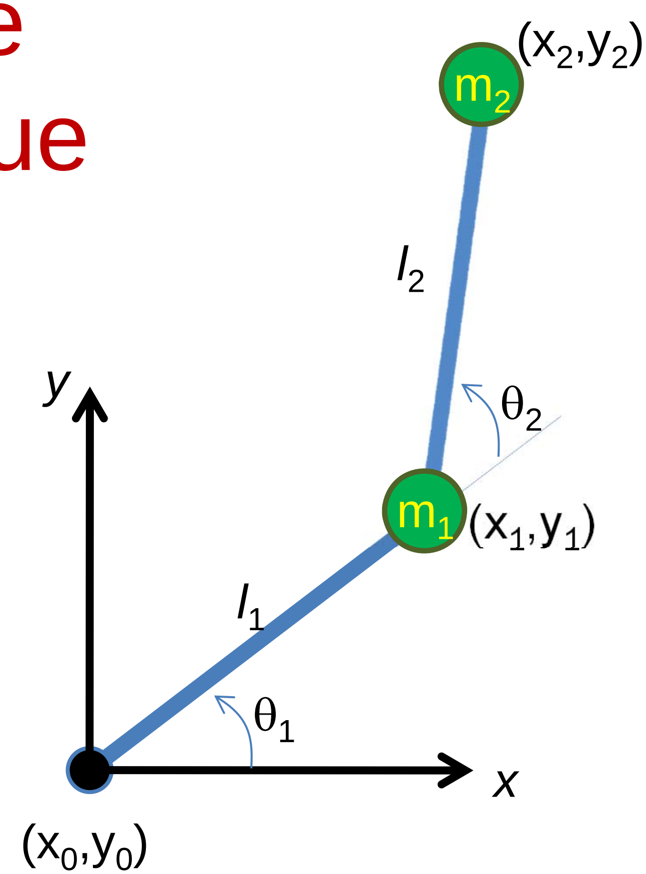
Lagrangian Manipulator Dynamics

Multi-Link Arm Dynamics

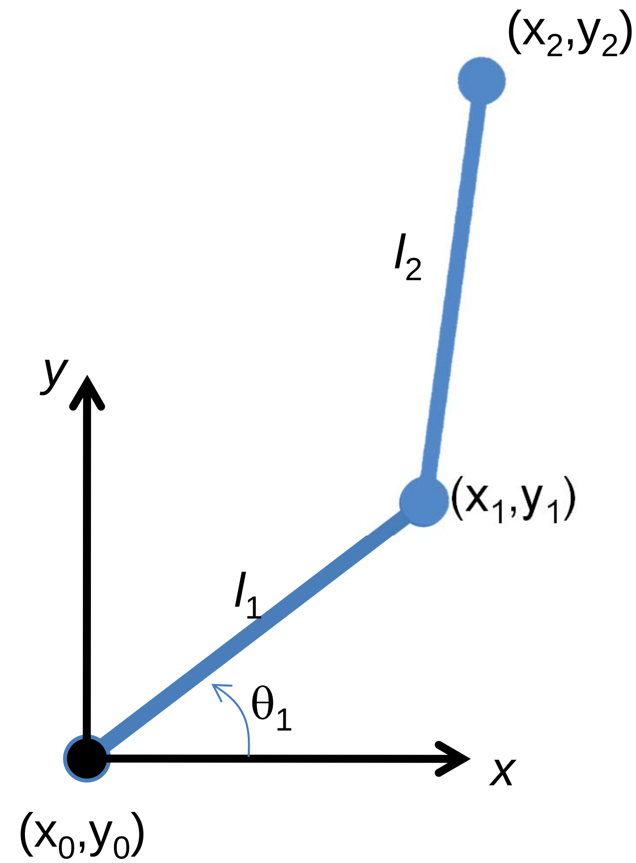
How can we extend the previous 1-link technique to multiple links??



