ. Then the frequency function is given by :

$$f(x) = N.\,p(x) = NP(X = x) = N.\frac{e^{-\lambda}.\lambda^x}{x!} ; x = 0, 1, 2, \ldots \qquad \ldots (*)$$

Putting $x = 3$ and 4 in (*), $f(3) = N.\dfrac{e^{-\lambda}\lambda^3}{3!}$ and $f(4) = N.\dfrac{e^{-\lambda}\lambda^4}{4!}$

We are given : $f(3) = \frac{2}{3} f(4) \Rightarrow N.\dfrac{e^{-\lambda}\lambda^3}{3!} = \dfrac{2}{3} N.\dfrac{e^{-\lambda}\lambda^4}{4!}$

$$\Rightarrow \quad \frac{1}{3!} = \frac{2}{3}.\frac{\lambda}{4!} \quad \Rightarrow \quad \lambda = \frac{1}{3!} \times \frac{3}{2} \times 4! = 6.$$

Mean of the Poisson distribution $= \lambda = 6$ and *s.d.* of the distribution $= \sqrt{\lambda} = \sqrt{6}$.

Example 8·39. *A manager accepts the work submitted by his typist only when there is no mistake in the work. The typist has to type on an average 20 letters per day of about 200 words each. Find the chance of her making a mistake : (i) if less than 1% of the letters submitted by her are rejected, (ii) if on 90% days all the letters submitted by her are accepted. [As the probability of making a mistake is small, you may use Poisson Distribution. Take e = 2·72.]*

Solution. (*i*) Assuming that the number of mistakes per page (X) follows Poisson distribution with parameter λ, probability of no mistake in a page is $p(0) = e^{-\lambda}$.

If less than 1% of the letters are rejected, then more than 99% of the letters are accepted, *i.e.*, probability of making no mistakes in a page is at least 0·99.

Hence $\qquad e^{-\lambda} \geq 0{\cdot}99.$

But $\lambda = np = 200p$, where p is the probability of making mistake in typing a word.

$\therefore \qquad e^{-200p} \geq 0{\cdot}99 \quad \Rightarrow \quad -200p.\,(\log 2{\cdot}72) \geq \log 0{\cdot}99$

$$\Rightarrow \quad -p\,(200 \times 0.4346) \geq -0.0044 \quad \Rightarrow \quad p \leq \frac{0.0044}{86.92} = 0.0000506.$$

(ii) The day's work of 20 letters of 200 words each is accepted, when there is no mistake in any of the $n = 20 \times 200 = 4{,}000$ words. Assuming Poisson distribution, probability of no mistake in the day's work $= e^{-\lambda}$, where $\lambda = np = 4{,}000p$. We want to find p such that :

$$\therefore \quad e^{-4000p} = 0.90 \quad \Rightarrow \quad -4{,}000\,p\,(\log 2.72) = \log 0.90$$

$$\Rightarrow \quad -p\,(4{,}000 \times 0.4346) = -0.0458 \quad \Rightarrow \quad p = \frac{0.0458}{1738.4} = 0.0000263.$$

Example 8·40. *Suppose that the number of telephone calls coming into a telephone exchange between 10 A.M. and 11 A.M. say, X_1 is a random variable with Poisson distribution with parameter 2. Similarly the number of calls arriving between 11 A.M. and 12 noon, say, X_2 has a Poisson distribution with parameter 6. If X_1 and X_2 are independent, what is the probability that more than 5 calls come in-between 10 A.M. and 12 noon ?*

Solution. We are given : $X_1 \sim P(2)$ and $X_2 \sim P(6)$. Let $X = X_1 + X_2$. By the additive property of Poisson distribution, X is also a Poisson variate with parameter (say) $\lambda = 2 + 6 = 8$.

