Intro to Dynamics

Dynamics Techniques

Euler-Lagrange

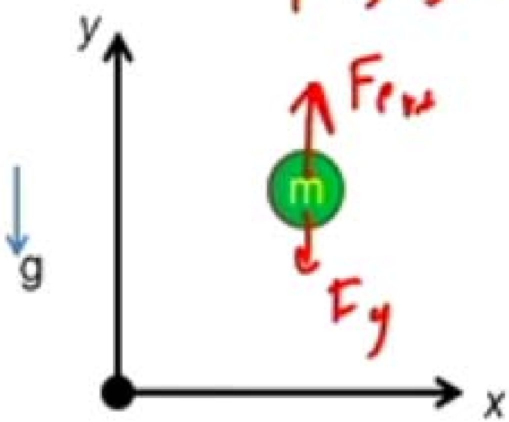
- Energy Based
- Derived from D'Alembert's Principle – Virtual Work

Newton-Euler

- Based on Newtonian Mechanics
    - Every action has an equal and opposite reaction.
    - Rate of change of the linear momentum = force
    - Rate of change of the angular momentum = torque

Particle Dynamics

FBD - free body diagram



F_{ext} = external force

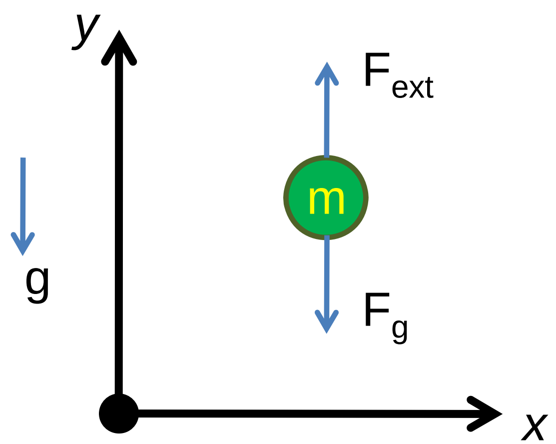
F_g = gravitational force

Eqn. of motion:

$$m\ddot{y} = \overset{F_{ext}}{f} - \overset{F_g}{mg}$$

Particle Dynamics

Draw FBD:



F_{ext} = External force, up direction

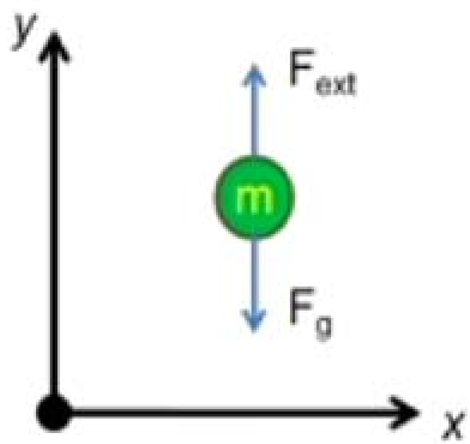
F_g = m*g = gravitational force

Eqn of motion:

$$m\ddot{y} = f - mg$$

$$\ddot{y} = \frac{1}{m} f - g$$

Particle Dynamics
Energy based Techniques



Kinetic Energy:

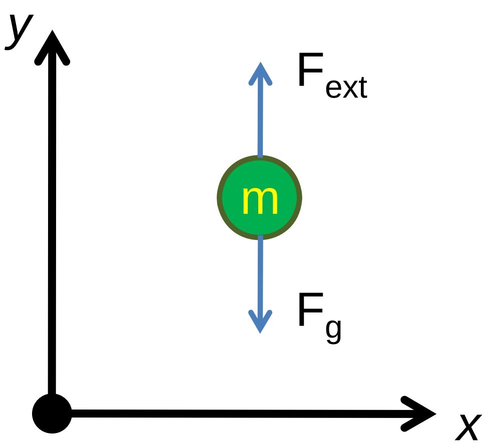
$$K = \frac{1}{2} m \dot{y}^2$$

Write '*m*a' in terms of Kinetic Energy:

$$m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2} m \dot{y}^2\right)$$

$$m\ddot{y} = \frac{d}{dt}\frac{\partial K}{\partial \dot{y}}$$

Particle Dynamics
Energy based Techniques



Kinetic Energy:

