EE601: Statistical Signal Analysis

Quiz #8

Dt: 12/11/2024 Time: 5:30 - 7 pm

$\boxed{\text{Q.1}}$ Let X number of candidates apear for an interview. Each of the apearing candidates is selected with prob p, independant of other selections. If $X \sim \text{Poisson}(\lambda)$, then find pmf of number of selected candidates. $\boxed{\text{7 Marks}}$

$\boxed{\text{Q.2}}$ Let joint density of random variables X & Y is:

$$f_{XY}(x,y) = \frac{e^{-x/y}\, e^{-y}}{y}, \quad 0 < x, y < \infty$$
$$= 0 \quad \text{otherwise.}$$

Find $E[P(X>1 \mid Y)]$. $\boxed{\text{7 Marks}}$

$\boxed{\text{Q.3}}$ Let $\{X_n\}_{n\geq 1}$ be a sequence of iid r.v.'s with marginal density $f(x) = e^{-x+\theta}$, for $x \geq \theta$ and zero otherwise. Prove or disprove:

$$\lim_{n\uparrow\infty} \min\{X_1, \dots, X_n\} = \theta \quad \text{w.p. 1.}$$

$\boxed{\text{6 Marks}}$

EE601: Statistical Signal Analysis

Quiz # 9

Dt: 12/11/2024 Time: 7 - 8:30pm

Q.1 You have invited 1000 guests for your wedding. Each guest gives a gift of expected value of Rs 2000 with variance Rs 800. Find the prob. (approximately) that the total value of gifts you receive is between Rs. 15 lakh and Rs. 22 lakh. **[7 Marks]**

Q.2 $(X_1, \dots, X_n) \sim$ iid Bernoulli(p), $p \in (0,1)$. If $\psi(p) = (1-p)^2$.

(a) Find unbiased estimator $\delta(\vec{x})$.

(b) Find $E[\delta(\vec{x}) / \sum_{k=1}^{n} X_k]$. **[6 Marks]**

Q.3 Let $(X_1, \dots, X_n)$ be independent random variables with $X_k \sim \text{Uniform}((0, \theta_k))$, s.t. $0 < \theta_1 \le \theta_2 \le \dots \le \theta_n < \infty$. Find

$$P(X_k = \min\{X_1, \dots, X_n\}).$$

[7 Marks]

Solutions for Quiz #8

Q.1) Let Y denote the number of selected candidates.

Note that:

$$P(Y=y \mid X=n) = \binom{n}{y} p^y (1-p)^{n-y} \quad \forall y \in \{0,\dots,n\}$$
$$= 0 \quad \text{otherwise.}$$

So,

$$P(Y=y) = \sum_{n=y}^{\infty} P(Y=y \mid X=n)\, P(X=n)$$
$$= \sum_{n=y}^{\infty} \binom{n}{y} p^y (1-p)^{n-y} \cdot e^{-\lambda} \frac{\lambda^n}{n!}$$
$$= e^{-\lambda} p^y \sum_{n=y}^{\infty} \binom{n}{y} \left[(1-p)\lambda\right]^{n-y} \frac{\lambda^y}{n!}$$
$$= e^{-\lambda} (\lambda p)^y \sum_{n=y}^{\infty} \frac{\cancel{n!}}{y!\,(n-y)!} \left[(1-p)\lambda\right]^{n-y} \cdot \frac{1}{\cancel{n!}}$$
$$= \frac{e^{-\lambda}(\lambda p)^y}{y!} \sum_{n=0}^{\infty} \frac{1}{n!} \left[(1-p)\lambda\right]^{n}$$
$$= \frac{e^{-\lambda}(\lambda p)^y}{y!}\, e^{-(1-p)\lambda}$$
$$= e^{-\lambda p} \frac{(\lambda p)^y}{y!}.$$

Thus, $Y \sim \text{Poisson}(\lambda p)$.

