High Frequency Trading Project

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1 Introduction

In traditional asset price modeling, the mid-price S_t often follows a Stochastic Differential Equation (SDE) where the drift is assumed to be constant. This constant drift represents the overall trend or direction of the price evolution over time, unaffected by the actions of individual traders or the volume of market activity. The standard SDE for an asset price is:

$$dS_t = \mu \, dt + \sigma \, dW_t, \tag{1}$$

where:

- μ is a constant drift parameter representing the expected change in the mid-price over time,
- σ is the volatility of the asset price,
- W_t is a standard Wiener process (Brownian motion) modeling the random fluctuations in the asset price.

Asset managers can apply the model in Derivative Pricing, Risk Management, Portfolio Optimization and Asset Allocation, and more. We will look at Utility Maximization and Optimal Control that use the asset price model extensively. Asset managers apply Optimal control to many tasks, we will look at 2 extensions of asset liquidation that we will solve.

2 Problem Formulations

2.1 Extension 1: Liquidation with Permanent Price Impact and risk aversion

In this extension, we assume the drift of the mid-price is not constant. Instead it is influenced by our agent that liquidates the asset. This assumption better reflects the real-world as large players in the market can move the price considerably. It is commonly assumed the contributions of our agent is a function of his trading speed and moreover follows a linear structure:

- $g(\nu_t) = b \, \nu_t$ represents the permanent price impact
- $f(\nu_t) = \kappa \, \nu_t$ represents the temporary price impact

where κ, b are positive constants and ν_t is the agent trading speed (note that we model positive speed as selling since the agent aims to liquidate).

The SDE now takes the form:

$$dS_t^\nu = -b \, \nu_t \, dt + \sigma \, dW_t, \tag{2}$$

We now add to our formulation, the agent's inventory and cash process.

The execution or LOB (Limit order book) price process $\hat{S}_t^\nu$:

$$\hat{S}_t^\nu = S_t^\nu - k \, \nu_t \tag{3}$$

The inventory process Q_t^ν:

$$dQ_t^\nu = -\nu_t \, dt \tag{4}$$

The Cash process X_t^ν:

$$dX_t^\nu = -\hat{S}_t^\nu \, dQ_t^\nu \tag{5}$$

To find the optimal trading speed ν_t we need to define our objective.
The cost functional given ν_t is defined as follows:

$$H^\nu(t,x,s,q) = \mathbb{E}_{t,x,s,q}\left[X_T^\nu + Q_T^\nu(S_T^\nu - \alpha Q_T^\nu) - \phi\int_t^T (Q_t^\nu)^2 du\right],$$

where

- t, x, s, q: These state variables that corresponds to the current time, the current cash, the current mid price of the asset and the current asset quantity.
- X_T^ν: The cash at final time T. The agent aims to maximize the cash accumulated by the end of the liquidation process.
- $Q_T^\nu \, S_T^\nu$: The theoretical value of the remaining inventory. Even if the agent fails to liquidate the entire position, the unsold inventory still holds market value, which must be considered
- $-\alpha(Q_T^\nu)^2$: The penalty for holding any remaining inventory at the final time. The quadratic penalty ensures that the agent is motivated to liquidate as much as possible by the end of the trading horizon.
- $-\phi\int_t^T (Q_t^\nu)^2 du$: The risk aversion term. It penalizes the agent for holding on to inventory throughout the execution.

The starting values for each state variable is:

- $t = 0$
- x= 0 as we start with zero money
- $Q_t^\nu = R$ where R is the initial quantity of the asset
- $S_t^\nu = S_0$ where S_0 is the initial value of the asset

2.1.1 Motivation for Temporary Price Impact

The temporary price impact reflects how the structure of the **order book** leads to price concessions during trading:

- When the agent trades at a low rate (ν_t is small), their trades are more likely to be executed close to the mid-price, since they are taking liquidity near the best available bid/ask prices.
- When the agent trades at a high rate (ν_t increases), they consume liquidity deeper into the **order book**, where prices are less attractive. As a result, the execution price $\hat{S}_t^\nu$ deviates further from the mid-price, reflecting the immediate, short-term price impact of aggressive trading.

This linear relationship captures the **temporary impact** of the agent's trades on the execution price. The more intensely the agent trades, the more their execution price moves away from the mid-price, due to the limited liquidity at attractive price levels. The coefficient κ quantifies the sensitivity of the execution price to the agent's trading rate.

It is important to note, however, that this model of temporary price impact is a rough approximation. In reality, the relationship between the trading rate and execution price may not be linear. Instead, the market impact may exhibit non-linear behavior, especially at extreme trading rates or in illiquid markets. Despite its simplicity, this linear model serves as a useful assumption for analyzing the basic idea of how trading intensity influences execution prices.

Temporary price impact is **short-lived**; it primarily affects the prices during execution and does not lead to a permanent shift in the mid-price, unlike permanent market impact.

2.2 Extension 2: Including the Impact of Market Orders

In the extended model, we assume that the drift of the mid-price is not constant. Instead, it is influenced by the flow of market orders (MOs) from all participants in the market, including a specific agent which we will be focusing on her control process. This assumption better reflects real-world market conditions where buy and sell pressure from traders directly impacts the asset's mid-price.

The SDE now takes the form:

$$dS_t^\nu = b\left(\mu_t^+ - \left(\nu_t + \mu_t^-\right)\right) dt + \sigma\, dW_t, \tag{6}$$

where:

- μ_t^+ represents the rate of buy market orders from other traders,
- μ_t^- represents the rate of sell market orders from other traders,
- ν_t is the liquidation rate of the agent we are focusing on, whose trades are modeled as sell orders (in the case of liquidation),
- b is a parameter that represents the permanent impact that trading has on the mid-price,
- $\sigma\, dW_t$ accounts for the random noise affecting the price evolution, as in the standard model.

All other problem dynamics like price and quantity processes are the same.

we now define the cost functional to this problem:

$$H^\nu(t, x, S, \mu, q) = \mathbb{E}_{t,x,S,\mu,q}\left[X_T^\nu + Q_T^\nu(S_T^\nu - \alpha Q_T^\nu) - \varphi \int_t^T (\nu_u - \rho\mu_u^-)^2 du\right],$$

where the agent seeks to maximize this function. Let's break down the key terms:

- t, x, S, q: These are the state variables that corresponds to the current time, the current cash, the current mid price of the asset and the current asset quantity.
- μ: The market's flow of buy and sell orders. μ^+ represents the rate of buy market orders, and μ^- represents the rate of sell market orders.
- X_T^ν: Cash at Final Time T - X_T^ν is the amount of cash the agent holds at the final time T. The agent aims to maximize the cash accumulated by the end of the liquidation process. This is critical, as the agent's primary goal is to sell off the assets and collect the proceeds in cash.
- $Q_T^\nu S_T^\nu$: Theoretical Value of Remaining Inventory - Q_T^ν represents the amount of unsold inventory at the final time T, and S_T^ν is the price of the asset. Even if the agent fails to liquidate the entire position, the unsold inventory still holds market value, which must be considered.
- $-\alpha(Q_T^\nu)^2$: This term penalizes the agent for holding any remaining inventory at the final time. The quadratic penalty ensures that the agent is motivated to liquidate as much as possible by the end of the trading horizon.
- $-\varphi \int_t^T (\nu_u - \rho\mu_u^-)^2 du$: This term penalizes the agent for deviating from a fraction ρ of the market's total sell volume. The penalty encourages the agent to trade in a way that does not stand out from the market, thereby avoiding signaling large liquidation intentions. A higher φ increases the sensitivity to deviations, forcing the agent to stay close to the market's trading volume.

2.2.1 Interpretation and Intuition

The drift term in this modified SDE reflects the net effect of buy and sell market orders on the mid-price. Specifically, the drift is now proportional to the difference between the rate of buy orders (μ_t^+) and the rate of sell orders (both from other traders μ_t^- and the agent ν_t).

- The term $\mu_t^+ - \mu_t^-$ represents the overall net order flow from other traders. If buy orders outpace sell orders, this would push the mid-price upward, and vice versa.
- The agent's liquidation rate ν_t subtracts from the drift, reflecting the fact that the agent's sell orders exert downward pressure on the mid-price.

By including the rates of buy and sell orders as stochastic processes, this model captures the dynamic and fluctuating nature of market order flows, leading to a more realistic representation of how asset prices evolve in response to trading activity.

3 Value Function

The value function represents the maximum possible value of the agent's performance criteria H^ν, optimized over the set of admissible trading strategies $\nu \in A$:

$$H = \sup_{\nu \in A} H^\nu$$

This value function result in the highest expected performance the agent can achieve by selecting an optimal trading strategy ν. ν represents the control process that the agent can choose to optimize the objective function.
For the first problem the value function will be:

$$H(t, x, s, q) = \sup_{\nu \in A} H^\nu(t, x, s, q)$$

And for the second one:

$$H(t, x, s, \mu, q) = \sup_{\nu \in A} H^\nu(t, x, s, \mu, q)$$

4 Hamilton-Jacobi-Bellman (HJB) Equation

The **Hamilton-Jacobi-Bellman (HJB) equation** is a partial differential equation that describes the evolution of the agent's value function. It takes the general form:

$$\partial_t H + \sup_{\nu \in A} (\mathcal{L}_t^\nu H + F) = 0.$$

Where F includes all of the integrands of the cost functional's intergrals, and $\mathcal{L}_t^\nu$ is the generator of H. The boundary condition at terminal time T is given by:

- The first problem:
$$H(T, x, s, q) = x + qS - \alpha q^2,$$
- The second problem:
$$H(T, x, S, \mu, q) = x + q(S - \alpha q),$$

where x is the cash, q is the remaining inventory, S is the asset price, and α is a penalty parameter for holding inventory at the final time.

