EE601: Statistical Signal Analysis

Quiz # 5

Dt: 19/10/2024 Time: 10:30 - 12

Q.1 A household uses light bulbs that have independent lifetimes with each having exponential distribution with average lifetime of 5 days. Bulbs are replaced instantaneously upon failure.

(a) Find prob that no more than 100 bulbs are used in an year? **[5 marks]**

(b) Find prob that exactly 10 bulbs are used in a month? **[5 Marks]**

Q.2 Let X be a random variable s.t. $E[X] < 0$. Prove or disprove: $E[e^{\theta X}] = 1 \Rightarrow \theta \geq 0$. **[5 Marks]**

Q.3 Consider random variables $X_1, \dots, X_n$ with joint density fn given by:

$$f(x_1, \dots, x_n) = 1 \quad \text{if } (x_1, \dots, x_n) \in (0,1)^n.$$

(a) Find $P(X_1 = \min\{X_1, \dots, X_n\})$. **[3 Marks]**

(b) Let D denote the distance of $(X_1, \dots, X_n)$ from origin. Find $E[D^2]$. **[2 Marks]**

Solutions

$\boxed{Q.1}$ Density of exp r.v. with parameter $\lambda > 0$ is $\lambda e^{-\lambda x}$ for $x \geq 0$ and 0 o.w.

Also, the corresponding expected value is $1/\lambda$.

Thus, $f_{X_i}(x) = \frac{1}{5} e^{-x/5}$ for $x \geq 0$,

$\qquad\qquad = 0 \qquad$ o.w.

i.e. $\lambda = 1/5$.

(a) P(No more than 100 lightbulb in an year)

$$= P\left(\sum_{k=1}^{100} X_k \geq 365\right) = P\left(\sum_{k=1}^{100} X_k > 365\right)$$

As shown in assignment, sum of n iid $\exp(\lambda)$ r.v.'s is Gamma(n, λ) random variable.

So, let $Y = \sum_{k=1}^{100} X_k$, then $Y \sim \text{Gamma}(100, 1/5)$

$$f_Y(y) = \frac{1}{99!} \left(\frac{1}{5}\right)^{100} y^{99} e^{-y/5}$$

Thus, $P(Y > 365) = \int_{365}^{\infty} f_Y(y)\, dy$.

(b) P(exactly 10 lightbulbs in a month)

$$= P\Big(\underbrace{\sum_{k=1}^{9} X_k}_{\tilde{Y}} \leq 30, \sum_{k=1}^{10} X_k > 30\Big) \qquad \tilde{Y} \sim \text{Gamma}(9, 1/5)$$

$$= P(\tilde{Y} \leq 30, \tilde{Y} + X_{10} > 30)$$

$$= \int_0^{30} P(y + X_{10} > 30 \mid \tilde{Y} = y)\, f_{\tilde{Y}}(y)\, dy$$

$$= \int_0^{30} P(X_{10} > 30 - y \mid \tilde{Y} = y)\, f_{\tilde{Y}}(y)\, dy$$

$$= \int_0^{30} P(X_{10} > 30-y)\, f_{\tilde{Y}}(y)\, dy \qquad \because \tilde{Y} \perp\!\!\!\perp X_{10}$$

$g(X_1,\ldots,X_9)$ & X_k's iid.

$$= \int_0^{30} e^{-(30-\cancel{y})/5} \frac{1}{8!}\left(\frac{1}{5}\right)^9 y^8 \cancel{e^{-y/5}}\, dy$$

$$= \left(\frac{1}{5}\right)^9 e^{-6} \cdot \frac{1}{8!} \int_0^{30} y^8\, dy$$

$$= \left(\frac{1}{5}\right)^9 e^{-6} \cdot \frac{1}{8!} \left.\frac{y^9}{9}\right|_0^{30}$$

$$= e^{-6} \frac{1}{9!} \cdot \left(\frac{1}{5}\right)^9 \cdot 30^9 = e^{-6} \frac{6^9}{9!}.$$

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$\boxed{Q.2}$ Need to prove if $E[X] < 0$ and $E[e^{\theta X}] = 1$, then $\theta \geq 0$.

Let $g(x) = e^{\theta x}$. Note that $g(\cdot)$ is a convex function $\forall \theta \in \mathbb{R}$. Thus by Jensen's inequality

$$1 = E[e^{\theta X}] \geq e^{\theta E X}$$

$\because e^{\theta EX} \leq 1$, we need $\theta EX \leq 0$

$\because EX < 0$, θ must be ≥ 0 for θEX to be ≤ 0.

— x — x — x —

$\boxed{Q.3}$ $f(x_1,\ldots,x_n) = 1$ if $(x_1,\ldots,x_n) \in (0,1)^n$.

Note that $(X_1,\ldots,X_n)$ are iid Uniform$((0,1))$ random variables.

(a) $P(X_1 = \min\{X_1,\ldots,X_n\}) = 1/n$

as $X_1,\ldots,X_n$ are iid and thus each one has equal chance of being the smallest.

(b) $D^2 = X_1^2 + X_2^2 + \cdots + X_n^2$.

$E[D^2] = E[X_1^2] + E[X_2^2] + \cdots + E[X_n^2]$ - by linearity of the expectatⁿ

$= n\,E[X_1^2]$ - $\because$ the r.v.'s are iid.

$E[X_1^2] = \int_0^1 x^2\,dx = \frac{1}{3}$.

$\Rightarrow E[D^2] = \frac{n}{3}$.

Note: How do we argue $(X_1, \ldots, X_n)$ are iid.

choose any k random variable from the given n

and their join distr. is

$f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = 1$ if $(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \in (0,1)^k$.

obtained by integrating out other variables.

Now, note that marginals (obtained by considering $k=1$) is Uniform$((0,1))$ for each r.v.

Thus,

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = f_{\sigma(1)}(x_{\sigma(1)}) \cdots f_{\sigma(k)}(x_{\sigma(k)})$$

$\Rightarrow X_1, \ldots, X_n$ are iid Uniform$((0,1))$.