18·1. INTRODUCTION

The main problems in statistical inference can be broadly classified into two areas :

(i) The area of estimation of population parameter(s) and setting up of confidence intervals for them, *i.e.*, the area of *point and interval estimation* and

(ii) *Tests of statistical hypothesis.*

The first topic has already been discussed in Chapter 17. In this chapter we shall discuss:

(*a*) The theory of testing of hypothesis initiated by J. Neyman and E.S. Pearson (§ 18·2),

(*b*) Sequential analysis propounded by A. Wald (§ 18·8) and

(*c*) Non-parametric tests (§ 18·7).

In Neyman-Pearson theory, we use statistical methods to arrive at decisions in certain situations where there is lack of certainty on the basis of a sample whose size is fixed in advance while in Wald's sequential theory the sample size is not fixed but is regarded as a random variable. Before taking up a detailed discussion of the topics in (*a*), (*b*) and (*c*), we shall explain below certain concepts which are of fundamental importance.

18·2. STATISTICAL HYPOTHESIS — SIMPLE AND COMPOSITE

A statistical hypothesis is some statement or assertion about a population or equivalently about the probability distribution characterising a population, which we want to verify on the basis of information available from a sample. If the statistical hypothesis specifies the population completely then it is termed as a *simple statistical hypothesis* otherwise it is called a *composite statistical hypothesis.*

For example, if $X_1, X_2, ..., X_n$ is a random sample of size n from a normal population with mean μ and variance σ^2, then the hypothesis $H_0 : \mu = \mu_0, \sigma^2 = \sigma_0^2$ is a simple hypothesis, whereas each of the following hypotheses is a composite hypothesis:

(*i*) $\mu = \mu_0$, (*ii*) $\sigma^2 = \sigma_0^2$, (*iii*) $\mu < \mu_0, \sigma^2 = \sigma_0^2$, (*iv*) $\mu > \mu_0, \sigma^2 = \sigma_0^2$

(*v*) $\mu = \mu_0, \sigma^2 < \sigma_0^2$, (*vi*) $\mu = \mu_0, \sigma^2 > \sigma_0^2$, (*vii*) $\mu < \mu_0, \sigma^2 > \sigma_0^2$.

A hypothesis which does not specify completely 'r' parameters of a population is termed as a *composite hypothesis with r degrees of freedom.*

18·2·1. Test of a Statistical Hypothesis. *A test of a statistical hypothesis is a two-action decision problem after the experimental sample values have been obtained, the two-actions being the acceptance or rejection of the hypothesis under consideration.*

18·2·2. Null Hypothesis. In hypothesis testing, a statistician or decision-maker should not be motivated by prospects of profit or loss resulting from the acceptance or rejection of the hypothesis. He should be completely impartial and should have no brief for any party or company nor should he allow his personal views to influence the decision. *Much, therefore, depends upon how the hypothesis is framed.* For example, let us consider the 'light-bulbs' problem. Let us suppose that the bulbs manufactured under some standard manufacturing process have an average life of μ hours and it is proposed to test a new procedure for manufacturing light bulbs. Thus, we have two populations of bulbs, those manufactured by standard process and those manufactured by the new process. In this problem the following three hypotheses may be set up :

(*i*) New process is better than standard process.

(*ii*) New process is inferior to standard process.

(*iii*) There is no difference between the two processes.

The first two statements appear to be biased since they reflect a preferential attitude to one or the other of the two processes. Hence the best course is to adopt *the hypothesis of no difference,* as stated in (*iii*). This suggests that the *statistician should take up the neutral or null attitude regarding the outcome of the test.* His attitude should be on the null or zero line in which the experimental data has the due importance and complete say in the matter. *This neutral or non-committal attitude of the statistician or decision-maker before the sample observations are taken is the keynote of the null hypothesis.*

Thus in the above example of light bulbs if μ_0 is the mean life (in hours) of the bulbs manufactured by the new process then the null hypothesis which is usually denoted by H_0, can be stated as follows : $H_0 : \mu = \mu_0$.

As another example let us suppose that two different concerns manufacture drugs for inducing sleep, drug A manufactured by first concern and drug B manufactured by second concern. Each company claims that its drug is superior to that of the other and it is desired to test which is a superior drug A or B ? To formulate the statistical hypothesis let X be a random variable which denotes the additional hours of sleep gained by an individual when drug A is given and let the random variable Y denote the additional hours of sleep gained when drug B is used. Let us suppose that X and Y follow the probability distributions with means μ_X and μ_Y respectively. Here our null *hypothesis would be that there is no difference between the effects of two drugs.* Symbolically, $H_0 : \mu_X = \mu_Y$.

18·2·3. Alternative Hypothesis. It is desirable to state what is called an *alternative hypothesis* in respect of every statistical hypothesis being tested because the acceptance or rejection of null hypothesis is meaningful only when it is being tested against a rival hypothesis which should rather be explicitly mentioned. *Alternative hypothesis* is usually denoted by H_1. For example, in the example of light bulbs, alternative hypothesis could be $H_1 : \mu > \mu_0$ or $\mu < \mu_0$ or $\mu \neq \mu_0$. In the example of drugs, the alternative hypothesis could be $H_1 : \mu_X > \mu_Y$ or $\mu_X < \mu_Y$ or $\mu_X \neq \mu_Y$.

In both the cases, the first two of the alternative hypotheses give rise to what are called *'one tailed'* tests and the third alternative hypothesis results in *'two tailed'* tests.

Important Remarks 1. In the formulation of a testing problem and devising a 'test of hypothesis' the roles of H_0 and H_1 are not at all symmetric. In order to decide which one of the two hypotheses should be taken as null hypothesis H_0 and which one as alternative hypothesis H_1, the intrinsic difference between the roles and the implifications of these two terms should be clearly understood.

2. If a particular problem cannot be stated as a test between two simple hypotheses, *i.e.*, simple null hypothesis against a simple alternative hypothesis, then the next best alternative is to formulate the problem as the test of a simple null hypothesis against a composite alternative hypothesis. In other words, one should try to structure the problem so that null hypothesis is simple rather than composite.

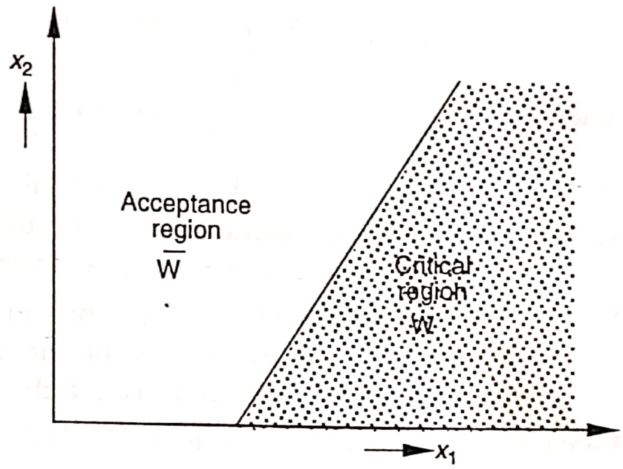
3. Keeping in mind the potential losses due to wrong decisions (which may or may not be measured in terms of money), the decision maker is somewhat conservative in holding the null hypothesis as true unless there is a strong evidence from the experimental sample observations that it is false. To him, the *consequences of wrongly rejecting a null hypothesis seem to be more severe than those of wrongly accepting it.* In most of the cases, the statistical hypothesis is in the form of a claim that a particular product or product process is superior to some existing standard. The null hypothesis H_0 in this case is that there is no difference between the new product or production process and the existing standard. In other words, null hypothesis nullifies this claim. The rejection of the null hypothesis wrongly which amounts to the acceptance of claim wrongly involves huge amount of pocket expenses towards a substantive overhaul of the existing set-up. The resulting loss is comparatively regarded as more serious than the opportunity loss in wrongly accepting H_0 which amounts to wrongly rejecting the claim, *i.e.*, in sticking to the less efficient existing standard. In the light-bulbs problem discussed earlier, suppose the research division of the concern, on the basis of the limited experimentation, claims that its brand is more effective than that manufactured by standard process. If in fact, the brand fails to be more effective the loss incurred by the concern due to an immediate *obsolescence* of the product, decline of the concern's image, etc., will be quite serious. On the other hand, the failure to bring out a superior brand in the market is an opportunity loss and is not a consideration to be as serious as the other loss.

18·2·4. Critical Region. Let $x_1, x_2, \ldots, x_n$ be the sample observations denoted by

O. All the values of O will be aggregate of a sample and they constitute a space, called *the sample space,* which is denoted by S.

Since the sample values $x_1, x_2, \ldots, x_n$ can be taken as a point in n-dimensional space, we specify some region of the n-dimensional space and see whether this point lies within this region or outside this region. We divide the whole sample space S into two disjoint parts W and $S - W$ or $\overline{W}$ or W'. The null hypothesis H_0 is rejected if the observed sample point falls in W and if it falls in W' we reject H_1 and accept H_0. *The region of rejection of H_0 when H_0 is true is that region of the outcome set where H_0 is rejected if the sample point falls in that region and is called critical region.* Evidently, the size of the critical region is α, the probability of committing type 1 error (discussed below).

Suppose if the test is based on a sample of size 2, then the outcome set or the sample space is the first quadrant in a two-dimensional space and a test criterion will enable us to separate our outcome set into two complementary subsets, W and $\overline{W}$. If the sample point falls in the subset W, H_0 is rejected, otherwise H_0 is accepted. This is shown in the adjoining diagram :



18·2·5. Two Types of Errors. The decision to accept or reject the null hypothesis H_0 is made on the basis of the information supplied by the observed sample observations. The conclusion drawn on the basis of a particular sample may not always be true in respect of the population. The four possible situations that arise in any test procedure are given in the following table.

