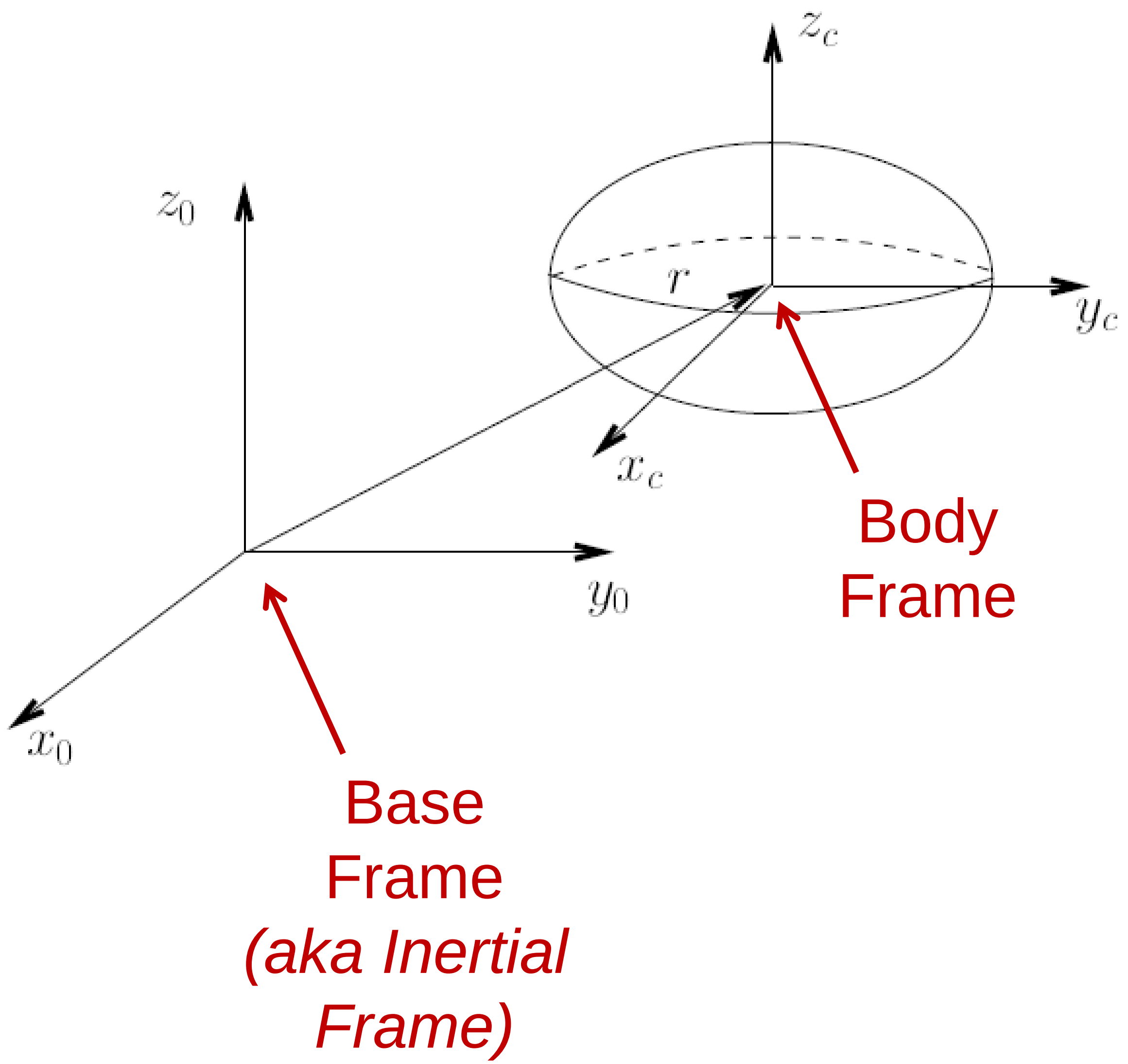
Advanced Dynamics Techniques for Robotic Manipulators