Hence the probability of x calls in-between 10 A.M. and 12 noon is given by :

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{x^{-8}\, 8^x}{x!};\ x = 0, 1, 2, \ldots$$

Probability that more than 5 calls come in-between 10 A.M. and 12 noon is :

$$P(X > 5) = 1 - P\,(X \leq 5) = 1 - \sum_{x=0}^{5} \frac{e^{-8}\, 8^x}{x!} = 1 - 0.1912 = 0.8088.$$

Example 8·41. *A Poisson distribution has a double mode at $x = 1$ and $x = 2$. What is the probability that x will have one or the other of these two values ?*

Solution. We know that if the Poisson distribution is bimodal, then the two modes are at the points $x = \lambda - 1$ and $x = \lambda$, where λ is the parameter of the Poisson distribution. Therefore, since we are given that the two modes are at the points $x = 1$ and $x = 2$, we find that $\lambda = 2$.

$$\therefore \quad P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-2}\, 2^x}{x!};\ x = 0, 1, 2, \ldots$$

$$\Rightarrow \quad P(X = 1) = e^{-2}\, 2 \quad \text{and} \quad P(X = 2) = \frac{e^{-2}.2^2}{2!} = e^{-2}\,.\,2.$$

Required probability $= P\,(X = 1) + P\,(X = 2) = 2e^{-2} + 2e^{-2} = 0.542$

Example 8·42. *If X is a Poisson variate such that*

$$P(X = 2) = 9P(X = 4) + 90\, P\,(X = 6) \qquad \ldots(*)$$

Find (i) λ, (ii)the mean of X, (iii) β_1, the coefficient of skewness.

Solution. If X is a Poisson variate with parameter λ, then

$$P(X = x) = \frac{e^{-\lambda}.\lambda^x}{x!};\ x = 0, 1, 2, \ldots, \lambda > 0$$

Hence (*) gives

$$\frac{e^{-\lambda}.\lambda^2}{2!} = e^{-\lambda}\left(9\,\frac{\lambda^4}{4!} + 90\,\frac{\lambda^6}{6!}\right) = \frac{e^{-\lambda}\lambda^2}{8}(3\,\lambda^2 + \lambda^4) \quad \Rightarrow \quad \lambda^4 + 3\,\lambda^2 - 4 = 0$$

Solving as a quadratic in λ^2, $\lambda^2 = \dfrac{-3 \pm \sqrt{9+16}}{2} = \dfrac{-3 \pm 5}{2}$. Since $\lambda > 0$, $\lambda^2 = 1 \Rightarrow \lambda = 1$.

Hence, Mean $= \lambda = 1$, and $\beta_1 =$ Coefficient of skewness $= \dfrac{1}{\lambda} = 1$.

Example 8·43. *If X and Y are independent Poisson variates such that*

$$P(X=1) = P(X=2) \quad \textit{and} \quad P(Y=2) = P(Y=3) \qquad \ldots (*)$$

Find the variance of $X - 2Y$.

Solution. Let $X \sim P(\lambda)$ and $Y \sim P(\mu)$. Then we have

$$P(X=x) = \frac{e^{-\lambda}.\lambda^x}{x!}, x = 0, 1, 2, \ldots ; \lambda > 0 \quad \text{and} \quad P(Y=y) = \frac{e^{-\mu}.\mu^y}{y!}, y = 0, 1, 2, \ldots ; \mu > 0$$

Using (*),
$$\lambda e^{-\lambda} = \frac{\lambda^2 e^{-\lambda}}{2!} \quad \text{and} \quad \frac{\mu^2 e^{-\mu}}{2} = \frac{\mu^3 e^{-\mu}}{3!} \qquad \ldots (**)$$

Solving (**), we obtain

$$\lambda e^{-\lambda}(\lambda - 2) = 0 \quad \text{and} \quad \mu^2 e^{-\mu}(\mu - 3) = 0 \Rightarrow \lambda = 2 \text{ and } \mu = 3, \text{ since } \lambda > 0, \mu > 0.$$

Now $\text{Var}(X) = \lambda = 2$, and $\text{Var}(Y) = \mu = 3$ $\qquad \ldots (***)$

$\therefore$ $\text{Var}(X - 2Y) = 1^2\,\text{Var}(X) + (-2)^2 . \text{Var}\,Y$,

covariance term vanishes since X and Y are independent.