DOUBLE DICHOTOMY RELATING TO DECISION AND HYPOTHESIS

		Decision From Sample	
		Reject H_0	*Accept H_0*
True State	*H_0 True*	Wrong (Type I Error)	Correct
	H_0 False (*H_1 True*)	Correct	Wrong (Type II Error)

From the above table it is obvious that in any testing problem we are liable to commit two types of errors.

Errors of Type I and Type II. The error of rejecting H_0 (accepting H_1) when H_0 is true is called *Type I error* and the error of accepting H_0 when H_0 is false (H_1 is true) is called *Type II* error. The probabilities of type I and type II errors are denoted by α and β respectively. Thus

$$\alpha = \text{Probability of type I error}$$
$$= \text{Probability of rejecting } H_0 \text{ when } H_0 \text{ is true.}$$

β = Probability of type II error
= Probability of accepting H_0 when H_0 is false.

Symbolically:

$$P(\mathbf{x} \in W \mid H_0) = \alpha, \text{ where } \mathbf{x} = (x_1, x_2, \ldots, x_n) \quad \Rightarrow \quad \int_W L_0 \, d\mathbf{x} = \alpha \quad \ldots(18\cdot1)$$

where L_0 is the likelihood function of the sample observations under H_0 and $\int d\mathbf{x}$ represents the n-fold integral $\int\int \ldots \int dx_1 \, dx_2 \ldots dx_n$.

Again $$P(\mathbf{x} \in \overline{W} \mid H_1) = \beta \quad \Rightarrow \quad \int_{\overline{W}} L_1 \, d\mathbf{x} = \beta \quad \ldots(18\cdot2)$$

where L_1 is the likelihood function of the sample observations under H_1. Since

$$\int_W L_1 \, d\mathbf{x} + \int_{\overline{W}} L_1 \, d\mathbf{x} = 1,$$

we get $$\int_W L_1 \, d\mathbf{x} = 1 - \int_{\overline{W}} L_1 \, d\mathbf{x} = 1 - \beta \quad \ldots(18\cdot2a)$$

$$\Rightarrow \quad P(\mathbf{x} \in W \mid H_1) = 1 - \beta \quad \ldots(18\cdot2b)$$

18·2·6. Level of Significance. α, the probability of type 1 error, is known as the level of significance of the test. It is also called the *size of the critical region.*

18·2·7. Power of the Test. $1 - \beta$, defined in (18·2*a*) and (18·2*b*) is called the *power function of the test hypothesis H_0 against the alternative hypothesis H_1*. The value of the power function at a *parameter* point is called the *power of the test* at that point.

Remarks 1. In quality control terminology, α and β are termed as *producer's risk and consumer's risk*, respectively.

2. An ideal test would be the one which properly keeps under control both the types of errors. But since the commission of an error of either type is a random variable, equivalently an ideal test should minimise the probability of both the types of errors, *viz.*, α and β. But unfortunately, for a fixed sample size n, α and β are so related (like producer's and consumer's risk in sampling inspection plans), that the reduction in one results in an increase in the other. Consequently, the simultaneous minimising of both the errors is not possible. Since type I error is deemed to be more serious than the type II error (*c.f.* Remark 3 § 18·2·3) the usual practice is to control α at a predetermined low level and subject to this constraint on the probabilities of type I error, *choose a test which minimises β or maximises the power function $1 - \beta$*. Generally, we choose $\alpha = 0{\cdot}05$ or $0{\cdot}01$.

18·3. STEPS IN SOLVING TESTING OF HYPOTHESIS PROBLEM

The major steps involved in the solution of a 'testing of hypothesis' problem may be outlined as follows :

1. Explicit knowledge of the nature of the population distribution and the parameter(s) of interest, *i.e.*, the parameter(s) about which the hypotheses are set up.

2. Setting up of the null hypothesis H_0 and the alternative hypothesis H_1 in terms of the range of the parameter values each one embodies.

3. The choice of a suitable statistic $t = t(x_1, x_2, \ldots, x_n)$ called the *test statistic* which will best reflect upon the probability of H_0 and H_1.

4. Partitioning the set of possible values of the test statistic t into two disjoint sets W (called the *rejection region* or *critical region*) and $\overline{W}$ (called the *acceptance region*) and framing the following test :

(i) Reject H_0 (*i.e.*, accept H_1) if the value of t falls in W.

(ii) Accept H_0 if the value of t falls $\overline{W}$.

5. After framing the above test, obtain experimental sample observations, compute the appropriate test statistic and take action accordingly.

18·4. OPTIMUM TEST UNDER DIFFERENT SITUATIONS

The discussion in § 18·3 and Remark 2, § 18·2·7 enables us to obtain the so called best test under different situations. In any testing problem the first two steps, *viz.*, the form of the population distribution, the parameter(s) of interest and the framing of H_0 and H_1 should be obvious from the description of the problem. The most crucial step is the choice of the '*best test*, *i.e.*, the best statistic 't' and the critical region W where *by best test we mean one which in addition to controlling α at any desired low level has the minimum type II error β or maximum power $1-\beta$, compared to β of all other tests having this* α. This leads to the following definition.

18·4·1. Most Powerful Test (MP Test). Let us consider the problem of testing a simple hypothesis : $H_0 : \theta = \theta_0$

against a simple alternative hypothesis : $H_1 : \theta = \theta_1$

> **Definition.** *The critical region W is the most powerful (MP) critical region of size α (and the corresponding test a most powerful test of level α) for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ if*
>
> $$P(x \in W \mid H_0) = \int_W L_0 \, dx = \alpha \qquad \ldots (18·3)$$
>
> and $$P(x \in W \mid H_1) \geq P(x \in W_1 \mid H_1) \qquad \ldots (18·3a)$$

for every other critical region W_1 satisfying (18·3).

18·4·2. Uniformly Most Powerful Test (UMP Test). Let us now take up the case of testing a simple null hypothesis against a composite alternative hypothesis, *e.g.*, of testing $H_0 : \theta = \theta_0$

against the alternative $H_1 : \theta \neq \theta_0$

In such a case, for a predetermined α, the best test for H_0 is called the *uniformly most powerful test of level* α.

> **Definition.** *The region W is called uniformly most powerful (UMP) critical region of size α [and the corresponding test as uniformly most powerful (UMP) test of level α] for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ i.e., $H_1 : \theta = \theta_1 \neq \theta_0$ if*
>
> $$P(x \in W \mid H_0) = \int_W L_0 \, dx = \alpha \qquad \ldots\ldots(18·4)$$
>
> and $$P(x \in W \mid H_1) \geq P(x \in W_1 \mid H_1) \text{ for all } \theta \neq \theta_0 \qquad \ldots\ldots(18·4a)$$
>
> *whatever the region W_1 satisfying (18·4) may be.*

18·5. NEYMAN J. AND PEARSON, E.S. LEMMA

This Lemma provides the most powerful test of simple hypothesis against a simple alternative hypothesis. The theorem, known as Neyman-Pearson Lemma, will be proved for density function $f(x, \theta)$ of a single continuous variate and a single parameter. However, by regarding x and θ as vectors, the proof can be easily generalised for any number of random variables $x_1, x_2, \ldots, x_n$ and any number of parameters $\theta_1, \theta_2, ..., \theta_k$. The variables $x_1, x_2, ..., x_n$ occurring in this theorem are understood to represent a random sample of size n from the population whose density function is $f(x, \theta)$. The lemma is concerned with a simple hypothesis $H_0 : \theta = \theta_0$ and a simple alternative $H_1 : \theta = \theta_1$.

Neyman Pearson Lemma *Let $k > 0$, be a constant and W be a critical region of size α such that*

$$W = \left\{ x \in S : \frac{f(x, \theta_1)}{f(x, \theta_0)} > k \right\}$$

$$\Rightarrow \qquad W = \left\{ x \in S : \frac{L_1}{L_0} > k \right\} \qquad \cdots (18\cdot5)$$

$$\text{and} \qquad \overline{W} = \left\{ x \in S : \frac{L_1}{L_0} < k \right\} \qquad \cdots (18\cdot5a)$$

where L_0 and L_1 are the likelihood functions of the sample observations $x = (x_1, x_2, ..., x_n)$ under H_0 and H_1 respectively. Then W is the most powerful critical region of the test hypothesis $H_0 : \theta = \theta_0$ against the alternative $H_1 : \theta = \theta_1$.

Proof. We are given

$$P(x \in W \mid H_0) = \int_W L_0 \, dx = \alpha. \qquad \text{—} (18\cdot6)$$

The power of the region is

$$P(x \in W \mid H_1) = \int_W L_1 \, dx = 1 - \beta, \text{ (say)}. \qquad \text{—} (18\cdot6a)$$

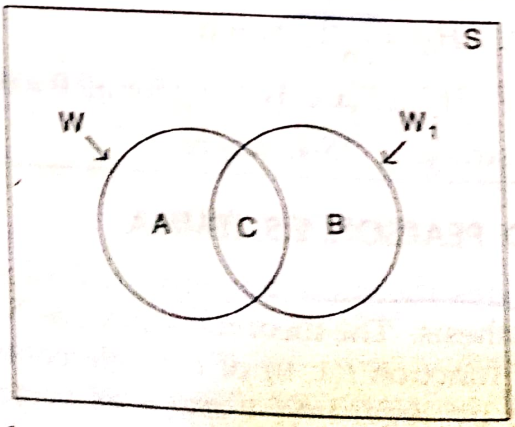
In order to establish the lemma, we have to prove that there exists no other critical region, of size less than or equal to α, which is more powerful than W. Let W_1 be another critical region of size $\alpha_1 \leq \alpha$ and power $1 - \beta_1$ so that we have

$$P(x \in W_1 \mid H_0) = \int_{W_1} L_0 \, dx = \alpha_1 \qquad \text{—} (18\cdot7)$$

$$\text{and} \qquad P(x \in W_1 \mid H_1) = \int_{W_1} L_1 \, dx = 1 - \beta_1 \qquad \text{—} (18\cdot7a)$$

Now we have to prove that $1 - \beta \geq 1 - \beta_1$

Let $\qquad W = A \cup C$ and $W_1 = B \cup C$



(C may be empty, *i.e.*, *W* and W_1 may be disjoint).