4.1 Generator of H

The generator $\mathcal{L}_t^\nu H$ describes how the value function H evolves in response to the dynamics of the system. It incorporates the stochastic dynamics of the asset price, the agent's trading and more. The generator is given by:

- The first problem:

$$\mathcal{L}_t^\nu H(t,x,s,q) = \frac{1}{2}\sigma^2 \frac{\partial^2 H}{\partial s^2} - b\nu \frac{\partial H}{\partial S} + (S-\kappa\nu)\nu \frac{\partial H}{\partial x} - \nu \frac{\partial H}{\partial q}$$

  In this case, the term $F(t,x,S,\mu,q,\nu)$ is given by:

$$F(t,x,s,q,\nu) = -\phi q^2,$$

- The second problem:

$$\mathcal{L}_t^\nu H(t,x,S,\mu,q) = \frac{1}{2}\sigma^2 \frac{\partial^2 H}{\partial S^2} + b\left((\mu^+ - \mu^-) - \nu\right) \frac{\partial H}{\partial S} + (S-\kappa\nu)\nu \frac{\partial H}{\partial x} - \nu \frac{\partial H}{\partial q} + \mathcal{L}^\mu H.$$

  In this case, the term $F(t,x,S,\mu,q,\nu)$ is given by:

$$F(t,x,S,\mu,q,\nu) = -\varphi(\nu - \rho\mu_t^-)^2,$$

  where φ is the weight applied to penalize the agent for deviating from a fraction ρ of the market's total sell volume.

  Notice that We keep the generator of μ in its general form because we do not have specific assumptions about the processes governing μ^+ or μ^-. Their forms could be subject to external market data, and different models might be more suitable depending on available information. Since μ^+ and μ^- are not directly controlled by ν, we place the generator $\mathcal{L}^\mu H$ outside the supremum in the HJB equation.

  This approach is consistent with the process we conducted during the lecture on targeting volume (refer to minute 28:00 in the first clip), where a similar treatment was applied.

4.2 Placing the Generator into the HJB Equation

Now, we place the generator $\mathcal{L}_t^\nu H$ into the HJB equation. This leads to the full form of the HJB equation:

- The first problem:

$$\partial_t H(t,x,s,q) + \sup_{\nu \in A}\left(\frac{1}{2}\sigma^2 \frac{\partial^2 H}{\partial s^2} - b\nu \frac{\partial H}{\partial S} + (S-\kappa\nu)\nu \frac{\partial H}{\partial x} - \nu \frac{\partial H}{\partial q} - \phi q^2\right) = 0.$$

  And by simplifying we get:

$$\left(\partial_t + \frac{1}{2}\sigma^2 \partial_{SS}\right) H - \phi q^2 + \sup_\nu \{(S-\kappa\nu)\nu\partial_x H - \nu\partial_q H - b\nu\partial_S H\} = 0.$$

- The second problem:

$$\partial_t H(t,x,S,\mu,q) + \sup_{\nu \in A}\left(\frac{1}{2}\sigma^2 \frac{\partial^2 H}{\partial S^2} + b\left((\mu^+ - \mu^-) - \nu\right) \frac{\partial H}{\partial S} + (S-\kappa\nu)\nu \frac{\partial H}{\partial x} - \nu \frac{\partial H}{\partial q} - \varphi(\nu - \rho\mu_t^-)^2\right) + \mathcal{L}^\mu H = 0.$$

  And by simplifying we get:

$$\left(\partial_t + \frac{1}{2}\sigma^2 \partial_{SS} + \mathcal{L}^\mu\right) H + \sup_\nu \left\{(S-\kappa\nu)\nu\partial_x H - \nu\partial_q H + b((\mu^+ - \mu^-) - \nu)\partial_S H - \varphi(\nu - \rho\mu^-)^2\right\} = 0.$$

For each problem, its equation now includes the full generator of the value function H, and represents the optimal control problem where the agent seeks to maximize their expected performance by adjusting their trading rate ν.

5 Next Steps to Solve the Problem

To solve this optimal control problem, the next steps involve finding the optimal strategy ν through the following process, as studied in the course:

1. **Solve the internal optimal problem in the supremum section**: This is a single-variable quadratic optimization problem. The objective here is to find the optimal value of ν that maximizes the expression inside the supremum.

2. **Substitute the optimal ν into the HJB equation**: After finding the optimal ν, we substitute it back into the HJB equation, simplifying the problem.

3. **Solve the partial differential equation (PDE)**: Finally, we solve the resulting PDE using an ansatz for the value function H.

6 Solution for Problem 1

6.1 Simplifying the expression

We want to get rid of the supremum and substitute the explicit expression.
Notice $-\left(\kappa\partial_x H\right)\nu^2 + \nu\left(S\partial_x H - \partial_q H - b\partial_S H\right)$ is a parabola in ν.
For the supremum to be finite, $-\kappa\partial_x H$ must be negative. This assumption is reasonable, as the $\kappa > 0$ and the value function must increase in x ($\partial_x H > 0$).
And we get:

$$\boxed{v^* = \frac{s\partial_x H - b\partial_s H - \partial_q H}{2k\partial_x H}.}$$

And the supremum will be:

$$\frac{(s\partial_x H - b\partial_s H - \partial_q H)^2}{4k\partial_x H}$$

Substituting the supremum we get:

$$\left(\partial_t + \frac{1}{2}\sigma^2\partial_{SS}\right)H - \phi q^2 + \frac{(s\partial_x H - b\partial_s H - \partial_q H)^2}{4k\partial_x H} = 0. \tag{7}$$

With end conditions:

$$H(T, x, s, q) = x + qs - \alpha q^2$$

6.2 Substituting Ansatz into the HJB Equation

We will try the Ansatz:

$$H(t, x, s, q) = x + sq + h(t, s, q)$$
$$h(T, s, q) = -\alpha q^2$$

And we get:

$$\partial_t h + \frac{1}{2}\sigma^2\partial_{ss}h - \phi q^2 + \frac{1}{4k}\left(\cancel{s} - bq - b\partial_s h - \cancel{s} - \partial_q h\right)^2 = 0$$
$$\partial_t h + \frac{1}{2}\sigma^2\partial_{ss}h - \phi q^2 + \frac{1}{4k}\left(bq + b\partial_s h + \partial_q h\right)^2 = 0$$

We use another Ansatz:

$$h(t, s, q) = h(t, q)$$

and get:

$$\partial_t h - \phi q^2 + \frac{1}{4k}\left(bq + \partial_q h\right)^2 = 0$$

$$h(T,q) = -\alpha q^2$$

We use a third Ansatz:

$$h(t,q) = q^2 h_2(t)$$

And get the following simplification:

$$\cancel{q^2}\partial_t h_2 - \phi \cancel{q^2} + \frac{1}{4k}\left(b\cancel{q} + 2\cancel{q}h_2\right)^2 = 0$$

$$\partial_t h_2 - \phi + \frac{1}{\kappa}\left(\frac{b}{2} + h_2\right)^2 = 0$$

$$h_2(T,q) = -\alpha$$

This is a Ricatti equation, and we have already a close form solution for it. But we will get the solution manually.

6.3 Solving the Ricatti equation

Define

$$\xi(t) = h_2(t) + \frac{b}{2}$$

and we get:

$$\frac{d\xi}{dt} - \phi + \frac{1}{\kappa}\xi^2 = 0$$

With terminal condition

$$\xi(T) = h_2(T) + \frac{b}{2} = \frac{b}{2} - \alpha$$

Arranging the terms and we get:

$$\frac{d\xi}{dt} = \phi - \frac{1}{\kappa}\xi^2$$

$$\frac{1}{\kappa\phi - \xi^2}d\xi = \frac{1}{\kappa}dt$$

Integrating both sides:

$$\int_{\xi(t)}^{\xi(T)} \frac{1}{\kappa\phi - \xi^2}d\xi = \int_t^T \frac{1}{\kappa}dt$$

And we get:

$$\frac{\kappa^{3/2}}{2\sqrt{\phi}}\left(\log\frac{\sqrt{\kappa\phi} + \xi(T)}{\sqrt{\kappa\phi} - \xi(T)} - \log\frac{\sqrt{\kappa\phi} + \xi(t)}{\sqrt{\kappa\phi} - \xi(t)}\right) = \kappa(T-t)$$

Define:

$$\gamma = \sqrt{\frac{\phi}{\kappa}}, \zeta = \frac{\alpha - \frac{1}{2}b + \sqrt{\phi\kappa}}{\alpha - \frac{1}{2}b - \sqrt{\phi\kappa}}$$

And using the new constants we get a simplified solution:

$$\xi(t) = \sqrt{\kappa\phi}\frac{1 + \zeta e^{2\gamma(T-t)}}{1 - \zeta e^{2\gamma(T-t)}}$$

6.4 Substituting into the optimal trading rate

Previously we have shown

$$\nu^* = \frac{s\partial_x H - b\partial_s H - \partial_q H}{2\kappa\partial_x H}.$$

Applying our Ansatz

$$H(t,x,s,q) = x + sq + q^2(\xi(t) - \frac{b}{2})$$

Gives:

$$\nu^* = \frac{\cancel{s} - \cancel{bq} - \cancel{s} - 2q\left(\xi(t) - \cancel{\frac{b}{2}}\right)}{2\kappa}$$

$$\nu^* = \frac{-2q\xi(t))}{2\kappa} = -\frac{\xi(t)}{\kappa}q$$

And we get the optimal trading speed as a closed loop solution:

$$\nu_t^* = \gamma\frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}} Q_t^{v^*}$$

6.5 Finding an explicit solution

We got a closed loop solution, where for each observation Q_t we can control our trading speed to be ν_t^*. To get and explicit solution for $v*$ we use the relation $dQ_t^{\nu^*} = -\nu_t dt$ and get:

$$\frac{dQ_t^{\nu^*}}{Q_t^{\nu^*}} = -\gamma\frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}} dt, \quad Q_0^{\nu^*} = R.$$

