

$$\mathcal{L}(f(x))(p) = \int_0^{+\infty} e^{-px} f(x) dx$$

$f(x)$	$\mathcal{L}(f(x))(p)$
$1, x \in [0, +\infty)$	$\frac{1}{p}, p > 0$
$x, x \in [0, +\infty)$	$\frac{1}{p^2}, p > 0$
$e^{ax}, x \in [0, +\infty)$	$\frac{1}{p-a}, p > a$
$\sin ax, x \in [0, +\infty)$	$\frac{a}{p^2+a^2}, p > 0$
$\cos ax, x \in [0, +\infty)$	$\frac{p}{p^2+a^2}, p > 0$
$x^n, x \in [0, +\infty)$	$\frac{n!}{p^{n+1}}, p > 0$
$u_a(x) = \begin{cases} 0, 0 \leq x < a \\ 1, x \geq a \end{cases}$	$\frac{e^{-ap}}{p}, p > 0$

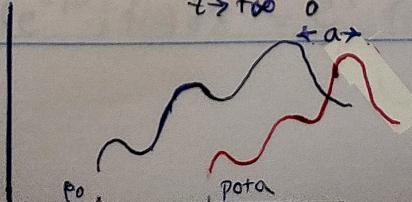
$$\rightarrow \mathcal{L}(af_1(x) + bf_2(x))(p) = a \mathcal{L}(f_1(x))(p) + b \mathcal{L}(f_2(x))(p)$$

### ΤΙΠΟΡΑΣΗ

Έστω  $\mathcal{L}(f(x))(p) = \eta(p), p > p_0$ , τότε  $\mathcal{L}(\underbrace{e^{ax} f(x)}_{g(x)})(p) = \eta(p-a), p > p_0 - a$

### ΑΠΟΔΕΙΞΗ:

$$\begin{aligned} \eta_g(p) &= \int_0^{+\infty} e^{-px} g(x) dx = \int_0^{+\infty} e^{-px} e^{-ax} f(x) dx = \int_0^{+\infty} e^{-(p-a)x} f(x) dx \\ &= \lim_{t \rightarrow +\infty} \int_0^t e^{-(p-a)x} f(x) dx \quad \text{διότι } p^* = p-a \\ &\quad \lim_{t \rightarrow +\infty} \int_0^t e^{-p^*x} f(x) dx = \eta_f(p^*) \\ &\Rightarrow \eta_g(p) = \eta_f(p-a), p > p_0 - a \end{aligned}$$



### ΠΡΟΤΑΣΗ

Έστω  $a > 0$  και  $\lambda(f(x))(p) = \eta(p)$ ,  $p \geq p_0$

$$\lambda(f(ax))(p) = \frac{1}{a} \eta\left(\frac{p}{a}\right)$$

### ΑΙΤΙΑΣΗ

$$\lambda(f(ax))(p) = \int_0^{+\infty} e^{-px} f(ax) dx \quad a > 0 \quad \text{δεικνύεται } g = ax \Rightarrow dg = adx$$

$$x = \frac{g}{a}$$

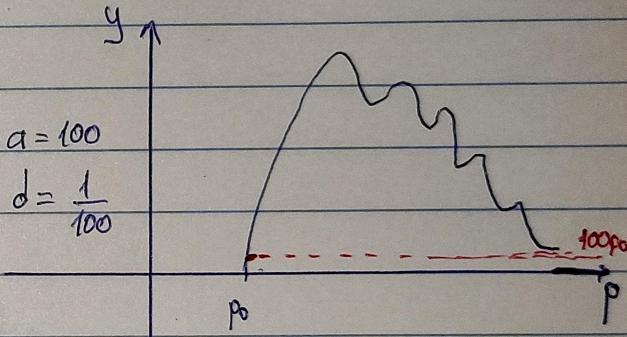
$$= \lim_{t \rightarrow +\infty} \int_0^{at} e^{-p\frac{g}{a}} f(g) \frac{1}{a} dg \quad t^* = at, \text{ αφού } a \rightarrow +\infty$$