If $\alpha_1 \lneqq \alpha$, we have

$$\int_{W_1} L_0\,dx \le \int_W L_0\,dx$$

$$\Rightarrow \quad \int_{B\cup C} L_0\,dx \le \int_{A\cup C} L_0\,dx$$

$$\Rightarrow \quad \int_B L_0\,dx \le \int_A L_0\,dx$$

$$\Rightarrow \quad \int_A L_0\,dx \ge \int_B L_0\,dx \qquad \ldots (18{\cdot}8)$$

Since $A \subset W$,

$$(18{\cdot}5) \quad \Rightarrow \quad \int_A L_1\,dx > k\int_A L_0\,dx \ge k\int_B L_0\,dx \qquad \text{[Using (18·8)]} \quad \ldots(18{\cdot}8a)$$

Also [18·5 (a)] implies

$$\frac{L_1}{L_0} \le k \;\forall\, x \in \overline{W}$$

$$\Rightarrow \quad \int_{\overline{W}} L_1\,dx \le k\int_{\overline{W}} L_0\,dx$$

This result also holds for any subset of $\overline{W}$, say $\overline{W} \cap W_1 = B$. Hence

$$\int_B L_1\,dx \le k\int_B L_0\,dx \le \int_A L_1\,dx \qquad \text{[From (18·8a)]}$$

Adding $\int_C L_1\,dx$ to both sides, we get

$$\int_{W_1} L_1\,dx \le \int_W L_1\,dx \quad \Rightarrow \quad 1-\beta \ge 1-\beta_1$$

Hence the Lemma.

Remark. Let W defined in (18·5) of the above theorem be the most powerful critical region of size α for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, and let it be independent of $\theta_1 \in \Theta_1 = \Theta - \Theta_0$, where Θ_0 is the parameter space under H_0. Then we say that C.R. W is the UMP CR of size α for testing : $H_0 : \theta = \theta_0$, against $H_1 : \theta \in \Theta_1$.

18·5·1. Unbiased Test and Unbiased Critical Region. Let us consider the testing of $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$: *The critical region W and consequently the test based on it is said to be unbiased if the power of the test exceeds the size of the critical region, i.e., if*

$$\text{Power of the test} \ge \text{Size of the C.R.} \qquad \ldots (18{\cdot}9)$$

$$\Rightarrow \quad 1-\beta \ge \alpha$$

$$\Rightarrow \quad P_{\theta_1}(W) \ge P_{\theta_0}(W)$$

$$\Rightarrow \quad P[\mathbf{x} : \mathbf{x} \in W \mid H_1] \ge P[\mathbf{x} : \mathbf{x} \in W \mid H_0] \qquad \ldots (18{\cdot}9a)$$

In other words, the critical region W is said to be unbiased if

$$P_\theta(W) \ge P_{\theta_0}(W), \;\forall\, \theta\,(\ne \theta_0) \in \Theta \qquad \ldots\ldots (18{\cdot}9b)$$

Theorem 18·2. *Every most powerful (MP) or uniformly most powerful (UMP) critical region (CR) is necessarily unbiased.*

(a) *If W be an MPCR of size α for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, then it is necessarily unbiased.*

(b) *Similarly if W be UMPCR of size α for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \in \Theta_1$, then it is also unbiased.*

Proof. (*a*) Since W is the MPCR of size α for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, by Neyman Pearson Lemma, we have ; for $\forall\, k > 0$,

$$W = \{x : L(x, \theta_1) \geq k\, L(x, \theta_0)\} = \{x : L_1 \geq k\, L_0\}$$

and $$W' = \{x : L(x, \theta_1) < k\, L(x, \theta_0)\} = \{x : L_1 < k\, L_0\},$$

where k is determined so that the size of the test is α *i.e.*,

$$P_{\theta_0}(W) = P[x \in W \mid H_0] = \int_W L_0\, dx = \alpha \qquad ...(i)$$

To prove that W is unbiased, we have to show that :

Power of $W \geq \alpha$ *i.e.*, $P_{\theta_1}(W) \geq \alpha$ ...(ii)

We have : $$P_{\theta_1}(W) = \int_W L_1\, dx \geq k \int_W L_0\, dx = k\alpha$$

[$\because$ On W, $L_1 \geq k L_0$ and Using (*i*)]

i.e., $$P_{\theta_1}(W) \geq k\alpha,\ \forall\, k > 0 \qquad ...(iii)$$

Also

$$1 - P_{\theta_1}(W) = 1 - P(x \in W \mid H_1) = P(x \in W' \mid H_1) = \int_{W'} L_1\, dx$$

$$< k \int_{W'} L_0\, dx = k\, P(x : x \in W' \mid H_0) \qquad [\because \text{On } W', L_1 < k L_0]$$

$$= k[1 - P(x : x \in W \mid H_0)] = k(1 - \alpha)$$

i.e., $$1 - P_{\theta_1}(W) \leq k(1-\alpha),\ \forall\, k > 0 \qquad \text{[Using (i)]} \quad ...(iv)$$

Case (*i*) $k \geq 1$. If $k \geq 1$, then from (*iii*), we get

$$P_{\theta_1}(W) \geq k\alpha \geq \alpha$$

$\Rightarrow$ W is unbiased CR.

Case (*ii*) $0 < k < 1$. If $0 < k < 1$, then from (*iv*), we get :

$$1 - P_{\theta_1}(W) < 1 - \alpha \Rightarrow P_{\theta_1}(W) > \alpha \Rightarrow W \text{ is unbiased C.R.}$$

Hence MP critical region is unbiased.

(*b*) If W is UMPCR of size α then also the above proof holds if for θ_1 we write θ such that $\theta \in \Theta_1$. So we have

$$P_\theta(W) > \alpha,\ \forall\, \theta \in \Theta_1 \Rightarrow W \text{ is unbiased CR.}$$

18·5·2. Optimum Regions and Sufficient Statistics. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a population with p.m.f. or p.d.f. $f(x, \theta)$, where the parameter θ may be a vector. Let T be a sufficient statistic for θ. Then by Factorization Theorem,

$$L(x, \theta) = \prod_{i=1}^{n} f(x_i, \theta) = g_\theta(t(x)) .\ h(x) \qquad ...(*)$$

where $g_\theta(t(x))$ is the marginal distribution of the statisitc $T = t(x)$.

By Neyman Pearson Lemma, the MPCR for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ is given by :

$$W = \{x : L(x, \theta_1) \geq k\, L(x, \theta_0)\},\ \forall\, k > 0 \qquad ...(**)$$

From (*) and (**), we get

$$W = \{x : g_{\theta_1}(t(x)) \cdot h(x) \geq k \cdot g_{\theta_0}(t(x)) \cdot h(x)\}, \forall k > 0$$
$$= \{x : g_{\theta_1}(t(x)) \geq k \cdot g_{\theta_0}(t(x))\}, \forall k > 0$$

Hence if $T = t(x)$ is sufficient statistic for θ then the MPCR for the test may be defined in terms of the marginal distribution of $T = t(x)$, rather than the joint distribution of $x_1, x_2, \ldots, x_n$.

Example 18·1. *Given the frequency function :*

$$f(x, \theta) = \begin{cases} \dfrac{1}{\theta}, & 0 \leq x \leq \theta \\ 0, & \text{elsewhere} \end{cases}$$

and that you are testing the null hypothesis $H_0 : \theta = 1$ against $H_1 : \theta = 2$, by means of a single observed value of x. What would be the sizes of the type 1 and type II errors, if you choose the interval (i) $0{\cdot}5 \leq x$, (ii) $1 \leq x \leq 1{\cdot}5$ as the critical regions ? Also obtain the power function of the test.

Solution. Here we want to test $H_0 : \theta = 1$, against $H_1 : \theta = 2$.

