~~A~~ 01/12/2025

Σειρές Fourier

Taylor $\{1, x, x^2, x^3, \dots, x^{n+1}\}$ $\boxed{f(x) = \sum_{n=0}^{\infty} a_n x^n} = a_0 + a_1 x + a_2 x^2 + \dots$

Σειρές Fourier $\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \dots\}$

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)}$$

Χρήση Ολοκληρώματος

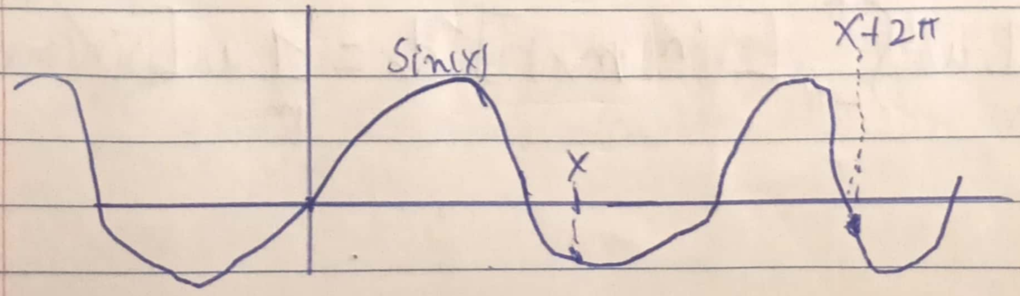
$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \delta_{mn} = \begin{cases} 1, & \text{αν } m=n \\ 0, & \text{~~διαφορετικά~~ αν } m \neq n \end{cases}$$

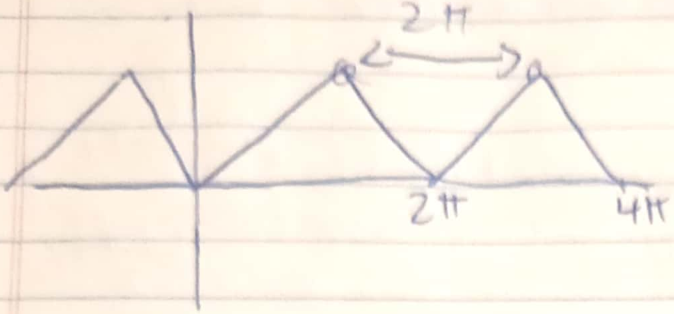
διαφορετικά

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \delta_{mn}$$

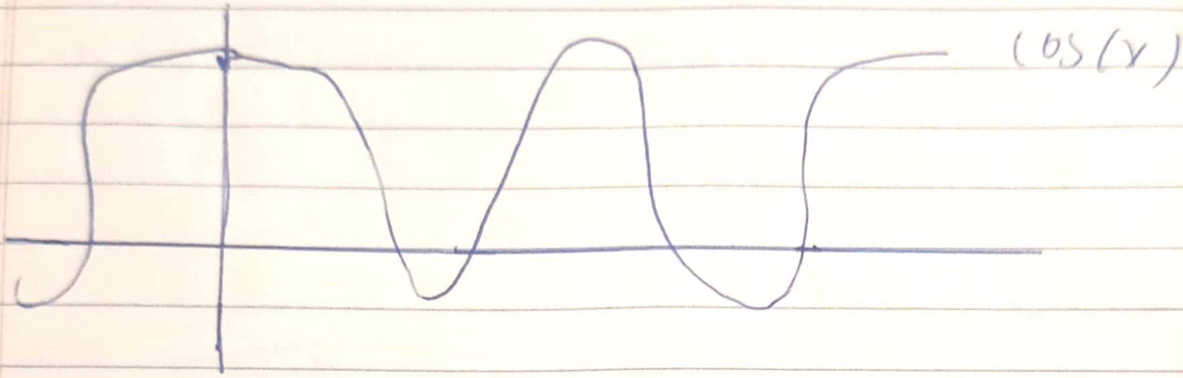
$f: \mathbb{R} \to \mathbb{R}$, f 2π-περιοδική

f 2π-περιοδική, αν $f(x+2\pi) = f(x)$





Μη παραγωγίσιμη σε όλα τα σημεία κ δεν υπάρχουν



•Θέλουμε να γράψουμε των f ως σειρά fourier

$$\boxed{f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx)} \quad *$$

Θέλουμε να υπολογίσουμε τις τιμές των συντελεστών $\{a_n\}_{n=0}^{\infty}$ και $\{b_n\}_{n=1}^{\infty}$