2-Link Arm Kinematics (Point 1)

$$x_1 = l_1 \cos(\theta_1)$$
$$y_1 = l_1 \sin(\theta_1)$$

$$\dot{x}_1 = -l_1 \dot{\theta}_1 \sin(\theta_1)$$
$$\dot{y}_1 = l_1 \dot{\theta}_1 \cos(\theta_1)$$



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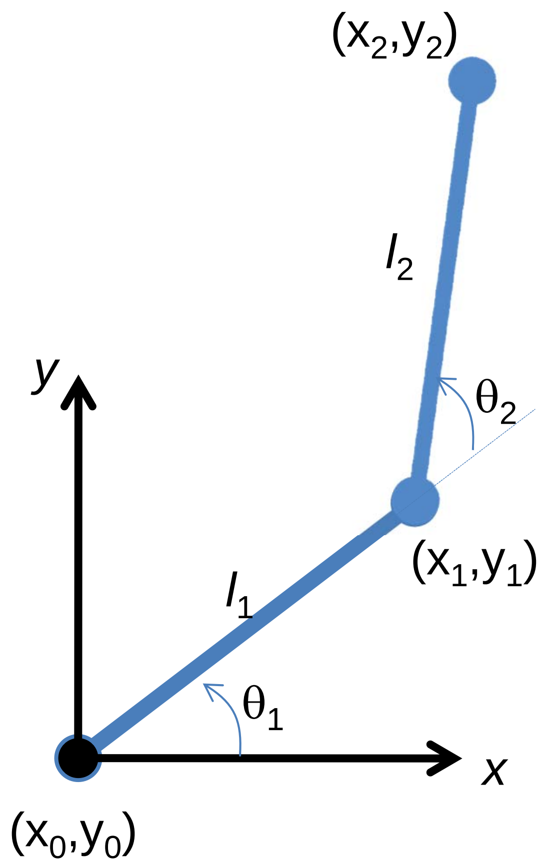
2-Link Arm Kinematics (Point 2)

$$x_2 = x_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$x_2 = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$

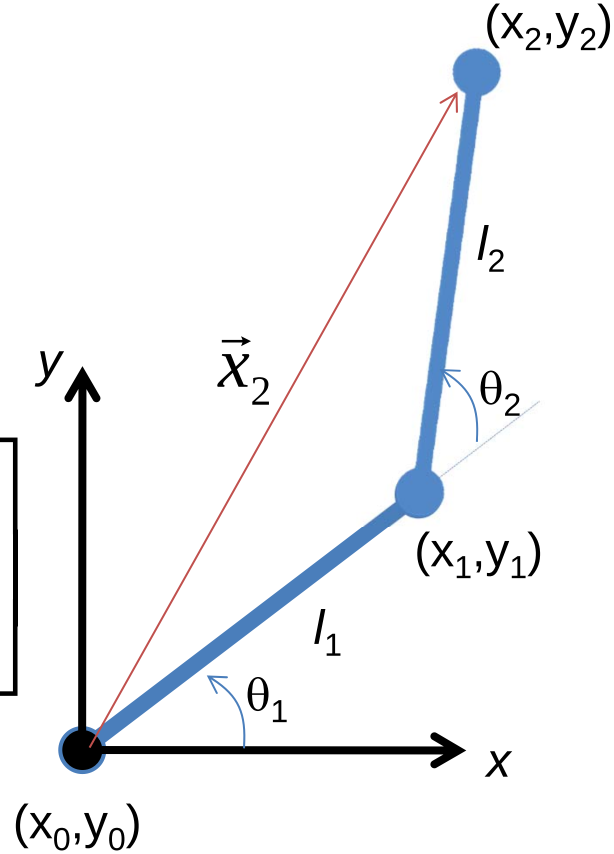
$$y_2 = y_1 + l_2 \sin(\theta_1 + \theta_2)$$
$$y_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$