Hence, on using (***), we get $\text{Var}(X - 2Y) = 2 + 4 \times 3 = 14$.

Example 8·44. *If X and Y are independent Poisson variates with means* λ_1 *and* λ_2 *respectively, find the probability that (i)* $X + Y = k$, and *(ii)* $X = Y$.

Solution. We have : $X \sim P(\lambda_1)$ and $Y \sim P(\lambda_2)$. Hence

$$P(X=x) = \frac{e^{-\lambda_1}\lambda_1^x}{x!}, x = 0, 1, 2, 3, \ldots ; \lambda_1 > 0 \quad \text{and} \quad P(Y=y) = \frac{e^{-\lambda_2}\lambda_2^y}{y!}, y = 0, 1, 2, 3, \ldots ; \lambda_2 > 0$$

(i)
$$P(X+Y=k) = \sum_{r=0}^{k} P(X = r \cap Y = k - r)$$

$$= \sum_{r=0}^{k} P(X=r)\,P(Y=k-r) \qquad [\because X \text{ and } Y \text{ are independent}]$$

$$= \sum_{r=0}^{k} \frac{e^{-\lambda_1}\lambda_1^r}{r!} . \frac{e^{-\lambda_2}\lambda_2^{k-r}}{(k-r)!} = e^{-(\lambda_1+\lambda_2)} \sum_{r=0}^{k} \frac{\lambda_1^r \lambda_2^{k-r}}{r!\,(k-r)!}$$

$$= e^{-(\lambda_1+\lambda_2)} \left[\frac{\lambda_2^k}{k!} + \frac{\lambda_1 \lambda_2^{k-1}}{1!\,(k-1)!} + \frac{\lambda_1^2 \lambda_2^{k-2}}{2!\,(k-2)!} + \ldots + \frac{\lambda_1^k}{k!} \right]$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \left[\lambda_2^k + {}^kC_1\,\lambda_2^{k-1}.\lambda_1 + {}^kC_2\,\lambda_2^{k-2}.\lambda_1^2 + \ldots + \lambda_1^k \right]$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \times (\lambda_1 + \lambda_2)^k ; k = 0, 1, 2, \ldots$$

which is the probability function of Poisson distribution with parameter $\lambda_1 + \lambda_2$.

Aliter. Since $X \sim P(\lambda_1)$ and $Y \sim P(\lambda_2)$ are independent, by the additive property of Poisson distribution $X + Y \sim P(\lambda_1 + \lambda_2)$. Hence

$$P(X + Y = k) = \frac{e^{-(\lambda_1 + \lambda_2)} \times (\lambda_1 + \lambda_2)^k}{k\,!}\,;\ k = 0, 1, 2, \dots$$

(ii) $$P(X = Y) = \sum_{r=0}^{\infty} P(X = r \cap Y = r)$$

$$= \sum_{r=0}^{\infty} P(X = r)\, P(Y = r) = e^{-(\lambda_1 + \lambda_2)} \sum_{r=0}^{\infty} \frac{(\lambda_1 \lambda_2)^r}{(r\,!)^2}$$

[$\because$ X and Y are independent]

Example 8·45. *Show that in Poisson distribution with unit mean, mean deviation about mean is (2/e) times the standard deviation.*

Solution. Let $X \sim P(\lambda)$. We are given : Mean $= \lambda = 1$.

$$\therefore \quad P(X = x) = \frac{e^{-\lambda}\lambda^x}{x\,!} = \frac{e^{-1}.\,1}{x\,!} = \frac{e^{-1}}{x\,!}\,;\ x = 0, 1, 2, \dots$$

Mean deviation about mean 1 is :

$$E(|X - 1|) = \sum_{x=0}^{\infty} |x - 1|\, P(X = x) = e^{-1} \sum_{x=0}^{\infty} \frac{|x - 1|}{x\,!} = e^{-1}\left(1 + \frac{1}{2\,!} + \frac{2}{3\,!} + \dots\right)$$