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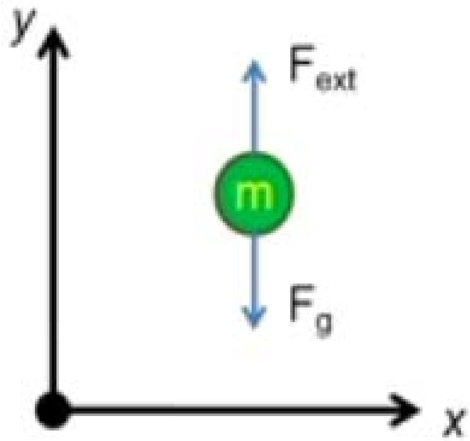
$$m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2} m \dot{y}^2\right) = \frac{d}{dt}\frac{\partial K}{\partial \dot{y}}$$

Particle Dynamics
Energy based Techniques



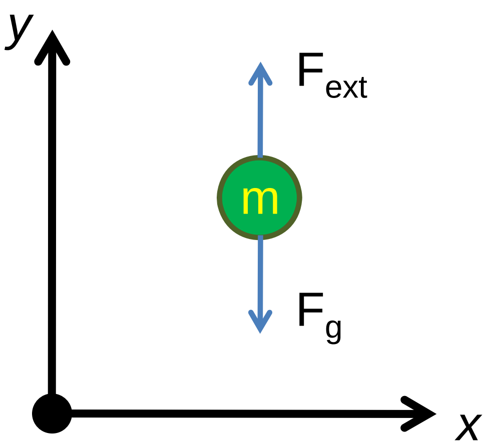
Potential Energy:

$$P = mgy$$

Write *'m*g'* in terms of Potential Energy:

$$mg = \frac{\partial P}{\partial y}$$

Particle Dynamics
Energy based Techniques



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Dynamics Techniques

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Particle Dynamics
Energy based Techniques

K & P terms:

$$m\ddot{y} = \frac{d}{dt}\frac{\partial K}{\partial \dot{y}} \qquad mg = \frac{\partial P}{\partial y}$$

Combining K & P:

$$f = \frac{d}{dt}\frac{\partial K}{\partial \dot{y}} + \frac{\partial P}{\partial y}$$

Define the Lagrangian:

$$L = K - P$$

Euler-Lagrange Equation

$$\Rightarrow \boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = f}$$

One mass example:

$$L = K - P = \frac{1}{2}m\dot{y}^2 - mgy$$

$$f = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2 - mgy\right) - \frac{\partial}{y}\left(\frac{1}{2}m\dot{y}^2 - mgy\right)$$

$$= \frac{d}{dt}(m\dot{y}) + mg$$

$$f = m\ddot{y} + mg$$

Euler-Lagrange Equation

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = f}$$

For the one mass example:

$$L = K - P = \frac{1}{2}m\dot{y}^2 - mgy$$

$$f = \frac{d}{dt}\frac{\partial\left(\frac{1}{2}m\dot{y}^2 - mgy\right)}{\partial \dot{y}} - \frac{\partial\left(\frac{1}{2}m\dot{y}^2 - mgy\right)}{\partial y}$$

$$= \frac{d}{dt}\left(m\dot{y}\right) + mg$$

$$= m\ddot{y} + mg$$

1-Link Arm Dynamics

Using Newton's 2nd Law (Rotational form):

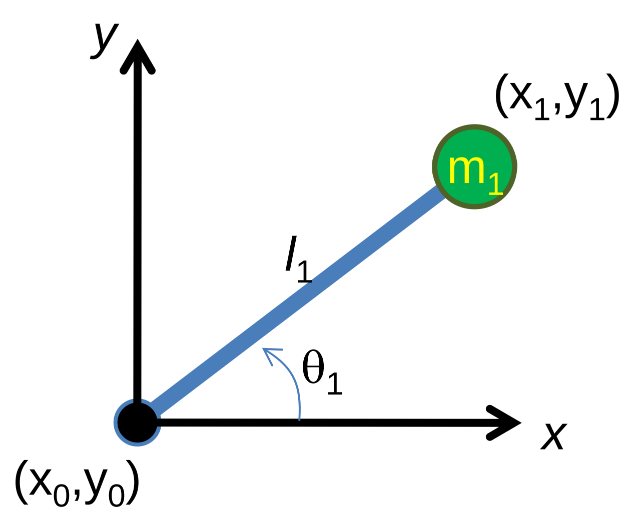
$$\tau = I\alpha \rightarrow \alpha = \frac{\tau}{I}$$