2-Link Arm Kinematics
Vector Notation

$$\vec{x}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$



2-Link Arm Endpoint Velocity

$$\dot{x}_2 = \dot{x}_1 - l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\sin\left(\theta_1 + \theta_2\right)$$

$$\dot{x}_2 = -l_1\dot{\theta}_1 \sin\left(\theta_1\right) - l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\sin\left(\theta_1 + \theta_2\right)$$

$$\dot{y}_2 = \dot{y}_1 + l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\cos\left(\theta_1 + \theta_2\right)$$

$$\dot{y}_2 = l_1\dot{\theta}_1 \cos\left(\theta_1\right) + l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\cos\left(\theta_1 + \theta_2\right)$$

2-Link Arm Dynamics
Lagrangian

$$L = K - P$$

$$L = \underbrace{\left(K_1 + K_2 + \ldots\right)}_{\text{KE of each mass}} - \underbrace{\left(P_1 + P_2 + \ldots\right)}_{\text{PE of each mass}}$$

2-Link Arm Dynamics
Lagrangian

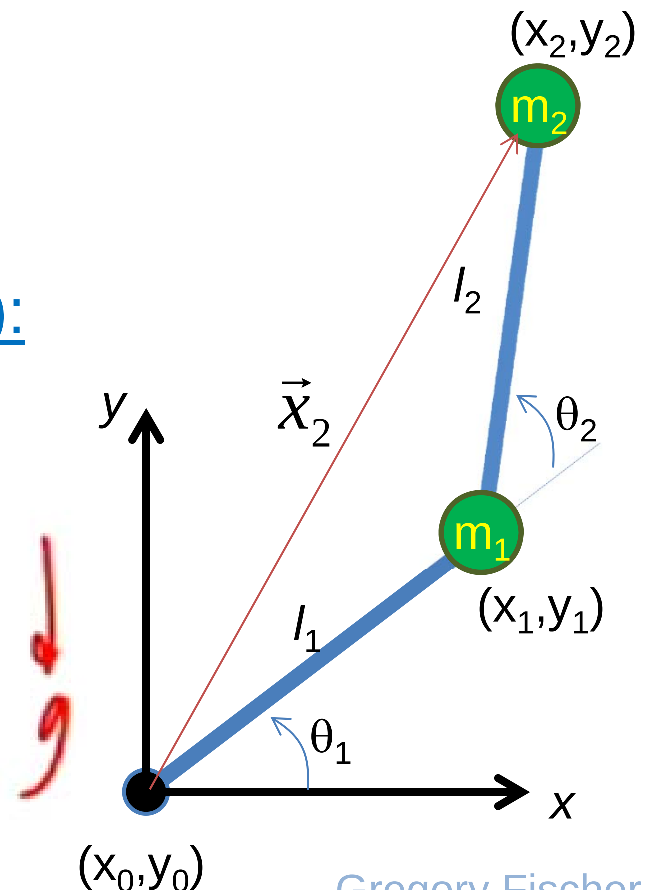
$$L = K - P$$

Separate into energy terms for <u>each mass</u>

$$L = \left(K_1 + K_2 + \ldots\right) - \left(P_1 + P_2 + \ldots\right)$$

2-Link Arm Dynamics
First Link

$$\dot{\vec{x}}_1 = \begin{bmatrix} -l_1\dot{\theta}_1 \sin(\theta_1) \\ l_1\dot{\theta}_1 \cos(\theta_1) \end{bmatrix}$$



Link 1 Energy Equations (From earlier):