Using integration we can get:

The optimal trading speed:

$$\nu_t^* = \gamma\frac{\zeta e^{\gamma(T-t)} + e^{-\gamma(T-t)}}{\zeta e^{\gamma(T)} + e^{-\gamma(T)}} R$$

The resulting Asset Quantity liquidation:

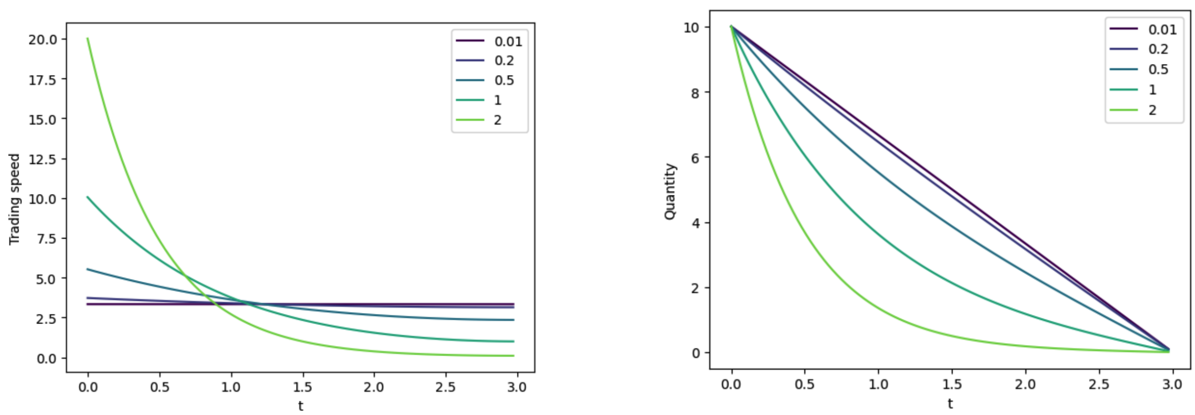
$$Q_t^{v^*} = \frac{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta e^{\gamma(T)} + e^{-\gamma(T)}} R$$

6.6 Asymptotic behaviour

We got an explicit solution for the trading speed, let us analyze the asymptotic behaviour of the solution. When $\alpha \to \infty$, we get $\zeta \to 1$, therefore:

$$\nu_t^* \to \gamma \frac{cosh(\gamma(T-t))}{sinh(\gamma T)} R, \quad Q_t^{v^*} \to \frac{sinh(\gamma(T-t))}{sinh(\gamma T)} R$$

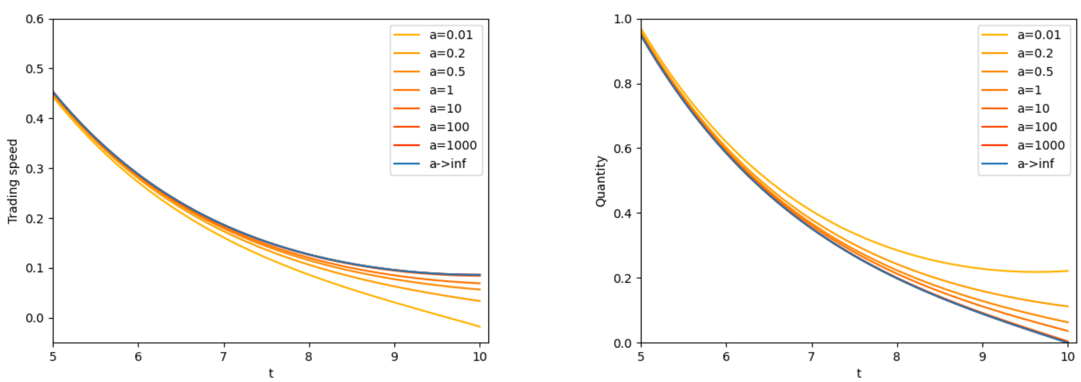
We now visualize the results. In this example we want to sell 10 units of an asset in 3 minutes and we plot v_t^* and $Q_t^{v^*}$ with different values of gamma.



(a) v_t^* plot through time

(b) $Q_t^{v^*}$ plot through time

We further visualize the change in behavior when gradually increasing *alpha* towards the limit, compared to the asymptotic case.



(c) v_t^* plot through time with increasing *alpha*

(d) $Q_t^{v^*}$ plot through time with increasing *alpha*

7 Solution for Problem 2

7.1 Substituting Ansatz into the HJB Equation

The initial suggested ansatz takes the form:

$$H(t, x, S, \mu, q) = x + qS + h(t, \mu, q),$$

We can see that it fits the final boundary condition at time T, i.e.,

$$H(T, x, S, \mu, q) = x + qS - \alpha q^2.$$

From this, we deduce that the terminal form of $h(t, \mu, q)$ must be the quadratic term $-\alpha q^2$.

We substitute the ansatz $H(t, x, S, \mu, q) = x + qS + h(t, \mu, q)$ into the HJB equation. The original HJB equation is:

$$\left(\partial_t + \frac{1}{2}\sigma^2 \partial_{SS} + \mathcal{L}^\mu\right) H + \sup_\nu \left\{(S - k\nu)\nu\partial_x H - \nu\partial_q H + b((\mu^+ - \mu^-) - \nu)\partial_S H - \varphi(\nu - \rho\mu^-)^2\right\} = 0.$$

Now, substituting $H(t, x, S, \mu, q) = x + qS + h(t, \mu, q)$ into the equation:

$$\begin{aligned}
&\left(\partial_t + \frac{1}{2}\sigma^2 \partial_{SS} + \mathcal{L}^\mu\right)(x + qS + h(t, \mu, q)) \\
&+ \sup_\nu \{(S - k\nu)\nu\partial_x(x + qS + h(t, \mu, q)) - \nu\partial_q(x + qS + h(t, \mu, q)) \\
&\quad + b((\mu^+ - \mu^-) - \nu)\partial_S(x + qS + h(t, \mu, q)) - \varphi(\nu - \rho\mu^-)^2\big\} = 0.
\end{aligned}$$

Simplifying the derivatives of H:

$$\partial_x H = 1, \quad \partial_S H = q, \quad \partial_q H = S + \partial_q h(t, \mu, q).$$

Substituting these into the equation:

$$\begin{aligned}
&\left(\partial_t + \frac{1}{2}\sigma^2 \partial_{SS} + \mathcal{L}^\mu\right)(x + qS + h(t, \mu, q)) \\
&+ \sup_\nu \left\{(S - k\nu)\nu - \nu(S + \partial_q h(t, \mu, q)) + b((\mu^+ - \mu^-) - \nu)q - \varphi(\nu - \rho\mu^-)^2\right\} = 0.
\end{aligned}$$

Now, simplify the terms inside the supremum:

$$\sup_\nu \left\{\cancel{S\nu} - k\nu^2 - \cancel{\nu S} - \nu\partial_q h(t, \mu, q) + b(\mu^+ - \mu^-)q - b\nu q - \varphi(\nu - \rho\mu^-)^2\right\}$$

One may notice that:

$$\partial_{SS} H = 0$$

And therefore:

$$(\partial_t + \mathcal{L}^\mu)\, h + b(\mu^+ - \mu^-)q + \sup_\nu \left(-k\nu^2 - (\partial_q h + bq)\nu - \varphi(\nu - \rho\mu^-)^2\right) = 0.$$

As we have explained in section 10 stage 1, we will now solve the internal optimization problem by differentiating the expression with respect to ν and setting it equal to zero:

$$f(\nu) = -k\nu^2 - (\partial_q h + bq)\nu - \varphi(\nu - \rho\mu^-)^2$$

Differentiate $f(\nu)$ with respect to ν:

$$f'(\nu) = -2k\nu - (\partial_q h + bq) - 2\varphi(\nu - \rho\mu^-)$$

Set the derivative equal to zero to find the critical point:

$$-2k\nu - (\partial_q h + bq) - 2\varphi(\nu - \rho\mu^-) = 0$$

Simplifying:

$$\nu(2k + 2\varphi) = -(\partial_q h + bq) + 2\varphi\rho\mu^-$$

Thus, the optimal ν^* is given by:

$$\nu^* = \frac{2\varphi\rho\mu^- - (\partial_q h + bq)}{2(k + \varphi)}$$

To confirm that this is a maximum, we take the second derivative of $f(\nu)$:

$$f''(\nu) = -2k - 2\varphi$$

Since $k > 0$ and $\varphi > 0$, we have $f''(\nu) < 0$, meaning the function is concave and ν^* is indeed a maximum.

7.2 Substituting ν^* into the HJB Equation

As explained in section 10, stage 2, after finding the optimal value for ν, we substitute ν^* back into the Hamilton-Jacobi-Bellman (HJB) equation to simplify and proceed with solving the problem.

