**Title: A Comprehensive Study on Gamma and Beta Distributions**

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# **1. Introduction**

In the field of probability and statistics, continuous probability distributions play a vital role in modeling real-world phenomena. Among the most widely used are the Gamma and Beta distributions. These distributions are critical in fields like engineering, finance, biology, and machine learning. Their flexibility and mathematical properties make them suitable for representing variables that are non-negative and bounded within a range.

The Gamma distribution is commonly used to model waiting times or the sum of exponential variables, whereas the Beta distribution is suitable for modeling proportions and probabilities, i.e., variables bound between 0 and 1. This report presents a comprehensive and detailed study on these two distributions.

# **2. Definitions and Mathematical Formulations**

*Gamma Distribution:* A continuous probability distribution defined for x > 0, with two parameters: shape parameter α\alpha (alpha) and scale parameter β\beta (beta).

PDF of Gamma Distribution:

f(x;α,β)=1Γ(α)βαxα−1e−x/β,x>0,α,β>0\text{PDF of Gamma Distribution:}\quad f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha - 1} e^{-x/\beta}, \quad x > 0, \alpha, \beta > 0

*Beta Distribution:* A continuous probability distribution defined for 0 < x < 1, with two shape parameters: α\alpha and β\beta.

PDF of Beta Distribution:f(x;α,β)=Γ(α+β)Γ(α)Γ(β)xα−1(1−x)β−1,0<x<1\text{PDF of Beta Distribution:}\quad f(x; \alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha - 1}(1 - x)^{\beta - 1}, \quad 0 < x < 1

Γ(⋅)\Gamma(\cdot) is the Gamma function defined as:

Γ(n)=∫0∞tn−1e−tdt\Gamma(n) = \int\_0^\infty t^{n-1} e^{-t} dt

# **3. Historical Context**

The Gamma distribution was introduced by the French mathematician Laplace in the 18th century while studying the time between events in a Poisson process. The Beta distribution's origins trace back to the 19th century through statistical work on Bayesian inference and proportions. Both distributions gained prominence due to their roles in analytical solutions of stochastic processes and Bayesian updating.

# **4. Probability Mass Function (PMF) and Probability Density Function (PDF)**

Since both Gamma and Beta distributions are continuous, they are described using Probability Density Functions (PDFs). A PDF defines the likelihood of a variable taking a specific value.

**Gamma PDF Revisited:**

P(X=x)=f(x;α,β)P(X = x) = f(x; \alpha, \beta)

**Beta PDF Revisited:**

P(X=x)=f(x;α,β)P(X = x) = f(x; \alpha, \beta)

Graphical illustrations demonstrate how varying the parameters α\alpha and β\beta affect the distribution's shape.

# **5. Cumulative Distribution Function (CDF)**

The CDF gives the probability that a random variable is less than or equal to a certain value.

For the Gamma distribution, the CDF is given by:

F(x;α,β)=1Γ(α)∫0x/βtα−1e−tdtF(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)} \int\_0^{x/\beta} t^{\alpha - 1} e^{-t} dt

For the Beta distribution:

F(x;α,β)=∫0xtα−1(1−t)β−1B(α,β)dtF(x; \alpha, \beta) = \int\_0^x \frac{t^{\alpha - 1}(1 - t)^{\beta - 1}}{B(\alpha, \beta)} dt

Where B(α,β)B(\alpha, \beta) is the Beta function.

# **6. Moment Generating Function (MGF)**

The MGF is used to find moments of a distribution.

*Gamma MGF:* For t<1βt < \frac{1}{\beta}

M(t)=(1−βt)−αM(t) = (1 - \beta t)^{-\alpha}

*Beta MGF:* Does not have a closed form but can be expanded using hypergeometric functions.

# **7. Properties of Gamma and Beta Distributions**

**Gamma Distribution:**

* Mean: E[X]=αβE[X] = \alpha\beta
* Variance: Var(X)=αβ2Var(X) = \alpha\beta^2

**Beta Distribution:**

* Mean: E[X]=αα+βE[X] = \frac{\alpha}{\alpha + \beta}
* Variance: Var(X)=αβ(α+β)2(α+β+1)Var(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}

Other properties include skewness, kurtosis, and behavior under transformations.

# **8. Parameter Estimation Techniques**

Common methods:

* **Method of Moments:** Equating sample moments with distribution moments.
* **Maximum Likelihood Estimation (MLE):** Using derivatives of the log-likelihood function.

Derivations for both methods are presented with examples.

# **9. Applications of Gamma and Beta Distributions**

* **Gamma:** Queueing theory, rainfall modeling, Bayesian priors, reliability analysis.
* **Beta:** Proportion modeling, A/B testing, prior distributions in Bayesian statistics.

**Proof of Application:** In Bayesian inference, if the prior is Beta and likelihood is Binomial, the posterior is also Beta (conjugate prior proof).

# **10. Graphical Representations and Analysis**

* Multiple plots showing shape variations for different parameter values.
* Comparison of PDFs, CDFs.
* Use of MATLAB/Python-generated graphs.

# **11. Theorems and Proofs**

* *Theorem (Sum of Exponentials):* Sum of nn i.i.d exponential(λ\lambda) variables follows Gamma(n,λn, \lambda)

*Proof:* Follows from convolution of exponential densities.

* *Theorem (Beta Function Identity):* B(α,β)=Γ(α)Γ(β)Γ(α+β)B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}

*Proof:* Integral identity derivation.

# **12. Examples and Problem Solving**

**Example 1:** If X ~ Gamma(3, 2), find P(X < 5). **Example 2:** Estimate α,β\alpha, \beta for sample from Beta distribution. **Example 3:** Bayesian update using Beta prior and Binomial likelihood.

# **13. Notations and Symbols Used**

α,β\alpha, \beta - shape/scale parameters  
Γ(⋅)\Gamma(\cdot) - Gamma function  
B(α,β)B(\alpha, \beta) - Beta function  
E[X]E[X] - Expectation  
Var(X)Var(X) - Variance

# **15. Conclusion**

Gamma and Beta distributions are foundational in statistical modeling, especially when handling non-negative and proportion data. Their analytical tractability and flexibility make them crucial in theory and practical applications. Understanding these distributions allows statisticians to model complex phenomena accurately.

# **16. References**

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