Robot Dynamics

Lecture 10

Extending Beyond Treating Links as Points

Generic Representation of a Rigid Body

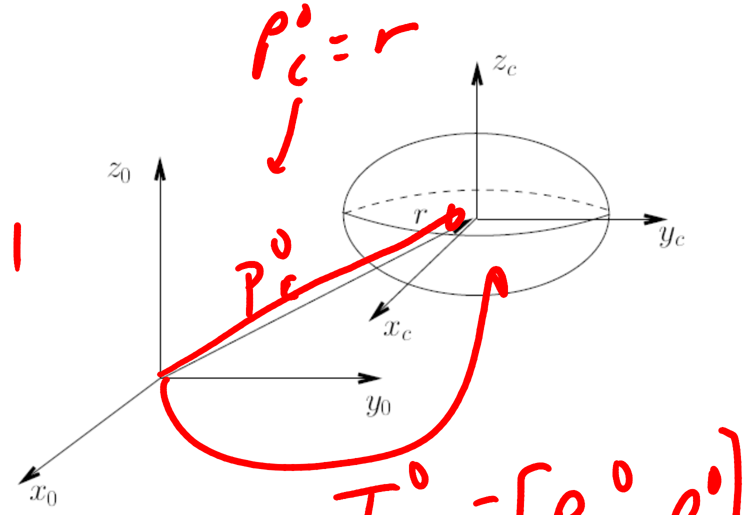


Kinetic Energy of Rigid Body

Total Kinetic Energy:

$$\mathcal{K} = \overbrace{\frac{1}{2}mv^Tv}^{\text{translational}} + \overbrace{\frac{1}{2}\omega^T\mathcal{I}\omega}^{\text{rotational}}$$

v^2 scalar



$T_c^0 = [R_c^0, p_c^0]$

Transforming Inertia Reference Frame:

$$\mathcal{I} = RIR^T$$

$K_{rot} = \frac{1}{2} \omega^T R I R^T \omega$

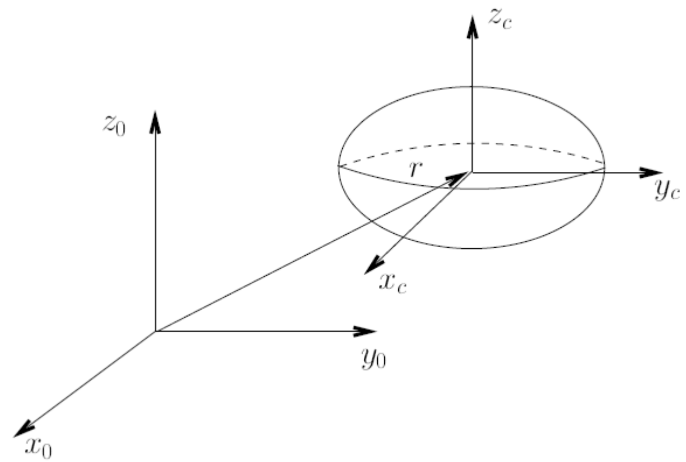
Aligned with Body Frame

Aligned with Inertial Frame

Kinetic Energy of Rigid Body

Total Kinetic Energy:

$$\mathcal{K} = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \mathcal{I} \omega$$



Transforming Inertia Reference Frame:

$$\mathcal{I} = R I R^T$$

Aligned with Body Frame

Aligned with Inertial Frame

Determining Angular Velocities

$$R(\theta)\, R(\theta)^T = I$$

define

$$S = \frac{d}{d\theta}R \; R(\theta)^T$$

$$\frac{dR}{d\theta} \cdot R(\theta)^T + R(\theta) \frac{d}{d\theta}R^T = 0$$

$$S + S^T = 0$$

$$S^T = R(\theta) \frac{d}{d\theta}R^T$$

S is skew symmetric

$$\frac{dR}{d\theta} = S\, R(\theta)$$

derivative of Rotation can be determined by multiplying by skew sym

Angular Velocities

Angular Velocity of Rotating Frame:

$$\dot{R}R^T = S(\omega_0)$$

$$\dot{R} = S(\omega_o)R.$$

Where:

$$\omega_0 = R\omega$$

w.r.t. Body Frame

w.r.t. Inertial Frame

Angular Velocities

Angular Velocity of Rotating Frame:

$$\dot{R}R^T \;=\; S(\omega_0).$$

$$\hat{w}_0 = S(\vec{w}_0)$$

$$\dot{R} = S(\omega_o)R.$$

$$= \begin{bmatrix} 0 & -w_z & +w_y \\ +w_z & 0 & -w_x \\ -w_y & +w_x & 0 \end{bmatrix}$$