——— x ——— x ——— x ———

Q.2) $$f_{XY}(x,y) = \frac{e^{-x/y} \cdot e^{-y}}{y}, \quad 0 < x, y < \infty$$
$$= 0 \quad \text{o.w.}$$

Approach I: Consider,

$$P(X>1 \mid Y=y) = \int_1^{\infty} f_X(x \mid Y=y)\, dx \quad \text{— (1)}$$

Now, $f_X(x|Y=y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$ — ②

Now, $f_Y(y) = \int_0^\infty f_{XY}(x,y)\, dx$

$= \int_0^\infty \dfrac{e^{-x/y} e^{-y}}{y}\, dx = \dfrac{e^{-y}}{y} \int_0^\infty e^{-x/y}\, dx$

$= \dfrac{e^{-y}}{\cancel{y}} \left. -\cancel{y} \cdot e^{-x/y} \right|_0^\infty = e^{-y}$. for $y \in (0,\infty)$.

$\leftarrow$ exp(1)

$= 0$ otherwise.

by ② $f_X(x|Y=y) = \frac{1}{y} e^{-x/y}$ for $0 < x, y < \infty$

$= 0$ o.w. $\leftarrow$ exp(1/y)

by ① $P(X>1|Y=y) = e^{-1/y}$.

$\Rightarrow P(X>1|Y) = e^{-1/Y} = g(Y)$

Thus, $E[P(X>1|Y)] = \int_0^\infty e^{-1/y} e^{-y}\, dy$

$= \int_0^\infty e^{-(y+1/y)}\, dy$.

Approach II: Note that

$P(X>1|Y) = E[1_{\{X>1\}}|Y]$

Thus, $E[P(X>1|Y)] = E[E[1_{\{X>1\}}|Y]]$

$= E[1_{\{X>1\}}]$

$= P(X>1)$

$= \int_{y=0}^\infty \int_{x=1}^\infty f_{XY}(x,y)\, dx\, dy$

$= \int_{y=0}^\infty \int_{x=1}^\infty \dfrac{e^{-x/y} e^{-y}}{y}\, dx\, dy$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} \, -ye^{-x/y}\Big|_1^{\infty} \, dy$$

$$= \int_0^{\infty} e^{-(y+1/y)} \, dy.$$

Q.3) $f(x) = e^{-x+\theta}$, for $x \geq \theta$
$= 0$ otherwise.

Let $Y_n = \min\{X_1, \dots, X_n\}$ & consider $\varepsilon > 0$

Now, $P(Y_n > \theta + \varepsilon)$

$$= P(\min\{X_1, \dots, X_n\} > \theta + \varepsilon)$$

$$= P(X_1 > \theta+\varepsilon, \dots, X_n > \theta+\varepsilon)$$

$$= [P(X_1 > \theta+\varepsilon)]^n$$

Now, $P(X_1 > \theta+\varepsilon)$

$$= 1 - \int_{\theta}^{\theta+\varepsilon} e^{-x+\theta} \, dx \qquad u = x - \theta$$

$$= 1 - \int_0^{\varepsilon} e^{-u} \, du \stackrel{\Delta}{=} p_\varepsilon \in (0,1) \quad \forall \varepsilon > 0.$$

Thus, $P(Y_n > \theta+\varepsilon) = p_\varepsilon^n$ — (1)

Approach I :

Note that

$$\lim_{n\uparrow\infty} P(Y_n > \theta+\varepsilon) = \lim_{n\uparrow\infty} p_\varepsilon^n = 0 \quad \forall \varepsilon > 0.$$

$\Rightarrow$ $Y_n \to \theta$ in prob/weakly. — (2)

Now, Y_n is monotone decreasing seq, that

$$Y_n \geq Y_{n+1} \quad \forall n.$$

As shown is tutorial:

If monotone seq converges weakly to a random variable, then the seq. converges strongly to the same random variable.

Thus, from (2) $Y_n \to \theta$ a.s.

