EE601: Statistical Signal Analysis
Quiz # 2

Date: 02/09/2023 Time: 11am-12:30pm

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Q.1 Let X be a random variable with distribution $F_X(\cdot)$, and $Y = aX+b$, where $a, b \in \mathbb{R}$. Find $F_Y(\cdot)$.

5 Marks

Q.2 Let X be a random variable with distribution $F_X(\cdot)$, and $Y = X(X+1)$. Find $F_Y(\cdot)$.

5 Marks

Q.3 Let the probability of a person having life span less than x years is $1-e^{-x/60}$ for $x \geq 0$ and 0 otherwise. Given that the person is "a" years old, find the probability that he will live for another "b" years. $(a, b \geq 0)$

5 Marks

Q.4 Consider a collection $\mathcal{A}$ of subsets of $\mathbb{R}$ as defined below:

$$\mathcal{A} = \{A : \text{Either } A \text{ or } A^c \text{ is countable}\}$$

(a) Show that if $A_1, A_2 \in \mathcal{A}$, then $A_1 \cup A_2 \in \mathcal{A}$. (Consider all possible cases for A_1 & A_2).

(b) Show that if $\{A_n\}_n \in \mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

(c) Show that $\mathcal{A}$ is σ-field. (ϕ is a countable set)

(d) Show that $\mathcal{A}$ is the smallest σ-field containing singletons i.e. containing collection $\{\{x\} : x \in \mathbb{R}\}$.

(e) Show that $\mathcal{A} \subsetneq \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-field. (strict subset)

1 Mark each

$\boxed{\text{Solutions}}$

Q.1 $Y = aX + b;\quad a, b \in \mathbb{R}.$

$$F_Y(y) = P(Y \le y) = P(aX + b \le y)$$

$$= P(aX \le y - b)$$

Three cases arise:

① $a = 0$

Then, $F_Y(y) = P(0 \le y - b)$

$$= 1 \quad \text{if } y \le b$$
$$= 0 \quad \text{o.w.}$$

② $a > 0$

Then, $F_Y(y) = P(X \le y - b/a)$

$$= F_X\left(\frac{y-b}{a}\right).$$

③ $a < 0$

Then, $F_Y(y) = P\left(X \ge \frac{y-b}{a}\right)$

$$= P\left(X = \frac{y-b}{a} \text{ or } X > \frac{y-b}{a}\right)$$

$$= P\left(X = \frac{y-b}{a}\right) + P\left(X > \frac{y-b}{a}\right) \quad [\sigma\text{-additivity of } P]$$

$$= P\left(X = \frac{y-b}{a}\right) + 1 - F_X\left(\frac{y-b}{a}\right).$$

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Q.2 $Y = X^2 + X$

$$F_Y(y) = P(Y \le y)$$
$$= P(X^2 + X - y \le 0)$$

Consider a polynomial $g(x) = x^2 + x - y$.

The roots of the polynomial are

$$\frac{-1 \pm \sqrt{1+4y}}{2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4y}.$$

Two cases arise:

① $1 + 4y \ge 0$, $y \ge -1/4$

② $1 + 4y < 0$, $y < -\frac{1}{4}$.

In case ① real roots

$\underbrace{-\frac{1}{2} - \frac{1}{2}\sqrt{1+4y}}_{r_1}$ and $\underbrace{-\frac{1}{2} + \frac{1}{2}\sqrt{1+4y}}_{r_2}$ exist.

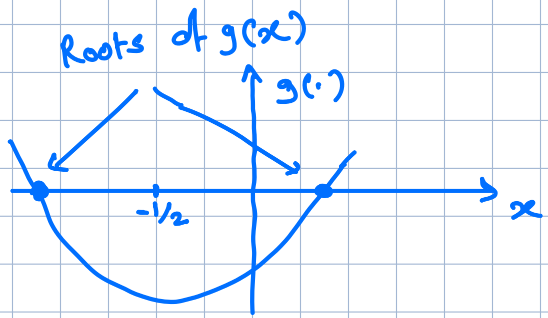
Now lets understand how fn $g(x)$ behaves.

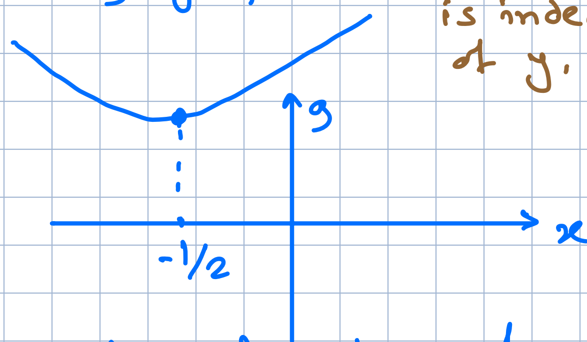
$$g(x) = x^2 + x + y$$
$$g'(x) = 2x + 1.$$

Note that $g'(x) < 0$ for $x < -\frac{1}{2}$ (fn strictly decreasing)
> 0 for $x > -\frac{1}{2}$ (fn strictly increasing)
$= 0$ for $x = 0$. (Global minima)

Thus, the $g(x)$ has following graphs.

behavior is independent of y.