$$= \frac{1}{a} \lim_{t^* \rightarrow +\infty} \int_0^{t^*} e^{-p\frac{g}{a}} f(g) dg$$

$$= \frac{1}{a} \lim_{t^* \rightarrow +\infty} \int_0^{t^*} e^{-\frac{p}{a} \cdot g} f(g) dg \quad \text{όπου } p^* = \frac{p}{a}$$

$$= \frac{1}{a} \lim_{t^* \rightarrow +\infty} \int_0^{t^*} e^{-p^* g} f(g) dg$$

$$\approx \frac{1}{a} \eta(p^*) \quad p^* \geq p_0$$



(αποτελεί η ευθεία)

$$\text{Άρα, } \lambda(f(ax))(p) = \frac{1}{a} \eta\left(\frac{p}{a}\right) \quad p \geq p_0$$

### ΠΡΟΤΑΣΗ

$f: [0, +\infty) \rightarrow \mathbb{R}$  συνάριστη στο  $[0, +\infty)$

$\exists f'$  καὶ τηγάνικα συνάριστη στο  $[0, +\infty)$

$|f(x)| \leq M e^{bx}$  για κάποια  $M, b > 0$  και για  $x \geq x_0$ ,

τότε  $\lambda(f'(x))(p) = p \lambda(f(x))(p) - f(0)$ ,  $p \geq p_0$ .

### ΑΙΤΙΑΣΗ

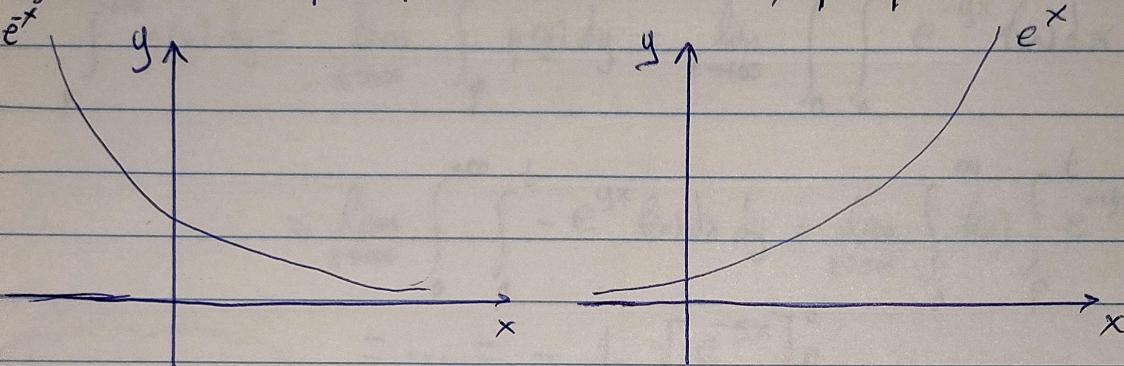
$$\lambda(f'(x))(p) = \int_0^{+\infty} e^{-px} f'(x) dx = \lim_{t \rightarrow +\infty} \int_0^t e^{-px} f'(x) dx$$

$$\cdot \int_0^t e^{-px} f'(x) dx = [e^{-px} f(x)]_0^t - \int_0^t (e^{-px})' f(x) dx = e^{-pt} f(t) - e^{-p \cdot 0} f(0) + p \int_0^t e^{-px} f(x) dx$$

$$\lim_{t \rightarrow +\infty} \int_0^t e^{-px} f(x) dx = \lim_{t \rightarrow +\infty} (e^{-pt} f(t) - f(0)) + p \lim_{t \rightarrow +\infty} \underbrace{\int_0^t e^{-px} f(x) dx}_{L(f(x))(p)}$$

$$\lim_{t \rightarrow +\infty} |e^{-pt} f(t)| = \lim_{t \rightarrow +\infty} (e^{-pt} |f(t)|) \leq M \lim_{t \rightarrow +\infty} e^{-pt} \cdot e^{p_0 t} \\ = M \lim_{t \rightarrow +\infty} e^{-(p-p_0)t} \quad \underline{p > p_0 \Rightarrow p-p_0 > 0} \quad 0.$$

Apa,  $L(f(x))(p) = p L(f(x))(p_0) - f(0)$ ,  $p > p_0$ .