$$K_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1$$

$$P_1 = m_1 g l_1 S_1$$

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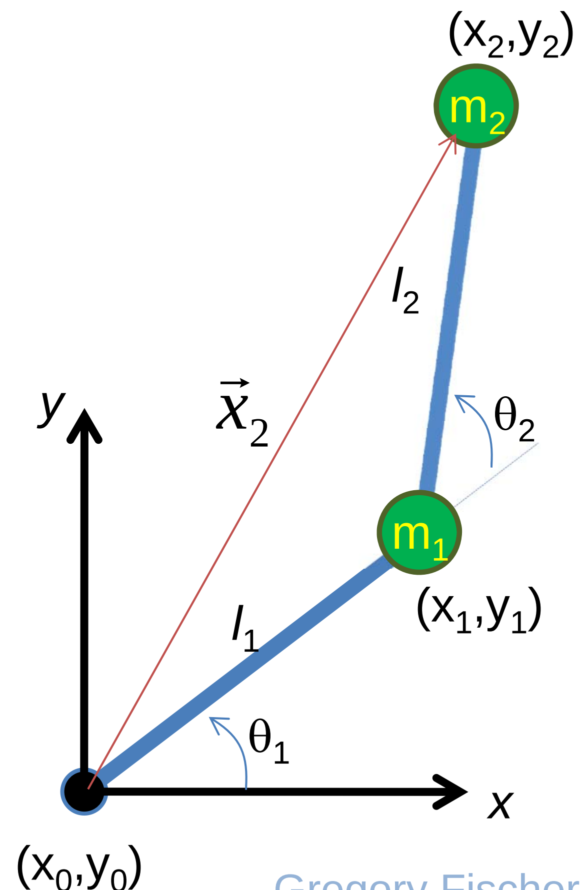
2-Link Arm Dynamics
First Link

$$\dot{\vec{x}}_1 = \begin{bmatrix} -l_1\dot{\theta}_1 \sin(\theta_1) \\ l_1\dot{\theta}_1 \cos(\theta_1) \end{bmatrix}$$



Link 1 Energy Equations (From earlier):

$$K_1 = \frac{1}{2} m_1 l_1^{\,2} \dot{\theta}_1^{\,2}$$

$$P_1 = m_1 g l_1 \sin(\theta_1)$$

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2-Link Arm Dynamics
Second Link

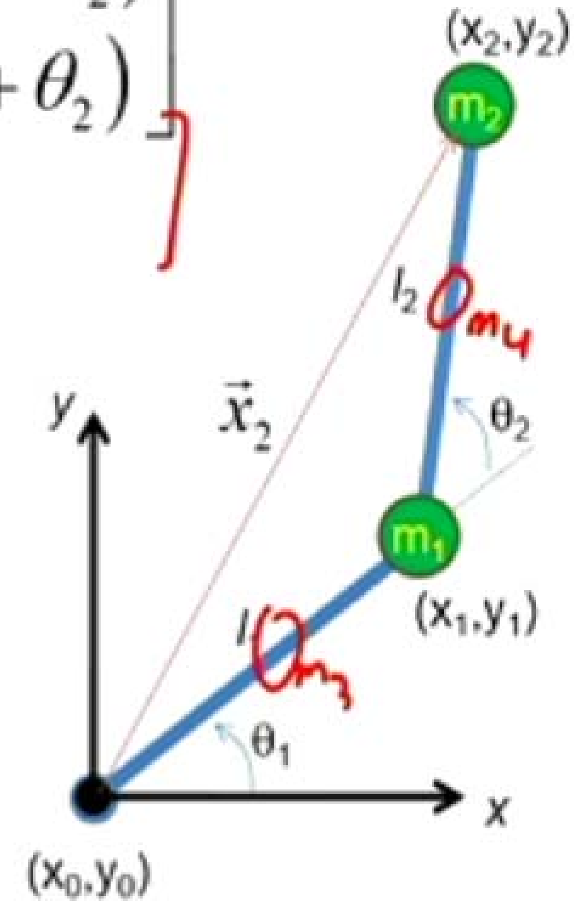
off of $T^{0}_{1,x}$ = [ ]

$$\dot{\vec{x}}_2 = \begin{bmatrix} -l_1\dot{\theta}_1\sin(\theta_1) - l_2(\dot{\theta}_1+\dot{\theta}_2)\sin(\theta_1+\theta_2) \\ l_1\dot{\theta}_1\cos(\theta_1) + l_2(\dot{\theta}_1+\dot{\theta}_2)\cos(\theta_1+\theta_2) \\ 0 \end{bmatrix}$$