(i) Here $\quad W = \{x : 0{\cdot}5 \leq x\} = \{x : x \geq 0{\cdot}5\}$

and $\quad \overline{W} = \{x : x \leq 0{\cdot}5\}$

$$\alpha = P\{x \in W \mid H_0\} = P\{x \geq 0{\cdot}5 \mid \theta = 1\} = P\{0{\cdot}5 \leq x \leq \theta \mid \theta = 1\}$$
$$= P\{0{\cdot}5 \leq x \leq 1 \mid \theta = 1\} = \int_{0{\cdot}5}^{1} [f(x, \theta)]_{\theta=1}\, dx = \int_{0{\cdot}5}^{1} 1.\, dx = 0{\cdot}5$$

Similarly, $\quad \beta = P\{x \in \overline{W} \mid H_1\} = P\{x \leq 0{\cdot}5 \mid \theta = 2\}$

$$= \int_0^{0{\cdot}5} [f(x, \theta)]_{\theta=2}\, dx = \int_0^{0{\cdot}5} \frac{1}{2}\, dx = 0{\cdot}25$$

Thus the sizes of type *I* and type *II* errors are respectively $\alpha = 0{\cdot}5$ and $\beta = 0{\cdot}25$ and power function of the test $= 1 - \beta = 0{\cdot}75$

(ii) $\quad W = \{x : 1 \leq x \leq 1{\cdot}5\}$

$$\alpha = P\{x \in W \mid \theta = 1\} = \int_1^{1{\cdot}5} [f(x, \theta)]_{\theta=1}\, dx = 0,$$

since under $H_0 : \theta = 1, f(x, \theta) = 0$, for $1 \leq x \leq 1{\cdot}5$.

$$\beta = P\{x \in \overline{W} \mid \theta = 2\} = 1 - P\{x \in W \mid \theta = 2\}$$
$$= 1 - \int_1^{1{\cdot}5} [f(x, \theta)]_{\theta=2}\, dx = 1 - \left| \frac{x}{2} \right|_1^{1{\cdot}5} = 0{\cdot}75$$

$\therefore$ Power Function $= 1 - \beta = 1 - 0{\cdot}75 = 0{\cdot}25$

Example 18·2. *If $x \geq 1$ is the critical region for testing $H_0 : \theta = 2$ against the alternative $\theta = 1$, on the basis of the single observation from the population,*

$$f(x, \theta) = \theta \exp(-\theta x), 0 \leq x < \infty,$$

obtain the values of type I and type II errors.

Solution. Here $\quad W = \{x : x \geq 1\}$ and $\overline{W} = \{x : x < 1\}$ and $H_0 : \theta = 2,\ H_1 : \theta = 1.$

α = Size of Type I error $= P[x \in W \mid H_0] = P[x \geq 1 \mid \theta = 2]$

$$= \int_1^\infty [f(x, \theta)]_{\theta = 2}\, dx = 2\int_1^\infty e^{-2x}\, dx = 2\left|\frac{e^{-2x}}{-2}\right|_1^\infty = \frac{1}{e^2}$$

$$\beta = \text{Size of type II error} = P[x \in \bar{W} \mid H_1\} = P\{x < 1 \mid \theta = 1\}$$

$$= \int_0^1 e^{-x}\, dx = \left|\frac{e^{-x}}{-1}\right|_0^1 = (1 - e^{-1}) = \frac{e-1}{e}.$$

Imp

Example 18·3. *Let p be the probability that a coin will fall head in a single toss in order to test $H_0 : p = \frac{1}{2}$ against $H_1 : p = \frac{3}{4}$. The coin is tossed 5 times and H_0 is rejected if more than 3 heads are obtained. Find the probability of type I error and power of the test.*

Solution. Here $H_0 : p = \frac{1}{2}$ and $H_1 : p = \frac{3}{4}$.

If the *r.v.* X denotes the number of heads in *n* tosses of a coin then $X \sim B(n, p)$ so that

$$P(X = x) = {}^nC_x\, p^x (1-p)^{n-x} = {}^5C_x\, p^x (1-p)^{5-x} \quad \ldots (*)$$

The critical region is given by : $W = \{x : x \geq 4\} \Rightarrow \bar{W} = \{x : x \leq 3\}$

$$\alpha = \text{Probability of type I error} = P[X \geq 4 \mid H_0]$$

$$= P\left(X = 4 \mid p = \tfrac{1}{2}\right) + P\left(X = 5 \mid p = \tfrac{1}{2}\right) = {}^5C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^{5-4} + {}^5C_5\left(\tfrac{1}{2}\right)^5 \quad \text{[From (*)]}$$

$$= 5\left(\tfrac{1}{2}\right)^5 + \left(\tfrac{1}{2}\right)^5 = 6\left(\tfrac{1}{2}\right)^5 = \frac{3}{16}$$

$$\beta = \text{Probability of Type II error} = P(x \in \bar{W} \mid H_1) = 1 - P(x \in W \mid H_1)$$

$$= 1 - \left[\left\{P\left(X = 4 \mid p = \tfrac{3}{4}\right) + P\left(X = 5 \mid p = \tfrac{3}{4}\right) = 1 - \left\{{}^5C_4\left(\tfrac{3}{4}\right)^4\left(\tfrac{1}{4}\right) + {}^5C_5\left(\tfrac{3}{4}\right)^5\right\}\right.\right.$$

$$= 1 - \left(\tfrac{3}{4}\right)^4\left\{\tfrac{5}{4} + \tfrac{3}{4}\right\} = 1 - \frac{81}{128} = \frac{47}{128}$$

$\therefore$ Power of the test $= 1 - \beta = \dfrac{81}{128}$.

Example 18·4. *Let $X \sim N(\mu, 4)$, μ unknown. To test $H_0 : \mu = -1$ against $H_1 : \mu = 1$, based on a sample of size 10 from this population, we use the critical region : $x_1 + 2x_2 + \ldots + 10x_{10} \geq 0$. What is its size ? What is the power of the test ?*

Solution. Critical Region $W = \{\mathbf{x} : x_1 + 2x_2 + \ldots + 10x_{10} \geq 0\}$.

Let $U = x_1 + 2x_2 + \ldots + 10x_{10}$

Since x_i's are *i.i.d.* $N(\mu, 4)$,

$$U \sim N[(1 + 2 + \ldots + 10)\,\mu,\ (1^2 + 2^2 + \ldots + 10^2)\,\sigma^2] = N(55\mu, 385\sigma^2)$$

$$\Rightarrow \quad U \sim N(55\mu, 385 \times 4) = N(55\mu, 1540) \quad \ldots (*)$$

The size 'α' of the critical region is : $\alpha = P(\mathbf{x} \in W \mid H_0) = P(U \geq 0 \mid H_0)$ ...(**)

Under $H_0 : \mu = -1$, $U \sim N(-55, 1540)$ [From (*)] $\Rightarrow Z = \dfrac{U - E(U)}{\sigma_U} = \dfrac{U + 55}{\sqrt{1540}}$

$\therefore$ Under H_0, when $U = 0$, $Z = \dfrac{55}{\sqrt{1540}} = \dfrac{55}{39{\cdot}2428} = 1{\cdot}4015$

$$\therefore \quad \alpha = P(Z \geq 1{\cdot}4015) = 0{\cdot}5 - P(Z \leq 1{\cdot}4015) \quad \text{[From (**)]}$$

$$= 0{\cdot}5 - 0{\cdot}4192 = 0{\cdot}1808 \quad \text{(From Normal Probability Tables)}$$

Alternatively, $\alpha = 1 - P(Z \le 1{\cdot}4015) = 1 - \Phi(1{\cdot}4015)$,

where $\Phi(\cdot)$ is the distribution function of standard normal variate.

Power of the test is : $1 - \beta = P(x \in W \mid H_1) = P(U \ge 0 \mid H_1)$

Under $H_1 : \mu = 1,\ U \sim N(55, 1540)$

$$\Rightarrow \quad Z = \frac{U - E(U)}{\sigma_U} = \frac{-55}{\sqrt{1540}} = -1{\cdot}40 \qquad \text{(when } U = 0\text{)}$$

$$\therefore \quad 1 - \beta = P(Z \ge -1{\cdot}40) = P(-1{\cdot}4 \le Z \le 0) + 0{\cdot}5$$
$$= P(0 \le Z \le 1{\cdot}4) + 0{\cdot}5 \qquad \text{(By symmetry)}$$
$$= 0{\cdot}4192 + 0{\cdot}5 = 0{\cdot}9192$$

Alternatively, $1 - \beta = 1 - P(Z \le -1{\cdot}40) = 1 - \Phi(-1{\cdot}40)$,

where $\Phi(.)$ is the distribution function of standard normal variate.

Example 18·5. *Let X have a p.d.f. of the form :*

$$f(x, \theta) = \begin{cases} \dfrac{1}{\theta} e^{-x/\theta} ; 0 < x < \infty,\ \theta > 0 \\ 0 \quad , \text{elsewhere.} \end{cases}$$

To test $H_0 : \theta = 2$, against $H_1 : \theta = 1$, use the random sample x_1, x_2 of size 2 and define a critical region : $W = \{(x_1, x_2) : 9{\cdot}5 \le x_1 + x_2\}$.

Find : (i) Power of the test.

(ii) Significance level of the test.

