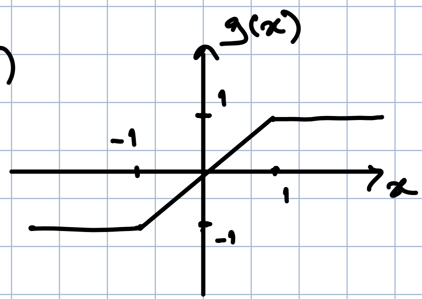
EE601: Statistical Signal Analysis

Quiz # 2

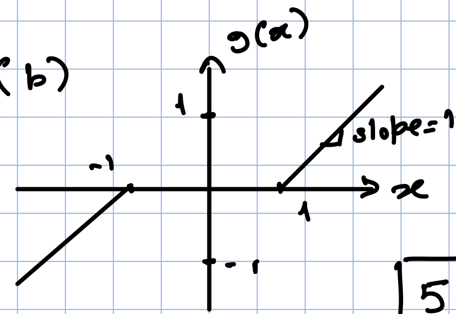
Date: 08/09/2023 Time: 10:30 – 12:00

Q.1 Consider $X \sim F_X$ and $Y = g(X)$. Find F_Y for $g(\cdot)$ shown below:

(a)



(b)



$\boxed{5+5\text{ M}}$

Q.2 Let $X \sim \exp(\lambda)$ for $\lambda > 0$ and $Y = \min\{X, X^2\}$. Find f_Y. $\boxed{5\text{M}}$

Q.3 Consider measurable space $(\Omega, \mathcal{P}(\Omega))$, where $\Omega = \{1,2,3,4,5,6\}$. Let $X : \Omega \to \mathbb{R}$ defined as follows: $X(\omega) = \max\{\omega - 4.5, 0\}$.

Find $X^{-1}(\mathcal{B})$. $\boxed{5\text{M}}$

Hint: You can assume that $X^{-1}(\mathcal{B})$ is the smallest σ-field generated by $X^{-1}(\{(-\infty, x] : x \in \mathbb{R}\})$.

Solutions

Q.1(a): Consider any $y < -1$,

$$P(Y \leq y) = P(g(X) < -1)$$
$$= P(\phi) = 0.$$

Now consider $y \in [-1, 1]$

$$P(Y \leq y) = P(g(X) \leq y)$$
$$= P(X \leq y) = F_X(y).$$

Now for $y > 1$

$$P(Y \leq y) = P(g(X) \leq y)$$
$$= P(\Omega) = 1.$$

$$\Rightarrow \quad F_Y(y) = \begin{cases} 0 & \text{if } y < -1 \\ F_X(y) & \text{if } y \in [-1, 1] \\ 1 & \text{if } y > 1. \end{cases}$$

Q.1(b):

$$g(x) = \begin{cases} x+1 & \text{if } x < -1 \\ 0 & \text{if } x \in [-1, 1] \\ x-1 & \text{if } x > 1. \end{cases}$$

Consider $y < 0$,

$$P(Y \leq y) = P(g(x) \leq y)$$
$$= P(X+1 \leq y)$$
$$= P(X \leq y-1) = F_X(y-1).$$

Consider $y = 0$.

$$P(Y \le y) = P(g(X) \le y)$$
$$= P(X \le 1) = F_X(1).$$

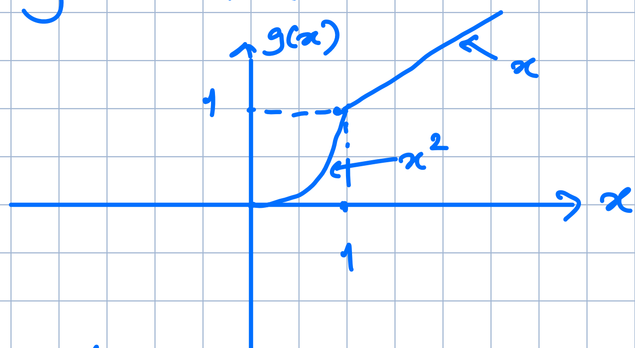
Consider $y > 0$:

$$P(Y \le y) = P(g(X) \le y)$$
$$= P(X - 1 \le y) = P(X \le y+1)$$
$$= F_X(y+1)$$

Q.2: $Y = \min(X, X^2)$

$$= X^2 \quad \text{if } X \in [0,1]$$
$$= X \quad \text{o.w.}$$

Enough to focus on $X \in \mathbb{R}_{0+}$ as $P(X<0) = 0$



Consider $y \in [0,1]$

$$F_Y(y) = P(Y \le y)$$
$$= P(X^2 \le y)$$
$$= P(X \le +\sqrt{y}) = F_X(\sqrt{y}) = 1 - e^{-\lambda\sqrt{y}}$$

Consider $y > 1$

$$F_Y(y) = P(Y \le y)$$
$$= P(X \le y) = F_X(y) = 1 - e^{-\lambda y}$$

$$F_Y(y) = 0 \quad \text{if } y < 0.$$

$\boxed{Q.3}$ $X(\omega) = 0$ if $\omega = 1, 2, 3, 4,$

$X(\omega) = 0.5$ if $\omega = 5$, and

$= 1.5$ if $\omega = 6.$

$X \in \{0, 0.5, 1.5\}.$

Easy to see that $X^{-1}(\{(-\infty, x] : x \in \mathbb{R}\})$, say $\mathcal{A}$,

$$= \{\underbrace{\phi}_{\forall x < 0}, \underbrace{\{1,2,3,4\}}_{\forall x \in [0, 0.5)}, \underbrace{\{1,2,3,4,5\}}_{\forall x \in [0.5, 1.5)}, \underbrace{\Omega}_{\forall x \geq 1.5}\}$$

The smallest σ-field containing $\mathcal{A}$ is the same as the smallest σ-field containing $\mathcal{A}'$, where

$$\mathcal{A}' = \{\{1,2,3,4\}, \{5\}, \{6\}\} \leftarrow \boxed{\text{Partition of } \Omega}$$

$$\sigma(\mathcal{A}') = \{\phi, \{1,2,3,4\}, \{5\}, \{6\}, \{1,2,3,4,5\}, \{1,2,3,4,6\}, \{5,6\}, \Omega\}$$

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