We have $$\frac{n}{(n+1)\,!} = \frac{(n+1) - 1}{(n+1)\,!} = \frac{1}{n\,!} - \frac{1}{(n+1)\,!}$$

$\therefore$ Mean deviation about mean

$$= e^{-1}\left\{1 + \left(1 - \frac{1}{2\,!}\right) + \left(\frac{1}{2\,!} - \frac{1}{3\,!}\right) + \left(\frac{1}{3\,!} - \frac{1}{4\,!}\right) + \dots\right\}$$

$$= e^{-1}(1 + 1) = \frac{2}{e} \times 1 = \frac{2}{e} \times \text{standard deviation},$$

since for the Poisson distribution, variance = mean = 1 (given) $\Rightarrow$ *s.d.* = 1.

Example 8·46. *Let $X_1, X_2, \dots, X_n$ be identically and independently distributed Bin (1, p) variates. Let $S_n = \sum_{j=1}^{n} X_j$, be a binomial (n, p) variate and $M_n(t)$ be the m.g.f. of S_n. Find*

$$\lim_{n \to \infty} M_n(t), \text{ using } np = \lambda \text{ (constant)}.$$

Solution. Since X_i, $i = 1, 2, \dots, n$ are *i.i.d.* binomial variates $B(1, p)$,

$$S_n = \sum_{j=1}^{n} X_j, \text{ is a binomial } B(n, p) \text{ variate.}$$

$$\therefore \quad M_n(t) = m.g.f. \text{ of } S_n = (q + pe^t)^n = \left[1 + (e^t - 1)p\right]^n$$

If we take $np = \lambda \quad \Rightarrow \quad p = \lambda/n$ and let $n \to \infty$, we get

$$\lim_{n\to\infty} M_n(t) = \lim_{n\to\infty} \left[1 + \frac{(e^t - 1)\,\lambda}{n}\right]^n = \exp\left[\lambda\,(e^t - 1)\right],$$

which is the *m.g.f.* of Poisson distribution with parameter λ.

Hence by uniqueness theorem of *m.g.f.*, $S_n = \sum_{j=1}^{n} X_j \to P(\lambda)$, as $n \to \infty$, with $np = \lambda$ (fixed).

Example 8·47. *(a) If X is a Poisson variate with mean m, show that the expectation of e^{-kX} is $\exp[-m(1-e^{-k})]$. Hence show that if $\bar{X}$ is the arithmetic mean of n independent random variables $X_1, X_2, \ldots, X_n$, each having Poisson distribution with parameter m, then $\exp(-\bar{X})$ as an estimate of e^{-m} is biased, although $\bar{X}$ is an unbiased estimate of m.*

(b) If X is a Poisson variate with mean m, what would be the expectation of $\exp(-kX)\,kX$, k being a constant.

Solution.

$$E(e^{-kX}) = \sum_{x=0}^{\infty} e^{-kx}\, p(x) = \sum_{x=0}^{\infty} e^{-kx} \cdot \frac{e^{-m} m^x}{x!}$$

$$= e^{-m} \sum_{x=0}^{\infty} \frac{(me^{-k})^x}{x!} = e^{-m}\left\{1 + me^{-k} + \frac{(me^{-k})^2}{2!} + \ldots\right\}$$

$$= \exp(-m) \cdot \exp(m e^{-k}) = \exp[-m(1-e^{-k})] \qquad \ldots (*)$$

We have $$E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i).$$

Since $X_i;\ i = 1, 2, \ldots, n$ are independent Poisson variates with parameter m, $E(X_i) = m$.

$$\therefore \quad E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} m = \frac{1}{n} nm = m.$$

Hence $\bar{X}$ is an unbiased estimate of m.

