(4) Let $\{X_n\}_{n\geq 1}$ be a sequence of independent r.v.'s such that
$$P\{X_n = 0\} = \frac{n-1}{n} \text{ \& } P\{X_n = \log n\} = \frac{1}{n}.$$
~~Sho~~ (a) Does X_n converge a.s.?
(b) Does X_n converge in m.s. sense?

Assignment :-

(a) Show that if $X_n \to X$ in probability, then $\exists$ a subsequence $\{n_k\}_{k\geq 1}$ such that $X_{n_k} \to X$ a.s. as $k \to \infty$.

Hint: Choose n_k s.t. $P\{|X_{n_k} - X| > 2^{-k}\} \leq 2^{-k}$.

(b) Show that if $X_n \to X$ in probability & $X_n \to Y$ a.s., then $P(X = Y) = 1$.

(c) Show that if $X_n \to X$ in probability & $X_n \leq Y$ for all n for some r.v. Y, then $P\{X \leq Y\} = 1$.

(d) Show that if $\{X_n\}_{n\geq 1}$ is a monotone sequence, then $X_n \to X$ in prob. $\Rightarrow$ $X_n \to X$ a.s.

Show that
$$\sum_{k=1}^{\infty} P(|X| \geq k) \leq E|X| \leq \sum_{k=0}^{\infty} P(|X| \geq k).$$