EE 601: Statistical Signal Analysis
Quiz #1

Dt: 22/08/2024 Time: 11:30 am - 1 pm

Q.1 Consider sample space Ω and let $A_1, A_2, \ldots, A_N$ be a partition of Ω. Let $\mathcal{A}$ denote collection of all possible unions of the sets from partition, i.e. $\mathcal{A}$ contains unions of all possible combination of k sets from the partition, for $k = 0, 1, \ldots, N$. Prove or disprove: $\mathcal{A}$ is the smallest σ-field containing $\{A_1, \ldots, A_N\}$.

For $\Omega = \{1, 2, \ldots, 6\}$, Find the smallest σ-field containg $\{\{1,2\}, \{3,4\}, \{5,6\}\}$.

6 + 2 Marks

Q.2 Consider probability space $(\Omega, \mathcal{F}, P)$. Events A and B are independent under P. Consider another event D such that $P(D) > 0$. Prove or disprove: Events A & B are conditionaly independent given D, i.e.,

$$P(A \cap B \mid D) = P(A \mid D) \cdot P(B \mid D).$$

5 Marks

Q.3 Let $\mathcal{A} = \{(x_1, x_2) : x_1 < x_2 \ \& \ x_1, x_2 \in \mathbb{R}\}$. Show that the smallest σ-field containing $\mathcal{A}$ is the Borel σ-field.

7 Marks

Solutions

① $\mathcal{A}$ is a σ-field.

Proof: By definition of $\mathcal{A}$, if $\exists A \in \mathcal{A}$, then $\exists k$ and

$$1 \le \alpha(1) < \alpha(2) \cdots < \alpha(k) \le N \quad \text{s.t. } A = \bigcup_{j=1}^{k} A_{\alpha(j)}.$$

Assuming $A \neq \phi$. If $A = \phi$, then $k = 0$.

We need to show:

ⓐ $\Omega \in \mathcal{A}$

ⓑ $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

ⓒ If $B_1, B_2, \ldots \in \mathcal{A}$, then $\bigcup B_n \in \mathcal{A}$.

ⓐ is obvious from def of $\mathcal{A}$ as $\Omega = \bigcup_{k=1}^{N} A_k$.

ⓑ Suppose $A \in \mathcal{A}$, then as stated above

$$A = \bigcup_{j=1}^{k} A_{\alpha(j)}. \quad \text{Now, note that}$$

$$A^c = \bigcup_{i \notin \{\alpha(j) : j=1,\ldots,k\}} A_i .$$

$\Rightarrow A^c \in \mathcal{A}$, by defⁿ of $\mathcal{A}$.

ⓒ Note that $\mathcal{A}$ can have at most 2^N elements.

Hence, it suffices to show that if

$B_1, B_2 \in \mathcal{A}$, then $B_1 \cup B_2 \in \mathcal{A}$.

Note that we can write

$$B_1 = \bigcup_{j=1}^{k_1} A_{\alpha(j)} \quad \text{and} \quad B_2 = \bigcup_{j=1}^{k_2} A_{\beta(j)}$$

Thus, $B_1 \cup B_2$ is also union of sets from $\{A_1, \ldots, A_N\}$. $\Rightarrow B_1 \cup B_2 \in \mathcal{A}$ by defⁿ.

$\Rightarrow \mathcal{A}$ is a σ-field containing $\{A_1, \ldots, A_N\}$.

To see that it is the smallest, observe that any σ-field containing $\{A_1, \dots, A_N\}$ must have there unions.

— x — x —

$\Omega = \{1,2,3,4,5,6\}$, $A_1 = \{1,2\}$, $A_2 = \{3,4\}$, $A_3 = \{5,6\}$.

$\{A_1, A_2, A_3\}$ is a partition of Ω.

Thus, from above result, the smallest σ-field can be given as

$\{\phi, \{1,2\}, \{3,4\}, \{5,6\}, \{1,2,3,4\}, \{1,2,5,6\},$
$\{3,4,5,6\}, \Omega\}$.

——x ——x ——x ——

(2) Consider prob space $(\Omega, \mathcal{F}, P)$ as described below.

$\Omega = \{1,2,3,4\}$

$\mathcal{F}$ = Power set of Ω

P = Uniform measure, i.e.

$P(\{\omega\}) = \frac{1}{4} \quad \forall\ \omega = 1,2,3,4.$

Let $A = \{1,2\}$ and $B = \{2,3\}$.

Note that $P(A) = P(B) = 1/2$ and

$P(A \cap B) = P(\{2\}) = \frac{1}{4}$.

$\Rightarrow$ A and B are independent under P.

Consider $D = \{2,4\}$. $P(D) = 1/2$.

$$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{P(\{2\})}{P(\{2,4\})} = \frac{1/4}{1/2} = \frac{1}{2}.$$

$P(B|D) = \frac{1}{2}$.

Now, $P(A\cap B|D) = \dfrac{P(A\cap B\cap D)}{P(D)} = \dfrac{P(\{2\})}{P(\{2,4\})} = 1/2$.

$\Rightarrow P(A\cap B|D) \neq P(A|D)\cdot P(B|D)$.

Thus, A & B are not conditionally independent given D. Other way to say this is:

Even though A & B are independent under P, they are not independent under $P_{|D}$.

Independence is a property of measure and of events themselves!!!

——x——x——x——

③ Let $\mathcal{F}$ denote the smallest σ-field containing $\{(x_1,x_2) : x_1 < x_2 \ \& \ x_1, x_2 \in \mathbb{R}\}$.

Now for any $x \in \mathbb{R}$, $(-n, x) \in \mathcal{F}$ for integer n.
(if $-n > x$, then $(-n,x) = \emptyset$).

Now, $\bigcup_{n=1}^{\infty} (-n, x) \in \mathcal{F}$ ($\because \mathcal{F}$ is σ-field).

$\Rightarrow (-\infty, x) \in \mathcal{F} \quad \forall x \in \mathbb{R}$.

Now, consider, $(-\infty, x+\frac{1}{n}) \in \mathcal{F} \quad \forall n$ integer.

Thus, $\bigcap_{n=1}^{\infty} (-\infty, x+\frac{1}{n}) \in \mathcal{F}$.

$\Rightarrow (-\infty, x] \in \mathcal{F} \quad \forall x \in \mathbb{R}$.

Thus, $\{(-\infty, x] : x \in R\} \subseteq \mathcal{F}$.

Thus, $\mathcal{F}$ is a σ-field containing $\{(-\infty, x] : x \in \mathbb{R}\}$ while $\mathcal{B}$ is the smallest σ-field containing $\{(-\infty, x] : x \in \mathbb{R}\}$.

Thus, $\mathcal{F} \supseteq \mathcal{B}$.

But now recall from the tutorial that

$$\{(x_1, x_2) : x_1 < x_2,\ x_1, x_2 \in \mathbb{R}\} \subseteq \mathcal{B}.$$

and $\mathcal{F}$ is the smallest σ-field containing .

Thus, $\mathcal{F} \subseteq \mathcal{B}$.

$\Rightarrow \mathcal{F} = \mathcal{B}$.