$$E[\exp(-\bar{X})] = E\left[\exp\left(\frac{-1}{n}\sum_{i=1}^{n} X_i\right)\right] = E(e^{-X_1/n} \cdot e^{-X_2/n} \ldots e^{-X_n/n})$$

$$= E(e^{-X_1/n})\, E(e^{-X_2/n}) \ldots E(e^{-X_n/n}),$$

(since $X_1, X_2, \ldots, X_n$ are independent) $\ldots (**)$

$$\therefore \quad E[\exp(-\bar{X})] = \prod_{i=1}^{n} E(e^{-X_i/n})$$

Using (*) with $k = 1/n$, we get

$$E(e^{-X_i/n}) = \exp[-m(1-e^{-1/n})] \text{(Since } X_i \text{ is a Poisson variate with parameter } m.)$$

$$\therefore E[\exp(-\bar{X})] = \prod_{i=1}^{n} [\exp\{-m(1-e^{-1/n})\}] \left[\exp\{-m(1-e^{-1/n})\}\right]^n$$

$$= \exp\{-mn(1-e^{-1/n})\} \neq e^{-m}$$

Hence $e^{-\bar{X}}$ is not an unbiased estimated of e^{-m}, though $\bar{X}$ is an unbiased estimate of m.

(b) $$E(e^{-kX}\,kX) = \sum_{x=0}^{\infty} e^{-kx}\, kx \cdot p(x, m) = k\sum_{x=1}^{\infty} e^{-kx}\, x\, \frac{e^{-m} m^x}{x!}$$

$$= ke^{-m} \sum_{x=1}^{\infty} \frac{(me^{-k})^x}{(x-1)!} = ke^{-m}\, me^{-k} \sum_{x=1}^{\infty} \frac{(me^{-k})^{x-1}}{(x-1)!}$$

$$= mke^{-m-k}\left\{1 + me^{-k} + \frac{(me^{-k})^2}{2!} + \ldots\right\}$$

$$= mk\, e^{-m-k} \cdot e^{me^{-k}} = mk \exp[\{m(e^{-k}-1)\} - k].$$

Consider the incomplete gamma integral ;

$$I_x = \frac{1}{x!}\int_{\lambda}^{\infty} e^{-t}\, t^x\, dt \;;\; (x \text{ is positive integer}) = \left| -\frac{e^{-t}t^x}{x!}\right|_{\lambda}^{\infty} + \frac{1}{(x-1)!}\int_{\lambda}^{\infty} e^{-t}\, t^{x-1}\, dt$$

$$= \frac{e^{-\lambda}\lambda^x}{x!} + I_{x-1} \qquad \ldots (**)$$

which is a reduction formula for I_x.

Repeated applications of (**) gives

$$I_x = \frac{e^{-\lambda}\lambda^x}{x!} + \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} + \ldots + \frac{e^{-\lambda}\lambda}{1!} + I_0$$

$$\text{But } I_0 = \int_{\lambda}^{\infty} e^{-t}\, dt = \left| -e^{-t}\right|_{\lambda}^{\infty} = e^{-\lambda}$$

$$\therefore \quad I_x = e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2!} + \ldots + \frac{\lambda^x}{x!}e^{-\lambda} = P(X=0) + P(X=1) + \ldots + P(X=x)$$

$$= P(X \le x) = F(x), \qquad \text{[From (***) and (*)]}$$

where $F(\cdot)$ is the distribution function of the r.v. X.

$$\Rightarrow \quad F(x) = \frac{1}{x!}\int_{\lambda}^{\infty} e^{-t}\, t^x\, dt = \frac{1}{\Gamma(x+1)}\int_{\lambda}^{\infty} e^{-t}\, t^x\, dt$$

($\because \Gamma(x+1) = x\,!$, since x is a positive integer.)

Remark. This result is of great practical utility. It enables us to represent the cumulative Poisson probabilities (which are generally tedious to compute numerically) in terms of incomplete gamma integral, the values of which are tabulated for different values of λ by Karl Pearson in his Tables of Incomplete Γ-Functions.