$$I = m_1 l_1^2$$

$$\alpha = \ddot{\theta}_1$$

$\tau =$ ext torque

$$\ddot{\theta}_1 = \frac{1}{m_1 l_1^2}\tau_1$$



1-Link Arm Dynamics

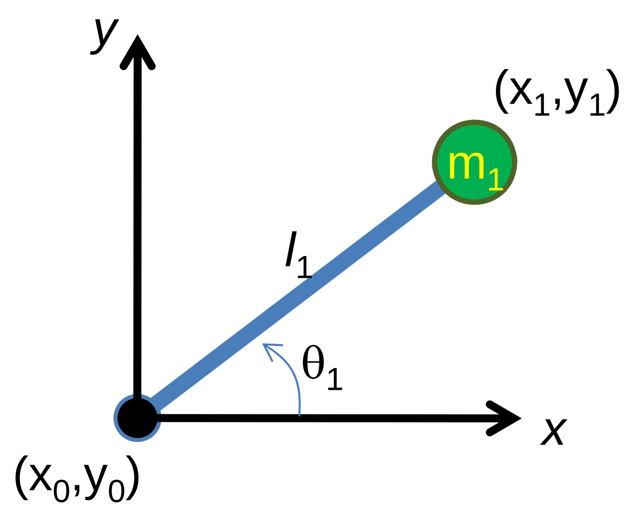
Using Newton's 2nd Law (Rotational form):

$$\alpha = \tau / \mathrm{I}$$

$$\mathrm{I} = m_1 l_1^{\ 2}$$

$$\alpha = \ddot{\theta}_1$$

$$\tau = \text{External Torque}$$

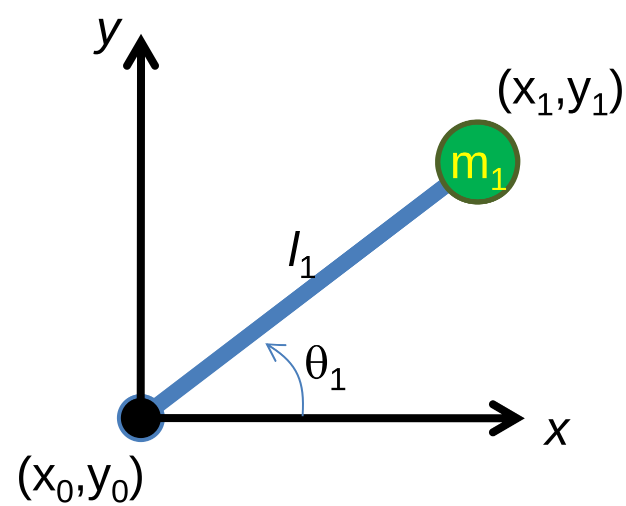


$$\ddot{\theta}_1 = \frac{1}{m_1 l_1^{\ 2}} \tau_1$$

1-Link Arm Dynamics

$$\ddot{\theta}_1 = \frac{\tau_1}{m_1 l_1^{\,2}}$$

$$\tau = u_{motor} + \tau_{disturbance}$$



$$\tau_{dist} = \tau_{grv} + \tau_{fric}$$

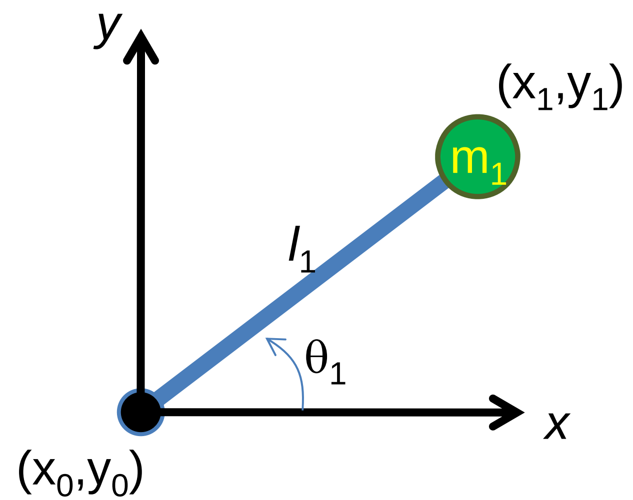
$$= -m_1 g l_1 c_1 - B_1 \dot{\theta}_1$$

