Newtonian Manipulator Dynamics

Dynamics Techniques

Euler-Lagrange

- Energy Based
- Derived from D'Alembert's Principle – Virtual Work

Newton-Euler

- Based on Newtonian Mechanics
  - Every action has an equal and opposite reaction.
  - Rate of change of the linear momentum = force
  - Rate of change of the angular momentum = torque

Applying Newtonian Mechanics to Newton-Euler Dynamics

1. Every action has an equal and opposite reaction. Thus, if body 1 applies a force f and torque τ to body 2, then body 2 applies a force of $-f$ and torque of $-\tau$ to body 1.
2. The rate of change of the linear momentum equals the total force applied to the body.
3. The rate of change of the angular momentum equals the total torque applied to the body.

What is the Newton-Euler Formulation?

- Based on balance of all forces acting on links

- Solved Recursively
  - Forward direction to propagate link velocities and accelerations
  - Reverse direction to propagate forces

    Find the forces and torques that correspond to a set of generalized coordinates and the 1st and 2nd derivatives

- Note that it is not a Closed Form solution

Basis of the Approach

- The method is based on:
  - Newton's 2nd Law of Motion Equation:

$$\boxed{F = m_i \dot{v}_C}$$

    and considering a 'rigid' link

  - Euler's Angular Force/ Moment Equation:

$$\boxed{N_{moment} = I_{CM_i}\dot{\omega}_i + \omega_i \times I_{CM_i}\omega_i}$$

General Approach

- Find a torque model for each Link Individually
- Move from Base to Tip to find Velocities and Accelerations
- Move from Tip to Base to compute force (f) and Moments (n)
- Determine the Torque:

$$\boxed{\tau_i = \xi_i \left(n_i\right)^T z_{i-1} + (1-\xi_i)\left(f_i\right)^T z_{i-1} + b_i(\dot{q}_i)}$$

ξ_i is the joint type parameter again:

1 if revolute

0 if prismatic

Newtonian Mechanics

Newton's 2nd Law:

rate of change of momentum

$$\frac{d(mv)}{dt} = f$$

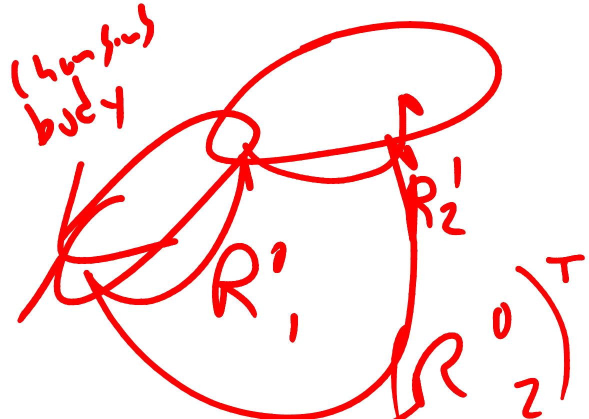
in inertial frame

ang momentum

$$\frac{d(I_0\omega_0)}{dt} = \tau_0$$

Where:

fixed, unchanging wrt body



$$I_0 = RIR^T$$

Angular Velocities

Angular Velocity of Rotating Frame:

$$\dot{R}R^T = S(\omega_0).$$

$$\dot{R} = S(\omega_o)R.$$

Where:

$$\omega_0 = R\omega$$

Angular Momentum

In Base (Inertial) Frame:

$$h = RIR^T R\omega = RI\omega.$$

const wrt body

Taking Derivative:

rate of change of momentum

$$\dot{h} = \dot{R}I\omega + RI\dot{\omega}.$$

Substituting:

$$\dot{h} = S(\omega_0)RI\omega + RI\dot{\omega}.$$

Angular Momentum

Now solve for rate of change of the angular momentum wrt the Body-fixed Frame:

$$
\begin{aligned}
R^T \dot{h} &= R^T S(\omega_0) R I \omega + I \dot{\omega} \\
&= S(R^T \omega_0) I \omega + I \dot{\omega} \\
&= S(\omega) I \omega + I \dot{\omega} = \underline{\omega \times (I\omega) + I\dot{\omega}}
\end{aligned}
$$

Which is the torque!