8·5·10. Recurrence Formula for the Probabilities of Poisson Distribution (*Fitting of Poisson Distribution*). For a Poisson distribution with parameter λ, we have

$$p(x) = \frac{e^{-\lambda}.\lambda^x}{x!};\; x = 0, 1, 2, \ldots, \infty \quad \text{and} \quad p(x+1) = \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!};\; x = 0, 1, 2, \ldots, \infty$$

$$\therefore \quad \frac{p(x+1)}{p(x)} = \frac{\lambda}{(x+1)} \quad \Rightarrow \quad p(x+1) = \frac{\lambda}{x+1}\, p(x), \qquad \ldots (8·21)$$

which is the required recurrence formula.

This formula provides us a very convenient method of graduating the given frequency distribution by a Poisson distribution. The only probability we need to calculate is $p(0)$ which is given by $p(0) = e^{-\lambda}$, where λ is estimated from the given frequency distribution by equating the mean of the distribution to λ. The other probabilities, *viz.*, $p(1), p(2) \ldots$ can now be easily obtained as explained below :

$$p(1) = [p(x+1)]_{x=0} = \left[\frac{\lambda}{x+1}\right]_{x=0} p(0),$$

$$p(2) = [p(x+1)]_{x=1} = \left[\frac{\lambda}{x+1}\right]_{x=1} p(1),$$

$$p(3) = [p(x+1)]_{x=2} = \left[\frac{\lambda}{x+1}\right]_{x=2} p(2), \text{ and so on.}$$

Example 8·55. *Fit a Poisson distribution to the following data :*

Number of mistakes per page :	0	1	2	3	4	*Total*
Number of pages :	109	65	22	3	1	200

Solution. If the above distribution is approximated by a Poisson distribution, then the parameter of Poisson distribution is given by :

λ = Mean =

$$m = \frac{0 \times 109 + 1 \times 65 + 2 \times 22 + 3 \times 3 + 4 \times 1}{200} = 0{\cdot}61.$$

By Poisson probability law, the frequency (number) of pages containing r mistakes is given by :

$$f(r) = N \,.\, p(r) = 200 \times \frac{e^{-0{\cdot}61}(0{\cdot}61)^r}{r\,!}$$

Putting $r = 0, 1, 2, \ldots$, we get the expected frequencies of Poisson distribution.

Also, $$p(0) = e^{-0{\cdot}61} = (2{\cdot}71828)^{-0{\cdot}61} = \frac{1}{\text{Antilog }(0{\cdot}61 \times \log 2{\cdot}71828)}$$

$$= \frac{1}{\text{Antilog }(0{\cdot}61 \times 0{\cdot}4343)} = \frac{1}{\text{Antilog }(0{\cdot}26492)} = \frac{1}{1{\cdot}841} = 0{\cdot}5432.$$

CALCULATIONS FOR EXPECTED POISSON FREQUENCIES

No. of Mistakes	*Expected Frequency*				
0	$f(0) = 200 \times e^{-0{\cdot}61}$	$= 200$	$\times\, 0{\cdot}5432$	$= 108{\cdot}64$	$\simeq 108$
1	$f(1) = 200 \times e^{-0{\cdot}61} \times 0{\cdot}61$	$= 108{\cdot}64$	$\times\, 0{\cdot}61$	$= 66{\cdot}27$	$\simeq 66$
2	$f(2) = 200 \times e^{-0{\cdot}61} \times \frac{(0{\cdot}61)^2}{2\,!}$	$= 66{\cdot}27$	$\times \frac{0{\cdot}61}{2}$	$= 20{\cdot}21$	$\simeq 20$
3	$f(3) = 200 \times e^{-0{\cdot}61} \times \frac{(0{\cdot}61)^3}{3\,!}$	$= 20{\cdot}21$	$\times \frac{0{\cdot}61}{3}$	$= 4{\cdot}11$	$\simeq 4$
4	$f(4) = 200 \times e^{-0{\cdot}61} \times \frac{(0{\cdot}61)^4}{4\,!}$	$= 4{\cdot}11$	$\times \frac{0{\cdot}61}{4}$	$= 0{\cdot}63$	$\simeq 1$

Example 8·56. *After correcting 50 pages of the proof of a book, the proof reader finds that there are, on the average, 2 errors per 5 pages. How many pages would one expect to find with 0, 1, 2, 3 and 4 errors, in 1,000 pages of the first print of the book ? (Given that $e^{-0{\cdot}4} = 0{\cdot}6703$)*

Solution. Let the random variable X denote the number of errors per page. Then the mean number of errors per page is given by : $\lambda = \frac{2}{5} = 0{\cdot}4$.