Where:

$$\omega_0 \;=\; R\omega$$

w.r.t. <u>Body</u> Frame

w.r.t. <u>Inertial</u> Frame

Combining Angular Velocities

$$R^0_2(t) = R^0_1(t) \cdot R^1_2(t)$$

$$\dot{R}^0_2(t) = \dot{R}^0_1(t) \cdot R^1_2(t) + R^0_1(t)\,\dot{R}^1_2(t)$$

$$\dot{R}^0_2 = S(\vec{w}^0_2) \cdot R^0_2$$

$$\hat{w}^0_2$$

$$\dot{R}^0_1 R^1_2 = S(w^0_a) R^0_1 R^1_2 = S(w^0_a) R^0_2$$

$$R^0_1 \dot{R}^1_2 = R^0_1 \cdot S(w^1_b) R^1_2$$

$$= R^0_1 S(w^1_b) R^{0\,T}_1 R^0_1 R^1_2$$

$$= S(R^0_1 w^1_b) R^0_1 R^1_2$$

$$= S(R^0_1 w^1_b) R^0_2$$

$$S(W^0_2)\,\cancel{R^0_2} = \left(S(W^0_a) + S(R^0_1 W^1_b)\right)\cancel{R^0_2}$$

$$S(a) + S(b) = S(a+b)$$

$$\boxed{W^0_2 = W^0_a + R^0_1 W^1_b}$$

Combining Angular Velocities

Series of Rotations:

$$R_n^0 = R_1^0 R_2^1 \cdots R_n^{n-1}$$

Determining the Derivative:

$$\dot{R}_n^0 = S(\omega_n^0) R_n^0$$

Solving for the Angular Velocity

$$\omega_n^0 = \omega_1^0 + R_1^0 \omega_2^1 + R_2^0 \omega_3^2 + R_3^0 \omega_4^3 + \cdots + R_{n-1}^0 \omega_n^{n-1}$$

Combining Angular Velocities

Series of Rotations:

$$R_n^0 = \underbrace{R_1^0 R_2^1}_{} \cdots R_n^{n-1}$$

Determining the Derivative:

$$\dot{R}_n^0 = S(\omega_n^0) R_n^0$$

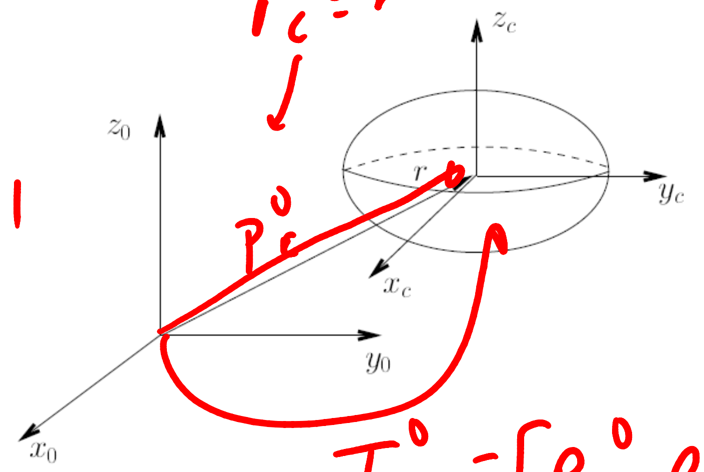
Solving for the Angular Velocity

from ex: $\omega_2^0 = \omega_1^0 + R_1^0 \omega_2^1$

$$\omega_n^0 = \omega_1^0 + R_1^0 \omega_2^1 + R_2^0 \omega_3^2 + R_3^0 \omega_4^3 + \cdots + R_{n-1}^0 \omega_n^{n-1}$$