Ολοκληρώσουμε την (*) $\cdot \cos(nx)$ από $-\pi$ μέχρι π

$$\int_{-\pi}^{\pi}f(x)\cos(nx)dx=\underbrace{\int_{-\pi}^{\pi}a_0\cos(mx)dx}_{\mathrm{I}}+$$

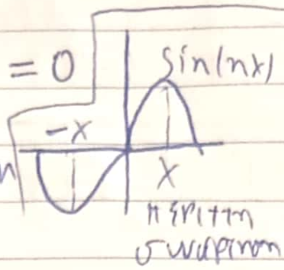
$$+\sum_{n=1}^{\infty}a_n\underbrace{\int_{-\pi}^{\pi}\cos(mx)\cos(nx)dx}_{\mathrm{II}}+\sum_{n=1}^{\infty}b_n\underbrace{\int_{-\pi}^{\pi}\cos(mx)\cdot\sin(nx)dx}_{\mathrm{III}}$$

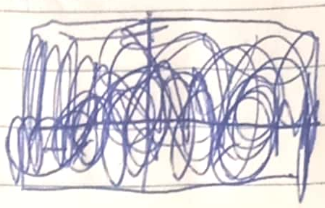
$$\mathrm{I}=\int_{-\pi}^{\pi}a_0\cos(mx)dx=a_0\int_{-\pi}^{\pi}\cos(mx)dx\overset{m\neq0}{=}$$

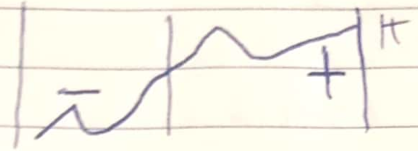
$$=\frac{1}{m}a_0\int_{-\pi}^{\pi}[\sin(mx)]'dx=\frac{1}{m}a_0[\overset{0}{\cancel{\sin(mx)}}]_{-\pi}^{\pi}=0$$

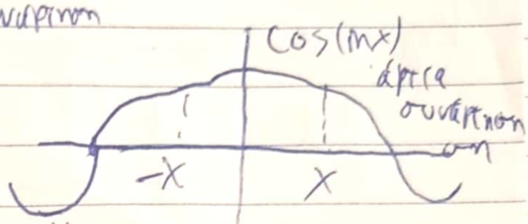
$II = \pi\delta_{mn}$ (Αποκλειστικά για διπλά sin και cos)

$III = \int_{-\pi}^{\pi} \underbrace{\cos(m\cancel{x})\sin(n\cancel{x})}_{\text{περιττή συνάρτηση}}\,dx = 0$









άρτια * περιττή -> περιττή

για $m \neq 0$ $\quad \int f(x)\cos(mx)dx = \sum_{n=1}^{\infty} a_n \pi, \; \delta_{mn} = a_m \pi$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx, \quad m \neq 0$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad n \neq 0}$$

<u>Τεκμηρίωση $a_n \to a_m$</u>

$$\sum_{n=1}^{\infty} a_n \pi \delta_{mn} = a_1\pi \overset{\to 0}{\cancel{\delta_{m1}}} + a_2\pi \overset{\to 0}{\cancel{\delta_{m2}}} + a_{n-1}\pi\cancel{\delta_{m(n-1)}} + a_n\pi\underbrace{\delta_{m\cdot n}}_{1} + a_{n+1}\pi\overset{\to 0}{\cancel{\delta_{m(n+1)}}} + \dots$$

Θέλουμε a_0, $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$

Εύρεση των $\{b_n\}_{n=1}$

περιττή συνάρτηση

$$\int_{-\pi}^{\pi} f(x)\sin(mx)dx = a_0 \overset{\to 0}{\cancel{\int_{-\pi}^{\pi}\sin(mx)dx}} + \cancel{\sum_{n=1}}$$