Free Body Diagram (FBD) of Generic Robot Link

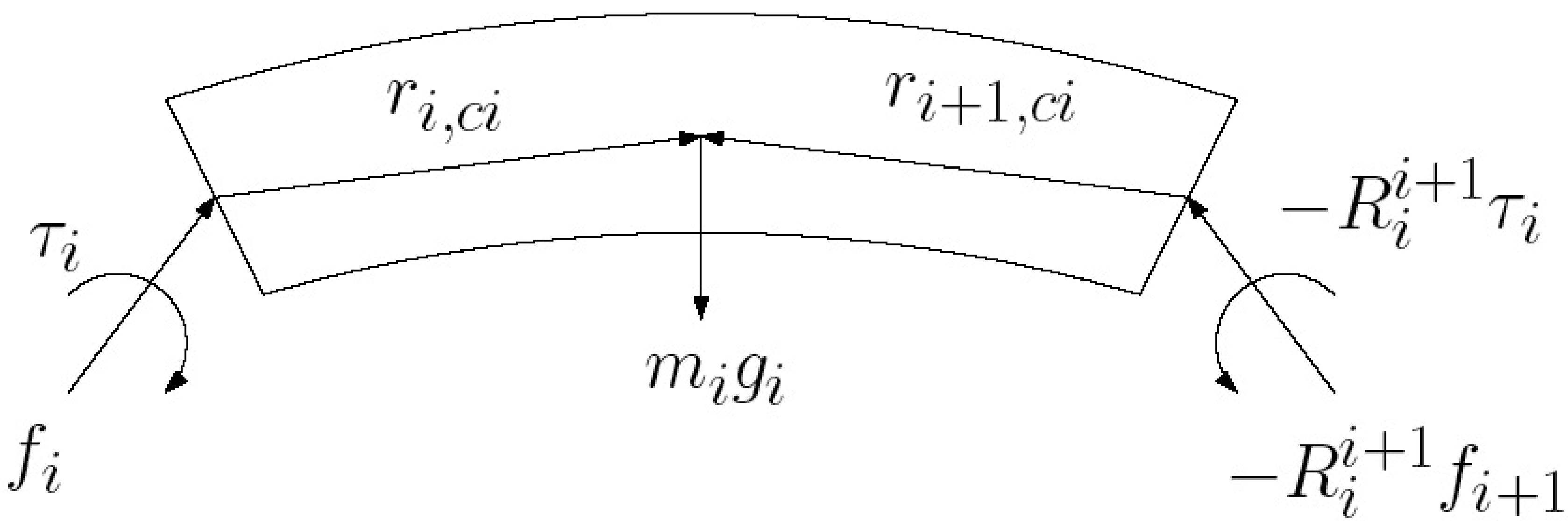
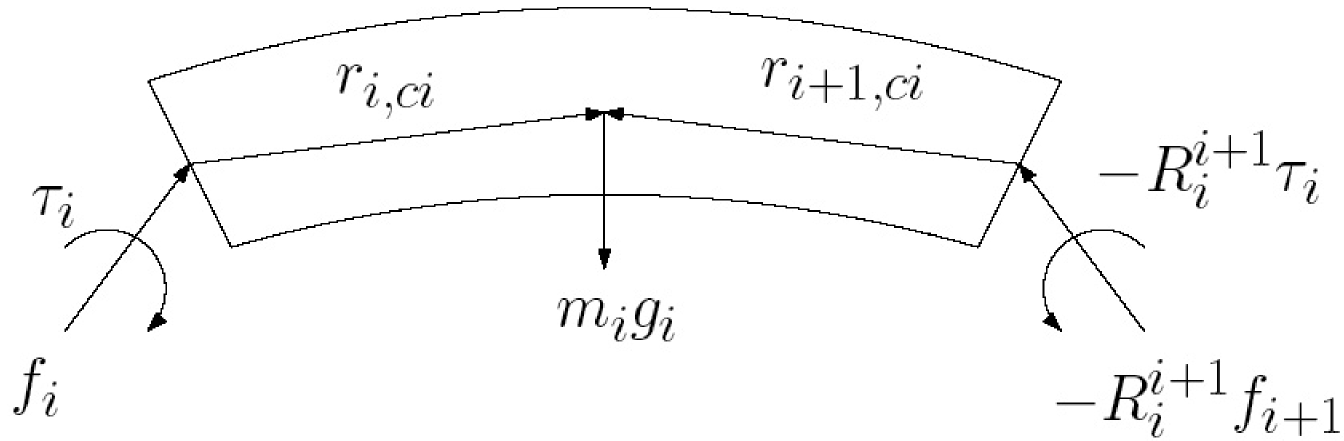


Figure 7.12: Forces and moments on link i.