eqn of motion:

$$\ddot{\theta}_1 = \frac{1}{m_1 l_1^2}\left(u - m_1 g l_1 c_1 - B_1 \dot{\theta}_1\right)$$

1-Link Arm Dynamics

$$\ddot{\theta}_1 = \frac{\tau_1}{m_1 l_1^{\ 2}}$$



$$\tau = u_{motor} + \tau_{disturbance}$$

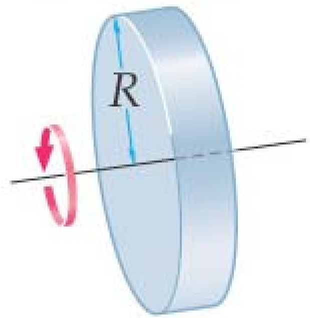
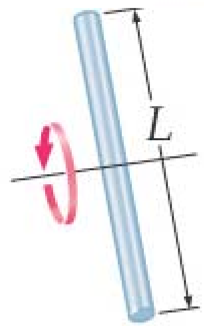
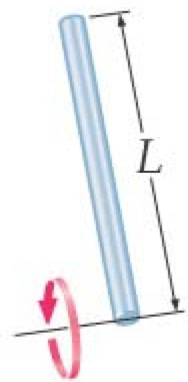
$$\tau_{dist} = \tau_{gravity} + \tau_{friction}$$

$$\tau_{dist} = -m_1 g l_1 \cos(\theta_1) - B_1 \dot{\theta}_1$$

Eqn. of motion:

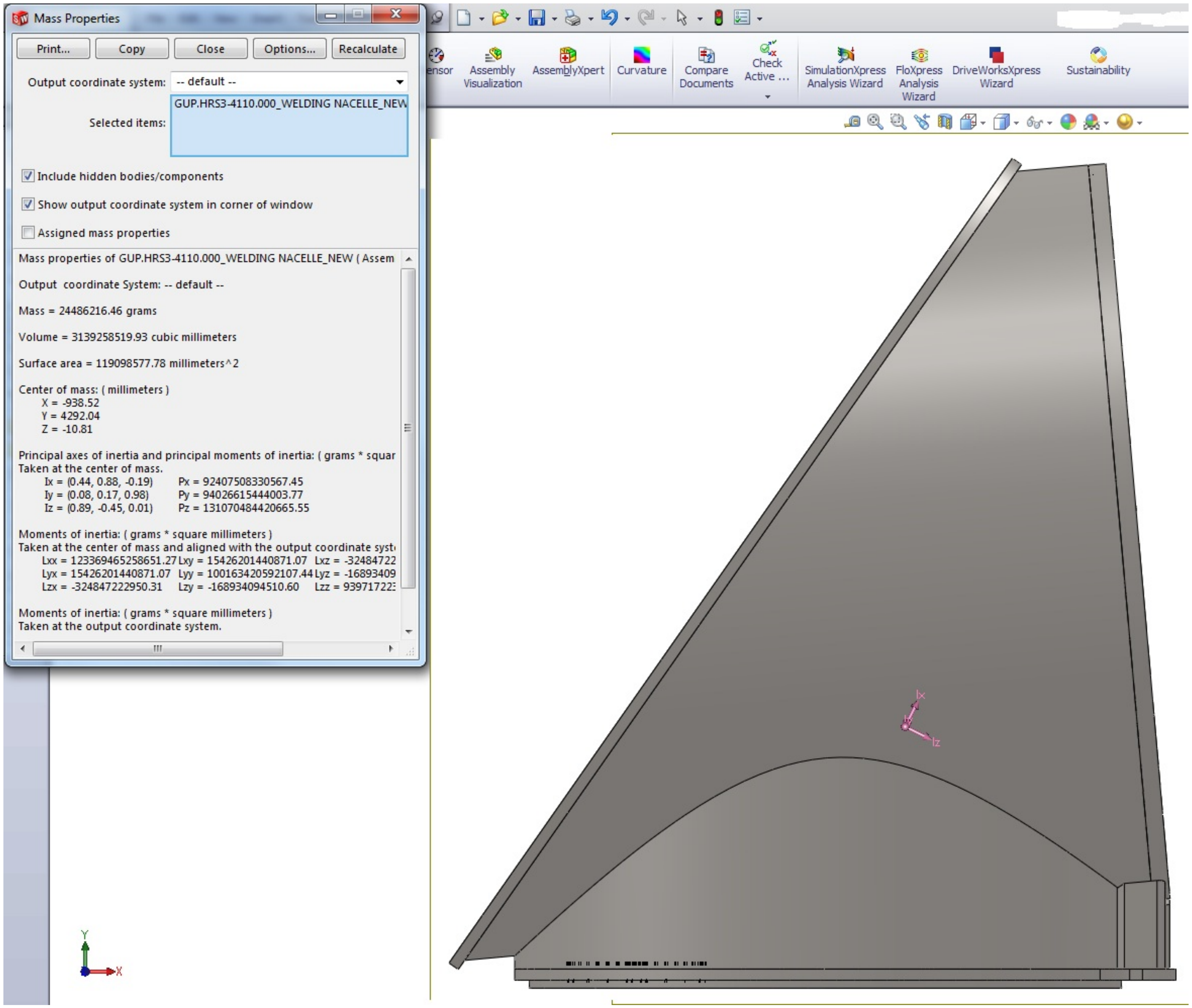
$$\boxed{\ddot{\theta}_1 = \frac{1}{m_1 l_1^{\ 2}} \left(u - m_1 g l_1 \cos(\theta_1) - B_1 \dot{\theta}_1 \right)}$$