ως γινόμενο άρτια με περιττή

$$+\sum_{n=1}^{\infty} a_n \overset{\to 0}{\cancel{\int_{-\pi}^{\pi}\cos(nx)\sin(mx)dx}} + \sum_{m=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin(mx)\sin(nx)dx$$

$$\int_{-\pi}^{\pi} f(x)\sin(mx)dx = \sum_{n=1}^{\infty} b_n \pi \delta_{mn} = b_m \pi$$

$$\Rightarrow b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad n=1,2,\dots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx; \quad n=1,2,\dots$$

$a_0 = ;$

Ολοκληρ την ⊛ απο $-\pi$ μέχρι π

$$\int_{-\pi}^{\pi} f(x)dx = \underbrace{\int_{-\pi}^{\pi} a_0 dx}_{2\pi a_0} + \cancel{\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos(nx)dx}^{\to 0} + \cancel{\sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}}$$

$$+ \cancel{\sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}}^{\to 0} \sin(nx)dx$$

$$\int_{-\pi}^{\pi}\cos(nx)dx = -\frac{1}{n}\int_{-\pi}^{\pi}(\sin(nx))'dx = \frac{1}{n}\left[\cancel{\sin(nx)}^{\to 0}\right]_{-\pi}^{\pi}$$

Άρα $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad n=1,2,\dots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \quad n=1,2,\dots$$

Σαρώθηκε με το CamScanner

$a = ;$

Ολοκληρ την (*) από $-\pi$ μέχρι π

$$\boxed{a = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

Έστω f - $2L$ περιοδική $f(x-2L) = f(x)$ $\forall x$

Ορίζουμε $q(x) = f\left(\frac{L}{\pi}x\right)$

η q είναι 2π- περιοδική

$$q(x+2\pi) = f\left(\frac{L}{\pi}(x+2\pi)\right) = f\left(\frac{L}{\pi}x + 2\pi\frac{L}{\pi}\right) =$$

$$= f\left(\frac{L}{\pi}x + 2L\right) = f\left(\frac{L}{\pi}x\right) = q(x)$$

$$q(\tilde{x}) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\tilde{x}) + \sum_{n=1}^{\infty} b_n\sin(n\tilde{x})$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} q(\tilde{x})\,d\tilde{x} \underset{d\tilde{x}=\frac{\pi}{L}dx}{\overset{\tilde{x}=\frac{\pi}{L}x}{=}} \frac{1}{2\pi}\cdot\frac{\pi}{L}\int_{-L}^{L} f\left(\frac{L}{\pi}\cdot\frac{\pi}{L}x\right)dx \overset{*}{=}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} q(\tilde{x})\cos(n\tilde{x})\,d\tilde{x} = \frac{1}{\pi}\cdot\frac{\pi}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{1}x\right)dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} q(\tilde{x})\sin(n\tilde{x})\,d\tilde{x} = \frac{1}{L}\int_{L}^{-L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$q(\tilde{x}) = f\left(\frac{L}{\pi}\tilde{x}\right) =$$

αλλάξουμε μεταβλητή από $\tilde{x}$ σε x ως $\tilde{x} = (\pi/L)\cdot x$

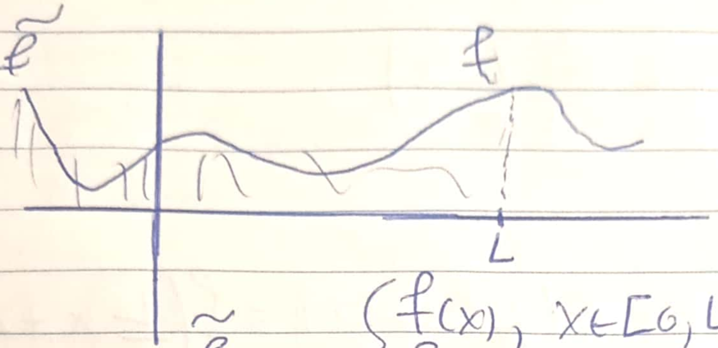
$$\underbrace{f\left(\frac{L}{\pi}\frac{\pi}{L}x\right)}_{f(x)} = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{L}\cdot x\right)$$

$\{1, \sin(\frac{\pi}{L}\cdot x), \cos(\frac{\pi}{L}\cdot x), \sin(\frac{2\pi}{L}\cdot x), \cos(\frac{2\pi}{L}\cdot x), \dots\}$