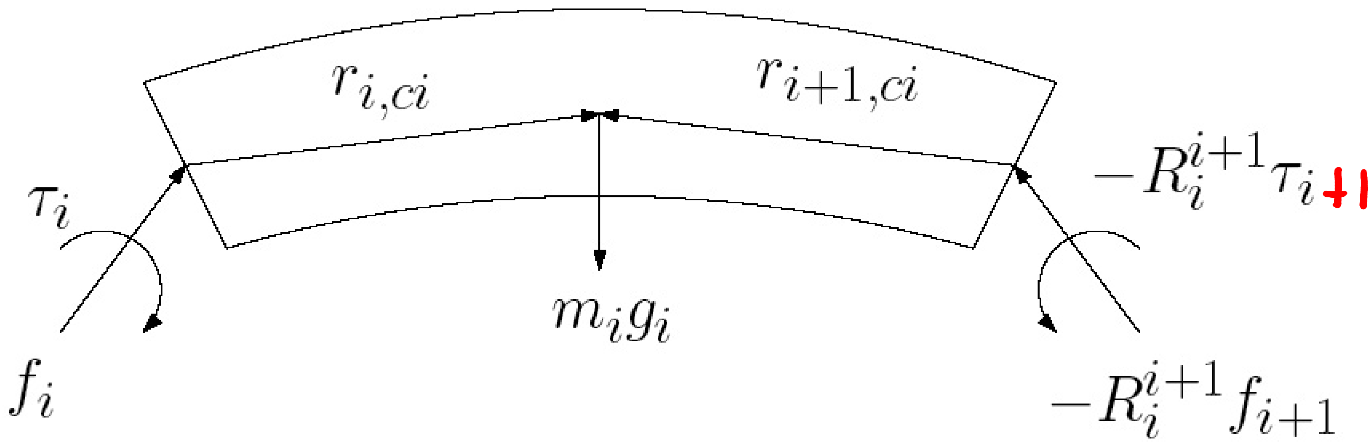
FBD



Where:

$$
\begin{aligned}
\boldsymbol{a}_{c,i} &= \text{the acceleration of the center of mass of link } i.\\
\boldsymbol{a}_{e,i} &= \text{the acceleration of the end of link } i \text{ (i.e., joint } i+1).\\
\boldsymbol{\omega}_i &= \text{the angular velocity of frame } i \text{ w.r.t. frame } 0.\\
\boldsymbol{\alpha}_i &= \text{the angular acceleration of frame } i \text{ w.r.t. frame } 0.\\
\boldsymbol{g}_i &= \text{the acceleration due to gravity (expressed in frame } i \text{).}\\
\boldsymbol{f}_i &= \text{the force exerted by link } i-1 \text{ on link } i.\\
\boldsymbol{\tau}_i &= \text{the torque exerted by link } i-1 \text{ on link } i.\\
R_i^{i+1} &= \text{the rotation matrix from frame } i+1 \text{ to frame } i.\\
m_i &= \text{the mass of link } i.\\
I_i &= \text{the inertia matrix of link } i \text{ about a frame parallel}\\
&\quad\ \text{to frame } i \text{ whose origin is at the center of mass of link } i.\\
\boldsymbol{r}_{i,ci} &= \text{the vector from joint } i \text{ to the center of mass of link } i.\\
\boldsymbol{r}_{i+1,ci} &= \text{the vector from joint } i+1 \text{ to the center of mass of link } i.\\
\boldsymbol{r}_{i,i+1} &= \text{the vector from joint } i \text{ to joint } i+1.
\end{aligned}
$$

