EE 601 – Statistical Signal Analysis

Quiz #1 (20 Marks)

Date: 19/08/2023 Time: 11am-12:30pm

Q.1 Consider prob. space $(\Omega, \mathcal{F}, P)$ and let $A, B \in \mathcal{F}$. Prove or dis-prove: If A and B are independent, then A^c and B^c are also independent. **5 Marks**

Q.2 Consider sequence of sets $\{A_n\}_{n \geq 1}$ such that

$$A_n = \left(\frac{1}{n}, 1 + \frac{1}{n}\right).$$

Does $\lim A_n$ exist? If yes, find the limiting set. Show your work in detail. **5+2 Marks**

Q.3 Suppose a biased coin (prob. of heads is $p \in (0,1)$) is tossed n-times independently.

(a) Describe the probability space corresponding to the experiment. **4 Marks**

(b) Find probability that there are at least k heads given that the first two coin tosses resulted in tails. **4 Marks**

Solutions

Q.1 $A, B \in \mathcal{F} \Rightarrow A^c, B^c \in \mathcal{F} \Rightarrow A^c \cap B^c \in \mathcal{F}.$

Now we need to show: $P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$
given $P(A \cap B) = P(A) \cdot P(B)$.

$$P(A^c \cap B^c) = P((A \cup B)^c)$$ as $A^c \cap B^c = (A \cup B)^c$ by D'Morgan's law

$$\begin{aligned}
&= 1 - P(A \cup B) \\
&= 1 - (P(A) + P(B) - P(A \cap B)) \\
&= 1 - P(A) - P(B) + P(A \cap B) \\
&= (1 - P(A)) - P(B)(1 - P(A)) \quad \text{as } A \,\&\, B \text{ are ind} \\
&= (1 - P(A))(1 - P(B)) \\
&= P(A^c)\, P(B^c).
\end{aligned}$$

This proves the required.

— x — x — x —

$\boxed{\text{Q.2}}$ $A_n = \left(\frac{1}{n}, 1 + \frac{1}{n}\right) \quad \forall n = 1, 2, \ldots$

For $n = 1$, $A_n = (1, 2)$

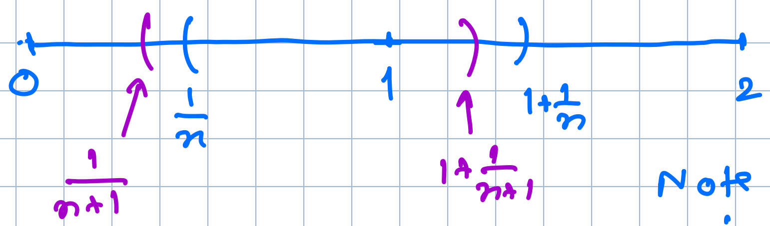
$n = 2$, $A_n = \left(\frac{1}{2}, \frac{3}{2}\right)$ and so on.

Clearly, A_n's are not monotone as neither
$A_1 \subseteq A_2$ nor $A_2 \subseteq A_1$.

Thus, to check if limit exists, we need to calulate $\limsup A_n$ and $\liminf A_n$ as they always exist.

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \quad \text{— (1)}$$

Consider, $\bigcup_{k=n}^{\infty} A_k = \bigcup_{k=n}^{\infty} \left(\frac{1}{k}, 1 + \frac{1}{k}\right)$



Note that A_k's are moving leftwards.

Thus in the union right boundary of A_n is retained while limiting left boundary is obtained. Thus,

$$\bigcup_{k=n}^{\infty} A_k = \left(0, 1+\frac{1}{n}\right) \quad - (2)$$

↑ note that 0 doesn't belong to any A_n, and hence is not the part of union.

Now, from (1) and (2),

$$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} \left(0, 1+\frac{1}{n}\right)$$

$$= (0, 1].$$

↑ Note that 1 belongs to every set and hence is a part of intersection.

Now consider,

$$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

Consider, $\displaystyle\bigcap_{k=n}^{\infty} A_k = \bigcap_{k=n}^{\infty} \left(\frac{1}{k}, 1+\frac{1}{k}\right)$

Note that: $\quad = \left(\frac{1}{n}, 1\right]$ (Refer to above fig and note that only these elements are in every set.)

Finally, $\displaystyle\liminf A_n = \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 1\right] = (0, 1].$

Thus, $\liminf A_n = \limsup A_n = (0, 1].$

Hence limit exists and equals $(0, 1]$.

——x——x——x——

Q.3 Coin is tossed n-times. Note that the experiment is n coin tosses. Hence, outcome space

$$\Omega = \{H, T\}^n, \text{ i.e.}$$

each outcome is n-touple $(\omega_1, \omega_2, \ldots, \omega_n)$, where $\omega_k \in \{H, T\}$ $\forall$ $k = 1, \ldots, n$.

Event space $\mathcal{F} = \mathcal{P}(\Omega)$ ← power set of Ω

(This can be done as Ω is finite).

Now, we have to define prob measure s.t. coin is unfair and tosses are independent

For $\omega = (\omega_1, \ldots, \omega_n)$, define $x_k(\omega) = 1$ if $\omega_k = H$

$= 0$ o.w.

Now, define: $\omega = (\omega_1, \ldots, \omega_n)$

$$P(\{\omega\}) = p^{\sum_{k=1}^{n} x_k(\omega)} (1-p)^{\underbrace{n - \sum_{k=1}^{n} x_k(\omega)}}$$

number of heads in ω.

# of tails in ω.

Note that: For example

$P(\text{first coin toss} = H, \text{second coin toss} = H)$

$$= P(\{\omega : \omega_1 = H, \omega_2 = H\})$$

$$= p^2 . \sum_{k=0}^{n-2} P(k \text{ heads in } \omega_3, \ldots, \omega_n)$$

$$= p^2 \underbrace{\sum_{k=0}^{n-2} \binom{n-2}{k} p^k (1-p)^{n-k-2}}_{=1}$$

$$= p^2 = P(\text{first coin toss} = H) \cdot P(\text{second coin toss} = H).$$

Can be proved for any given combination.

Now, P(at least k heads | first two tosses are heads)

$$= P\big(\underbrace{\{\omega : \text{at least } k \text{ heads}\}}_{A} \mid \underbrace{\{\omega : \omega_1 = H, \omega_2 = H\}}_{B}\big)$$

$$= P(A|B) = \frac{P(A\cap B)}{P(B)} \quad \text{- by def}^{n}.$$

$P(B) = p^2$ as shown above.

Now consider $A \cap B = \phi$ if $k = 0, 1$

$= \{\omega : k-2 \text{ heads in last } n-2 \text{ tosses and } \omega_1 = H \text{ \& } \omega_2 = H\}$ if $k = 2, \ldots, n$

$$\Rightarrow P(A\cap B) = P(\phi) = 0 \quad \text{if } k = 0, 1$$

$$= p^2 \sum_{k=0}^{n-2} p^k (1-p)^{n-2-k} \binom{n-2}{k}$$

$$\Rightarrow P(A|B) = 0 \quad \text{if } k = 0, 1,$$

$$= \sum_{k=0}^{n-2} \binom{n-2}{k} p^k (1-p)^{n-k-2} \quad \text{for } k = 2, \ldots, n.$$

— x — x — x —