Kinetic Energy of Rigid Body

$p_c^0 = r$



$T_c^0 = [R_c^0, p_c^0]$

Total Kinetic Energy:

translational

rotational

$$\mathcal{K} = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \mathcal{I} \omega$$

v^2 scalar

Transforming Inertia Reference Frame:

$$K_{rot} = \frac{1}{2} \omega^T R I R^T \omega$$

$$\mathcal{I} = R I R^T$$

Aligned with Body Frame

Aligned with Inertial Frame

Using to Calculate Linear Velocity

Position wrt Base/Origin:

$$p^0 = R_1^0(t)p^1.$$

Velocity wrt Base/Origin:

$$\begin{aligned}\dot{p}^0 &= \dot{R}_1^0(t)p^1 + R_1^0(t)\dot{p}^1 \\ &= S(\omega^0)R_1^0(t)p^1 \\ &= S(\omega^0)p^0 = \omega^0 \times p^0\end{aligned}$$

Using to Calculate Linear Velocity

Position wrt Base/Origin:

$$p^0 = R_1^0(t)p^1.$$

Velocity wrt Base/Origin:

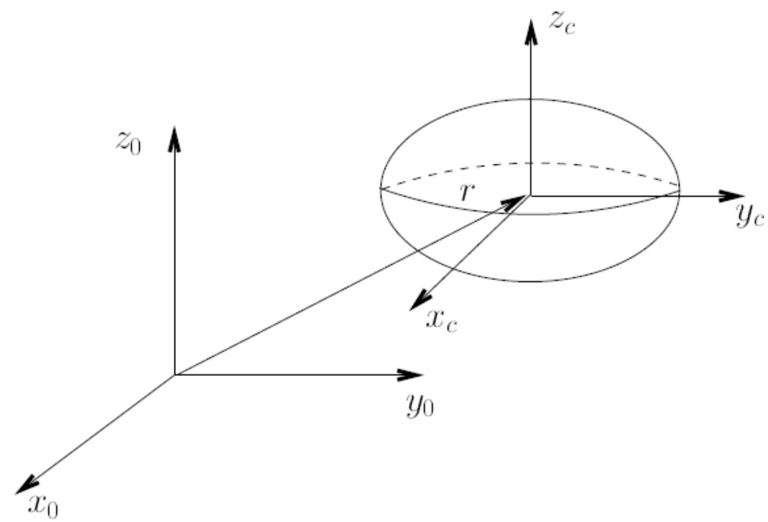
$$\begin{aligned}\dot{p}^0 &= \dot{R}_1^0(t)p^1 + \cancel{R_1^0(t)\dot{p}^1}\\ &= S(\omega^0)R_1^0(t)p^1\\ &= S(\omega^0)p^0 = \omega^0 \times p^0\end{aligned}$$

$\vec{v} = \vec{w} \times \vec{r}$

$\dot{R} R^T = S(w)$

$\dot{R} = S(w) R$

Getting Back to Kinetic Energy



$$\mathcal{K} = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \mathcal{I} \omega$$

Angular Component

Linear Component

Inertia Tensor

In *<u>Inertial</u>* *Frame*

$$S(w) = \hat{w} = \dot{R} R^T$$

$$\mathcal{I} = R I R^T$$

constant body-attached inertia

inertia tensor
in the inertial frame

Inertia Tensor

In <u>Body</u> Frame

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}.$$

where

$$\begin{aligned}
I_{xx} &= \int\int\int (y^2 + z^2)\rho(x, y, z)dx\ dy\ dz \\
I_{yy} &= \int\int\int (x^2 + z^2)\rho(x, y, z)dx\ dy\ dz \\
I_{zz} &= \int\int\int (x^2 + y^2)\rho(x, y, z)dx\ dy\ dz \\
I_{xy} = I_{yx} &= -\int\int\int xy\rho(x, y, z)dx\ dy\ dz \\
I_{xz} = I_{zx} &= -\int\int\int xz\rho(x, y, z)dx\ dy\ dz \\
I_{yz} = I_{zy} &= -\int\int\int yz\rho(x, y, z)dx\ dy\ dz
\end{aligned}$$