Force Balance



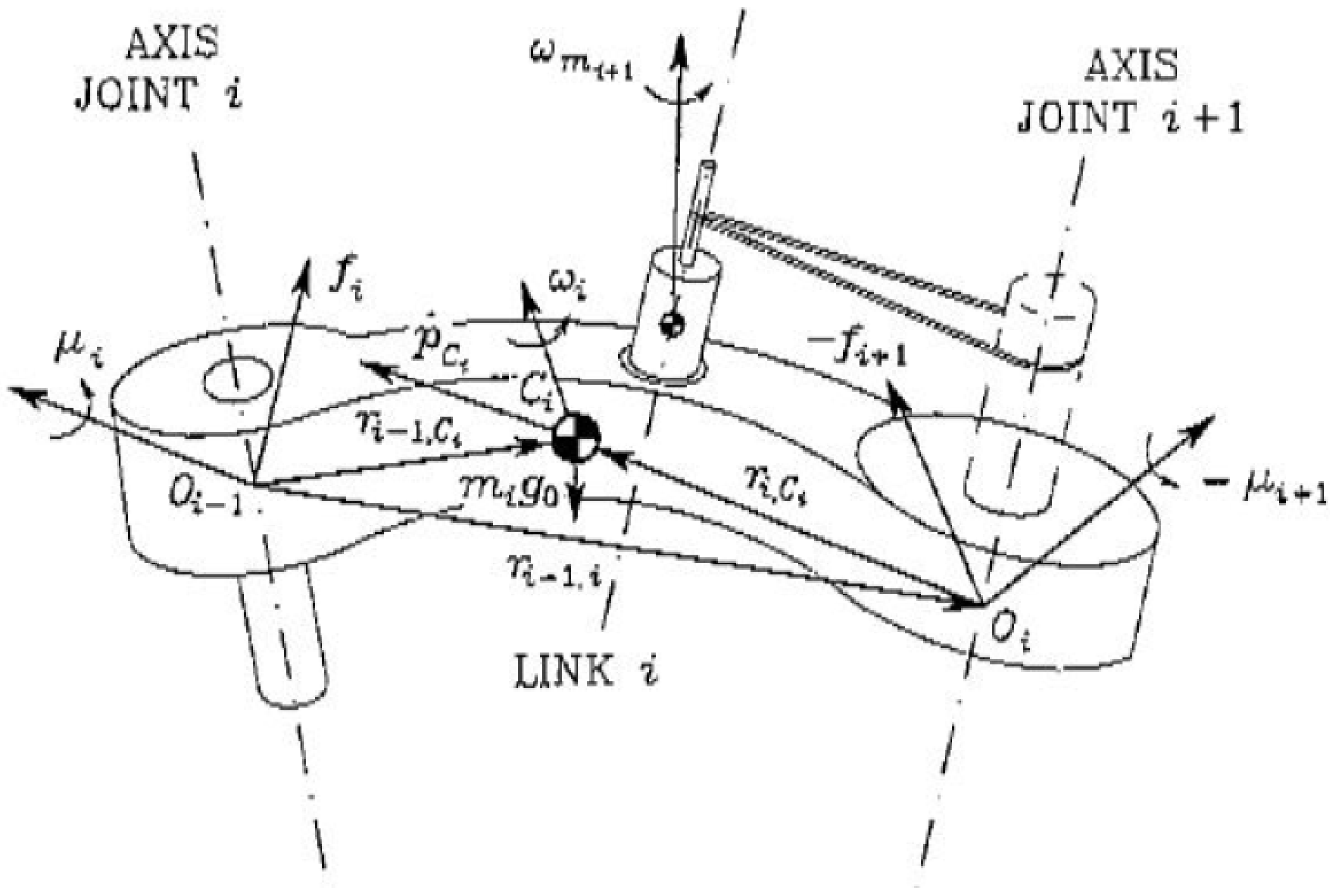
Force Balance Equation for Link *i*:

$$\boldsymbol{f}_i - R_i^{i+1}\boldsymbol{f}_{i+1} + m_i\boldsymbol{g}_i = m_i\boldsymbol{a}_{c,i}.$$

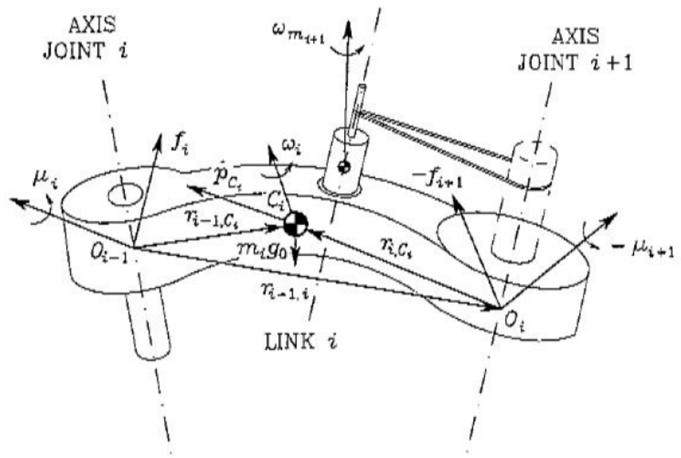
Torque/Moment Balance:

$$\begin{aligned}&\boldsymbol{\tau}_i - R_i^{i+1}\boldsymbol{\tau}_{i+1} + \boldsymbol{f}_i \times \boldsymbol{r}_{i,ci} - (R_i^{i+1}\boldsymbol{f}_{i+1}) \times \boldsymbol{r}_{i+1,ci}\\ = \; &\boldsymbol{\alpha}_i + \boldsymbol{\omega}_i \times (I_i\boldsymbol{\omega}_i).\end{aligned}$$

