EE601: Statistical Signal Analysis.

Quiz # 4

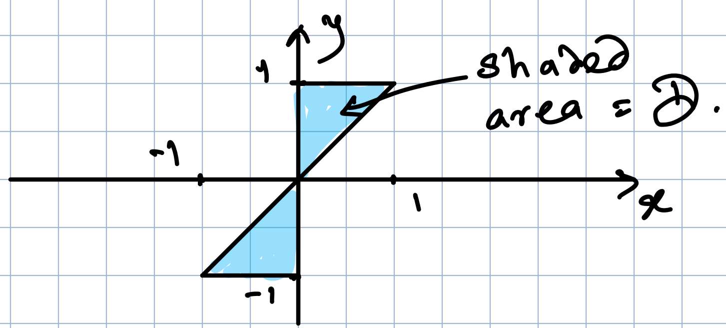
Date: 05/11/2024 Time: 10:30 - 12 pm

Q.1 Let $X \sim \text{Uniform}(0,1)$ & $Y \sim \text{Poisson}(\lambda)$ be independent random variables. Find the expected area of a square with side length XY.

Q.2 Let X_1 & X_2 be i.i.d. random variables. Find $E[X_1 \mid X_1 + X_2 = x]$.

Q.3 Let X & Y be iid $\text{Uniform}((0,\theta))$ for $\theta > 0$. Find $F_X(x \mid \max(X,Y) = u)$.

Q.4 Let $f_{XY}(x,y) = c$ for $(x,y) \in \mathcal{D}$. Let $Z = X+Y$ and $W = X-Y$. Find $f_{ZW}(\cdot,\cdot)$.



Solution

$\boxed{Q.1}$ Area of square $= X^2Y^2$

$\Rightarrow E[\text{Area}] = E[X^2Y^2]$ $\because X \perp\!\!\!\perp Y \Rightarrow X^2 \perp\!\!\!\perp Y^2$

$= E[X^2]\,E[Y^2]$ $\because$ Independent random variables are un-correlated

$$E[X^2] = \int_0^1 x^2\,dx = \left.\frac{x^3}{3}\right|_0^1 = 1/3.$$

Moment generating fn of Y

$$\Phi_Y(t) = E[e^{tY}]$$

$$= \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t\lambda)^k}{k!} = e^{-\lambda} \cdot e^{e^t\lambda}$$

$$= e^{-\lambda(1-e^t)}$$

$$\Phi_Y'(t) = e^{-\lambda} e^{e^t\lambda} \cdot \lambda e^t.$$

$$\Phi''(t) = e^{-\lambda}\lambda\left[e^{e^t\lambda} e^t + e^t \cdot e^{e^t\lambda} \cdot \lambda e^t\right]$$

$$\Phi''(t)\big|_{t=0} = \lambda e^{-\lambda}\left[e^{\lambda} + e^{\lambda}\cdot\lambda\right]$$

$$E[Y^2] = \lambda + \lambda^2.$$

$$E[\text{Area}] = \frac{\lambda}{3}(1+\lambda).$$

$\boxed{Q.2}$ Note that because of symmetry

$$E[X_1 | X_1 + X_2 = x] = E[X_2 | X_1 + X_2 = x]$$

Thus, $2E[X_1 | X_1+X_2=x] = E[X_1 | X_1+X_2=x] + E[X_2 | X_1+X_2=x]$

$$= E[X_1 + X_2 \mid X_1 + X_2 = x]$$ - by linearity of the expectation

$$= x.$$

$$\Rightarrow E[X_1 \mid X_1 + X_2 = x] = x/2.$$

—— x —— x —— x —— x ——

$\boxed{Q.3}$ Let $Z = \max(X, Y)$.

$$F_Z(z) = P(\max(X,Y) \le z)$$
$$= P(X \le z, Y \le z)$$
$$= \left(\frac{z}{\theta}\right)^2 \quad \text{if } z \in (0, \theta)$$
$$= 0 \quad \text{if } z \le 0$$
$$= 1 \quad \text{if } z \ge 1.$$

$$f_Z(z) = \frac{2z}{\theta^2} \quad \text{if } z \in (0, \theta)$$
$$= 0 \quad \text{o.w.}$$

Now,

$$F_X(x \mid \max(X,Y) = u) = F_X(x \mid Z = u)$$
$$= \lim_{\Delta u \downarrow 0} \frac{P(X \le x, u < Z \le u + \Delta u)}{P(u < Z \le u + \Delta u)}$$
$$= \lim_{\Delta u \downarrow 0} \frac{P(X \le x, u < Z \le u + \Delta u)/\Delta u}{P(u < Z \le u + \Delta u)/\Delta u}.$$

Note that $\lim_{\Delta u \downarrow 0} P(u < Z \le u + \Delta u)/\Delta u = f_Z(u)$.