Parameter Identification

Interesting Example

https://www.youtube.com/watch?v=jEEKv8Wg6zY&index=3&list=PLHOd8iRtfY3Kvs90RcJ40Qd3T0JyzuWCY

Using Manipulator Jacobian to Determine Velocities

$$\dot{\vec{x}} = J(q)\dot{\vec{q}}$$

$$J(\vec{q}) = \begin{bmatrix} \dfrac{\partial \vec{x}_t(\vec{q})}{\partial q_1} & \dfrac{\partial \vec{x}_t(\vec{q})}{\partial q_2} & \cdots & \dfrac{\partial \vec{x}_t(\vec{q})}{\partial q_n} \\ \xi_1 \vec{z}_0(\vec{q}) & \xi_2 \vec{z}_1(\vec{q}) & \cdots & \xi_n \vec{z}_{n-1}(\vec{q}) \end{bmatrix} = \begin{bmatrix} A(\vec{q}) \\ B(\vec{q}) \end{bmatrix}$$

$$\xi_k = \begin{cases} 0 & \text{Prismatic Joint k} \\ 1 & \text{Revolute Joint k} \end{cases}$$

Calculating Kinetic Energy

Jacobian Matrix:

$$J(\vec{q}) = \begin{bmatrix} \dfrac{\partial \vec{x}_t(\vec{q})}{\partial q_1} & \dfrac{\partial \vec{x}_t(\vec{q})}{\partial q_2} & \cdots & \dfrac{\partial \vec{x}_t(\vec{q})}{\partial q_n} \\ \\ \xi_1 \vec{z}_0(\vec{q}) & \xi_2 \vec{z}_1(\vec{q}) & \cdots & \xi_n \vec{z}_{n-1}(\vec{q}) \end{bmatrix} = \begin{bmatrix} A(\vec{q}) \\ B(\vec{q}) \end{bmatrix}$$

Translational Velocity:

$$v_i = J_{v_i}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Angular Velocity:

$$\omega_i = J_{\omega_i}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Calculating Kinetic Energy

Jacobian Matrix:

$$J(\vec{q}) = \begin{bmatrix} \frac{\partial \vec{x}_t(\vec{q})}{\partial q_1} & \frac{\partial \vec{x}_t(\vec{q})}{\partial q_2} & \cdots & \frac{\partial \vec{x}_t(\vec{q})}{\partial q_n} \\ \xi_1 \vec{z}_0(\vec{q}) & \xi_2 \vec{z}_1(\vec{q}) & \cdots & \xi_n \vec{z}_{n-1}(\vec{q}) \end{bmatrix} = \begin{bmatrix} A(\vec{q}) \\ B(\vec{q}) \end{bmatrix}$$

Translational Velocity:

A(q)

$$v_i = J_{v_i}(q)\dot{q}$$

B(q)

Angular Velocity:

$$\omega_i = J_{\omega_i}(q)\dot{q}$$

Calculating Kinetic Energy for Multi-Link Arm

$$\mathcal{K} = \frac{1}{2} m v^T v + \frac{1}{2} \boldsymbol{\omega}^T \mathcal{I} \boldsymbol{\omega}$$

$$K = \frac{1}{2} \dot{\boldsymbol{q}}^T \sum_{i=1}^{n} \left[m_i J_{v_i}(\boldsymbol{q})^T J_{v_i}(\boldsymbol{q}) + J_{\omega_i}(\boldsymbol{q})^T R_i(\boldsymbol{q}) I_i R_i(\boldsymbol{q})^T J_{\omega_i}(\boldsymbol{q}) \right] \dot{\boldsymbol{q}}$$

$$K = \frac{1}{2} \dot{\boldsymbol{q}}^T D(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