Recall that the optimal value of ν derived from the supremum is:

$$\nu^* = \frac{2\varphi\rho\mu^- - (\partial_q h + bq)}{2(k + \varphi)}.$$

We now substitute this expression for ν^* into the previously derived HJB equation:

$$(\partial_t + \mathcal{L}^\mu)h + b(\mu^+ - \mu^-)q + \sup_\nu \left(-k\nu^2 - (\partial_q h + bq)\nu - \varphi(\nu - \rho\mu^-)^2\right) = 0.$$

7.2.1 Step 1: Expanding the Expression Inside the Supremum

We first expand the term $-\varphi(\nu - \rho\mu^-)^2$:

$$-\varphi(\nu - \rho\mu^-)^2 = -\varphi(\nu^2 - 2\nu\rho\mu^- + \rho^2(\mu^-)^2)$$

Substituting this back into the equation:

$$(\partial_t + \mathcal{L}^\mu)h + b(\mu^+ - \mu^-)q + \sup_\nu \left(-k\nu^2 - (\partial_q h + bq)\nu - \varphi(\nu^2 - 2\nu\rho\mu^- + \rho^2(\mu^-)^2)\right) = 0$$

7.2.2 Step 2: Grouping the Terms

We now group the terms involving ν^2, ν, and the constant terms:

$$(\partial_t + \mathcal{L}^\mu)h + b(\mu^+ - \mu^-)q + \sup_\nu \left(-(k + \varphi)\nu^2 - (\partial_q h + bq - 2\varphi\rho\mu^-)\nu - \varphi\rho^2(\mu^-)^2\right) = 0$$

7.2.3 Step 3: Substituting ν^* into the Equation

Recall that the optimal value for ν, ν^*, is given by:

$$\nu^* = \frac{2\varphi\rho\mu^- - (\partial_q h + bq)}{2(k+\varphi)}$$

Now, we substitute ν^* back into the equation:

$$-(k+\varphi)(\nu^*)^2 = -\frac{(2\varphi\rho\mu^- - (\partial_q h + bq))^2}{4(k+\varphi)}$$

$$-(\partial_q h + bq - 2\varphi\rho\mu^-)\nu^* = -\frac{(\partial_q h + bq - 2\varphi\rho\mu^-)(2\varphi\rho\mu^- - (\partial_q h + bq))}{2(k+\varphi)} = 2\frac{(2\varphi\rho\mu^- - (\partial_q h + bq))^2}{4(k+\varphi)}$$

Thus, the full HJB equation becomes:

$$(\partial_t + \mathcal{L}^\mu)h + b(\mu^+ - \mu^-)q - \frac{(2\varphi\rho\mu^- - (\partial_q h + bq))^2}{4(k+\varphi)} + 2\frac{(2\varphi\rho\mu^- - (\partial_q h + bq))^2}{4(k+\varphi)} - \varphi\rho^2(\mu^-)^2 = 0$$

$$(\partial_t + \mathcal{L}^\mu)h + b(\mu^+ - \mu^-)q + \frac{(2\varphi\rho\mu^- - (\partial_q h + bq))^2}{4(k+\varphi)} - \varphi\rho^2(\mu^-)^2 = 0$$

7.2.4 Comparison to the Case Without Impact from Other Traders (9.8)

We now compare the form of the partial differential equation derived above to the case where the volume of other traders does not affect the mid-price, as presented in Chapter 9.2.1, specifically equation (9.8). For convenience:

$$0 = (\partial_t + \mathcal{L}^\mu)h + \frac{1}{4(k+\varphi)}(\partial_q h + bq - 2\varphi\rho\mu)^2 - \varphi\rho^2(\mu)^2$$

The primary difference between these two cases is the appearance of the term $b(\mu^+ - \mu^-)q$ in our current equation, which captures the impact of the net order flow from other traders on the mid-price. This term is absent in the previous model, where the mid-price drift was unaffected by other traders' buy and sell market orders.

Additionally, the terms $\frac{(2\varphi\rho\mu^- - (\partial_q h + bq))^2}{4(k+\varphi)}$ and $-\varphi\rho^2(\mu^-)^2$ in the current equation reflect the fact that the agent's penalty only applies to deviations from the negative order flow. This is consistent with the objective of penalizing the agent for deviating from trading a fraction ρ of the market's total negative order flow, rather than penalizing based on the entire market flow, as was assumed in the simpler case of equation (9.8).

7.3 Substituting Ansatz into the HJB Equation

$$h(t,\mu,q) = h_0(t,\mu) + h_1(t,\mu)q + h_2(t,\mu)q^2,$$

The terminal conditions for h_0, h_1, and h_2 are derived from the terminal condition of $h(t,\mu,q)$, which is given as:

$$h(T,\mu,q) = -\alpha q^2.$$

This implies the following boundary conditions for the components of the ansatz:

$$h_0(T,\mu) = 0, \quad h_1(T,\mu) = 0, \quad h_2(T,\mu) = -\alpha.$$

The condition $h_0(T,\mu) = 0$ reflects that there is no constant component in the terminal value of $h(t,\mu,q)$, while $h_1(T,\mu) = 0$ shows that the linear term in q also vanishes. The quadratic term is determined by $h_2(T,\mu) = -\alpha$, enforcing the final inventory penalty $-\alpha q^2$.

Given this structure, the next step involves solving the system of ordinary differential equations (ODEs) for $h_0(t,\mu)$, $h_1(t,\mu)$, and $h_2(t,\mu)$, which results from substituting the ansatz into the full HJB equation. The solution of these ODEs allows us to fully determine the optimal trading strategy and the corresponding value function for the agent.

7.3.1 Step 1: Calculate $\partial_q h$

Since $h(t,\mu,q) = h_0(t,\mu) + h_1(t,\mu)q + h_2(t,\mu)q^2$, we compute the derivative with respect to q:

$$\partial_q h = h_1(t,\mu) + 2h_2(t,\mu)q.$$

7.3.2 Step 2: Substitute $\partial_q h$ into the HJB equation

Now substitute $\partial_q h = h_1(t,\mu) + 2h_2(t,\mu)q$ into the term $(2\varphi\rho\mu^- - (\partial_q h + bq))^2$:

$$(2\varphi\rho\mu^- - (h_1(t,\mu) + 2h_2(t,\mu)q + bq))^2 = (2\varphi\rho\mu^- - h_1(t,\mu) - (b + 2h_2(t,\mu))q)^2.$$

Expand this square:

$$= (2\varphi\rho\mu^- - h_1(t,\mu))^2 - 2(2\varphi\rho\mu^- - h_1(t,\mu))(b + 2h_2(t,\mu))q + (b + 2h_2(t,\mu))^2 q^2.$$

7.3.3 Step 3: Grouping terms

Now, substitute this back into the HJB equation:

$$\begin{aligned}
&(\partial_t + \mathcal{L}^\mu)(h_0(t,\mu) + h_1(t,\mu)q + h_2(t,\mu)q^2) + b(\mu^+ - \mu^-)q \\
&\quad + \frac{1}{4(k+\varphi)} \big[(2\varphi\rho\mu^- - h_1(t,\mu))^2 \\
&\quad -2(2\varphi\rho\mu^- - h_1(t,\mu))(b + 2h_2(t,\mu))q \\
&\quad +(b + 2h_2(t,\mu))^2 q^2\big] - \varphi\rho^2(\mu^-)^2 = 0.
\end{aligned}$$

The terms are set to zero because each power of q (constant, linear, and quadratic) represents an independent component in the equation. Therefore, to satisfy the overall equation, each term involving a different power of q must equal zero separately. This leads to three differential equations.

Collect terms by powers of q:

Constant term:

$$(\partial_t + \mathcal{L}^\mu)h_0(t,\mu) + \frac{(2\varphi\rho\mu^- - h_1(t,\mu))^2}{4(k+\varphi)} - \varphi\rho^2(\mu^-)^2 = 0.$$

Linear term q:

$$(\partial_t + \mathcal{L}^\mu)h_1(t,\mu) + b(\mu^+ - \mu^-) + \frac{-(2\varphi\rho\mu^- - h_1(t,\mu))(b + 2h_2(t,\mu))}{2(k+\varphi)} = 0.$$

Quadratic term q^2:

$$(\partial_t + \mathcal{L}^\mu)h_2(t,\mu) + \frac{(b + 2h_2(t,\mu))^2}{4(k+\varphi)} = 0.$$

7.3.4 Quadratic Term Solution for $h_2(t,\mu)$

We substitute the following ansatz for $h_2(t,\mu)$:

$$h_2(t,\mu) = -\left(\frac{T-t}{k+\varphi} + \frac{1}{\alpha - \frac{1}{2}b}\right)^{-1} - \frac{1}{2}b.$$

Step 1: Compute $\partial_t h_2(t,\mu)$

First, calculate the time derivative of $h_2(t,\mu)$:

$$\partial_t h_2(t,\mu) = -\frac{1}{\left(\frac{T-t}{k+\varphi} + \frac{1}{\alpha - \frac{1}{2}b}\right)^2} \cdot \frac{1}{k+\varphi}.$$

Step 2: Substitute into the quadratic equation
Next, substitute $h_2(t,\mu)$ and $\partial_t h_2(t,\mu)$ into the quadratic equation:

$$-\frac{1}{\left(\frac{T-t}{k+\varphi}+\frac{1}{\alpha-\frac{1}{2}b}\right)^2}\cdot\frac{1}{k+\varphi}+\frac{(b+2\left(-\left(\frac{T-t}{k+\varphi}+\frac{1}{\alpha-\frac{1}{2}b}\right)^{-1}-\frac{1}{2}b\right))^2}{4(k+\varphi)}=0.$$

Simplifying the second term:

$$b+2h_2(t,\mu)=b-2\left(\frac{T-t}{k+\varphi}+\frac{1}{\alpha-\frac{1}{2}b}\right)^{-1}-b=-2\left(\frac{T-t}{k+\varphi}+\frac{1}{\alpha-\frac{1}{2}b}\right)^{-1}.$$

Thus, the quadratic term becomes:

$$\frac{\left(-2\left(\frac{T-t}{k+\varphi}+\frac{1}{\alpha-\frac{1}{2}b}\right)^{-1}\right)^2}{4(k+\varphi)}=\frac{4}{4(k+\varphi)}\cdot\frac{1}{\left(\frac{T-t}{k+\varphi}+\frac{1}{\alpha-\frac{1}{2}b}\right)^2}.$$

Step 3: Combine the terms
Now, combining both terms from the quadratic equation:

$$-\frac{1}{\left(\frac{T-t}{k+\varphi}+\frac{1}{\alpha-\frac{1}{2}b}\right)^2}\cdot\frac{1}{k+\varphi}+\frac{4}{4(k+\varphi)}\cdot\frac{1}{\left(\frac{T-t}{k+\varphi}+\frac{1}{\alpha-\frac{1}{2}b}\right)^2}=0.$$

Both terms are identical, so they cancel out, and we obtain:

$$0=0.$$

Thus, the solution for $h_2(t,\mu)$ satisfies the quadratic equation, confirming that the ansatz is correct.