Example Moments of Inertia

Object	Axis of Rotation		Moment of Inertia
Solid Disk	Central axis of disk		$\frac{1}{2}MR^2$
Thin Rod	Axis through mid point		$\frac{1}{12}ML^2$
Thin Rod	Axis at one end		$\frac{1}{3}ML^2$

See table on myWPI

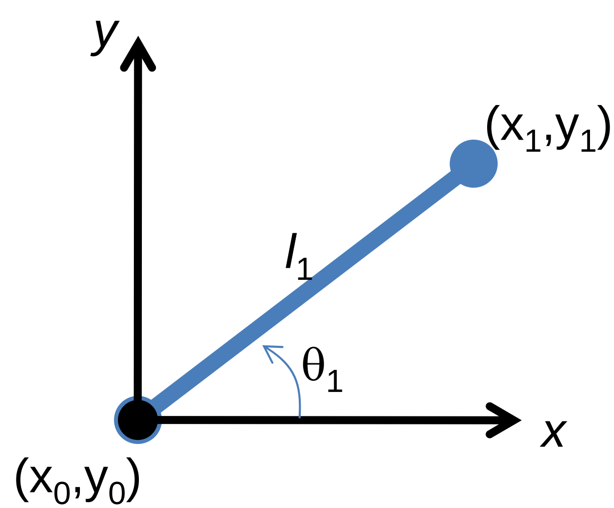
Example Moments of Inertia



1-Link Arm Kinematics

$$x_1 = l_1 \cos(\theta_1)$$

$$y_1 = l_1 \sin(\theta_1)$$



$$\dot{x}_1 = (-l_1 \sin(\theta_1))\dot{\theta}_1$$

$$\dot{y}_1 = (l_1 \cos(\theta_1))\dot{\theta}_1$$

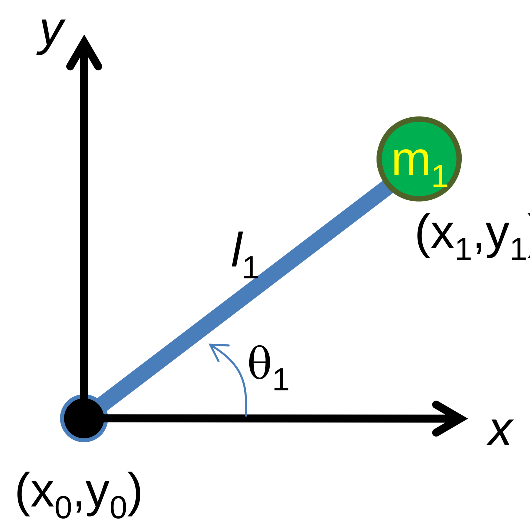
1-Link Arm Dynamics

$$\dot{\vec{x}}_1 = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} -l_1\dot{\theta}_1 \sin(\theta_1) \\ l_1\dot{\theta}_1 \cos(\theta_1) \end{bmatrix}$$