Calculating Kinetic Energy for Multi-Link Arm

$$\mathcal{K} = \frac{1}{2} m v^T v + \frac{1}{2} \boldsymbol{\omega}^T \mathcal{I} \boldsymbol{\omega}$$

$$K = \frac{1}{2} \dot{\boldsymbol{q}}^T \sum_{i=1}^{n} \left[m_i J_{v_i}(\boldsymbol{q})^T J_{v_i}(\boldsymbol{q}) + J_{\omega_i}(\boldsymbol{q})^T R_i(\boldsymbol{q}) I_i R_i(\boldsymbol{q})^T J_{\omega_i}(\boldsymbol{q}) \right] \dot{\boldsymbol{q}}$$

$$K = \frac{1}{2} \dot{\boldsymbol{q}}^T D(\boldsymbol{q}) \dot{\boldsymbol{q}}$$

Calculating Potential Energy for Multi-Link Arm

For a Given Link:

$$P_i = g^T r_{ci} m_i$$

For Sum of All Links:

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} g^T r_{ci} m_i$$

And, back to the equations of motion…

Kinetic Energy:

$$K = \frac{1}{2}\dot{\boldsymbol{q}}^T D(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$= \frac{1}{2}\sum_{i,j}^{n} d_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j$$

Potential Energy:

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} g^T r_{ci} m_i$$

Solving for the Lagrangian

Same approach as before

$$L = K - P$$

$$= \frac{1}{2}\sum_{i,j} d_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j - P(\boldsymbol{q})$$

Solving the Lagrange Equation

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}\dot{q}_j$$

$$\vec{\tau}_k = \frac{d}{dt}\frac{\partial L}{\partial \vec{\dot{q}}_k} - \frac{\partial L}{\partial \vec{q}_k}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} &= \sum_i d_{kj}\ddot{q}_j + \sum_j \frac{d}{dt} d_{kj}\dot{q}_j \\ &= \sum_j d_{kj}\ddot{q}_j + \sum_{i,j} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j\end{aligned}$$

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k}.$$

Rewriting the Lagrange Equation

$$\vec{\tau}_k = \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{q}}_k} - \frac{\partial L}{\partial \vec{q}_k}$$

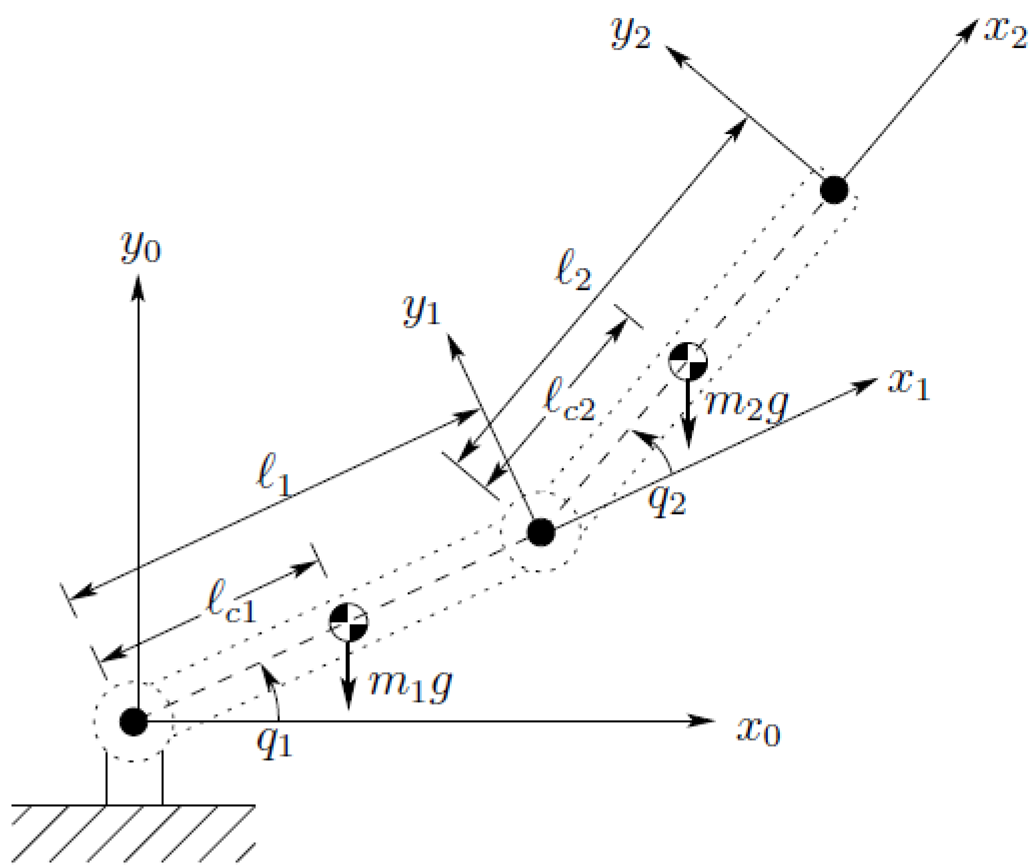
$$= \sum_i d_{kj}(\boldsymbol{q})\ddot{q}_j + \sum_{i,j} c_{ijk}(\boldsymbol{q})\dot{q}_i\dot{q}_j + \phi_k(\boldsymbol{q})$$