Παράδειγμα

$$f'(x) + 3f(x) = e^{-2x}, \quad f(0) = 0$$

$$L(f'(x) + 3f(x))(p) = L(f'(x))(p) + 3L(f(x))(p) \\ = pL(f(x))(p) - \cancel{f(0)} + 3L(f(x))(p) \\ = (p+3)L(f(x))(p)$$

$$L(e^{-2x})(p) = \frac{1}{p+2} \quad (e^{ax} \leftrightarrow \frac{1}{p-a}) \quad \text{Apa} \quad (p+3)L(f(x))(p) = \frac{1}{p+2}$$

$$\Rightarrow L(f(x))(p) = \frac{1}{(p+2)(p+3)} \quad p \neq -2, -3$$

$$L^{-1}\left(\frac{1}{p+2}\right) = e^{-2x} \quad \frac{1}{(p+2)(p+3)} = \frac{1}{p+2} - \frac{1}{p+3}$$

$$L^{-1}\left(\frac{1}{p+3}\right) = e^{-3x} \quad \text{Apa, } L^{-1}(L(f(x))) = f'(x) = e^{-2x} - e^{-3x}$$

DEFINITION

$f: [0, +\infty) \rightarrow \mathbb{R}$  uara qifhaa euvexis ero  $[0, +\infty)$   $\exists M > 0, p \in \mathbb{R}$  uar  
 $x_0 \geq 0 \quad \lim_{x \rightarrow 0^+} \frac{f(x)}{x} = l \in \mathbb{R}$  t.w.  $|f(x)| \leq M \cdot e^{p_0 \cdot x}, x \geq x_0$ . Far  $L(f(x))(p) = \eta(p)$

tore  $L\left(\frac{f(x)}{x}\right)(p) = \int_p^{+\infty} \eta(y) dy$

AMMADAH:

$$\begin{aligned} \int_p^{+\infty} \eta(y) dy &= \lim_{t \rightarrow +\infty} \int_p^t \eta(y) dy = \lim_{t \rightarrow +\infty} \int_0^t \int_0^{+\infty} e^{-yx} f(x) dx dy \\ &= \lim_{t \rightarrow +\infty} \int_0^{+\infty} \int_p^t -e^{-yx} f(x) dx dy = \lim_{t \rightarrow +\infty} \int_0^\infty f(x) \int_p^t e^{-yt} dy dx \\ &= \dots = -\frac{1}{x} [e^{-yx}]_p^t \\ (e^{-yx})' &= x e^{-yx} \\ \Rightarrow e^{-yx} &= -\frac{1}{x} (e^{-yx}) \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{x} e^{-px} + \frac{1}{x} e^{-tx} \\ \dots &= -\int_0^{+\infty} f(x) \frac{1}{x} e^{-px} dx - \lim_{t \rightarrow +\infty} \int_0^{+\infty} \frac{f(x)}{x} e^{-tx} dx \\ &= L\left(\frac{f(x)}{x}\right)(p) - \lim_{t \rightarrow +\infty} \int_0^{+\infty} \frac{f(x)}{x} e^{-tx} dx = 0. \end{aligned}$$