Approach II:

Define, $B_n^{\varepsilon} = \{\omega : Y_n(\omega) > \theta + \varepsilon\}$

Now: $\sum_{n=1}^{\infty} P(B_n^{\varepsilon}) = \sum_{n=1}^{\infty} p_{\varepsilon}^{n} < \infty.$

Thus, by BC Lemma I,

B^{ε} $P(\limsup B_n^{\varepsilon}) = 0. \quad \forall \varepsilon > 0.$

Now, $\{\omega : Y_n \not\to \theta\} = \bigcup_{m=1}^{\infty} B^{1/m}$

$\Rightarrow P(Y_n \not\to \theta) \leq \sum_{m=1}^{\infty} P(B^{1/m}) = 0$

$\Rightarrow P(Y_n \not\to \theta) = 0 \Rightarrow P(Y_n \to \theta) = 1.$

Solutions for Quiz #9

Q.1) Let X_k denote the value of gift from k^{th} guest. Let $n = 1000$, $\mu = 2000$ & $\sigma^2 = 400$.

$x_1 = 15,00,000$ & $x_2 = 22,00,000$.

$$P\left(x_1 \le \sum_{k=1}^{n} X_k \le x_2\right)$$

$$= P\left(\frac{x_1 - n\mu}{\sqrt{n}\,\sigma} \le \frac{\Sigma X_k - n\mu}{\sqrt{n}\,\sigma} \le \frac{x_2 - n\mu}{\sqrt{n}\,\sigma}\right)$$

$\rightarrow Z \sim G(0,1)$ in distribution.

$$= \int_{\frac{x_1 - n\mu}{\sqrt{n}\,\sigma}}^{\frac{x_2 - n\mu}{\sqrt{n}\,\sigma}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

———x———x———x———

Q.2) Define $\delta(\vec{x}) = (1-x_1)(1-x_2)$.

$$E_p[\delta(\vec{x})] = (1-p)^2, \quad \forall p$$

$\Rightarrow \delta(\vec{x})$ is an unbiased estimator for $\psi(p)$.

Now, consider

$$E[\delta(\vec{x}) \mid \Sigma X_k = t]$$

$$= P(\delta(\vec{x}) = 1 \mid \Sigma X_k = t)$$

$$= \frac{P(X_1 = 0, X_2 = 0, \Sigma X_k = t)}{P(\Sigma X_k = t)}$$

$$= \frac{(1-p)^2\, P\left(\sum_{k=3}^{n} X_k = t\right)}{P(\Sigma X_k = t)} \quad \forall t = 0, 1, \dots, n-2.$$

$$= \frac{\cancel{(1-p)^2} \binom{n-2}{t} \cancel{p^t}\, \cancel{(1-p)^{n-t-2}}}{\binom{n}{t} \cancel{p^t}\, \cancel{(1-p)^{n-t}}}$$

$$= \frac{(n-t)(n-t-1)}{n(n-1)}.$$

Thus, $E[\delta(\vec{x}) \mid \Sigma X_k] = \dfrac{(n-\Sigma X_k)(n-\Sigma X_k-1)}{n(n-1)}.$

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Q.3) $P(X_k = \min\{X_1, \dots, X_n\})$

$$= \int_0^{\theta_k} P(X_k = \min\{X_1, \dots, X_n\} \mid X_k = x)\, f_{X_k}(x)\, dx$$

$$= \int_0^{\theta_k} P(X_1 \geq x, \dots, X_{k-1} \geq x, X_{k+1} \geq x, \dots, X_n \geq x)\, f_{X_k}(x)\, dx$$

$$= \int_0^{\theta_1} \prod_{i \neq k} \frac{x-\theta_i}{\theta_i} \cdot \frac{1}{\theta_k}\, dx.$$

$$= \frac{1}{\theta_k} \prod_{i \neq k} \int_0^{\theta_1} \left(\frac{x}{\theta_i} - 1\right) dx$$

$$= \frac{1}{\theta_k} \prod_{i \neq k} \left(\frac{\theta_1^2}{2\theta_i} - \theta_1\right) = \frac{\theta_1^{n-1}}{\theta_k} \prod_{i \neq k} \left(\frac{\theta_1}{2\theta_i} - 1\right)$$

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