↳ για a_0

$\underline{*}$ $\frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

Έστω $f: [0, L] \to \mathbb{R}$



f 2π περίοδος
$f: \mathbb{R} \to \mathbb{R}$
$\tilde{f}|_{[0,L]} = f$

$$\tilde{f}(x) = \begin{cases} f(x), & x \in [0, L] \\ f(-x), & x \in [-L, 0) \\ f(x+2L) & \text{αλλού} \end{cases}$$

Η $\tilde{f}$ ονομάζεται άρτια επέκταση της f στο $\mathbb{R}$

$\tilde{f}$ είναι άρτια και 2π περιοδική

$$\tilde{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L}\tilde{f}(x)\,dx = 2\cdot\frac{1}{2L}\int_{0}^{L}\tilde{f}(x)\,dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L}\underset{\substack{\uparrow \\ \text{άρτια}}}{\tilde{f}(x)}\,\underbrace{\cos\left(\frac{n\pi}{L}x\right)}_{\text{άρτια}}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$b_n=\frac{1}{L}\int_{-L}^{L}\underset{\text{άρτια}}{\underset{\uparrow}{\tilde{f}(x)}}\sin\left(\frac{n\pi}{L}x\right)dx=0$$

$x\in\mathbb{R}$, $\tilde{f}(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi}{L}x\right)$

$$a_0=\frac{1}{L}\int_0^L f(x)\,dx$$

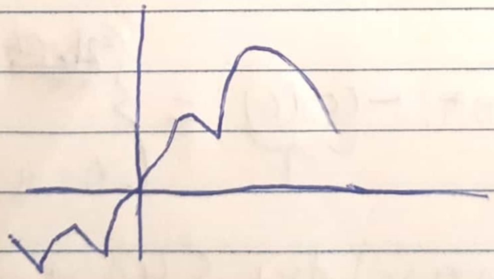
$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

για $x\in[0,L]$ η (~) γραμμένη ως

$$\boxed{\begin{aligned} f(x)&=a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi}{L}x\right)\\ a_0&=\frac{1}{L}\int_0^L f(x)\,dx\\ a_n&=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx\end{aligned}}$$

Έστω $f:[0,L]\to\mathbb{R}$ τ.ω $f(0)=0$

(αν $f(0)=A\neq 0$, θα μπορούσαμε να ορίσουμε την $g(x)=f(x)-A$)



$$\tilde{f}=\begin{cases} f(x), & x\in[0,L]\\ -f(-x), & x\in[-L,0)\\ f(x+2L), & x \text{ διαφορετικά}\end{cases}$$

$$\tilde{f}(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} \underset{\uparrow\ \text{περιττή}}{\tilde{f}(x)}\, dx = 0, \quad a_n = \frac{1}{L}\int_{-L}^{L} \underset{\uparrow\ \text{περιττή}}{\tilde{f}(x)} \underset{\uparrow\ \text{άρτια}}{\cos\left(\frac{n\pi x}{L}\right)} dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underset{\uparrow\ \text{περιττή}}{\tilde{f}(x)} * \underset{\uparrow\ \text{περιττή}}{\sin\left(\frac{n\pi x}{L}\right)}}_{\text{άρτια}} dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Αν $f(x)=0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Αν $f(0) = A \neq 0$

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

$$, b_n = \frac{2}{L}\int_0^L (f(x) - A)\cdot \sin\left(\frac{n\pi x}{L}\right) dx \overset{*}{=}$$

$$= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx - \frac{2A}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \int_0^L \sin\left(\frac{n\pi x}{L}\right) dx = \frac{L}{n\pi}\int_0^L \left[\cos\left(\frac{n\pi x}{L}\right)\right] dx =$$

$$= \frac{L}{n\pi}\left(\cos(n\pi) - \underbrace{\cos(0)}_{1}\right) = \begin{cases} -\frac{2L}{n\pi}, & \text{αν } n \text{ περιττός} \\ 0, & \text{αν } n \text{ άρτιος} \end{cases}$$

$$b_n \overset{*}{=} \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{2}\right) dx + \begin{cases} 4A/n\pi, & n \text{ περιττός} \\ 0, & n \text{ άρτιος} \end{cases}$$