Solution. We are given the critical region :

$$W = \{(x_1, x_2) : 9{\cdot}5 \le x_1 + x_2\} = \{(x_1\ x_2) : x_1 + x_2 \ge 9{\cdot}5\}$$

Size of the critical region *i.e.*, the significance level of the test is given by :

$$\alpha = P(x \in W \mid H_0) = P[x_1 + x_2 \ge 9{\cdot}5 \mid H_0] \qquad \ldots(*)$$

In sampling from the given exponential distribution,

$$\frac{2}{\theta}\sum_{i=1}^{n} x_i \sim \chi^2_{(2n)} \quad \Rightarrow \quad U = \frac{2}{\theta}(x_1 + x_2) \sim \chi^2_{(4)},\ (n = 2) \qquad [\text{c.f. Example 18·8}]$$

$$\therefore \quad \alpha = P\left[\frac{2}{\theta}(x_1 + x_2) \ \ge\ \frac{2}{\theta} \times 9{\cdot}5 \mid H_0\right] \qquad [\text{From } (*)]$$
$$= P[\chi^2_{(4)} \ge 9{\cdot}5] \qquad (\because \text{Under } H_0,\ \theta = 2)$$
$$\Rightarrow \quad \alpha = 0{\cdot}05 \qquad [\text{From Probability Tables of } \chi^2\text{-distribution}]$$

Power of the test is given by

$$1 - \beta = P(x \in W \mid H_1) = P(x_1 + x_2 \ge 9{\cdot}5 \mid H_1)$$
$$= P\left[\frac{2}{\theta}(x_1 + x_2) \ge \frac{2}{\theta} \times 9{\cdot}5 \mid H_1\right]$$
$$= P[\chi^2_{(4)} \ge 19] \qquad (\because \text{Under } H_1,\ \theta = 1)$$

Example 18·6. *Use the Neyman-Pearson Lemma to obtain the region for testing $\theta = \theta_0$ against $\theta = \theta_1 > \theta_0$ and $\theta = \theta_1 < \theta_0$, in the case of a normal population $N(\theta, \sigma^2)$, where σ^2 is known. Hence find the power of the test.*

Solution.

$$L = \prod_{i=1}^{n} f(x_i, \theta) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2\right\}$$

Using Neyman-Pearson Lemma, best critical region (B.C.R.) is given by (for $k > 0$)

$$\frac{L_1}{L_0} = \frac{\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta_1)^2\right\}}{\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta_0)^2\right\}} \geq k$$

$$\Rightarrow \quad \exp\left[-\frac{1}{2\sigma^2}\left\{\sum_{i=1}^{n}(x_i-\theta_1)^2 - \sum_{i=1}^{n}(x_i-\theta_0)^2\right\}\right] \geq k$$

$$\Rightarrow \quad \exp\left[-\frac{n}{2\sigma^2}(\theta_1^2-\theta_0^2) + \frac{1}{\sigma^2}(\theta_1-\theta_0)\sum_{i=1}^{n}x_i\right] \geq k$$

$$\Rightarrow \quad -\frac{n}{2\sigma^2}(\theta_1^2-\theta_0^2) + \frac{1}{\sigma^2}(\theta_1-\theta_0)\sum_{i=1}^{n}x_i \geq \log k$$

(since $\log x$ is an increasing function of x)

$$\Rightarrow \quad \bar{x}(\theta_1-\theta_0) \geq \frac{\sigma^2}{n}\log k + \frac{\theta_1^2-\theta_0^2}{2}$$

Case (*i*) If $\theta_1 > \theta_0$, the B.C.R. is determined by the relation (right-tailed test):

$$\bar{x} > \frac{\sigma^2}{n}\cdot\frac{\log k}{\theta_1-\theta_0} + \frac{\theta_1+\theta_0}{2}$$

$$\Rightarrow \quad \bar{x} > \lambda_1, \text{ (say).}$$

$\therefore$ B.C.R. is: $W = \{\mathbf{x} : \bar{x} > \lambda_1\}$ ...(18·10)

Case (*ii*) If $\theta_1 < \theta_0$, the B.C.R. is given by the relation (left handed test)

$$\bar{x} < \frac{\sigma^2}{n}\cdot\frac{\log k}{\theta_1-\theta_0} + \frac{\theta_1+\theta_0}{2} = \lambda_2, \text{ (say).}$$

Hence B.C.R. is: $W_1 = \{\mathbf{x} : \bar{x} \leq \lambda_2\}$ ...(18·11)

The constants λ_1 and λ_2 are so chosen as to make the probability of each of the relations (18·10) and (18·11) equal to α when the hypothesis H_0 is true. The sampling distribution of $\bar{x}$, when H_i is true is $N\left(\theta_i, \frac{\sigma^2}{n}\right)$, $(i = 0, 1)$. Therefore, the constants λ_1 and λ_2 are determined from the relations:

$$P[\bar{x} > \lambda_1 \mid H_0] = \alpha \quad \text{and} \quad P[\bar{x} < \lambda_2 \mid H_0] = \alpha$$

$$\therefore \quad P(\bar{x} > \lambda_1 \mid H_0) = P\left[Z > \frac{\lambda_1-\theta_0}{\sigma/\sqrt{n}}\right] = \alpha\,; \; Z \sim N(0, 1)$$

$$\Rightarrow \quad \frac{\lambda_1-\theta_0}{\sigma/\sqrt{n}} = z_\alpha \quad \Rightarrow \quad \lambda_1 = \theta_0 + \frac{\sigma}{\sqrt{n}}z_\alpha \qquad ...(18·12)$$

where z_α is the upper α-point of the standard normal variate given by:

$$P(Z > z_\alpha) = \alpha \qquad ...(*)$$

Also $\quad P(\bar{x} < \lambda_2 \mid H_0) = \alpha \quad \Rightarrow \quad P(\bar{x} \geq \lambda_2 \mid H_0) = 1 - \alpha$

$$\Rightarrow \quad P\left(Z \geq \frac{\lambda_2 - \theta_0}{\sigma/\sqrt{n}}\right) = 1 - \alpha \quad \Rightarrow \quad \frac{\lambda_2 - \theta_0}{\sigma/\sqrt{n}} = z_{1-\alpha}$$

$$\Rightarrow \quad \lambda_2 = \theta_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha} \qquad \ldots(18\cdot12a)$$

Note. By symmetry of normal distribution, we have $z_{1-\alpha} = -z_\alpha$.

Power of the test. By definition, the power of the test in case (*i*) is :

$$1 - \beta = P[\mathbf{x} \in W \mid H_1] = P[\bar{x} \geq \lambda_1 \mid H_1]$$

$$= P\left(Z \geq \frac{\lambda_1 - \theta_1}{\sigma/\sqrt{n}}\right) \qquad \left[\because \text{Under } H_1, Z = \frac{\bar{x} - \theta_1}{\sigma/\sqrt{n}} \sim N(0, 1)\right]$$

$$= P\left(Z \geq \frac{\theta_0 + \frac{\sigma}{\sqrt{n}} z_\alpha - \theta_1}{\sigma/\sqrt{n}}\right) \qquad [\text{Using } (18\cdot12)]$$

$$= P\left(Z \geq z_\alpha - \frac{\theta_1 - \theta_0}{\sigma/\sqrt{n}}\right) \qquad (\because \theta_1 > \theta_0)$$

$$= 1 - P(Z \leq \lambda_3) \qquad \left\{\lambda_3 = z_\alpha - \frac{\theta_1 - \theta_0}{\sigma/\sqrt{n}}, \text{ say.}\right\}$$

$$= 1 - \Phi(\lambda_3), \qquad \ldots(18\cdot13)$$

where $\Phi(.)$ is the distribution function of standard normal variate.

Similarly in case (*ii*), $(\theta_1 < \theta_0)$, the power of the test is

$$1 - \beta = P(\bar{x} < \lambda_2 \mid H_1) = P\left(Z < \frac{\lambda_2 - \theta_1}{\sigma/\sqrt{n}}\right)$$

$$= P\left(Z < \frac{\theta_0 + \frac{\sigma}{\sqrt{n}} z_{1-\alpha} - \theta_1}{\sigma/\sqrt{n}}\right) \qquad [\text{Using } (18\cdot12a)]$$

$$= P\left(Z < z_{1-\alpha} + \frac{\theta_0 - \theta_1}{\sigma/\sqrt{n}}\right) = \Phi(\lambda_4), \qquad (\because \theta_0 > \theta_1) \ldots(18\cdot13a)$$

where $\quad \lambda_4 = z_{1-\alpha} + \dfrac{\sqrt{n}(\theta_0 - \theta_1)}{\sigma} = \dfrac{\sqrt{n}(\theta_0 - \theta_1)}{\sigma} - z_\alpha \qquad \ldots(18\cdot13b)$

UMP Critical Region. (18·10) provides best critical region for testing $H_0 : \theta = \theta_0$ against the hypothesis $\theta = \theta_1$, provided $\theta_1 > \theta_0$ while (18·11) defines the best critical region for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, provided $\theta_1 < \theta_0$. Thus, the best critical region for testing simple hypothesis $H_0 : \theta = \theta_0$ against the simple hypothesis $\theta = \theta_1 + c$, $c > 0$ will not serve as best critical region for testing simple hypothesis $H_0 : \theta = \theta_0$ against simple alternative hypothesis $H_1 : \theta = \theta_0 - c, c > 0$.

Hence in this problem, no uniformly most powerful test exists for testing the simple hypothesis, $H_0 : \theta = \theta_0$ against the composite alternative hypothesis, $H_1 : \theta \neq \theta_0$.

However, for each alternative hypothesis, $H_1 : \theta = \theta_1 > \theta_0$ or $H_1 : \theta = \theta < \theta_0$, *a UMP* test exists and is given by (18·10) and (18·11) respectively.

Remark. In particular, if we take $n = 2$, then the B.C.R. for testing $H_0 : \theta = \theta_0$, against $H_1 : \theta = \theta_1$ $(> \theta_0)$ is given by : [From (18·10) and (18·12)]

$$W = \{x : (x_1 + x_2)/2 \geq \theta_0 + \sigma z_\alpha/\sqrt{2}\} \qquad [\because \bar{x} = (x_1 + x_2)/2]$$
$$= \{x : x_1 + x_2 \geq 2\theta_0 + \sqrt{2}\ \sigma z_\alpha\}$$
$$= \{x : x_1 + x_2 \geq C\}, \text{ (say)} \qquad \dots(^{**})$$

where $C = 2\theta_0 + \sqrt{2}\,\sigma z_\alpha = 2\theta_0 + \sqrt{2}\,\sigma \times 1{\cdot}645$, if $\alpha = 0{\cdot}05$.