<u>and</u>

$$\lim_{\Delta u \downarrow 0} \frac{P(X \le x,\, u < Z \le u+\Delta u)}{\Delta u} = \frac{\partial}{\partial z} F_{XZ}(x,z)\Big|_{z=u}.$$

Compute $F_{XZ}(x,z)$

$$= P(X \le x,\, Z \le z)$$

$$= P(X \le x,\, \max(X,Y) \le z)$$

$$= P(X \le x,\, X \le z,\, Y \le z)$$

$$= P(X \le \min(x,z),\, Y \le z).$$

Two cases: $x \le z$ and $x > z$.

① $x \le z$, then

$$F_{XZ}(x,z) = P(X \le x,\, Y \le z)$$

$$= F_X(x)\, F_Y(z)$$

$\frac{\partial F_{XZ}}{\partial z}\Big|_{z=u}$

$= 0$ $\qquad$ $= 0$ if $x \le 0$

$= \frac{x}{\theta^2}$ if $0 < x \le u < \theta$ $\qquad$ $= \frac{xz}{\theta^2}$ if $0 < x \le z < \theta$

$= 0$ $\qquad$ $= \frac{x}{\theta}$ if $0 < x < \theta \le z$

$= 0$ $\qquad$ $= 1$ if $\theta < x$.

② $x > z$, then

$$F_{XZ}(x,z) = P(X \le z,\, Y \le z)$$

$\frac{\partial F_{XZ}}{\partial z}\Big|_{z=u}$

$= 0$ $\qquad$ $= 0$ if $z < 0$

$= \frac{2u}{\theta^2}$ if $0 < u < \theta$ & $x > z$ $\qquad$ $= z^2/\theta^2$ if $0 < z < \theta$

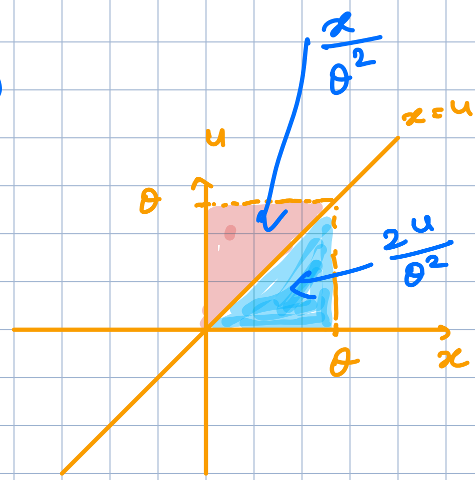
$= 0$ if $u > 1$. $\qquad$ $= 1$ if $z > 1$.

Thus,

$$\frac{\partial F_{XZ}}{\partial z}\Big|_{z=u} = \frac{x}{\theta^2} \quad \text{if } 0 < x \le u < \theta$$

$$= \frac{2u}{\theta^2} \quad \text{if } u \in (0,\theta) \;\&\; x > u.$$

$$= 0 \quad \text{o.w.}$$



Thus,

$$F_X(x \mid Z=u) = \frac{x}{2u} \quad \text{if } 0 < x \le u < \theta$$

$$= 1 \quad \text{if } u \in (0,\theta) \;\&\; x > u.$$

——— x ——— x ——— x ———

$\boxed{\text{Q.4}}$ Note that $c=1$. Thus,

$$f_{XY}(x,y) = 1 \quad \text{if } 0 < x \le y < 1 \text{ or } -1 < y \le x < 0.$$

Now, $Z = X+Y$ & $W = X-Y$.

$$J(x,y) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \; ; \quad |J(x,y)| = 2.$$

$z = x+y$
$w = x-y$ finding roots.

$$x = \frac{z+w}{2} \quad \& \quad y = \frac{z-w}{2}.$$

$$f_{ZW}(z,w) = \frac{1}{2} f_{XY}\left(\frac{z+w}{2}, \frac{z-w}{2}\right)$$

$$= \frac{1}{2} \quad \text{if } 0 < \frac{z+w}{2} \le \frac{z-w}{2} < 1$$

$$\underbrace{\text{or } -1 < \frac{z-w}{2} \le \frac{z+w}{2} < 0}_{\text{can be further simplified.}}$$

e.g.

$$0 < \frac{g+w}{2} \leq \frac{g-w}{2} < 1$$

$$\Rightarrow \quad 0 < g+w \leq g-w < 2 .$$

$$\Rightarrow \quad -w < g < 2+w \quad \& \quad w < 0$$

$$g+w \leq g-w$$

$$\Rightarrow \quad w \leq -w$$

$$\Rightarrow \quad w < 0 .$$

$$g+w > 0 \Rightarrow g > -w .$$

$$g-w < 2$$

$$\Rightarrow \quad g < 2+w .$$