Another View



Another View



m_i mass of augmented link,

$\bar{I}_i$ inertia tensor of augmented link,

I_{m_i} moment of inertia of rotor,

r_{i-1,C_i} vector from origin of frame $(i-1)$ to center of mass C_i,

r_{i,C_i} vector from origin of frame i to center of mass C_i,

$r_{i-1,i}$ vector from origin of frame $(i-1)$ to origin of frame i.

$\dot{p}_{C_i}$ linear velocity of center of mass C_i,

$\dot{p}_i$ linear velocity of origin of frame i,

ω_i angular velocity of link,

ω_{m_i} angular velocity of rotor,

$\ddot{p}_{C_i}$ linear acceleration of center of mass C_i,

$\ddot{p}_i$ linear acceleration of origin of frame i,

$\dot{\omega}_i$ angular acceleration of link,

$\dot{\omega}_{m_i}$ angular acceleration of rotor,

g_0 gravity acceleration.

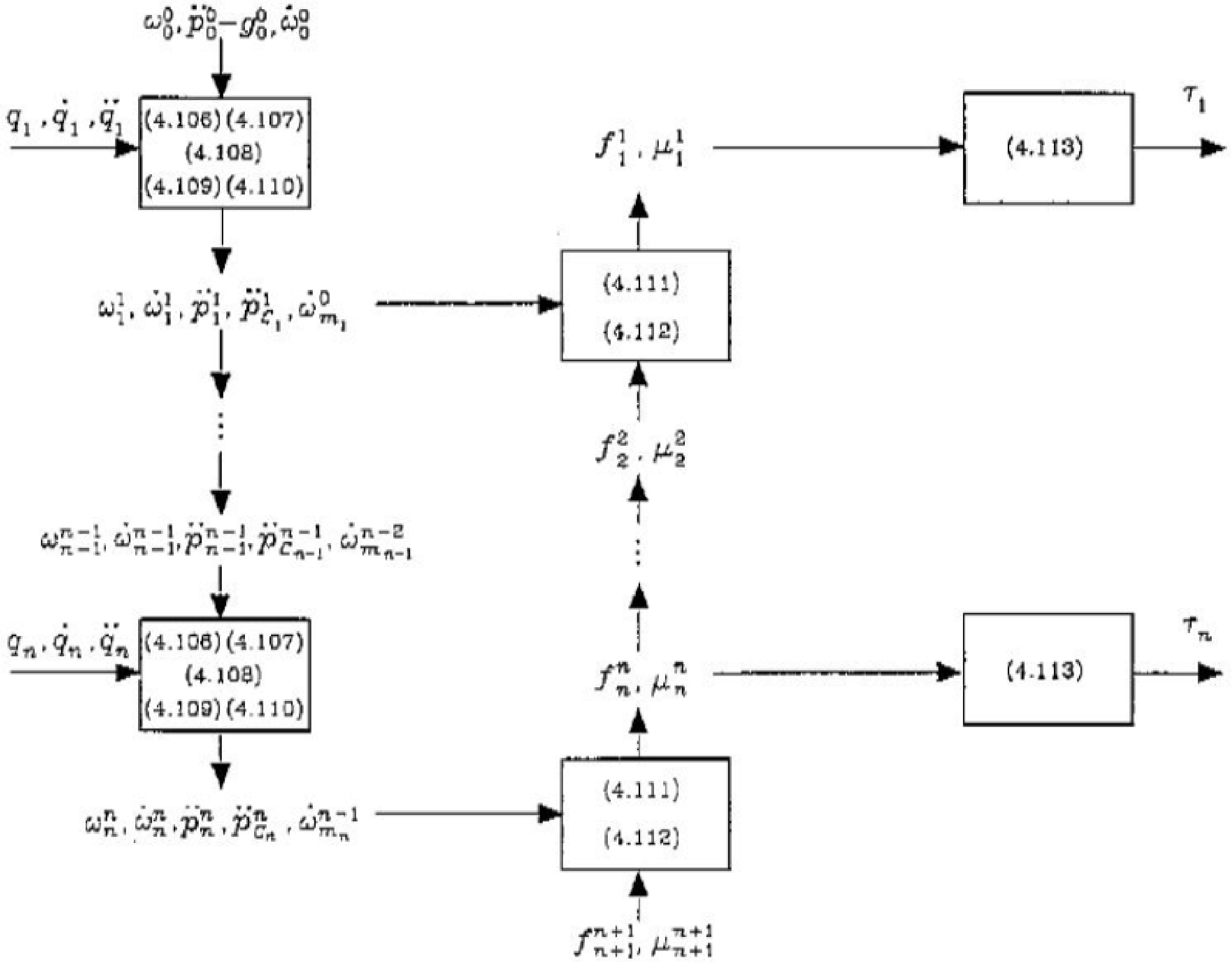
f_i force exerted by link $i-1$ on link i,