No real roots, so fn. never crosses 0.

Thus, for $y \geq -\frac{1}{4}$

$$F_Y(y) = P(X \in [r_1, r_2])$$
$$= F_X(r_2) - F_X(r_1) + P(X = r_1)$$

for $y < -\frac{1}{4}$, $F_Y(y) = 0$.

$\boxed{\text{Q.3}}$ Let X denote the life span of the person. Then,

$$F_X(x) = 1 - e^{-x/60} \quad \text{for } x \geq 0$$
$$= 0 \quad \text{o.w.}$$

We need to find:

$$P(X > a+b \mid X > a) = \frac{P(X > a+b,\ X > a)}{P(X > a)}.$$

$$= \frac{P(X > a+b)}{P(X > a)} = \frac{1 - F_X(a+b)}{1 - F_X(a)}$$

$$= \frac{e^{-(a+b)/60}}{e^{-a/60}} = e^{-b/60}.$$

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$\boxed{Q.4}$ (a) If $A_1, A_2 \in \mathcal{A}$, then $A_1 \cup A_2 \in \mathcal{A}$.

Three cases to be considered.

(i) both A_1 and A_2 are countable.

clearly, $A_1 \cup A_2$ is countable.

$\Rightarrow A_1 \cup A_2 \in \mathcal{A}$.

(ii) both A_1 and A_2 have countable complement, i.e. A_1^c and A_2^c are countable.

Note that $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$

$\because$ intersection of countable sets is countable,

$(A_1 \cup A_2)^c$ is countable $\Rightarrow$ (by def^n of $\mathcal{A}$)

$A_1 \cup A_2 \in \mathcal{A}$.

(iii) One set is countable and other has countable complement.

w.l.o.g. A_1 & A_2^c are countable.

Note that $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$

$\because$ A_2^c is countable, and $A_1^c \cap A_2^c \subseteq A_2^c$,

$A_1^c \cap A_2^c$ is also countable $\Rightarrow (A_1 \cup A_2)^c$ is countable $\Rightarrow A_1 \cup A_2 \in \mathcal{A}$ (by def^n).

(b) $\{A_1, A_2, \dots\} \in \mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Three cases arise.

(i) All A_n's are countable.

Then $\bigcup_{n=1}^{\infty} A_n$ is a countable union of countable sets $\Rightarrow$ it is countable

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

(ii) All A_n's are s.t. A_n^c is countable.

Then $\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$

insection of countable sets

$\Rightarrow \bigcap_{n=1}^{\infty} A_n^c$ is countable

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$ (by def^n).

(iii) At least one A_n is s.t. A_n^c is countable.

Divide $\{A_n\}$ into two sub-sequences

$\{A_{n_i}\}_i$ and $\{A_{n_j}\}_j$ such that

A_{n_i} is countable for every i

A_{n_j} is s.t. $A_{n_j}^c$ is countable for every j.

by (i) $\bigcup_i A_{n_i} = A' \in \mathcal{A}$

by (ii) $\bigcup_j A_{n_j} = A'' \in \mathcal{A}$.

Now, $\bigcup_{n=1}^{\infty} A_n = A' \cup A'' \in \mathcal{A}$ (by part (a)).

(c) Show that $\mathcal{A}$ is a σ-field.

(i) $\because \mathcal{X}^c = \phi$ is countable $\Rightarrow \mathcal{X} \in \mathcal{A}$.

(ii) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ (by def^n of $\mathcal{A}$)

(iii) countable union $\in \mathcal{A}$ by part (b).

(d) Let $\mathcal{S} = \{\{x\} : x \in \mathbb{R}\}$ and $\sigma(\mathcal{S})$ be the smallest σ-field containing $\mathcal{S}$. We need to show $\sigma(\mathcal{S}) = \mathcal{A}$.

Note that $\{x\}$ is a countable set $\forall x \in \mathbb{R}$

$\Rightarrow \mathcal{S} \subseteq \mathcal{A}$.

$\because \mathcal{A}$ is a σ-field, $\sigma(\mathcal{S}) \subseteq \mathcal{A}$.

Now, since every countable set is a countable union of singletons, every countable set must belong to $\sigma(\mathcal{S})$. Moreover, since every countable $A \in \sigma(\mathcal{S})$, $A^c \in \sigma(\mathcal{S})$ as $\sigma(\mathcal{S})$ is a σ-field.

$\Rightarrow \mathcal{A} \subseteq \sigma(\mathcal{S})$.

$\Rightarrow \mathcal{A} = \sigma(\mathcal{S})$.

(e) $\mathcal{A} \subsetneq \mathcal{B}$.

Note that $\{x\} \in \mathcal{B}$ $\forall x \in \mathbb{R}$.

$\Rightarrow \sigma(\mathcal{S}) \subseteq \mathcal{B}$.

Now consider an interval $(-\infty, x]$

Note that $(-\infty, x]^c = (x, +\infty)$

Both of these sets are not countable

$\Rightarrow (-\infty, x] \notin \mathcal{A}$.

Hence $\mathcal{A}$ is a strict subset of $\mathcal{B}$

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