7.3.5 Feynman-Kac Theorem

Consider the stochastic differential equation (SDE):

$$dX_t=\mu(t,X_t)dt+\sigma(t,X_t)dW_t,$$

where W_t is a Brownian motion. Now, define the function:

$$f(t,x)=\mathbb{E}_{t,x}\left[\int_t^T e^{-\int_t^s g(u,X_u)du}\gamma(s,X_s)ds+e^{-\int_t^T g(u,X_u)du}h(X_T)\right],$$

with the assumptions:

$$\mathbb{E}[|h(X_T)|]<\infty,\quad \mathbb{E}\left[\int_0^T|g(s,X_s)|ds\right]<\infty,$$

and $\int_0^T\gamma(t,X_t)dt$ is bounded from below almost surely.
Then, the function $f(t,x)$ satisfies the partial differential equation (PDE):

$$\partial_t f(t,x)+\mathcal{L}_t^X f(t,x)+g(t,x)f(t,x)=\gamma(t,x)$$

with boundary condition:

$$f(T,x)=h(x).$$

Here, $\mathcal{L}_t^X$ represents the infinitesimal generator of X_t, which takes the form:

$$\mathcal{L}_t^X f=\mu(t,x)\frac{\partial f}{\partial x}+\frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}.$$

7.4 Using Feynman-Kac Theorem to Derive h_1

We now want to rewrite the differential equation governing $h_1(t,\mu)$ in a form that corresponds to the Feynman-Kac theorem. Equation (13.3.2) is:

$$(\partial_t + \mathcal{L}^\mu)h_1(t,\mu) + b(\mu^+ - \mu^-) + \frac{-(2\varphi\rho\mu^- - h_1(t,\mu))(b + 2h_2(t,\mu))}{2(k+\varphi)} = 0.$$

Our goal is to transform this equation into the Feynman-Kac form, which is:

$$\frac{\partial h_1}{\partial t} + \mathcal{L}^\mu h_1 + g(t,\mu)h_1 = \gamma(t,\mu)$$

7.4.1 Step 1: Identifying Terms in Equation (13.3.2)

1. **Operator**: The differential operator $(\partial_t + \mathcal{L}^\mu)$ acting on $h_1(t,\mu)$ is analogous to the drift and diffusion terms in the Feynman-Kac PDE. We assume μ^+, μ^- are in the form of Itô processes.

2. **Coefficient of** $h_1(t,\mu)$: We extract the terms multiplying $h_1(t,\mu)$. In equation (13.3.2), the relevant term is:

$$\frac{(b + 2h_2(t,\mu))}{2(k+\varphi)},$$

which corresponds to the $g(t,\mu)$ term in the Feynman-Kac equation. Thus:

$$g(t,\mu) = \frac{b + 2h_2(t,\mu)}{2(k+\varphi)} = \frac{\frac{1}{2}b + h_2(t,\mu)}{k+\varphi}.$$

3. **Source Terms**: The remaining terms in equation (13.3.2) are independent of $h_1(t,\mu)$, and they correspond to the source function $\gamma(\mu,t)$ in the Feynman-Kac formula. These terms are:

$$-\gamma(t,\mu) = b(\mu^+ - \mu^-) - \frac{2\varphi\rho\mu^-(b + 2h_2(t,\mu))}{2(k+\varphi)}.$$

7.4.2 Step 2: Rewriting Equation (13.3.2) in Feynman-Kac Form

We now rewrite equation (13.3.2) in the Feynman-Kac structure:

$$(\partial_t + \mathcal{L}^\mu)h_1(t,\mu) + \underbrace{\frac{\frac{1}{2}b + h_2(t,\mu)}{k+\varphi}}_{g(t,\mu)} h_1(t,\mu) = \underbrace{-\left[b(\mu^+ - \mu^-) - \frac{2\varphi\rho\mu^-(b + 2h_2(t,\mu))}{2(k+\varphi)}\right]}_{\gamma(t,\mu)}$$

7.4.3 Step 3: Defining $\tilde{h}_2$

$$\tilde{h}_2(t,\mu) := \frac{\frac{1}{2}b + h_2(t,\mu)}{k+\varphi}$$

7.4.4 Step 4: Applying the Feynman-Kac Formula

First, one may note that the terminal condition of h1 as discussed earlier, $h_1(T,x) \equiv 0$. Using the Feynman-Kac theorem, we now have the solution for $h_1(t,\mu)$ as a conditional expectation:

$$h_1(t,\mu) = \mathbb{E}_{t,\mu}\left[\int_t^T e^{\int_t^s \tilde{h}_2(u,\mu_u)du}\left(-2\varphi\rho\tilde{h}_2(s,\mu_s)\mu_s^- + b(\mu_s^+ - \mu_s^-)\right)ds + \cancel{e^{\int_t^T \tilde{h}_2(u,\mu_u)du}} * 0\right].$$

To simplify the following integral expression:

$$e^{\int_t^s \tilde{h}_2(u,\mu_u)du} = \exp\left\{-\frac{1}{k+\varphi}\int_t^s \left(\frac{T-u}{k+\varphi}+\frac{1}{\alpha-\frac{1}{2}b}\right)^{-1} du\right\} = \frac{(T-s)+\zeta}{(T-t)+\zeta},$$

where

$$\zeta = \frac{k+\varphi}{\alpha-\frac{1}{2}b},$$

we proceed as follows:

- **Simplifying the integrand**: We begin with the expression inside the integral:

$$\left(\frac{T-u}{k+\varphi}+\frac{1}{\alpha-\frac{1}{2}b}\right)^{-1}.$$

Combining these two terms into a single fraction gives:

$$\left(\frac{(T-u)(\alpha-\frac{1}{2}b)+(k+\varphi)}{(k+\varphi)(\alpha-\frac{1}{2}b)}\right)^{-1} = \frac{(k+\varphi)(\alpha-\frac{1}{2}b)}{(T-u)(\alpha-\frac{1}{2}b)+(k+\varphi)}.$$

- **Simplifying the integral**: After simplifying the integrand, the integral becomes:

$$-\cancel{\frac{1}{k+\varphi}}\int_t^s \frac{\cancel{(k+\varphi)}(\alpha-\frac{1}{2}b)}{(T-u)(\alpha-\frac{1}{2}b)+(k+\varphi)}du.$$

$$-\int_t^s \frac{(\alpha-\frac{1}{2}b)du}{(T-u)(\alpha-\frac{1}{2}b)+(k+\varphi)} = \int_t^s \frac{-(\alpha-\frac{1}{2}b)du}{(T)(\alpha-\frac{1}{2}b)+(-(\alpha-\frac{1}{2}b))u+(k+\varphi)} = \ln\left(\frac{(T-s)(\alpha-\frac{1}{2}b)+(k+\varphi)}{(T-t)(\alpha-\frac{1}{2}b)+(k+\varphi)}\right).$$

- **Placing**:

$$\exp\left(\ln\left(\frac{(T-s)(\alpha-\frac{1}{2}b)+(k+\varphi)}{(T-t)(\alpha-\frac{1}{2}b)+(k+\varphi)}\right)\right) = \frac{(T-s)(\alpha-\frac{1}{2}b)+(k+\varphi)}{(T-t)(\alpha-\frac{1}{2}b)+(k+\varphi)} = \frac{(T-s)+\zeta}{(T-t)+\zeta},$$

where $\zeta = \frac{k+\varphi}{\alpha-\frac{1}{2}b}$.

Now, as one can see, h_1 is given as follows:

$$h_1(t,\mu) = \mathbb{E}_{t,\mu}\left[\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\left(-2\varphi\rho\tilde{h}_2(s,\mu_s)\mu_s^- + b(\mu_s^+-\mu_s^-)\right)ds\right].$$

7.4.5 Step 5: Calculate the explicit expression of h_1

Starting with the equation for $\tilde{h}_2(s,\mu_s)$:

$$\tilde{h}_2(s,\mu_s) = \frac{\frac{1}{2}b+h_2(s,\mu_s)}{k+\varphi}$$

Substitute the expression for $h_2(t,\mu)$:

$$h_2(s,\mu_s) = -\left(\frac{T-s}{k+\varphi}+\frac{1}{\alpha-\frac{1}{2}b}\right)^{-1} - \frac{1}{2}b$$

$$\tilde{h}_2(s,\mu_s) = \frac{\frac{1}{2}b+\left[-\left(\frac{T-s}{k+\varphi}+\frac{1}{\alpha-\frac{1}{2}b}\right)^{-1}-\frac{1}{2}b\right]}{k+\varphi}$$

$$\tilde{h}_2(s,\mu_s) = \frac{-\left(\frac{T-s}{k+\varphi} + \frac{1}{\alpha-\frac{1}{2}b}\right)^{-1}}{k+\varphi}$$

Now simplify the term inside the inverse:

$$\frac{T-s}{k+\varphi} + \frac{1}{\alpha-\frac{1}{2}b} = \frac{(T-s)(\alpha-\frac{1}{2}b)+(k+\varphi)}{(k+\varphi)(\alpha-\frac{1}{2}b)}$$

$$\left(\frac{(T-s)(\alpha-\frac{1}{2}b)+(k+\varphi)}{(k+\varphi)(\alpha-\frac{1}{2}b)}\right)^{-1} = \frac{(k+\varphi)(\alpha-\frac{1}{2}b)}{(T-s)(\alpha-\frac{1}{2}b)+(k+\varphi)}$$

Substitute back:

$$\tilde{h}_2(s,\mu_s) = -\frac{\cancel{(k+\varphi)}(\alpha-\frac{1}{2}b)}{(T-s)(\alpha-\frac{1}{2}b)+(k+\varphi)} \cdot \frac{1}{\cancel{k+\varphi}}$$