Where:

$$\phi_k = \frac{\partial P}{\partial q_k}$$

$$= D(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q})$$

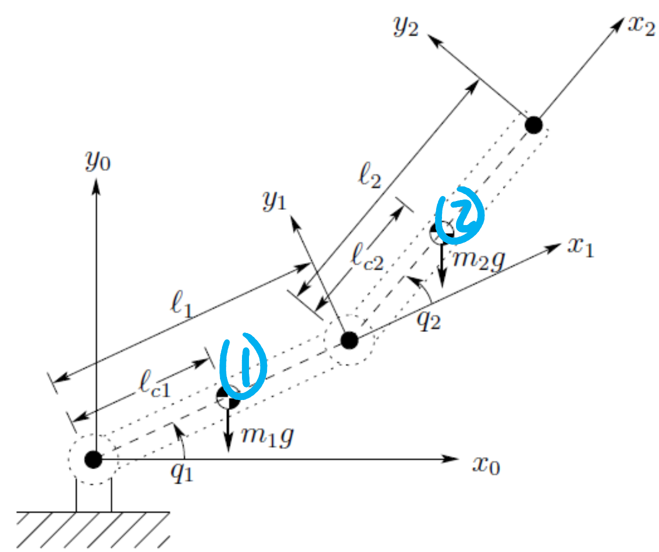
Example Robot



Example Robot Velocities

Velocity 1:

$$v_{c1} = J_{v_{c1}}\dot{q}$$



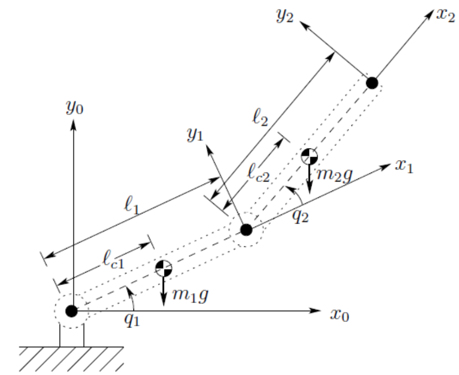
$$J_{v_{c1}} = \begin{bmatrix} -\ell_{c1}\sin q_1 & 0 \\ \ell_{c1}\cos q_1 & 0 \\ 0 & 0 \end{bmatrix}$$

Velocity 2:

$$v_{c2} = J_{v_{c2}}\dot{q}$$

$$J_{v_{c2}} = \begin{bmatrix} -\ell_1\sin q_1 - \ell_{c2}\sin(q_1+q_2) & -\ell_{c2}\sin(q_1+q_2) \\ \ell_1\cos q_1 + \ell_{c2}\cos(q_1+q_2) & \ell_{c2}\cos(q_1+q_2) \\ 0 & 0 \end{bmatrix}$$

Example Robot Kinetic Energy



Translational Component:

$$\frac{1}{2} m_1 \boldsymbol{v}_{c1}^T \boldsymbol{v}_{c1} + \frac{1}{2} m_2 \boldsymbol{v}_{c2}^T \boldsymbol{v}_{c2}$$

$$\frac{1}{2} \dot{\boldsymbol{q}}^T \left\{ m_1 J_{\boldsymbol{v}_{c1}}^T J_{\boldsymbol{v}_{c1}} + m_2 J_{\boldsymbol{v}_{c2}}^T J_{\boldsymbol{v}_{c2}} \right\} \dot{\boldsymbol{q}}$$