Calculate magnitude of velocity^2

$$V_2^2 = \dot{\vec{X}}_2 \cdot \dot{\vec{X}}_2$$

$$V_2^2 = X_2^2 + Y_2^2 + Z_2^2$$



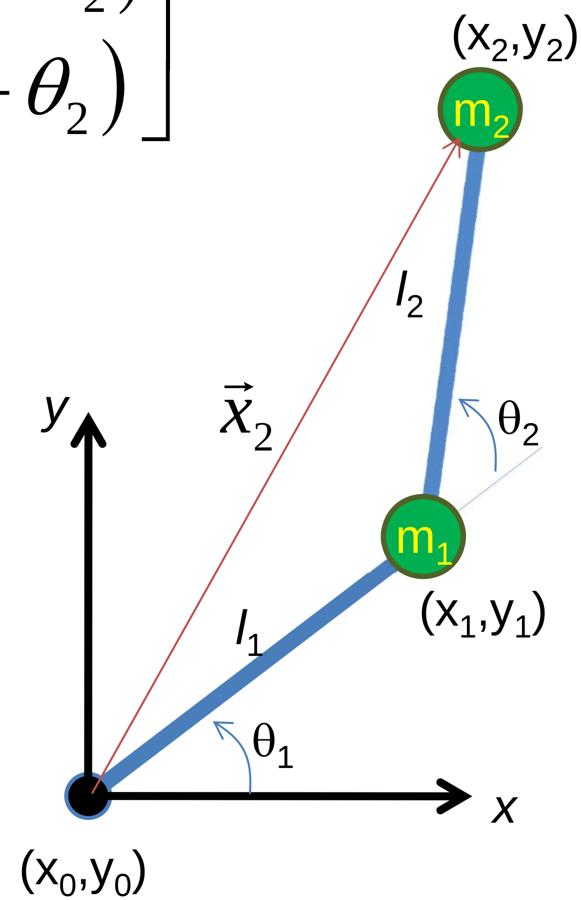
2-Link Arm Dynamics
Second Link

$$\dot{\vec{x}}_2 = \begin{bmatrix} -l_1\dot{\theta}_1 \sin(\theta_1) - l_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 + \theta_2) \\ l_1\dot{\theta}_1 \cos(\theta_1) + l_2(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2) \end{bmatrix}$$

Calculate magnitude of velocity^2

$$v_2^{\ 2} = \dot{\vec{x}}_2 \bullet \dot{\vec{x}}_2$$

$$v_2^{\ 2} = x_2^{\ 2} + y_2^{\ 2}$$



2-Link Arm Dynamics
Second Link

$$v_2{}^2 = l_1{}^2\dot{\theta}_1{}^2 + l_2{}^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + 2l_1l_2\left(\dot{\theta}_1{}^2 + \dot{\theta}_1\dot{\theta}_2\right)\cos\left(\dot{\theta}_2\right)$$

Link 2 Energy Equations:

$$K_2 = \frac{1}{2}m_2\left[l_1{}^2\dot{\theta}_1{}^2 + l_2{}^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + 2l_1l_2\left(\dot{\theta}_1{}^2 + \dot{\theta}_1\dot{\theta}_2\right)\cos\left(\dot{\theta}_2\right)\right]$$

$$= \frac{1}{2}m_2l_1{}^2\dot{\theta}_1{}^2 + \frac{1}{2}m_2l_2{}^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + m_2l_1l_2\left(\dot{\theta}_1{}^2 + \dot{\theta}_1\dot{\theta}_2\right)\cos\left(\dot{\theta}_2\right)$$

$$P_2 = m_2g\left[l_1\sin\left(\theta_1\right) + l_2\sin\left(\theta_1 + \theta_2\right)\right]$$

