Q.1 Define moment generating function (MGF) for r.v. X as $M_X(t) = E[e^{tX}]$.

(a) Does moment generating function always exist? Give an example where MGF does not exist. State conditions under which MGF exists?

(b) When MGF exists in some neighbourhood of zero, i.e. for $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, then explain how we can find moments of the random variable.

Q.2 Find MGF and characteristic fn (CF) for:

(a) Bernoulli(p); $X = 1$ w.p. p, $= 0$ w.p. $(1-p)$ for $p \in [0,1]$

(b) Binomial$(n;p)$: $X = k$ w.p. $\binom{n}{k} p^k (1-p)^{n-k}$, $\forall\, k = 0, 1, \ldots, n$, where for n integer and $p \in [0,1]$

(c) Poisson(λ): $X = k$ w.p. $e^{-\lambda}\frac{\lambda^k}{k!}$, for $k \in \{0, 1, \ldots\}$ & parameter $\lambda > 0$.

(d) Geometric(p): $X = k$ w.p. $(1-p)^{k-1} p$, for $k \in \{1, 2, \ldots\}$ & parameter $p \in [0,1]$.

(e) Exp(λ): $f_X(x) = \lambda e^{-\lambda x}$ for $x \in [0, \infty)$ where parameter $\lambda > 0$.

(f) Gamma$(n;\lambda)$: $f_X(x) = \lambda^n \frac{x^{n-1}}{(n-1)!} e^{-\lambda x}$ for $x \in [0, \infty)$ parameters $n \in \{1, 2, \ldots\}$ and $\lambda > 0$.

(g) Uniform((a,b)) : $f_X(x) = \frac{1}{b-a}$ for $x \in (a,b)$

$= 0$ otherwise.

(h) Uniform($\{a_1, \dots, a_n\}$) : $P(X = a_k) = \frac{1}{n}, \forall k \in \{1, \dots, n\}$.

(i) $G(\mu, \sigma^2)$ or $N(\mu, \sigma^2)$: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$, for $x \in \mathbb{R}$

Gaussian normal where parameters $\mu \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}_+$.

(j) $\chi^2(n)$: $f_X(x) = \frac{x^{n/2 - 1} \cdot e^{-x/2}}{2^{n/2} \Gamma(n/2)}$ $\forall x \in \mathbb{R}_+$

chi-square

Gamma fn

$= 0$ o.w.

(k) X s.t. $f_X(x) = c_\alpha x^{-\alpha}$ for $x \geq 1$

$= 0$ o.w.

parameter $\alpha \geq 2$.

$\boxed{\text{Q.3}}$ Let X be a discrete random variable with prob. mass fn. (pmf) as given below:

$$P(X = x) = kx \quad \text{for } x = 1, 2, 4$$
$$= k(x-1) \quad \text{for } x = 3, 5, 6$$
$$= 0 \quad \text{otherwise}$$

(a) Find k.

(b) Find $E[X]$ and $E[X^2]$.

$\boxed{\text{Q.4}}$ Let X be Poisson(λ). Find $E[X]$ and var(X).

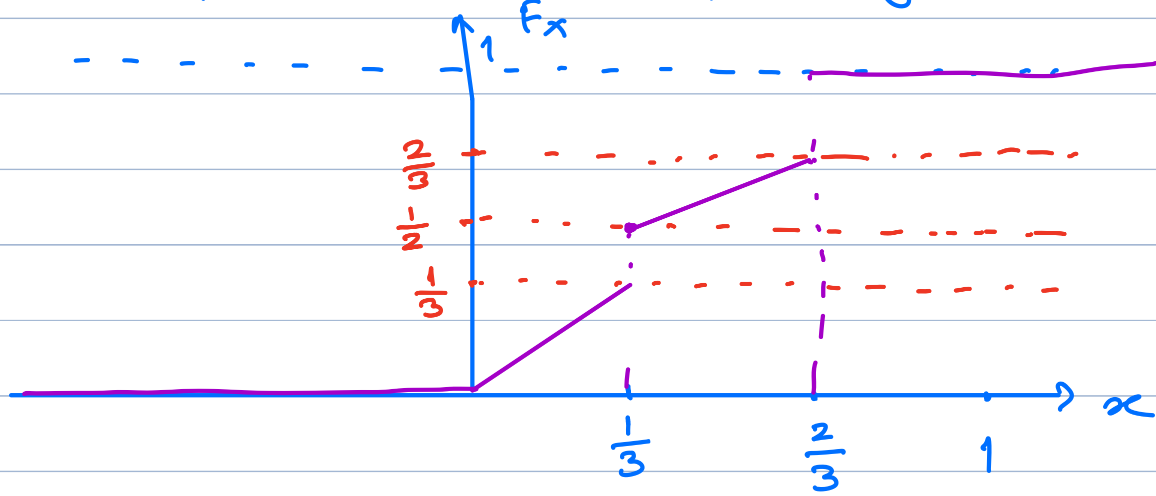
Q.5 Let $X \sim F_X$ (X has distribution $F_X(\cdot)$), and $Y = 1\{X > 0\}$. Show that Y is a random variable and find $F_Y(\cdot)$.

Q.6 Let $X \sim F_X$, and $Y = F_X(X)$. Show that Y is a random variable and find $F_Y(\cdot)$. (Assume that F_X is strictly increasing fn).

Q.7 Let $X \sim \text{Uniform}([0,1])$ and $Y = -\log X$. Show that Y is a random variable and find $F_Y(\cdot)$.

Q.8 Let $X \sim G(0,4)$ and $Y = 3X^2$. Find $E[Y]$ & $var(Y)$.

Q.9 Let $X \sim F_X$ where F_X is given bleow:



Write F_X as a convex combination of two distribution functions F_Y and F_Z, where F_Y is continuous and F_Z is discrete.

$\boxed{Q.10}$ Let X be a random variable from $(\Omega, \mathcal{F}, P)$ to $(\mathcal{X}, \mathcal{B}, P_X)$. Show the following:

ⓐ If $A = X^{-1}(B)$, then $A^c = X^{-1}(B^c)$.

ⓑ If $A_n = X^{-1}(B_n)$ $\forall n = 1, \dots$, then $\bigcup_{n=1}^{\infty} A_n = X^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right)$.

ⓒ If $A_1 = X^{-1}(B_1)$ and $A_2 = X^{-1}(B_2)$ and $B_1 \cap B_2 = \phi$, then $A_1 \cap A_2 = \phi$.

ⓓ Show that $\{A : \exists B \in \mathcal{B} \text{ s.t. } X^{-1}(B) = A\}$ is a σ-field on Ω and is a subset of $\mathcal{F}$.

Remark: $\{A : \exists B \in \mathcal{B} \text{ s.t. } X^{-1}(B) = A\}$ is called σ-field generated by X. Moreover, X can provide information only about this σ-field. This σ-field is denoted by $X^{-1}(\mathcal{B})$.

ⓔ Show that P_X is a prob. measure of $(\mathcal{X}, \mathcal{B})$.