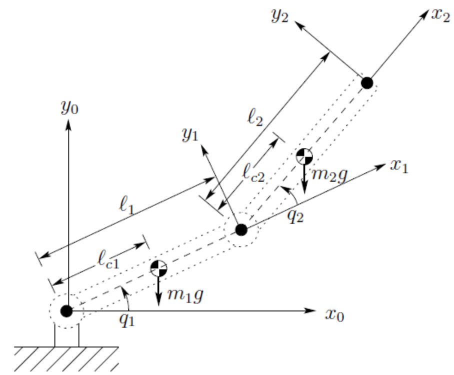
Example Robot Kinetic Energy

<u>Rotational Component:</u>

$K = \hat{z} = \vec{e}_z$ unit vec in z dir.

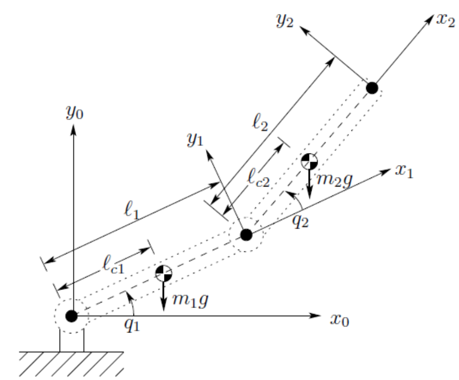
$$\boldsymbol{\omega}_1 = \dot{q}_1 \boldsymbol{k}$$

$$\boldsymbol{\omega}_2 = (\dot{q}_1 + \dot{q}_2)\boldsymbol{k}$$



$$\frac{1}{2}\dot{\boldsymbol{q}}^T \left\{ I_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I_2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \dot{\boldsymbol{q}}$$

Example Robot Kinetic Energy



Translational Component:

$$\frac{1}{2}\dot{\boldsymbol{q}}^{T}\left\{m_1 J_{\boldsymbol{v}_{c1}}^T J_{\boldsymbol{v}_{c1}} + m_2 J_{\boldsymbol{v}_{c2}}^T J_{\boldsymbol{v}_{c2}}\right\}\dot{\boldsymbol{q}}$$

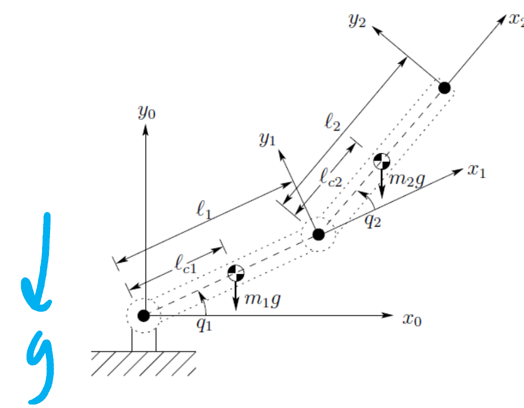
Rotational Component:

$$\frac{1}{2}\dot{\boldsymbol{q}}^T\left\{I_1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + I_2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right\}\dot{\boldsymbol{q}}$$

Combined

$$D(\boldsymbol{q}) \;=\; m_1 J_{\boldsymbol{v}_{c1}}^T J_{\boldsymbol{v}_{c1}} + m_2 J_{\boldsymbol{v}_{c2}}^T J_{\boldsymbol{v}_{c2}} + \begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$$

Example Robot Potential Energy



$$
\begin{aligned}
P_1 &= m_1 g \ell_{c1} \sin q_1 \\
P_2 &= m_2 g(\ell_1 \sin q_1 + \ell_{c2} \sin(q_1 + q_2)) \\
P &= P_1 + P_2 = (m_1 \ell_{c1} + m_2 \ell_1) g \sin q_1 + m_2 \ell_{c2} g \sin(q_1 + q_2)
\end{aligned}
$$