2-Link Arm Lagrangian

$$L = \left(K_1 + K_2\right) - \left(P_1 + P_2\right)$$

$$K_1 + K_2 = \frac{1}{2}\left(m_1 + m_2\right) l_1{}^2 \dot{\theta}_1{}^2 + \frac{1}{2} m_2 l_2{}^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + m_2 l_1 l_2 \left(\dot{\theta}_1{}^2 + \dot{\theta}_1 \dot{\theta}_2\right) \cos\left(\theta_2\right)$$

$$P_1 + P_2 = \left(m_1 + m_2\right) g l_1 \sin\left(\theta_1\right) + m_2 g l_2 \sin\left(\theta_1 + \theta_2\right)$$

2-Link Arm Lagrangian

$$L = \left(K_1 + K_2\right) - \left(P_1 + P_2\right)$$

$$L = \frac{1}{2}\left(m_1 + m_2\right) l_1{}^2 \dot{\theta}_1{}^2 + \frac{1}{2} m_2 l_2{}^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2$$
$$+ m_2 l_1 l_2 \left(\dot{\theta}_1{}^2 + \dot{\theta}_1 \dot{\theta}_2\right) \cos\left(\theta_2\right)$$
$$- \left(m_1 + m_2\right) g l_1 \sin\left(\theta_1\right) - m_2 g l_2 \sin\left(\theta_1 + \theta_2\right)$$

Lagrange’s Equation

Scalar Form:

$$\tau = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta}$$

Vector Form (n=2 in this case):

Generalized forces (τ_i)

$$\vec{\tau} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{\theta}}} - \frac{\partial L}{\partial \vec{\theta}}$$

Generalized joint coordinates (q_i)

$n \times 1$ vector

Lagrange's Equation

Scalar Form:

$$\tau = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta}$$

Vector Form (n=2 in this case):

$$\vec{\tau} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{\theta}}} - \frac{\partial L}{\partial \vec{\theta}}$$

Generalized Joint Coordinates *(q_i)*

Generalized Forces *(τ_i)*

Lagrange's Equation
Link 1 Components

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1{}^2 \dot{\theta}_1 + m_2 l_2{}^2 \left(\dot{\theta}_1 + \dot{\theta}_2\right) + m_2 l_1 l_2 \left(2\dot{\theta}_1 + \dot{\theta}_2\right)\cos(\theta_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1{}^2 \ddot{\theta}_1 + m_2 l_2{}^2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2 l_1 l_2 \left(2\ddot{\theta}_1 + \ddot{\theta}_2\right)\cos(\theta_2)$$
$$- m_2 l_1 l_2 \left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2{}^2\right)\sin(\theta_2)$$

$$\frac{\partial L}{\partial \theta_1} = -(m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_1 + \theta_2)$$

Lagrange's Equation
Link 2 Components

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2{}^2 \left(\dot{\theta}_1 + \dot{\theta}_2 \right) + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2{}^2 \left(\ddot{\theta}_1 + \ddot{\theta}_2 \right) + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 l_1 l_2 \left(\dot{\theta}_1{}^2 + \dot{\theta}_1 \dot{\theta}_2 \right) \sin(\theta_2) - m_2 g l_2 \cos(\theta_1 + \theta_2)$$

Solve Lagrange's Equation
Joint 1

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1}$$

Plug in:

$$\tau_1 = \left[(m_1+m_2)l_1^{\,2}\ddot{\theta}_1 + m_2 l_2^{\,2}\left(\ddot{\theta}_1+\ddot{\theta}_2\right) + m_2 l_1 l_2\left(2\ddot{\theta}_1+\ddot{\theta}_2\right)\cos(\theta_2) - m_2 l_1 l_2\left(2\dot{\theta}_1\dot{\theta}_2+\dot{\theta}_2^{\,2}\right)\sin(\theta_2)\right]$$
$$-\left[-(m_1+m_2)gl_1\cos(\theta_1) - m_2 g l_2\cos(\theta_1+\theta_2)\right]$$