Thus, the simplified expression for $\tilde{h}_2(s,\mu_s)$ is:

$$\tilde{h}_2(s,\mu_s) = -\frac{\alpha-\frac{1}{2}b}{(T-s)(\alpha-\frac{1}{2}b)+(k+\varphi)} = \frac{-1}{T-s+\zeta}$$

Now, we may subtitute $\tilde{h}_2$ to the equation of h_1:

$$h_1(t,\mu) = \mathbb{E}_{t,\mu}\left[\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\left(-2\varphi\rho * \frac{-1}{T-s+\zeta} * \mu_s^- + b(\mu_s^+ - \mu_s^-)\right) ds\right].$$

Now, using the linearity of the integral + interchanging the expectation and the integral we get:

$$h_1(t,\mu) = 2\varphi\rho\int_t^T \frac{\mathbb{E}_{t,\mu}\left[\mu_s^-\right]}{(T-t)+\zeta} ds + b\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\mathbb{E}_{t,\mu}\left[\mu_s^+ - \mu_s^-\right] ds$$

7.5 Derive ν^*

One may note that in section 11 we have got:

$$\nu^* = \frac{2\varphi\rho\mu^- - (\partial_q h + bq)}{2(k+\varphi)}$$

And that in order to get the expression of ν^*, we just need the derivative of h by q, which means that we need only $h1$ and $h2$. In section 13.1 we have got:

$$\partial_q h = h_1(t,\mu) + 2h_2(t,\mu)q.$$

Substitute the expression for $\partial_q h$:

$$\nu^* = \frac{2\varphi\rho\mu^- - (h_1(t,\mu) + 2h_2(t,\mu)q + bq)}{2(k+\varphi)}$$

Simplify:

$$\nu^* = \frac{2\varphi\rho\mu^- - h_1(t,\mu) - (2h_2(t,\mu)+b)q}{2(k+\varphi)}$$

Substitute the explicit expressions for $h_1(t,\mu)$ and $h_2(t,\mu)$:

$$h_1(t,\mu) = 2\varphi\rho\int_t^T \frac{\mathbb{E}_{t,\mu}\left[\mu_s^-\right]}{(T-t)+\zeta} ds + b\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\mathbb{E}_{t,\mu}\left[\mu_s^+ - \mu_s^-\right] ds$$

$$h_2(t,\mu) = -\left(\frac{T-t}{k+\varphi} + \frac{1}{\alpha - \frac{1}{2}b}\right)^{-1} - \frac{1}{2}b$$

Thus, the final expression for ν^* is:

$$\nu^* = \frac{2\varphi\rho\mu^- - \left[2\varphi\rho\int_t^T \frac{\mathbb{E}_{t,\mu}\left[\mu_s^-\right]}{(T-t)+\zeta} ds + b\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\mathbb{E}_{t,\mu}\left[\mu_s^+ - \mu_s^-\right] ds\right] - \left(2\left[-\left(\frac{T-t}{k+\varphi} + \frac{1}{\alpha-\frac{1}{2}b}\right)^{-1} - \frac{1}{2}b\right] + b\right) q}{2(k+\varphi)}$$

$$\nu^* = \frac{2\varphi\rho\mu^- - 2\varphi\rho\int_t^T \frac{\mathbb{E}_{t,\mu}\left[\mu_s^-\right]}{(T-t)+\zeta} ds - b\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\mathbb{E}_{t,\mu}\left[\mu_s^+ - \mu_s^-\right] ds}{2(k+\varphi)} + \frac{2\left(\frac{T-t}{k+\varphi} + \frac{1}{\alpha-\frac{1}{2}b}\right)^{-1} q}{2(k+\varphi)}$$

$$\nu^* = \frac{\varphi\rho\mu^- - \varphi\rho\int_t^T \frac{\mathbb{E}_{t,\mu}\left[\mu_s^-\right]}{(T-t)+\zeta} ds}{(k+\varphi)} - \frac{b\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\mathbb{E}_{t,\mu}\left[\mu_s^+ - \mu_s^-\right] ds}{2(k+\varphi)} + (T-t+\zeta)^{-1} q$$

$$\nu_t^* = \varphi\rho[\frac{\mu_t^- - \int_t^T \frac{\mathbb{E}_{t,\mu}\left[\mu_s^-\right]}{(T-t)+\zeta} ds}{(k+\varphi)}] - \frac{b\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\mathbb{E}_{t,\mu}\left[\mu_s^+ - \mu_s^-\right] ds}{2(k+\varphi)} + \frac{Q_t^{\nu^*}}{(T-t+\zeta)}$$

■

7.6 Qualitative analysis of ν^*

$$\nu_t^* = \underbrace{\varphi\rho\left[\frac{\mu_t^-}{(k+\varphi)}\right]}_{A} - \underbrace{\varphi\rho\frac{\int_t^T \mathbb{E}_{t,\mu}\left[\mu_s^-\right] ds}{(\kappa+\varphi)((T-t)+\zeta)}}_{B} - \underbrace{\frac{b\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\mathbb{E}_{t,\mu}\left[\mu_s^+ - \mu_s^-\right] ds}{2(k+\varphi)}}_{C} + \underbrace{\frac{Q_t^{\nu^*}}{(T-t+\zeta)}}_{D}$$

In this section, we analyze the four key components of the optimal liquidation rate ν_t^*, denoted as A, B, C, and D, under various parameter conditions.

7.6.1 Term A: Immediate Reaction to Sell Volume

$$A = \varphi\rho\left[\frac{\mu_t^-}{(k+\varphi)}\right]$$

Interpretation: This term reflects the agent's immediate response to the current sell market orders (μ_t^-). The liquidation rate increases in proportion to μ_t^-, capturing the need to react quickly to market conditions.

Dependence on φ: As $\varphi \to \infty$, this term approaches $\rho\mu_t^-$, implying that the agent becomes more reactive to the current sell volume. When φ is small, this term diminishes, indicating that the agent places less emphasis on immediate sell volume and more on future dynamics.

7.6.2 Term B: Future Expected Sell Volume

$$B = \varphi\rho \frac{\int_t^T \mathbb{E}_{t,\mu}\left[\mu_s^-\right] ds}{(\kappa+\varphi)((T-t)+\zeta)}$$

Interpretation: This term captures the expected future sell volume over the remaining time horizon $[t, T]$. The agent takes into account future market conditions, but this influence decays over time.

Dependence on φ: As $\varphi \to \infty$, B vanishes because both $\kappa + \varphi$ and ζ grow large. Hence, the agent focuses solely on current market conditions. When φ is small, this term plays a more significant role, reflecting a cautious liquidation strategy that incorporates future expectations.

Edge Case with $\kappa \to \infty$: When $\kappa \to \infty$, the agent's reliance on future expected sell volume diminishes, as it is more important for the agent to not liquidate large positions at once, far more than he cares about other factors and penalty terms.

7.6.3 Term C: Reaction to Net Buy-Sell Volume

$$C = \frac{b \int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta} \mathbb{E}_{t,\mu}\left[\mu_s^+ - \mu_s^-\right] ds}{2(k+\varphi)}$$

Interpretation: This term reflects the difference between future expected buy and sell volumes. If future buy orders exceed sell orders ($\mu_s^+ > \mu_s^-$), liquidation slows down; otherwise, liquidation accelerates.

Dependence on φ: As $\varphi \to \infty$, this term diminishes, indicating that the agent reacts less to net buy-sell volumes and more to immediate market conditions.

Impact of $\alpha \to \infty$: When $\alpha \to \infty$, the scaling factor ζ decreases, reducing the influence of future differences in buy-sell volume on liquidation decisions.

7.6.4 Term D: TWAP term

$$D = \frac{Q_t^{\nu^*}}{(T-t+\zeta)}$$

Interpretation: This term reflects the pressure to liquidate the remaining inventory $Q_t^{\nu^*}$ as time progresses. As t approaches T, the agent is forced to increase liquidation to ensure that all inventory is sold by the terminal time.

Dependence on φ: Although $\zeta = \frac{\kappa+\varphi}{\alpha - \frac{1}{2}b}$, this term does eventually vanish as $\varphi \to \infty$. This is due to the fact that ζ grows large, causing the decay factor $(T - t + \zeta)^{-1}$ to shrink. Thus, with large φ, the agent delays liquidation based on the remaining time horizon and market conditions.

Impact of $\alpha \to \infty$: As $\alpha \to \infty$, ζ becomes small, meaning that the time decay effect becomes bigger. Therefore, the agent liquidates more steadily over time rather than accelerating liquidation near the end of the trading horizon.

7.6.5 Overall Qualitative Analysis

- **As** $\varphi \to \infty$: The optimal strategy ν_t^* becomes dominated by the immediate market sell volume A, and the influence of future volumes B and C diminishes. Inventory decay D also vanishes due to the large ζ. Thus, the agent focuses more on current market conditions, ignoring future expectations.

- **As** $\alpha \to \infty$: The scaling factor ζ decreases, reducing the impact of future sell volumes B and C, and making the liquidation more dependent on immediate inventory levels and market orders.

- **As** $\kappa \to \infty$: ν_t^* diminishes, meaning that the agent is so harmed from liquidate, so he prefers to absorb the damage of the other penalty factors instead of liquidate.

7.7 Example - Stochastic Mean-Reverting Volume

In order to illustrate the optimal startegy, we provide a demonstration of a volume process. It will help us to replace the general form solution with closed form formula which we could analyze deeper. The assumptions of the current example are as follows:

- Buy and Sell trading volumes arrive independently.
- $d\mu_t^{\pm} = \underbrace{-\kappa\mu_t^{\pm}dt}_{Y} + \underbrace{\eta_{1+N_{t^-}^{\pm}}dN_t^{\pm}}_{Z}$.
- $N_t^{\pm}$ are independent Poisson processes with intensity λ.
- $\{\eta_1^{\pm}, \eta_2^{\pm}, ...\}$ are i.i.d random variables, with distribution function F - representing jumps in trading volume. All are independent of $N_t^{\pm}$ and of W_t (the Brownian motion which drives the mid-price).