YILDIZIYE co qaydijafha  $\int_0^{+\infty} e^{-x} \frac{\sin x}{x} dx = j$

$$\int_0^{+\infty} e^{-px} f(x) dx$$

(da sus taw  
kufiyeen uara ca suo ofkaa)

$$\int_0^{+\infty} e^{-x} \frac{\sin x}{x} dx = L\left(\frac{\sin x}{x}\right) \quad (1) \quad (p=1)$$

$$\int_0^{+\infty} e^{-px} f(x) dx = L(f(x))(p)$$

Apa, o keragymkaadiis  $L\left(\frac{\sin x}{x}\right)(p) = \int_p^{+\infty} L(\sin x)(y) dy = \int_p^{+\infty} \frac{1}{y^2 + 1} dy$

$$= \lim_{t \rightarrow +\infty} \int_p^t \frac{1}{y^2+1} dy$$

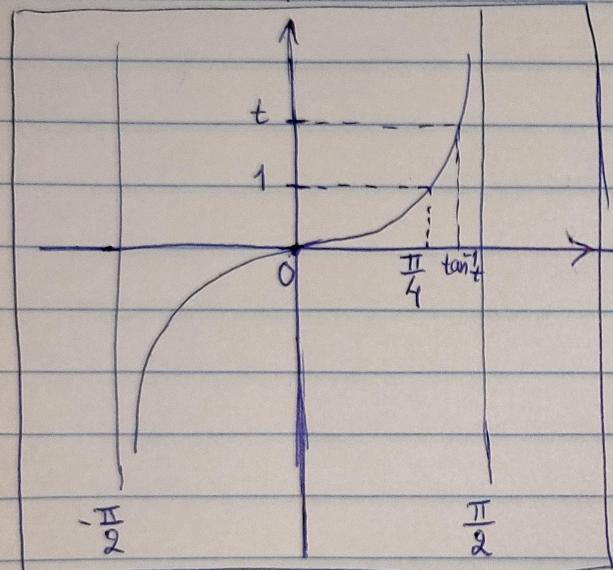
$$* \text{ Aho zo } L\left(\frac{f(x)}{g(x)}\right)(p) = \int_p^{+\infty} L(f(x))(y) dy$$

$$** y = \tan \theta \quad dy = \frac{1}{\cos^2 \theta} d\theta$$

$$\frac{\sin \theta}{\cos \theta} \quad \frac{d}{d\theta} = \frac{\sin \theta}{\cos \theta}$$

$$\text{Jia } p=1 \quad \int_{\frac{\pi}{4}}^{\tan^{-1} t} \frac{1}{\tan^2 \theta + 1} \cdot \frac{1}{\cos^2 \theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\tan^{-1} t} d\theta = \tan^{-1} t - \frac{\pi}{4}$$



$$L(\sin x)(p) = \lim_{t \rightarrow +\infty} \tan^{-1} t - \frac{\pi}{4}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\text{Apa, } \int_0^{+\infty} e^{-x} \frac{\sin x}{x} dx = \frac{\pi}{4}.$$

Späť na Matematický  
Analýz

### ZYNELEZH (Convolution)

Esou  $f, g: [0, +\infty) \rightarrow \mathbb{R}$  uvažte funkce  $g(x)$ 's  $(f * g)(x)$ . Síverou aho zo ej's o funkcií  $g$ .

$$(f * g)(x) = \int_0^x f(t) g(x-t) dt = \int_0^x g(t) f(x-t) dt = (g * f)(x)$$

Ytôžiame da esou  $f, g: [0, +\infty) \rightarrow \mathbb{R}$ ,  $\mu_1, \mu_2 > 0$  uvažme  $p_1, p_2 \in \mathbb{R}$ ,  $x_1, x_2 \geq 0$  t.w.

$$|f(x)| \leq \mu_1 e^{p_1 x}, \quad x \geq x_1$$

$$|g(x)| \leq \mu_2 e^{p_2 x}, \quad x \geq x_2$$

$$\text{Záverečne, } L((f * g)(x))(p) = L(f(x))(p) L(g(x))(p), \quad p > \max(p_1, p_2)$$