Group Terms:

$$\tau_1 = \left[(m_1+m_2)l_1^{\,2} + m_2 l_2^{\,2} + 2m_2 l_1 l_2\cos(\theta_2)\right]\ddot{\theta}_1 + \left[m_2 l_2^{\,2} + m_2 l_1 l_2\cos(\theta_2)\right]\ddot{\theta}_2$$
$$+\left[-m_2 l_1 l_2\left(2\dot{\theta}_1\dot{\theta}_2+\dot{\theta}_2^{\,2}\right)\sin(\theta_2)\right] + \left[(m_1+m_2)gl_1\cos(\theta_1) + m_2 g l_2\cos(\theta_1+\theta_2)\right]$$

Solve Lagrange's Equation
Joint 1

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1}$$

Plug in:

$$\tau_1 = \left[(m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) + m_2 l_1 l_2\left(2\ddot{\theta}_1+\ddot{\theta}_2\right)\cos(\theta_2) - m_2 l_1 l_2\left(2\dot{\theta}_1\dot{\theta}_2+\dot{\theta}_2^{\,2}\right)\sin(\theta_2)\right]$$
$$-\left[-(m_1+m_2)g l_1\cos(\theta_1) - m_2 g l_2\cos(\theta_1+\theta_2)\right]$$

Group Terms:

$$\tau_1 = \left[(m_1+m_2)l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2\cos(\theta_2)\right]\ddot{\theta}_1 + \left[m_2 l_2^2 + m_2 l_1 l_2\cos(\theta_2)\right]\ddot{\theta}_2$$
$$+\left[-m_2 l_1 l_2\left(2\dot{\theta}_1\dot{\theta}_2+\dot{\theta}_2^{\,2}\right)\sin(\theta_2)\right] + \left[(m_1+m_2)g l_1\cos(\theta_1) + m_2 g l_2\cos(\theta_1+\theta_2)\right]$$

Solve Lagrange's Equation
Joint 2

$$\tau_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2}$$

Plug in:

$$\tau_2 = \left[m_2 l_2{}^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2)\right]$$
$$- \left[-m_2 l_1 l_2 \left(\dot{\theta}_1{}^2 + \dot{\theta}_1 \dot{\theta}_2\right)\sin(\theta_2) - m_2 g l_2 \cos(\theta_1 + \theta_2)\right]$$

Group Terms:

$$\tau_2 = \left[m_2 l_2{}^2 + m_2 l_1 l_2 \cos(\theta_2)\right]\ddot{\theta}_1 + \left[m_2 l_2{}^2\right]\ddot{\theta}_2$$
$$+ \left[m_2 l_1 l_2 \dot{\theta}_1{}^2 \sin(\theta_2)\right] + \left[m_2 g l_2 \cos(\theta_1 + \theta_2)\right]$$

Solve Lagrange's Equation
Joint 2

$$\tau_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2}$$

Plug in:

$$\tau_2 = \left[m_2 l_2{}^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2)\right]$$
$$- \left[- m_2 l_1 l_2 \left(\dot{\theta}_1{}^2 + \dot{\theta}_1 \dot{\theta}_2\right)\sin(\theta_2) - m_2 g l_2 \cos(\theta_1 + \theta_2)\right]$$

Group Terms:

$$\tau_2 = \left[m_2 l_2{}^2 + m_2 l_1 l_2 \cos(\theta_2)\right]\ddot{\theta}_1 + \left[m_2 l_2{}^2\right]\ddot{\theta}_2$$
$$+ \left[m_2 l_1 l_2 \dot{\theta}_1{}^2 \sin(\theta_2)\right] + \left[m_2 g l_2 \cos(\theta_1 + \theta_2)\right]$$