$$V_1^2 = \dot{\vec{x}}_1 \cdot \dot{\vec{x}}_1$$

$$= \dot{x}_1^2 + \dot{y}_1^2$$

$$= l_1^2 \dot{\theta}_1^2$$



1-Link Arm Dynamics

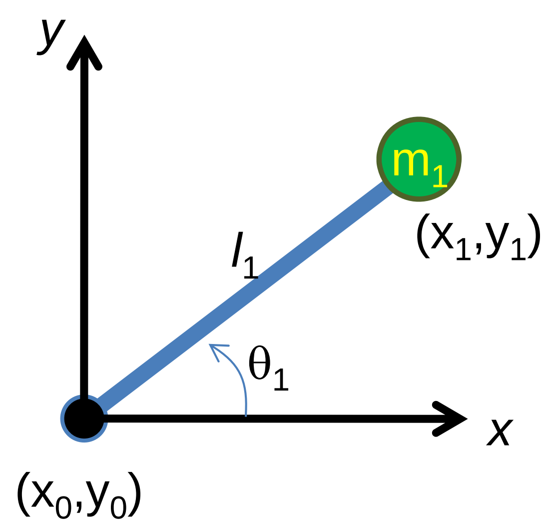
$$\dot{\vec{x}}_1 = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} -l_1\dot{\theta}_1 \sin(\theta_1) \\ l_1\dot{\theta}_1 \cos(\theta_1) \end{bmatrix}$$

$$v_1^{\ 2} = \dot{\vec{x}}_1 \bullet \dot{\vec{x}}_1$$ Dot Product

$$= \dot{x}_1^{\ 2} + \dot{y}_1^{\ 2}$$

$$= \ldots$$

$$= l_1^{\ 2}\dot{\theta}_1^{\ 2}$$



1-Link Arm Dynamics

Solve for Link 1 Energy Equations:

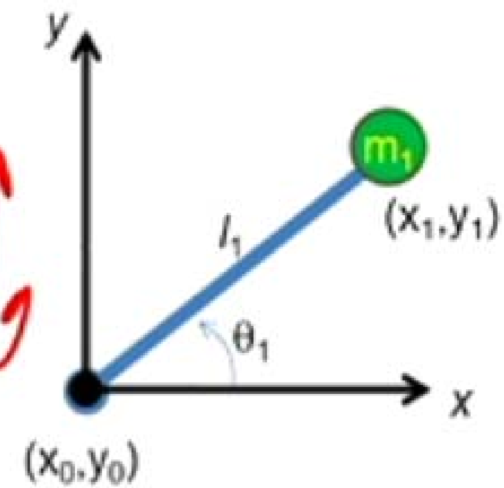
$I = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$

$K_{Rot} = \frac{1}{2} I \omega^2$ $\dot{\theta}_1^2$

$K_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$

$P_1 = m_1 g l_1 S_1$

g



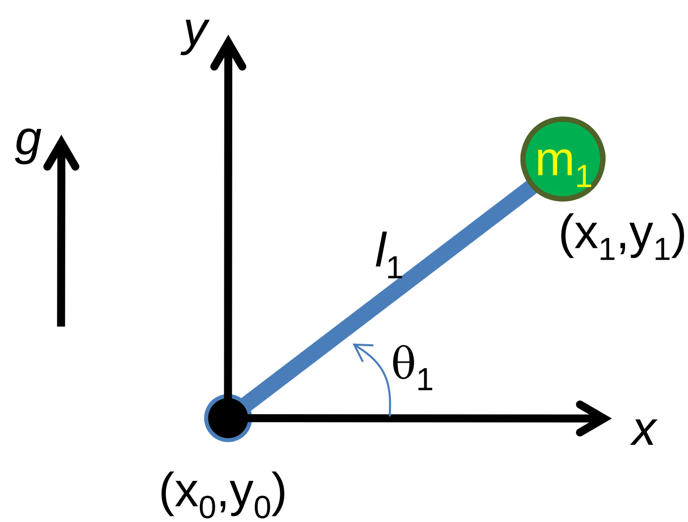
1-Link Arm Dynamics

Link 1 Energy Equations:

$$K_{Rot} = \frac{1}{2} I \omega^2$$

$$K_1 = \frac{1}{2} m_1 l_1^{\,2} \dot{\theta}_1^{\,2}$$

$$P_1 = m_1 g l_1 \sin(\theta_1)$$



1-Link Arm Lagrangian

$$L = K_1 - P_1$$

$$K_1 = \frac{1}{2} m_1 l_1^{\,2} \dot{\theta}_1^{\,2}$$

$$P_1 = m_1 g l_1 \sin(\theta_1)$$

$$L = \frac{1}{2} m_1 l_1^{\,2} \dot{\theta}_1^{\,2} - m_1 g l_1 S_1.$$

1-Link Arm Lagrangian

$$L = K_1 - P_1$$

$$K_1 = \frac{1}{2} m_1 l_1^{\,2} \dot{\theta}_1^{\,2}$$

$$P_1 = m_1 g l_1 \sin(\theta_1)$$

$$L = \frac{1}{2} m_1 l_1^{\,2} \dot{\theta}_1^{\,2} - m_1 g l_1 \sin(\theta_1)$$

Lagrange’s Equation

Scalar Form:

$$\tau = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta}$$

Lagrange's Equation
Link 1 Components

$$L = \frac{1}{2} m_1 l_1{}^2 \dot{\theta}_1{}^2 - m_1 g l_1 \sin(\theta_1)$$

$$\frac{\partial L}{\partial \dot{\theta}_1}$$

Typical 3 steps

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)$$

$$\frac{\partial L}{\partial \theta_1}$$

Lagrange's Equation
Link 1 Components

$$L = \frac{1}{2} m_1 l_1{}^2 \dot{\theta}_1{}^2 - m_1 g l_1 \sin(\theta_1)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 \leftarrow$$

$$\frac{\partial L}{\partial \theta_1} = -m_1 g l_1 c_1 \leftarrow$$

Lagrange's Equation
Link 1 Components

$$L = \frac{1}{2} m_1 l_1^{\,2} \dot{\theta}_1^{\,2} - m_1 g l_1 \sin(\theta_1)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^{\,2} \dot{\theta}_1$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^{\,2} \ddot{\theta}_1$$

$$\frac{\partial L}{\partial \theta_1} = -m_1 g l_1 \cos(\theta_1)$$

Solve Lagrange’s Equation
Joint 1

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1}$$

$$\tau_1 = m_1 l_1^2 \ddot{\theta}_1 + m_1 g l_1 c_1$$

Solve Lagrange's Equation
Joint 1

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1}$$

Plug in:

$$\tau_1 = m_1 l_1^2 \ddot{\theta}_1 + m_1 g l_1 \cos(\theta_1)$$

Solve Lagrange's Equation
Joint 1

$$\tau_1 = m_1 l_1^2 \ddot{\theta}_1 + m_1 g l_1 \cos(\theta_1)$$

Solve for equation of motion:

$$\ddot{\theta}_1 = \frac{1}{m_1 l_1^2}\left(\tau_1 - m_1 g l_1 c_1\right)$$

Solve Lagrange's Equation Joint 1

$$\tau_1 = m_1 l_1^{\,2} \ddot{\theta}_1 + m_1 g l_1 \cos(\theta_1)$$

Solve for equation of motion:

$$\ddot{\theta}_1 = \frac{1}{m_1 l_1^{\,2}} \left[\tau_1 - m_1 g l_1 \cos(\theta_1) \right]$$