Using Poisson probability law, probability of x errors per page is given by :

$$P(X = x) = p(x) = \frac{e^{-\lambda}\lambda^x}{x\,!} = \frac{e^{-0{\cdot}4}(0{\cdot}4)^x}{x\,!}\,;\; x = 0, 1, 2, \ldots$$

Expected number of pages with x errors per page in a book of $N = 1{,}000$ pages are :

$$f(x) = N \times P(X = x) = 1{,}000 \times \frac{e^{-0{\cdot}4}(0{\cdot}4)^x}{x\,!}\,;\; x = 0, 1, 2, \ldots$$

Using the recurrence formula (8·21), various probabilities can be easily calculated as shown in the following table :

CALCULATIONS FOR EXPECTED POISSON FREQUENCIES

No. of errors per page (x)	Probability $p(x)$	Expected number of pages, $f(x) = Np(x)$
0	$p(0) = e^{-0.4} = 0.6703$	$670.3 \simeq 670$
1	$p(1) = \frac{0.4}{0+1} p(0) = 0.26812$	$268.12 \simeq 268$
2	$p(2) = \frac{0.4}{1+1} p(1) = 0.053624$	$53.624 \simeq 54$
3	$p(3) = \frac{0.4}{2+1} p(2) = 0.0071298$	$7.1298 \simeq 7$
4	$p(4) = \frac{0.4}{3+1} p(3) = 0.00071298$	$0.71298 \simeq 1$

Example 8·57. *Fit a Poisson distribution to the following data which gives the number of doddens in a sample of clover seeds.*

No. of doddens (x)	:	0	1	2	3	4	5	6	7	8
Observed frequency (f)	:	56	156	132	92	37	22	4	0	1

Solution. $\text{Mean} = \frac{1}{N} \sum fx = \frac{986}{500} = 1.972$

Taking the mean of the given distribution as the mean of the Poisson distribution we want to fit, we get $\lambda = 1.972$, and

$$p(x) = \frac{e^{-\lambda} . \lambda^x}{x!} ; x = 0, 1, 2, \dots, \infty \quad \Rightarrow \quad p(0) = e^{-\lambda} = e^{-1.972}, \text{ so that}$$

$$\log_{10} p(0) = -1.972 \log_{10} e = -1.972 \times 0.43429 = -0.856419 = \bar{1}.143581$$

$$\therefore \qquad p(0) = \text{Antilog } (\bar{1}.1436) = 0.1392$$

Using the recurrence formula (8·21), the various probabilities, *viz.*, $p(1)$, $p(2)$, ..., can be easily calculated as shown in the following table :

CALCULATIONS FOR EXPECTED POISSON FREQUENCIES

x	$\frac{\lambda}{x+1}$	$p(x)$	Expected frequency $N.p(x) = 500.\ p(x)$
0	1·972	0·13920	$69.6000 \simeq 70$
1	0·986	0·27455	$137.2512 \simeq 137$
2	0·657	0·27006	$135.3296 \simeq 135$
3	0·493	0·17793	$88.9566 \simeq 89$
4	0·394	0·10964	$43.8556 \simeq 44$
5	0·328	0·03459	$17.2966 \simeq 17$
6	0·281	0·01137	$5.6846 \simeq 6$
7	0·247	0·00320	$1.6013 \simeq 2$
8	0·219	0·00078	$0.3942 \simeq 0$

Remark. In rounding the figures to the nearest integer it has to be kept in mind that the total of the observed and the expected frequencies should be same.