$-f_{i+1}$ force exerted by link $i+1$ on link i,

μ_i moment exerted by link $i-1$ on link i with respect to origin of frame $i-1$,

$-\mu_{i+1}$ moment exerted by link $i+1$ on link i with respect to origin of frame i.

Approach Flow



From Robotics: Modeling, Planning, and Control, Siciliano et al

Forward Recursion
Solving for velocities

In Inertial Frame:

$$\boldsymbol{\omega}_i^{(0)} = \boldsymbol{\omega}_{i-1}^{(0)} + \boldsymbol{z}_{i-1}\dot{q}_i$$

Angular velocity equals angular velocity of previous link plus added angular velocity of current link

Forward Recursion
Solving for velocities

In Link Coordinates:

Given:

$\omega_0=\alpha_0=a_{c,0}=a_{e,0}=0$

Solve for i=1, 2, … n to find velocities

$$\boldsymbol{\omega}_i = (R^i_{i-1})^T \boldsymbol{\omega}_{i-1} + \boldsymbol{b}_i \dot{q}_i$$

$$\boldsymbol{b}_i = (R^i_0)^T \boldsymbol{z}_{i-1}$$

Forward Recursion
Solving for accelerations

Solve for i=1, 2, … n to find angular accelerations:

$$\boldsymbol{\alpha}_i \quad = \quad (R_0^i)^T \dot{\boldsymbol{\omega}}_i^{(0)}$$

Taking time derivative of angular velocity:

ang accel wrt inertial frame

$$\dot{\boldsymbol{\omega}}_i^{(0)} \quad = \quad \dot{\boldsymbol{\omega}}_{i-1}^{(0)} + \boldsymbol{z}_{i-1}\ddot{q}_i + \boldsymbol{\omega}_i^{(0)} \times \boldsymbol{z}_{i-1}\dot{q}_i$$

And wrt Frame *i*

$$\boldsymbol{\alpha}_i \quad = \quad (R_{i-1}^i)^T \boldsymbol{\alpha}_{i-1} + \boldsymbol{b}_i\ddot{q}_i + \boldsymbol{\omega}_i \times \boldsymbol{b}_i\dot{q}_i$$

Forward Recursion
Solving for accelerations

And for the linear velocities & accelerations:

$$\boldsymbol{v}_{c,i}^{(0)} = \boldsymbol{v}_{e,i-1}^{(0)} + \boldsymbol{\omega}_i^{(0)} \times \boldsymbol{r}_{i,ci}^{(0)}$$

$$\boldsymbol{a}_{c,i}^{(0)} = \boldsymbol{a}_{e,i-1}^{(0)} \times \boldsymbol{r}_{i,ci}^{(0)} + \boldsymbol{\omega}_i^{(0)} \times (\boldsymbol{\omega}_i^{(0)} \times \boldsymbol{r}_{i,ci}^{(0)})$$

relate inertial
frame to
link
frame

$$\boldsymbol{a}_{c,i} = (R_0^i)^T \boldsymbol{a}_{c,i}^{(0)}$$

ctr of
mass

$$a_{c,i} = (R_{i-1}^i)^T a_{e,i-1} + \dot{\omega}_i \times r_{i,ci} + \omega_i \times (\omega_i \times r_{i,ci}).$$

end

$$a_{e,i} = (R_{i-1}^i)^T a_{e,i-1} + \dot{\omega}_i \times r_{i,i+1} + \omega_i \times (\omega_i \times r_{i,i+1}).$$