Similarly, the B.C.R. for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ $(< \theta_0)$ with $n = 2$ and $\alpha = 0{\cdot}05$ is given by [From (18·11) and (18·12a)] :

$$W_1 = \{x : (x_1 + x_2)/2 \leq \theta_0 - \sigma z_\alpha/\sqrt{2}\}$$
$$= \{x : (x_1 + x_2) \leq 2\theta_0 - \sqrt{2}\,\sigma \times 1{\cdot}645\}$$
$$= \{x : x_1 + x_2 \leq C_1\}, \text{ (say)}, \qquad \dots(^{***})$$

where $C_1 = 2\theta_0 - \sqrt{2}\,\sigma z_\alpha = 2\theta_0 - \sqrt{2}\,\sigma \times 1{\cdot}645$, if $\alpha = 0{\cdot}05$

The B.C.R. for testing $H_0 : \theta = \theta_0$ against the two tailed alternative $H_1 : \theta = \theta_1$ $(\neq \theta_0)$, is given by : $W_2 = \{x : (x_1 + x_2 \geq C) \cup (x_1 + x_2 \leq C_1)\}$ $\dots(^{****})$

The regions in (**), (***), and (****) are given by the shaded portions in the following figures (*i*), (*ii*) and (*iii*) respectively.

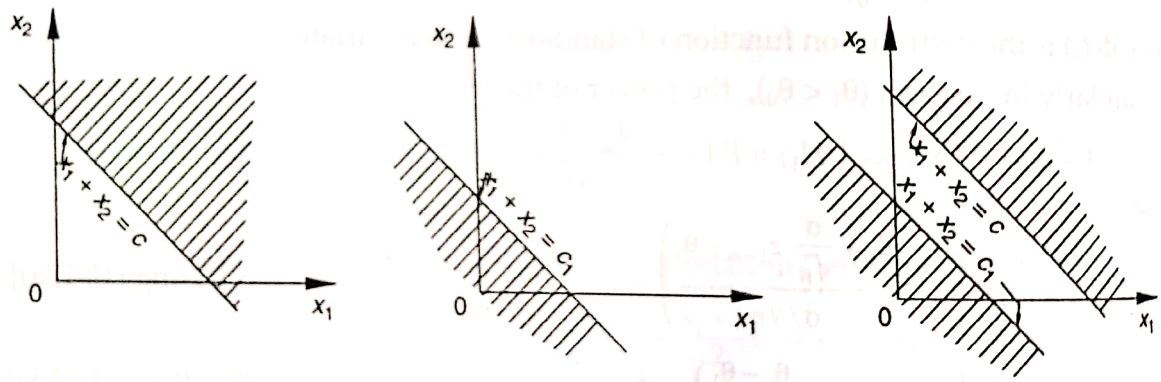


Fig. (*i*) BCR for $\left.\right\}$: $H_0 : \theta = \theta_0$, $H_1 : \theta = \theta_1$ $(> \theta_0)$

Fig. (*ii*) BCR for $\left.\right\}$: $H_0 : \theta = \theta_0$, $H_1 : \theta = \theta_1$ $(< \theta_0)$

Fig. (*iii*) BCR for $\left.\right\}$: $H_0 : \theta = \theta_0$, $H_1 : \theta = \theta_1$ $(\neq \theta_0)$

Example 18·7. *Show that for the normal distribution with zero mean and variance σ^2, the best critical region for $H_0 : \sigma = \sigma_0$ against the alternative $H_1 : \sigma = \sigma_1$ is of the form :*

$$\sum_{i=1}^{n} x_i^2 \leq a_\alpha, \text{ for } \sigma_0 > \sigma_1$$

$$\text{and } \sum_{i=1}^{n} x_i^2 \geq b_\alpha, \text{ for } \sigma_0 < \sigma_1$$

Show that the power of the best critical region when $\sigma_0 > \sigma_1$ is $F\left(\dfrac{\sigma_0^2}{\sigma_1^2} \cdot \chi^2_{\alpha,n}\right)$, where $\chi^2_{\alpha,n}$ is lower 100 α-per cent point and F is the distribution function of the χ^2-distribution with n degrees of freedom.

Solution. Here we are given :

$$f(x, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

The best critical region (B.C.R.), according to Neyman-Pearson Lemma, is given by (for $k_\alpha > 0$), $\frac{L_0}{L_1} \le \frac{1}{k_\alpha} = A_\alpha$, (say).

$$\Rightarrow \quad \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} x_i^2\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\right\} \le A_\alpha$$

$$\Rightarrow \quad n \log\left(\frac{\sigma_1}{\sigma_0}\right) - \frac{1}{2}\sum_{i=1}^{n} x_i^2\left(\frac{\sigma_1^2 - \sigma_0^2}{\sigma_0^2\,\sigma_1^2}\right) \le \log A_\alpha$$

(since $\log x$ is an increasing function of x).

$$\Rightarrow \quad \frac{\sigma_0^2 - \sigma_1^2}{2\sigma_0^2\,\sigma_1^2}\sum_{i=1}^{n} x_i^2 \le \left\{\log A_\alpha - n\log\left(\frac{\sigma_1}{\sigma_0}\right)\right\} \qquad \ldots(*)$$

Case (i). If $\sigma_0 > \sigma_1 \Rightarrow \sigma_1 < \sigma_0$, then B.C.R. is given by

$$\sum_{i=1}^{n} x_i^2 \le \left[\log A_\alpha - n\log\left(\frac{\sigma_1}{\sigma_0}\right)\right]\frac{2\sigma_0^2\,\sigma_1^2}{\sigma_0^2 - \sigma_1^2} = a_\alpha, \text{ (say)}.$$

i.e., $$W = \left(x : \sum_{i=1}^{n} x_i^2 \le a_\alpha\right) \qquad \ldots(18{\cdot}14)$$

Case (ii). If $\sigma_0 < \sigma_1$, then B.C.R. is given by

$$\sum_{i=1}^{n} x_i^2 \ge \left[\log A_\alpha - n\log\left(\frac{\sigma_1}{\sigma_0}\right)\right]\cdot\frac{2\sigma_0^2\,\sigma_1^2}{\sigma_0^2 - \sigma_1^2} = b_\alpha, \text{ (say)}.$$

i.e., $$W_1 = \left(x : \sum_{i=1}^{n} x_i^2 \ge b_\alpha\right) \qquad \ldots(18{\cdot}14a)$$

The constants a_α and b_α are so chosen that the size of the critical region is α. Thus, a_α is determined so that

$$P[x \in W \mid H_0] = \alpha$$

$$\Rightarrow \quad P\left[\sum_{i=1}^{n} x_i^2 \le a_\alpha \,\middle|\, H_0\right] = \alpha$$

$$\Rightarrow \quad P\left(\sum_{i=1}^{n} \frac{x_i^2}{\sigma_0^2} \le \frac{a_\alpha}{\sigma_0^2} \,\middle|\, H_0\right) = \alpha \qquad \ldots(*)$$

Since under H_0, $\chi_{(n)}^2 = \sum_{i=1}^{n} \frac{x_i^2}{\sigma_0^2}$, is a χ^2-variate with n *d.f.*,

$$\therefore \quad P\left[\chi_{(n)}^2 \le \frac{a_\alpha}{\sigma_0^2}\right] = \alpha \Rightarrow \frac{a_\alpha}{\sigma_0^2} = \chi^2_{\alpha, n} \Rightarrow \sigma_0^2\,\chi^2_{\alpha, n} = a_\alpha \qquad \ldots(18{\cdot}15)$$

where $\chi^2_{\alpha, n}$ is the lower 100 α-per cent point of chi-square distribution with n *d.f.* given by :

$$P\,(\chi^2 \le \chi^2_{\alpha, n}) = \alpha \qquad \ldots(18{\cdot}15a)$$

Hence the B.C.R. for testing $H_0 : \sigma = \sigma_0$ against $H_1 : \sigma = \sigma_1 \ (< \sigma_0)$, is given by [From (18·14) and (18·15)] :

$$W = \left[\mathbf{x} : \sum_{i=1}^{n} x_i^2 \leq \sigma_0^2 \chi^2_{\alpha, n} \right\} \quad \ldots(18·15b)$$

where $\chi^2_{\alpha, n}$ is defined in (18·15a).

Also by definition, the power of the test is :

$$1 - \beta = P[\mathbf{x} \in W \mid H_1] = P\left(\sum_{i=1}^{n} x_i^2 \leq a_\alpha \mid H_1 \right)$$

$$= P\left(\frac{\sum_{i=1}^{n} x_i^2}{\sigma_0^2} \leq \frac{a_\alpha}{\sigma_0^2} \middle| H_1 \right) = P\left(\frac{\sum_{i=1}^{n} x_i^2}{\sigma_0^2} \leq \chi^2_{\alpha, n} \mid H_1 \right) \quad \text{[From (18·15)]}$$

$$= P\left(\frac{\sum_{i=1}^{n} x_i^2}{\sigma_1^2} \leq \frac{\sigma_0^2}{\sigma_1^2} \chi^2_{\alpha, n} \middle| H_1 \right) = P\left(\chi^2_{(n)} \leq \frac{\sigma_0^2}{\sigma_1^2} \chi^2_{\alpha, n} \right)$$

since under H_1, $\Sigma x_i^2 / \sigma_1^2$ is a χ^2-variate with n *d.f.*

$$\text{Hence, power of the test} = F\left(\frac{\sigma_0^2}{\sigma_1^2} \cdot \chi^2_{\alpha, n} \right), \quad \ldots(18·15c)$$

where F(·) is the distribution function of chi-square distribution with n d.f.