Solve Lagrange's Equation Joint 1

$$\tau_1 = m_1 l_1^{\,2} \ddot{\theta}_1 + m_1 g l_1 \cos(\theta_1)$$

replace w/ $\tau_1 = u_1 - B_1 \dot{\theta}_1$

eqn of motion:

$$\ddot{\theta}_1 = \frac{1}{m_1 l_1^2} \left(u_1 - B_1 \dot{\theta}_1 - m_1 g l_1 c_1 \right)$$

Solve Lagrange's Equation
Joint 1

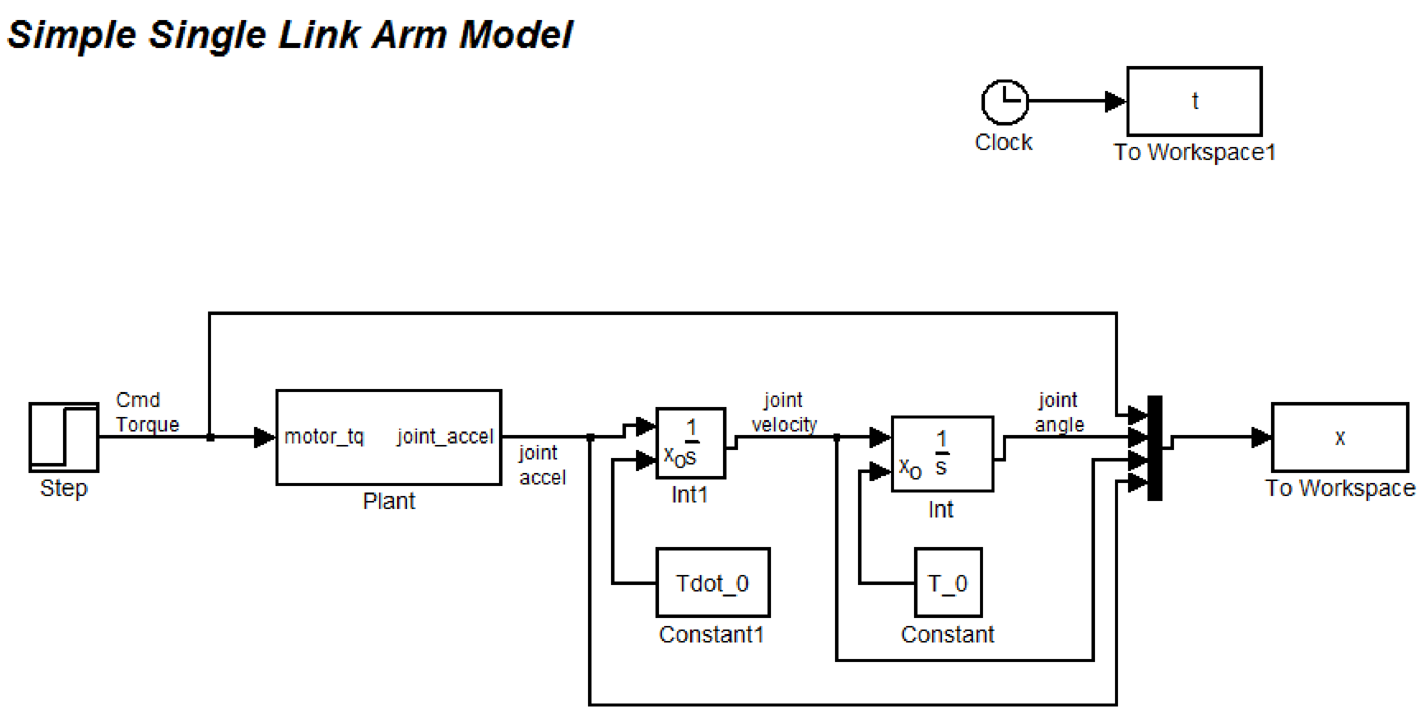
$$\tau_1 = m_1 l_1^{\,2} \ddot{\theta}_1 + m_1 g l_1 \cos(\theta_1)$$

Replace:
*Tau = u – B*theta_d*

Solve for equation of motion:

$$\ddot{\theta}_1 = \frac{1}{m_1 l_1^{\,2}} \left[u_1 - B\dot{\theta}_1 - m_1 g l_1 \cos(\theta_1) \right]$$

Matlab Example



test_SingArm_Dyn.m
Single_Link_Arm_Model_Dynamics_Only.mdl