Forward Recursion
Solving for accelerations

Transforming to be wrt the link frame:

$$\boldsymbol{a}_{c,i} \;=\; (R_{i-1}^{i})^{T}\boldsymbol{a}_{e,i-1} + \dot{\boldsymbol{\omega}}_i \times \boldsymbol{r}_{i,ci} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \boldsymbol{r}_{i,ci})$$

$$\boldsymbol{a}_{e,i} \;=\; (R_{i-1}^{i})^{T}\boldsymbol{a}_{e,i-1} + \dot{\boldsymbol{\omega}}_i \times \boldsymbol{r}_{i,i+1} + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \boldsymbol{r}_{i,i+1})$$

Backward Recursion

Given:

Known trajectory

$f_{n+1}=\tau_{n+1}=0$

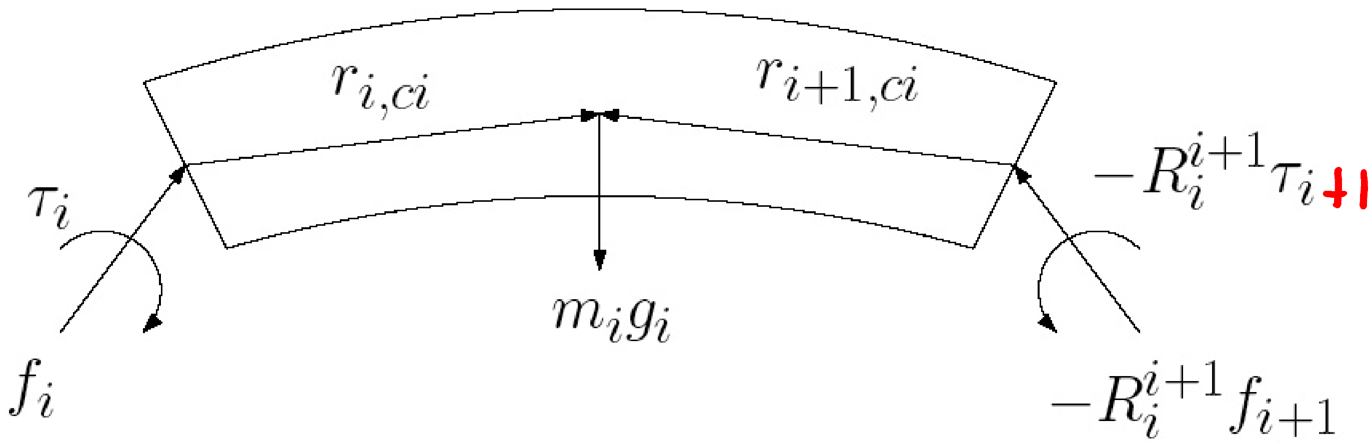
Solve for i=n, n-1, … 1 to find forces and torques

$$f_i = R_i^{i+1} f_{i+1} + m_i a_{c,i} - m_i g_i$$

$I\dot{\omega}_i$

$$\tau_i = R_i^{i+1}\tau_{i+1} - f_i \times r_{i,ci} + (R_i^{i+1} f_{i+1}) \times r_{i+1,ci} + \alpha_i + \omega_i \times (I_i \omega_i)$$

Force Balance



Force Balance Equation for Link *i*:

Solve for

$$f_i - R_i^{i+1} f_{i+1} + m_i g_i \quad = \quad m_i a_{c,i}.$$

Torque/Moment Balance:

$$\tau_i - R_i^{i+1}\tau_{i+1} + f_i \times r_{i,ci} - (R_i^{i+1} f_{i+1}) \times r_{i+1,ci}$$
$$= \quad \alpha_i + \omega_i \times (I_i \omega_i).$$

Multi-link Arm Dynamics

General Form:

$$\vec{\tau} = M(\vec{q})\ddot{\vec{q}} + V(\vec{q}, \dot{\vec{q}}) + G(\vec{q}) + \vec{\tau}_d$$

Where:

$\vec{\tau}$ = Generalized joint forces/torques

$\vec{q}$ = Generalized coordinates (angles/translation)

$M(\vec{q})$ = Inertia term

$V(\vec{q}, \dot{\vec{q}})$ = Coriolis/Centripital Coupling term

$G(\vec{q})$ = Gravity term

$\vec{\tau}_d$ = External disturbances (friction, ...)