One might see that as we have defined the buy and sell processes match the definition of Jump Ornstein-Uhlenbeck process. Where Y is the mean-reversion term (the reversion is to zero) and Z is the Compound-Poisson process term ("in charge" for the jumps of the process). κ is the mean-reversion rate, determining how fast would the process revert to zero.

7.7.1 Explicit Form of $\mu_t^{\pm}$

To solve the given Stochastic Differential Equation (SDE):

$$d\mu_t^{\pm} = -\kappa\mu_t^{\pm}dt + \eta_{1+N_{t^-}^{\pm}}dN_t^{\pm},$$

we aim to find an explicit form for $\mu_t^{\pm}$, which represents the buy or sell trading rates following a mean-reverting jump Ornstein-Uhlenbeck process.

7.7.2 Step 1: Substituting a New Variable

We substitute a new variable $\tilde{\mu}_t^{\pm}$ such that:

$$\mu_t^{\pm} = e^{-\kappa t}\tilde{\mu}_t^{\pm}.$$

This substitution helps to eliminate the mean-reversion term. Taking the differential of $\mu_t^{\pm}$ yields:

$$d\mu_t^{\pm} = e^{-\kappa t}d\tilde{\mu}_t^{\pm} - \kappa e^{-\kappa t}\tilde{\mu}_t^{\pm}dt.$$

Substituting this expression back into the original SDE:

$$e^{-\kappa t}d\tilde{\mu}_t^{\pm} - \kappa\cancel{e^{-\kappa t}\tilde{\mu}_t^{\pm}dt} = -\kappa\cancel{e^{-\kappa t}\tilde{\mu}_t^{\pm}dt} + \eta_{1+N_{t^-}^{\pm}}dN_t^{\pm}.$$

Now, dividing both sides by $e^{-\kappa t}$, we arrive at the simplified equation:

$$d\tilde{\mu}_t^{\pm} = e^{\kappa t}\eta_{1+N_{t^-}^{\pm}}dN_t^{\pm}.$$

Step 2: Solving the SDE
We now solve this simplified SDE for $\tilde{\mu}_t^{\pm}$. The general solution to the SDE can be written as:

$$\tilde{\mu}_s^{\pm} = \tilde{\mu}_t^{\pm} + \int_t^s e^{\kappa u}\eta_{1+N_{u^-}^{\pm}}dN_u^{\pm},$$

where t is some initial time. Using the relationship $\mu_t^{\pm} = e^{-\kappa t}\tilde{\mu}_t^{\pm} \Rightarrow e^{\kappa t}\mu_t^{\pm} = \tilde{\mu}_t^{\pm}$, we substitute back to get the solution for $\mu_t^{\pm}$:

$$\mu_s^{\pm} = e^{-\kappa s}\left(\underbrace{e^{\kappa t}\mu_t^{\pm}}_{\tilde{\mu}_t^{\pm}} + \int_s^t e^{\kappa u}\eta_{1+N_{u^-}^{\pm}}\, dN_u^{\pm}\right).$$

This simplifies to:

$$\mu_s^{\pm} = e^{-\kappa(s-t)}\mu_t^{\pm} + \int_s^t e^{-\kappa(s-u)}\eta_{1+N_{u^-}^{\pm}}\, dN_u^{\pm}.$$

Step 3: Interpretation of the Solution
The solution for $\mu_t^{\pm}$ has two key components:

- The first term, $e^{-\kappa(t-s)}\mu_s^{\pm}$, captures the mean-reverting behavior of the process. As time progresses, this term decays exponentially, illustrating how the process tends to revert to its long-term mean (which is zero in this case).
- The second term, $\int_s^t e^{-\kappa(t-u)}\eta_{1+N_{u^-}^{\pm}}\, dN_u^{\pm}$, reflects the jumps caused by the Poisson process. Each jump $dN_u^{\pm}$, occurring at time u, is weighted by the factor $e^{-\kappa(t-u)}$, which decreases over time, capturing the effect of mean-reversion after each jump until time t.

7.7.3 Calculation of $\mathbb{E}[\mu_t^{\pm}]$

The expression of ν_t^* which we have got only requires the expectation of the volume process $\mu_t^{\pm}$. And therefore, we will get its explicit expression.

$$\begin{aligned}
\mathbb{E}_{t,\mu}\left[\mu_s^{\pm}\right] &= \underbrace{e^{-\kappa(s-t)}\mu_t^{\pm}}_{\mathbb{E}[e^{-\kappa(s-t)}\mu_t^{\pm}|\mathcal{F}_t]} + \int_t^s e^{-\kappa(s-u)}\mathbb{E}[\eta]\underbrace{\lambda du}_{\mathbb{E}[dN_u]} \\
&= e^{-\kappa(s-t)}\mu_t^{\pm} + \lambda\mathbb{E}[\eta]\underbrace{\frac{1}{\kappa}\left(1 - e^{-\kappa(s-t)}\right)}_{\int_t^s e^{-\kappa(s-u)}du} \\
&= e^{-\kappa(s-t)}\left(\mu_t^{\pm} - \frac{\lambda\mathbb{E}[\eta]}{\kappa}\right) + \frac{\lambda\mathbb{E}[\eta]}{\kappa}.
\end{aligned}$$

$$\mathbb{E}_{t,\mu}[\mu_s^+ - \mu_s^-] = e^{-\kappa(s-t)}\left(\mu_t^+ - \frac{\lambda\mathbb{E}[\eta]}{\kappa}\right) + \cancel{\frac{\lambda\mathbb{E}[\eta]}{\kappa}} - \left[e^{-\kappa(s-t)}\left(\mu_t^- - \frac{\lambda\mathbb{E}[\eta]}{\kappa}\right) + \cancel{\frac{\lambda\mathbb{E}[\eta]}{\kappa}}\right] = e^{-\kappa(s-t)}(\mu_t^+ - \mu_t^-)$$

7.7.4 Substituting the Expressions for $\mathbb{E}_{t,\mu}[\mu_s^{\pm}]$ into ν_t^*

$$\nu_t^* = \varphi\rho\left[\frac{\mu_t^- - \overbrace{\int_t^T \frac{e^{-\kappa(s-t)}\left(\mu_t^- - \frac{\lambda\mathbb{E}[\eta]}{\kappa}\right) + \frac{\lambda\mathbb{E}[\eta]}{\kappa}}{(T-t)+\zeta}ds}^{D}}{(k+\varphi)}\right] - \frac{b\int_t^T \frac{(T-s)+\zeta}{(T-t)+\zeta}\left(e^{-\kappa(s-t)}\left(\mu_t^+ - \mu_t^-\right)\right)ds}{2(k+\varphi)}$$

$$+\frac{Q_t^{\nu^*}}{(T-t)+\zeta}.$$

Simplifying D:

$$D = \int_t^T \frac{e^{-\kappa(s-t)}\left(\mu_t^- - \frac{\lambda\mathbb{E}[\eta]}{\kappa}\right) + \frac{\lambda\mathbb{E}[\eta]}{\kappa}}{(T-t)+\zeta} ds = \int_t^T \frac{e^{-\kappa(s-t)}\left(\mu_t^- - \frac{\lambda\mathbb{E}[\eta]}{\kappa}\right)}{(T-t)+\zeta} ds + \int_t^T \frac{\frac{\lambda\mathbb{E}[\eta]}{\kappa}}{(T-t)+\zeta} ds$$

$$D = \left[\frac{\left(\mu_t^- - \frac{\lambda\mathbb{E}[\eta]}{\kappa}\right)}{(T-t+\zeta)}\right] \int_t^T e^{-\kappa(s-t)} ds + \frac{\frac{\lambda\mathbb{E}[\eta]}{\kappa}}{(T-t)+\zeta}(T-t)$$

$$D = \left[\frac{\left(\mu_t^- - \frac{\lambda\mathbb{E}[\eta]}{\kappa}\right)}{(T-t+\zeta)}\right] \left[\frac{e^{-\kappa(s-t)}}{-\kappa}\right]_{s=t}^{s=T} + \frac{\frac{\lambda\mathbb{E}[\eta]}{\kappa}}{(T-t)+\zeta}(T-t)$$

$$D = \left[\frac{\left(\mu_t^- - \frac{\lambda\mathbb{E}[\eta]}{\kappa}\right)}{(T-t+\zeta)}\right] \left[\frac{e^{-\kappa(T-t)} - e^{-\kappa(\cancel{t}-\cancel{t})}}{-\kappa}\right] + \frac{\frac{\lambda\mathbb{E}[\eta]}{\kappa}}{(T-t)+\zeta}(T-t)$$

$$D = \left[\frac{\left(\mu_t^- - \frac{\lambda\mathbb{E}[\eta]}{\kappa}\right)}{(T-t+\zeta)}\right] \left[\frac{1 - e^{-\kappa(T-t)}}{\kappa}\right] + \frac{\frac{\lambda\mathbb{E}[\eta]}{\kappa}}{(T-t)+\zeta}(T-t)$$