***Remarks* 1.** Similarly, for testing $H_0 : \sigma = \sigma_0$ against $H_1 : \sigma = \sigma_1 \ (> \sigma_0)$, b_α in (16·14a) is determined so that :

$$P\left[\mathbf{x} : \in W_1 \mid H_0 \right] = \alpha$$

$$\Rightarrow \quad P\left(\mathbf{x} : \sum_{i=1}^{n} x_i^2 \geq b_\alpha \mid H_0 \right) = \alpha$$

$$\Rightarrow \quad P\left(\mathbf{x} : \frac{\Sigma x_i^2}{\sigma_0^2} \geq \frac{b_\alpha}{\sigma_0^2} \middle| H_0 \right) = \alpha$$

$$\Rightarrow \quad P\left[\mathbf{x} : \chi^2_{(n)} \geq \frac{b_\alpha}{\sigma_0^2} \right] = \alpha$$

$$\Rightarrow \quad P\left(\mathbf{x} : \chi^2_{(n)} \leq \frac{b_\alpha}{\sigma_0^2} \right) = 1 - \alpha$$

$$\therefore \quad \frac{b_\alpha}{\sigma_0^2} = \chi^2_{1-\alpha, n} \quad \Rightarrow \quad b_\alpha = \sigma_0^2 \cdot \chi^2_{1-\alpha, n} \quad \ldots(18·16)$$

where $\chi^2_{\alpha, n}$ is defined in (18·15a).

Hence the B.C.R. for testing $H_0 : \sigma = \sigma_0$ against $H_1 : \sigma = \sigma_1 \ (> \sigma_0)$, is given by :

$$W_1 = \left(\mathbf{x} : \sum_{i=1}^{n} x_i^2 \geq \sigma_0^2 \cdot \chi^2_{1-\alpha, n} \right) \quad \ldots(18·16a)$$

The power of the test in this case is given by

$$1 - \beta = P(\mathbf{x} \in W_1 \mid H_1) = P\left[\sum_{i=1}^{n} x_i^2 \geq \sigma_0^2 \chi^2_{1-\alpha, n} \mid H_1 \right]$$

$$= P\left(\frac{\sum_{i=1}^{n} x_i^2}{\sigma_1^2} \geq \frac{\sigma_0^2}{\sigma_1^2} \chi^2_{1-\alpha, n} \middle| H_1 \right) = P\left(\chi^2_{(n)} \geq \frac{\sigma_0^2}{\sigma_1^2} \cdot \chi^2_{1-\alpha, (n)} \right), \quad \ldots(18·16b)$$

since under H_1, $\sum_{i=1}^{n} x_i^2 \mid \sigma_1^2$ is a χ^2-variate with n d.f.

$$\therefore \quad 1-\beta = 1 - P\left\{\chi^2_{(n)} \le \frac{\sigma_0^2}{\sigma_1^2}\cdot \chi^2_{1-\alpha,\, n}\right\} = 1 - F\left(\frac{\sigma_0^2}{\sigma_1^2}\cdot \chi^2_{1-\alpha,\, n}\right), \quad \ldots(18{\cdot}16c)$$

where $F(.)$ is the distribution function of chi-square distribution with n *d.f.*

2. *Graphical representation of the B.C.R. for the particular case n = 2.*

For $n = 2$, the B.C.R. for testing $H_0 : \sigma = \sigma_0$, against $H_1 : \sigma = \sigma_1 \ (< \sigma_0)$ is given by [From (18·15b)]

$$W = \left\{x : \sum_{i=1}^{2} x_i^2 \le \sigma_0^2 \cdot \chi^2_{\alpha,\, 2}\right\} = \{x : x_1^2 + x_2^2 \le a^2\},$$

where $a^2 = \sigma_0^2 \chi^2_{\alpha,\, 2}$. Thus the B.C.R. is the interior of the circle with centre (0, 0) and radius 'a' and is shown as the shaded region in Figure (i) below:

Similarly, from (18·16a), the B.C.R. for testing $H_0 : \sigma = \sigma_0$, against $H_1 : \sigma = \sigma_1 \ (> \sigma_0)$ for $n = 2$ is given by :

$$W_1 = \{x : x_1^2 + x_2^2 \ge \sigma_0^2\, \chi^2_{1-\alpha,\, 2}\} = \{x : x_1^2 + x_2^2 \ge b^2\}$$

where $b^2 = \sigma_0^2 \cdot \chi^2_{1-\alpha,\, 2}$. Thus, B.C.R. is the exterior of the circle with centre (0, 0) and radius b and is shown as the shaded region in Figure (ii) below:

Similarly the B.C.R. for testing $H_0 : \sigma = \sigma_0$ against two tailed alternative

$$H_1 : \sigma_1 = \sigma_1 \ (\ne \sigma_0), \text{ for } n = 2 \text{ is given by :}$$

$$W_2 = W \cup W_1 = \{x : x_1^2 + x_2^2 \le a^2\} \cup \{x : x_1^2 + x_2^2 \ge b^2\}$$

and is shown as the shaded region in the Figure (iii) below.

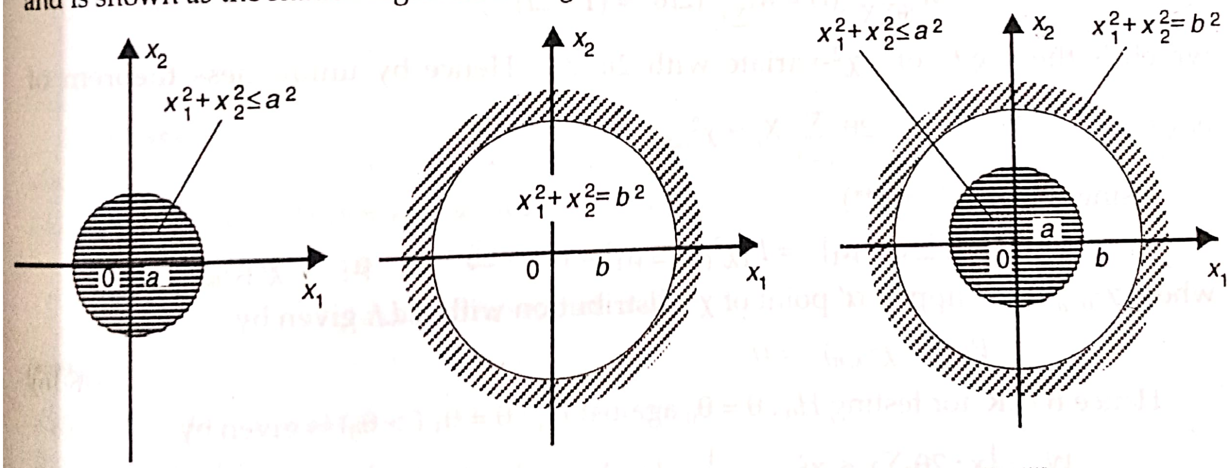


Fig. (i) Fig. (ii) Fig (iii)

3. (18·14) defines an *UMP* test for testing simple hypothesis $H_0 : \sigma = \sigma_0$ against simple alternative hypothesis $H_1 : \sigma = \sigma_1 \ (< \sigma_0)$ whereas (18·14a) defines an *UMP* test for testing simple hypothesis H_0 against the simple alternative hypothesis $H_1 : \sigma = \sigma_1 \ (> \sigma_0)$. However no *UMP* test exists for testing simple hypothesis $H_0 : \sigma = \sigma_0$ against the composite alternative hypothesis $H_1 : \sigma \ne \sigma_0$.

Example 18·8. *Given a random sample $x_1, x_2, \ldots, x_n$ from the distribution with p.d.f.*

$$f(x, \theta) = \theta\, e^{-\theta x},\ x > 0$$

show that there exists no UMP test for testing

$$H_0 : \theta = \theta_0 \text{ against } H_1 : \theta \neq \theta_0.$$

Solution. $L = \prod_{i=1}^{n} f(x_i, \theta) = \theta^n \cdot \exp\left[-\theta \sum_{i=1}^{n} x_i\right]$

Consider $H_1 : \theta = \theta_1, (\theta_1 \neq \theta_0)$

The best critical region, using Neyman-Perason Lemma is given by :

$$\theta_1^n \exp[-\theta_1 \Sigma x_i] \geq k \cdot \theta_0^n \exp[-\theta_0 \Sigma x_i] \; ; \; k > 0$$

$$\Rightarrow \quad exp\,[(\theta_0 - \theta_1) \Sigma x_i] \geq k \cdot \left(\frac{\theta_0}{\theta_1}\right)^n$$

$$\Rightarrow \quad (\theta_0 - \theta_1) \Sigma x_i \geq \log \left\{k \cdot \left(\frac{\theta_0}{\theta_1}\right)^n\right\} = k_1, \text{ (say)}. \quad \ldots(*)$$

Case (*i*) If $\theta_1 > \theta_0$, then *B.C.R.* is given by

$$\Sigma x_i \leq \frac{k_1}{\theta_0 - \theta_1} = \lambda_1, \text{ (say)}.$$

Case (*ii*) If $\theta_1 < \theta_0$, then *B.C.R.* is given by

$$\Sigma x_i \geq \frac{k_1}{\theta_0 - \theta_1} = \lambda_2, \text{ (say)}.$$

The constants λ_1 and λ_2 are so determined that

$$\left.\begin{aligned} P[\Sigma x_i \leq \lambda_1 \mid H_0] &= \alpha \\ \Rightarrow P[2\theta \Sigma x_i \leq 2\theta\, \lambda_1 \mid H_0] &= \alpha \end{aligned}\right\} \ldots (**) \quad \text{and} \quad \left.\begin{aligned} P[\Sigma x_i \geq \lambda_2 \mid H_0] &= \alpha \\ \Rightarrow P[2\theta\, \Sigma x_i \geq 2\theta\, \lambda_2 \mid H_0] &= \alpha \end{aligned}\right\} \ldots (***)$$