Solve Lagrange's Equation

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1}$$

$$\tau_1 = \left[(m_1 + m_2)l_1^{\,2} + m_2 l_2^{\,2} + 2m_2 l_1 l_2 \cos(\theta_2)\right]\ddot{\theta}_1 + \left[m_2 l_2^{\,2} + m_2 l_1 l_2 \cos(\theta_2)\right]\ddot{\theta}_2$$
$$+ \left[-m_2 l_1 l_2 \left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^{\,2}\right)\sin(\theta_2)\right] + \left[(m_1 + m_2) g l_1 \cos(\theta_1) + m_2 g l_2 \cos(\theta_1 + \theta_2)\right]$$

$$\tau_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_2}$$

$$\tau_2 = \left[m_2 l_2^{\,2} + m_2 l_1 l_2 \cos(\theta_2)\right]\ddot{\theta}_1 + \left[m_2 l_2^{\,2}\right]\ddot{\theta}_2$$
$$+ \left[m_2 l_1 l_2 \dot{\theta}_1^{\,2} \sin(\theta_2)\right] + \left[m_2 g l_2 \cos(\theta_1 + \theta_2)\right]$$

Solve Lagrange’s Equation Vector Form

nx1 $\vec{\tau}$ vector of generalized forces

Joint accel.

$$\underbrace{\vec{\tau}=\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}}_{n\times 1}=\overbrace{\begin{bmatrix}(m_1+m_2)l_1^{\,2}+m_2l_2^{\,2}+2m_2l_1l_2\cos(\theta_2) & m_2l_2^{\,2}+m_2l_1l_2\cos(\theta_2)\\ m_2l_2^{\,2}+m_2l_1l_2\cos(\theta_2) & m_2l_2^{\,2}\end{bmatrix}}^{\text{Inertia Matrix } M(q) \text{ (mass matrix)}}\underbrace{\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}}_{n\times 1}$$

$$+\underbrace{\begin{bmatrix}-m_2l_1l_2\left(2\dot{\theta}_1\dot{\theta}_2+\dot{\theta}_2^{\,2}\right)\sin(\theta_2)\\ m_2l_1l_2\dot{\theta}_1^{2}\sin(\theta_2)\end{bmatrix}}_{\text{Coriolis / Centripital coupling terms}}+\underbrace{\begin{bmatrix}(m_1+m_2)gl_1\cos(\theta_1)+m_2gl_2\cos(\theta_1+\theta_2)\\ m_2gl_2\cos(\theta_1+\theta_2)\end{bmatrix}}_{\text{gravity vector}}$$

Solve Lagrange's Equation Vector Form

Joint Torque Vector

Inertia Matrix

Joint Angle Acceleration Vector

$$\vec{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} (m_1 + m_2) l_1^{\,2} + m_2 l_2^{\,2} + 2 m_2 l_1 l_2 \cos(\theta_2) & m_2 l_2^{\,2} + m_2 l_1 l_2 \cos(\theta_2) \\ m_2 l_2^{\,2} + m_2 l_1 l_2 \cos(\theta_2) & m_2 l_2^{\,2} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -m_2 l_1 l_2 \left(2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^{\,2}\right) \sin(\theta_2) \\ m_2 l_1 l_2 \dot{\theta}_1^{\,2} \sin(\theta_2) \end{bmatrix} + \begin{bmatrix} (m_1 + m_2) g l_1 \cos(\theta_1) + m_2 g l_2 \cos(\theta_1 + \theta_2) \\ m_2 g l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

Coriolis/Centripital Coupling Vector

Gravity Vector

Multi-link Arm Dynamics

General Form:

$$\vec{\tau} = M(\vec{q})\ddot{\vec{q}} + V(\vec{q}, \dot{\vec{q}}) + G(\vec{q}) + \vec{\tau}_d$$

Where:

$\vec{\tau}$ = Generalized joint forces/torques

$\vec{q}$ = Generalized coordinates (angles/translation)

$M(\vec{q})$ = Inertia term

$V(\vec{q}, \dot{\vec{q}})$ = Coriolis/Centripital Coupling term

$G(\vec{q})$ = Gravity term

$\vec{\tau}_d$ = External disturbances (friction, ...)