EE601: Statistical Signal Analysis
Quiz # 3

Date: 17/09/2024 Time: 11 - 1 pm

Q.1 Let $X \sim \text{Poisson}(\lambda)$ for $\lambda > 0$, i.e.

$$P(X = k) = e^{-\lambda} \lambda^k / k! \,, \quad \forall\, k = 0, 1, 2, \ldots$$

Define, $Y = 2X \mathbb{1}_A$, where $A = \{\omega : X(\omega) \geq 10\}$.

Find prob. mass fn of Y. **5 Marks**

Q.2 Let $X \sim \text{Gaussian}(0,1)$. Define, $Y = aX + b$, where $a, b \in \mathbb{R}$. Find $f_Y(\cdot)$. **5 Marks**

Q.3 Let X be a random variable such that

$$f_X(x) = e^{-\lambda x} \frac{\lambda^k \cdot x^{k-1}}{k!} \quad \text{for } x \geq 0,$$

$$= 0 \quad \text{o.w.}$$

Here, $\lambda > 0$ and k is an integer. If $E[X] = 4$ and $\text{var}(X) = 8$. Then, find values of λ and k. **5 Marks**

Q.4 Let $X \sim \exp(\lambda)$, for $\lambda > 0$. Let $t > 0$ be a real number. Find $E[X \mid X > t]$. **5 Marks**

Solution

$\boxed{Q.1}$ $Y = 2X\,1_A$, where $A = \{\omega : X(\omega) \geq 10\}$.

Note that: $Y(\omega) = 2X(\omega)\,1_A(\omega)$, where

$$1_A(\omega) = 1 \quad \text{if } \omega \in A$$
$$= 0 \quad \text{o.w.}$$

Thus, $Y = 2X$ if $X \geq 10$

$= 0$ o.w.

$\because$ $X(\omega) \in \{0, 1, 2, \dots\}$ $\forall\, \omega$,

$Y(\omega) \in \{0, 20, 22, \dots\}$ $\forall\, \omega$.

Now, $P(Y=0) = P(X < 10) = \sum_{k=0}^{9} P(X=k)$

$$= \sum_{k=0}^{9} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{9} \frac{\lambda^k}{k!}$$

For any odd integer y:

$$P(Y=y) = 0.$$

For any even integer y satisfying $0 < y < 20$

$$P(Y=y) = 0.$$

For any even integer $y \geq 20$

$$P(Y=y) = P(2X=y) = P\left(X = \frac{y}{2}\right)$$

$$= e^{-\lambda} \frac{\lambda^{y/2}}{(y/2)!}.$$

———— x ———— x ———— x ————

$\boxed{Q.2}$ $X \sim \text{Gaussian}(0,1)$

$Y = aX + b$. Define $g(x) = ax + b$, $a, b \in \mathbb{R}$.

Now, recall that

$$f_Y(y) = \sum_{n=1}^{\infty} \frac{f_X(x_n)}{|g'(x_n)|}, \text{ where } x_n\text{'s are}$$

roots of $y = g(x)$.

Thus,

$$f_Y(y) = \frac{f_X\left(\frac{y-b}{a}\right)}{|a|},$$ as $y = ax+b$ has only one root $x = \frac{y-b}{a}$ and $g'(x) = a \ \forall x$.

$$\Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi a^2}} \cdot e^{-((y-b)/a)^2/2}$$

Note that $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

$$\Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi a^2}} e^{-(y-b)^2/2a^2}$$

$\Rightarrow Y \sim \text{Gaussian}(b, a^2)$.

$\boxed{Q.3}$ $f_X(x) = e^{-\lambda x} \frac{\lambda^k x^{k-1}}{k!} \ \forall x \geq 0$

$= 0$ o.w.

As pointed out during quiz, $f_X(x)$ should have $(k-1)!$ instead of $k!$.

Now, consider MGF of X.

$$M_X(t) = E[e^{tX}]$$
$$= \int_0^{\infty} e^{tx} e^{-\lambda x} \frac{\lambda^k x^{k-1}}{k!} dx$$
$$= \int_0^{\infty} e^{-(\lambda - t)x} \frac{\lambda^k x^{k-1}}{k!} dx$$

$$= \frac{\lambda^k}{(\lambda-t)^k} \int_0^\infty e^{-(\lambda-t)x} \frac{(\lambda-t)^k x^{k-1}}{k!} dx$$

$\lambda - t > 0$
$\forall\ t \in (-\infty, \lambda)$.

Thus, integrant is a prob. density fn with parameter $(\lambda-t)$ instead of λ for $t \in (-\infty, \lambda)$.

$$\Rightarrow\quad m_X(t) = \frac{\lambda^k}{(\lambda-t)^k} \quad \forall\ t \in (-\infty, \lambda).$$

Thus, MGF is defined in $(-\lambda, \lambda)$ interval around 0.

Thus, $E[X] = \frac{d}{dt} m_X(t) \Big|_{t=0}$ &

$E[X^2] = \frac{d^2}{dt^2} m_X(t) \Big|_{t=0}$.

$$\frac{d}{dt}(\lambda-t)^{-k}$$
$$-k(\lambda-t)^{-(k+1)} \cdot (-1)$$
$$= k(\lambda-t)^{-(k+1)}$$

$$\frac{d}{dt} m_X(t) = \frac{\lambda^k k}{(\lambda-t)^{k+1}} \Big|_{t=0}$$

$$\Rightarrow\quad E[X] = \frac{k}{\lambda}.$$

$$\frac{d^2}{dt^2} m_X(t) = \frac{d}{dt} \cdot \frac{k\lambda^k}{(\lambda-t)^{k+1}} = \frac{k(k+1)\lambda^k}{(\lambda-t)^{k+2}}$$

$$\Rightarrow\quad E[X^2] = \frac{k(k+1)}{\lambda^2}.$$

Now, $\operatorname{var}(X) = E[X^2] - E^2[X]$

$$= \frac{k(k+1)}{\lambda^2} - \frac{k^2}{\lambda^2} = \frac{k}{\lambda^2}.$$

$$\Rightarrow\quad \frac{k}{\lambda} = 4 \quad \text{and} \quad \frac{k}{\lambda^2} = 8$$

$$\Rightarrow\quad \lambda = \frac{1}{2} \quad \text{and} \quad k = 2.$$

——— x ——— x ———

$\boxed{Q.4}$ $X \sim \exp(\lambda)$, $\lambda > 0$.

Note that $\because X$ is non-negative r.v.

$$E[X \mid X > t] = \int_0^\infty F_X^c(x \mid X > t)\, dx$$

$$= \int_0^\infty P(X > x \mid X > t)\, dx$$

Now consider,

$$P(X > x \mid X > t) = \frac{P(X > x, X > t)}{P(X > t)}.$$

$$= \frac{P(X > \max\{x, t\})}{P(X > t)}$$

$$= \frac{P(X > x)}{P(X > t)} = \frac{e^{-\lambda x}}{e^{-\lambda t}} = e^{-\lambda(x-t)} \text{ if } x > t$$

$$= \frac{P(X > t)}{P(X > t)} = 1 \quad \text{if } x \le t.$$

$$\Rightarrow E[X] = \int_0^\infty P(X > x \mid X > t)\, dx$$

$$= \int_0^t dx + \int_t^\infty e^{-\lambda(x-t)}\, dx$$

$$= t + \int_0^\infty e^{-\lambda x}\, dx$$

$$= t + \frac{1}{\lambda}.$$

———— x ———— x ———— x ————