Substitute D back into ν_t^*:

$$\nu_t^* = \frac{\varphi\rho}{(k+\varphi)}\left[\mu_t^- - \frac{(\frac{1-e^{-\kappa(T-t)}}{\kappa})(\mu_t^- - \frac{\lambda\mathbb{E}[\eta]}{\kappa}) + \frac{\lambda\mathbb{E}[\eta](T-t)}{\kappa}}{(T-t)+\zeta}\right] - \frac{b(\mu_t^+ - \mu_t^-)}{2(k+\varphi)(T-t+\zeta)} \underbrace{\int_t^T (T+\zeta-s)e^{-\kappa(s-t)} ds}_{W}$$

$$+\frac{Q_t^{\nu^*}}{(T-t)+\zeta}.$$

Simplifying W:

$$W = \int_t^T (T+\zeta-s)e^{-\kappa(s-t)} ds = \int_t^T (T+\zeta)e^{-\kappa(s-t)} ds - \int_t^T se^{-\kappa(s-t)} ds$$

$$W = (T+\zeta)\left[\frac{e^{-\kappa(T-t)} - e^{-\kappa(\cancel{t}-\cancel{t})}}{-\kappa}\right] - \int_t^T (s)e^{-\kappa(s-t)} ds$$

Using integration by parts:

$$W = (T+\zeta)\left[\frac{e^{-\kappa(T-t)} - 1}{-\kappa}\right] - \left[\left[\frac{se^{-\kappa(s-t)}}{-\kappa}\right]_{s=t}^{s=T} - \int_t^T \frac{e^{-\kappa(s-t)}}{-\kappa} ds\right]$$

$$W = (T+\zeta)\left[\frac{e^{-\kappa(T-t)} - 1}{-\kappa}\right] + \left[\frac{se^{-\kappa(s-t)}}{\kappa}\right]_{s=t}^{s=T} + \int_t^T \frac{e^{-\kappa(s-t)}}{-\kappa} ds$$

$$W = (T+\zeta)\left[\frac{e^{-\kappa(T-t)} - 1}{-\kappa}\right] + \left[\frac{Te^{-\kappa(T-t)} - t}{\kappa}\right] + \frac{1}{\kappa^2}\left[e^{-\kappa(T-t)} - 1\right]$$

$$W = \frac{-\kappa\zeta e^{-\kappa(T-t)} + \kappa(T+\zeta-t) + (e^{-\kappa(T-t)} - 1)}{\kappa^2}$$

$$W = \frac{(1-\kappa\zeta)e^{-\kappa(T-t)} + \kappa(T+\zeta-t) - 1}{\kappa^2}$$

Now, substitute W back into ν_t^* expression:

$$\nu_t^* = \frac{\varphi\rho}{(k+\varphi)}\left[\mu_t^- - \frac{(\frac{1-e^{-\kappa(T-t)}}{\kappa})(\mu_t^- - \frac{\lambda\mathbb{E}[\eta]}{\kappa}) + \frac{\lambda\mathbb{E}[\eta](T-t)}{\kappa}}{(T-t)+\zeta}\right] - \frac{b(\mu_t^+ - \mu_t^-)}{2(k+\varphi)(T-t+\zeta)}\left[\frac{(1-\kappa\zeta)e^{-\kappa(T-t)} + \kappa(T+\zeta-t) - 1}{\kappa^2}\right] - \frac{}{2(}$$

$$+\frac{Q_t^{\nu^*}}{(T-t)+\zeta}.$$

This expression for ν_t^* provides the optimal liquidation rate based on the mean-reverting jump processes of buy and sell volumes, incorporating both current market orders and expectations of future orders.

7.8 Simulating the Solution

In this section we provide some simulations of the derived optimal strategy. We focus on the case of ensuring that all inventory is liquidated (large $\alpha = 100$).

For the simulations, we use the following modeling parameters:

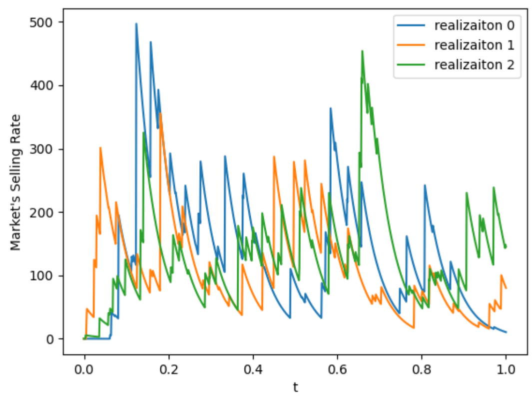
$$\begin{array}{ccccc} \sigma = 0.5, & \mu_0 = \dfrac{\psi}{\kappa}, & \eta \sim \text{Exp}(0.02), & \lambda = 50, & \kappa = 20 \\ T = 1, & k = 0.5, & b = 3.6, & \rho = 0.05 & \phi = 5 \end{array}$$

and in Figure 1 we show three sample paths of the selling rate of other market participants μ_t^-, the optimal trading rate ν_t^*, the difference between the optimal rate and the target rate $\nu_t^* - \rho\mu_t^-$, and the agent's inventory $Q_t^{v^*}$. In the bottom right panel, the dotted line is TWAP. Note that ν_t^* and μ_t^- are strongly correlated, as can be further seen in Figure 2. In Figure 3, we average over 1000 realizations and show the mean values and intervals between the 25% and 75% percentiles.

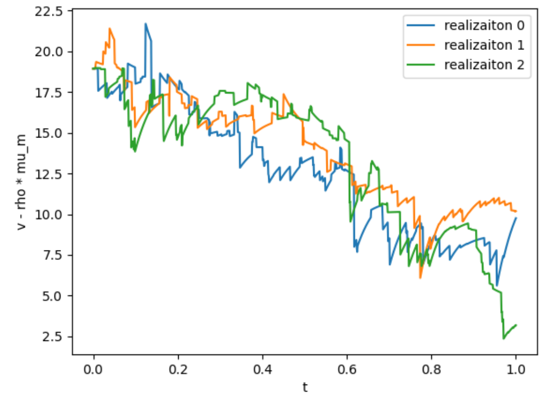
References

The project is an extended solution to the problem presented in:

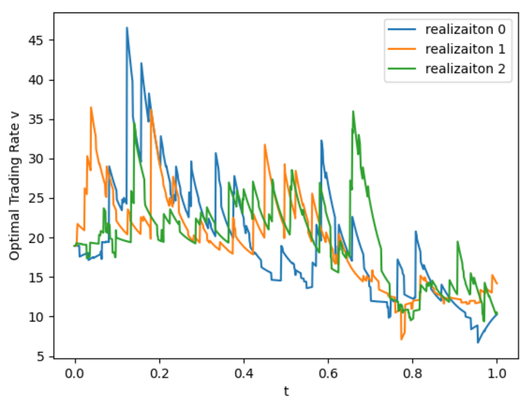
- Eyal Neuman's lecture notes.
- Á. Cartea, S. Jaimungal and J. Penalva - Algorithmic and high-frequency trading.



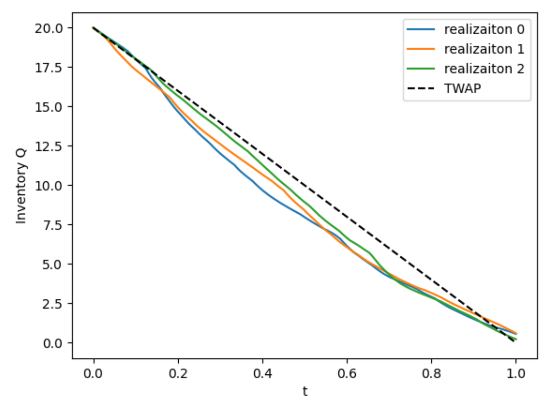
(e) Market's selling rate μ_t^-



(f) $\nu_t^* - \rho\mu_t^-$



(g) Optimal trading rate ν_t^*



(h) Inventory $Q_t^{v^*}$

Figure 1: Three sample paths/realizations of the market's selling rate, the optimal trading rate, the difference between the optimal trading rate and the targeted rate, and the agent's inventory.

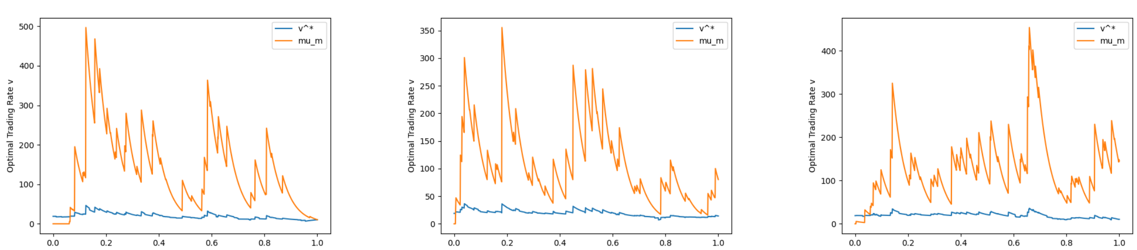


Figure 2: Three sample paths/realizations of the market's selling rate together with the optimal trading rate. Evidently, the two are strongly correlated.

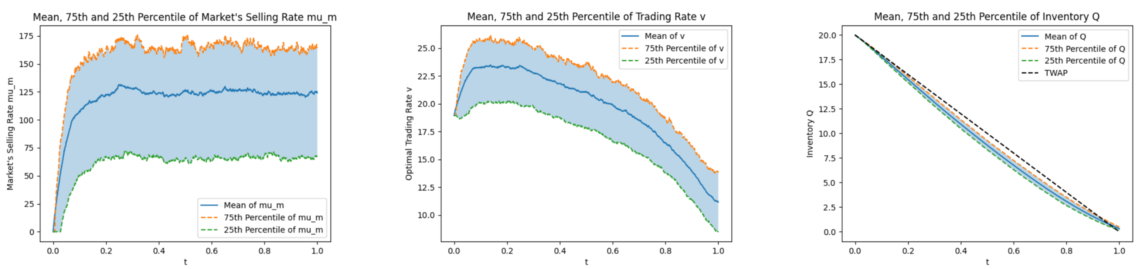


Figure 3: Statistics of 1000 sample paths/realizations of the market's selling rate, the optimal trading rate and the agent's inventory. The shadowed area is the interval between the 25% and 75% percentiles and the centered line is the mean value.