But in random sampling from the given exponential distribution,

$$M_{\Sigma X_i}(t) = \prod_{i=1}^{n} M_{X_i}(t) = \{M_{X_i}((t)\}^n = \left(1 - \frac{t}{\theta}\right)^{-n}$$

$$\Rightarrow \quad M_{2\theta \Sigma X_i}(t) = M_{\Sigma x_i}(2t\theta) = (1 - 2t)^{-n},$$

which is the *m.g.f.* of a χ^2-variate with $2n$. *d.f.* Hence by uniqueness theorem of *m.g.f.*'s, $2\theta \sum_{i=1}^{n} X_i \sim \chi^2_{(2n)}$

Using this result in (**)

$$P[2\theta_0\, \Sigma x_i \leq \mu_1] = P[\chi^2_{(2n)} \leq \mu_1] = \alpha \quad \Rightarrow \quad \mu_1 = \chi^2_{1-\alpha, 2n}$$

where $\chi^2_{\alpha, n}$ is the upper 'α' point of χ^2-distribution with *n.d.f.* given by

$$P(\chi^2 > \chi^2_{\alpha, n}) = \alpha \quad \ldots(****)$$

Hence B.C.R. for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1 \; (> \theta_0)$ is given by

$$W_0 = \{x : 2\theta_0 \Sigma x_i \leq \chi^2_{1-\alpha, 2n}\} = \{x : \Sigma x_i \leq \frac{1}{2\theta_0} \chi^2_{1-\alpha, 2n}\}$$

and since it is independent of θ_1, W_0 is *U.M.P.C.R.* for $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1 \; (> \theta_0)$.

Similarly from (***) and (****), we get

$$P[2\theta_0 \Sigma x_i \geq \mu_2] = P(\chi^2_{(2n)} \geq \mu_2) = \alpha \quad \Rightarrow \quad \mu_2 = \chi^2_{\alpha, 2n}$$

Hence B.C.R. for testing H_0 against $H_1 : \theta = \theta_1 \; (< \theta_0)$ is given by :

$$W_1 = \{x : 2\theta_0 \Sigma x_i \geq \chi^2_{\alpha, 2n}\} = \left\{x : \Sigma x_i \geq \frac{1}{2\theta_0} \chi^2_{\alpha, 2n}\right\}$$

and since it is independent of θ_1, W_1 is also *UPM C.R.* for $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1\ (< \theta_0)$.

However, since the two critical regions W_0 and W_1 are different, there exists no critical region of size α which is *U.M.P.* for $H_0 : \theta = \theta_0$ against the two tailed alternative, $H_1 : \theta \neq \theta_0$.

Power of the test. The power of the test for testing $H_0 : \theta = \theta_0$, against $H_1 : \theta = \theta_1\ (> \theta_0)$ is given by

$$1 - \beta = P[\mathbf{x} \in W_0 \mid H_1] = P\left(\sum_{i=1}^{n} x_i \leq \frac{1}{2\theta_0}\chi^2_{1-\alpha,\,2n} \,\middle|\, H_1\right)$$

$$= P\left(2\theta_1 \sum_{i=1}^{n} x_i \leq \frac{\theta_1}{\theta_0}\chi^2_{1-\alpha,\,2n} \,\middle|\, H_1\right)$$

$$= P\left\{\chi^2_{(2n)} \leq \frac{\theta_1}{\theta_0}\chi^2_{1-\alpha,\,2n}\right\}, \qquad \ldots(*)$$

since under H_1, $2\theta_1 \sum_{i=1}^{n} x_i \sim \chi^2_{(2n)}$.

Similarly the power of the test for testing $H_0 : \theta = \theta_0$, against $H_1 : \theta = \theta_1\ (< \theta_0)$ is given by :

$$1 - \beta = P(\mathbf{x} \in W_1 \mid H_1) = P\left(\sum_{i=1}^{n} x_i \geq \frac{1}{2\theta_0}\chi^2_{\alpha,\,2n} \,\middle|\, H_1\right)$$

$$= P\left(2\theta_1 \sum_{i=1}^{n} x_i \geq \frac{\theta_1}{\theta_0}\chi^2_{\alpha,\,2n} \,\middle|\, H_1\right)$$

$$= P\left\{\chi^2_{(2n)} \geq \frac{\theta_1}{\theta_0}\chi^2_{\alpha,\,2n}\right\} \qquad \ldots(**)$$

Remark. The graphic representation of the B.C.R. for $H_0 : \theta = \theta_0$ against different alternatives $H_1 : \theta = \theta_1\ (> \theta_0)$, $H_1 : \theta = \theta_1\ (< \theta_0)$ and $H_1 : \theta = \theta_1\ (\neq \theta_0)$ for $n = 2$, can be done similarly as in Example 18·6, for the mean of normal distribution.

Example 18·9. *For the distribution*

$$dF = \begin{cases} \beta \exp\{-\beta(x-\gamma)\}\, dx, & x \geq \gamma \\ 0, & x < \gamma \end{cases}$$

show that for a hypothesis H_0 that $\beta = \beta_0$, $\gamma = \gamma_0$ and an alternative H_1 that $\beta = \beta_1$, $\gamma = \gamma_1$, the best critical region is given by

$$\bar{x} \leq \frac{1}{\beta_1 - \beta_0}\left\{\gamma_1\beta_1 - \gamma_0\beta_0 - \frac{1}{n}\log k + \log\frac{\beta_1}{\beta_0}\right\}$$

provided that the admissible hypothesis is restricted by the condition $\gamma_1 \leq \gamma_0$, $\beta_1 \geq \beta_0$

Solution.

$$f(x\,;\beta,\gamma) = \begin{cases} \beta \exp\{-\beta(x-\gamma)\}, & x \geq \gamma \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore \quad \prod_{i=1}^{n} f(x_i\,;\beta,\gamma) = \begin{cases} \beta^n \exp\left\{-\beta\sum_{i=1}^{n}(x_i-\gamma)\right\}; & x_1, x_2, \ldots, x_n \geq \gamma \\ 0, & \text{otherwise} \end{cases}$$

Using Neyman-Pearson Lemma, B.C.R. for $k > 0$, is given by

$$\frac{\beta_1^n \exp\left\{-\beta_1 \sum_{i=1}^{n} (x_i - \gamma_1)\right\}}{\beta_0^n \exp\left\{-\beta_0 \sum_{i=1}^{n} (x_i - \gamma_0)\right\}} \geq k$$

$$\Rightarrow \quad \left(\frac{\beta_1}{\beta_0}\right)^n \exp\left\{-\beta_1 \sum_{i=1}^{n} (x_i - \gamma_1) + \beta_0 \sum_{i=1}^{n} (x_i - \gamma_0)\right\} \geq k$$

$$\Rightarrow \quad \left(\frac{\beta_1}{\beta_0}\right)^n \exp\left[-\beta_1 n\, (\bar{x} - \gamma_1) + \beta_0 n(\bar{x} - \gamma_0)\right] \geq k$$

$$\Rightarrow \quad n \log (\beta_1/\beta_0) - n\bar{x}\,(\beta_1 - \beta_0) + n\beta_1\gamma_1 - n\beta_0\gamma_0 \geq \log k$$

(since $\log x$ is an increasing function of x).

$$\Rightarrow \quad \bar{x}(\beta_1 - \beta_0) \leq \left\{\gamma_1\beta_1 - \gamma_0\beta_0 - \frac{1}{n}\log k + \log\left(\frac{\beta_1}{\beta_0}\right)\right\}$$

$$\therefore \quad \bar{x} \leq \frac{1}{\beta_1 - \beta_0}\left\{\gamma_1\beta_1 - \gamma_0\beta_0 - \frac{1}{n}\log k + \log\left(\frac{\beta_1}{\beta_0}\right)\right\} \text{ provided } \beta_1 > \beta_0.$$

Example 18·10. *Examine whether a best critical region exists for testing the null hypothesis $H_0 : \theta = \theta_0$ against the alternative hypothesis $H_1 : \theta > \theta_0$ for the parameter θ of the distribution :*

$$f(x, \theta) = \frac{1+\theta}{(x+\theta)^2},\ 1 \leq x < \infty$$

Solution. $\prod_{i=1}^{n} f(x_i, \theta) = (1+\theta)^n \prod_{i=1}^{n} \frac{1}{(x_i+\theta)^2}$

By Neyman-Pearson Lemma, the B.C.R. for $k > 0$, is given by

$$(1+\theta_1)^n \prod_{i=1}^{n} \frac{1}{(x_i+\theta_1)^2} \geq k\,(1+\theta_0)^n \prod_{i=1}^{n} \frac{1}{(x_i+\theta_0)^2}$$

$$\Rightarrow \quad n \log (1+\theta_1) - 2\sum_{i=1}^{n} \log (x_i+\theta_1) \geq \log k + n \log (1+\theta_0) - 2\sum_{i=1}^{n} \log (x_i+\theta_0)$$

$$\Rightarrow \quad 2\sum_{i=1}^{n} \log\left(\frac{x_i+\theta_0}{x_i+\theta_1}\right) \geq \log k + n \log\left(\frac{1+\theta_0}{1+\theta_1}\right)$$

Thus the test criterion is $\sum_{i=1}^{n} \log\left(\frac{x_i+\theta_0}{x_i+\theta_1}\right)$, which cannot be put in the form of a function of the sample observations, not depending on the hypothesis. Hence no B.C.R. exists in this case.

18·6. LIKELIHOOD RATIO TEST

Neyman-Pearson Lemma based on the magnitude of the ratio of two probability ... simple