EE601: Statistical Signal Analysis

Quiz #6

Date: 26/10/2024 Time: 11:30 - 1 pm

Q.1 Consider a non-negative r.v. X.

(a) Prove or disprove: $\sum_{k=0}^{\infty} P(X \geq k) \geq E[X] \geq \sum_{k=1}^{\infty} P(X \geq k)$.

Let $\{X_n\}_{n\geq 1}$ be a sequence of iid random variables such that $F_{X_1}(x) = F_X(x) \; \forall x \in \mathbb{R}$, and $E[X_1] \in (0,\infty)$.

(b) Prove or disprove:

$$P\left(\limsup_{n\uparrow\infty} \{\omega : X_n(\omega) \geq k\}\right) = 0 \quad \forall k \in (0,\infty).$$

(c) Prove or disprove:

$$P\left(\limsup_{n\uparrow\infty} \{\omega : X_n(\omega) \geq n\}\right) = 0.$$

Q.2 Let $\{A_n\}_{n\geq 1}$ be monotone non-decreasing seq. of events. Prove or disprove: If $\lim_{n\uparrow\infty} P(1_{A_n} = 1) = 1$, then $1_{A_n} \to 1$ a.s.

Solutions

$\boxed{Q.1}$ (a) $\because$ X is a non-negative r.v.

$$E[X] = \int_0^\infty F_X^c(x)\,dx$$
$$= \sum_{n=0}^{\infty} \int_n^{n+1} F_X^c(x)\,dx$$

Note that

$$F_X^c(n) \geq F_X^c(x) \geq F_X^c(n+1) \quad \forall x \in [n, n+1].$$

$$F_X^c(n) \geq \int_n^{n+1} F_X^c(x)\,dx \geq F_X^c(n+1)$$

Thus,

$$\sum_{n=0}^{\infty} F_X^c(n+1) \leq E[X] \leq \sum_{n=0}^{\infty} F_X^c(n)$$

$$\Rightarrow \sum_{n=1}^{\infty} F_X^c(n) \leq E[X] \leq \sum_{n=0}^{\infty} F_X^c(n)$$

$$\Rightarrow \sum_{n=1}^{\infty} P(X \geq n) \leq E[X] \leq \sum_{n=0}^{\infty} P(X \geq n).$$

(b) $\because$ $E[X] > 0$, $\exists\ \varepsilon > 0$ & $p > 0$ s.t.

$$P(X \geq \varepsilon) = p.$$

Now, consider $A_n \overset{\text{def}}{=} \{\omega : X_n(\omega) \geq \varepsilon\}$

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} p = \infty.$$

Now, note that $\because$ $\{X_n\}$ are iid, $\{A_n\}$ are also independent. Thus, by BC Lemma 2,

$$P(\limsup A_n) = 1.$$

This shows that the given statement is false.

(c) Define, $A_n = \{\omega : X_n(\omega) \geq n\}$

consider,

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} P(X_n \geq n)$$

$$= \sum_{n=1}^{\infty} P(X \geq n)$$

$$\leq E[X] < \infty$$

Thus, by BC Lemma I,

$$P(\limsup A_n) = 0$$

This proves the required.

Q.2 We know

$$\lim_{n \uparrow \infty} P(\mathbb{1}_{A_n} = 1) = 1.$$

$$\Rightarrow \lim_{n \uparrow \infty} P(\mathbb{1}_{A_n} = 0) = 0$$

$$\Rightarrow \lim_{n \uparrow \infty} P(|\mathbb{1}_{A_n} - 1| > \varepsilon) = 0 \quad \forall \varepsilon > 0.$$

$$\Rightarrow \mathbb{1}_{A_n} \to 1 \text{ in prob.}$$

Moreover, note that $\{\mathbb{1}_{A_n}\}_{n \geq 1}$ is a monotone seq. As shown in tutorial problem, if a monotone seq converges in prob, then it also converges a.s. Thus,

$$\mathbb{1}_{A_n} \to 1 \text{ a.s.}$$
