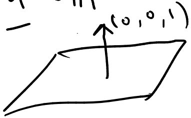
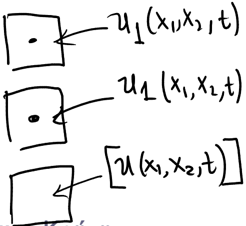
$\underline{u^p} \in \mathbb{R}^{N \times N \times T}$

$\underline{u^s} \in \mathbb{R}^{N \times N \times T}$

$u^p + u^s$

$n = 1$





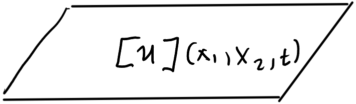
MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

12η Διάλεξη - 5.5.2022

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \nabla \phi(\underset{\sim}{x},t) + \nabla \times \underset{\sim}{\psi}(\underset{\sim}{x},t)$$



Μετασχηματισμός Fourier

- Χρονικός $\quad t \longrightarrow \omega$

$$f(x,t) \xrightarrow{\mathcal{F}_{t\to\omega}} \hat{f}(x,\omega) = \int_{-\infty}^{+\infty} f(x,t)\, e^{i\omega t}\, dt$$

$$\hat{f}(x,\omega) \xrightarrow{\mathcal{F}^{-1}_{\omega\to t}} f(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(x,\omega)\, e^{-i\omega t}\, d\omega$$

- Χωρικός $\quad x \longrightarrow k_x$

$$f(x,t) \xrightarrow{\mathcal{F}_{x\to k_x}} \hat{f}(k_x,t) = \int_{\mathbb{R}} f(x,t)\, e^{-ik_x x}\, dx$$

$$\hat{f}(k_x,t) \xrightarrow{\mathcal{F}^{-1}_{k_x\to x}} f(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(k_x,t)\, e^{ik_x x}\, dk_x$$

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \nabla\phi(\underset{\sim}{x},t) + \nabla\times\underset{\sim}{\psi}(\underset{\sim}{x},t)$$

$$\nabla^2\phi - \alpha^{-2}\ddot{\phi} = 0 \qquad\qquad \nabla^2\underset{\sim}{\psi} - \beta^{-2}\ddot{\underset{\sim}{\psi}} = \underset{\sim}{0}$$

$\underline{t \to w}$

$$\int_{\mathbb{R}} \ddot{\phi}(\underset{\sim}{x},t)\, e^{iwt}\, dt = (iw)^2 \int_{\mathbb{R}} \phi(\underset{\sim}{x},t)\, e^{iwt}\, dt = -w^2\, \hat{\phi}(\underset{\sim}{x},w)$$

$$\int_{\mathbb{R}} \left(\nabla^2\phi(\underset{\sim}{x},t) - \alpha^{-2}\ddot{\phi}(x,t)\right) e^{iwt}\, dt = \nabla^2 \underbrace{\int_{\mathbb{R}} \phi(\underset{\sim}{x},t)\, e^{iwt}\, dt}_{\hat{\phi}(\underset{\sim}{x},w)} - \alpha^{-2}\cdot(-w^2)\,\hat{\phi}(x,w)$$

$$\hat{\phi}(\underset{\sim}{x},w) \doteq \phi(\underset{\sim}{x},w)$$

άρα $$\boxed{\nabla^2\phi(\underset{\sim}{x},w) + \alpha^{-2}w^2\,\phi(\underset{\sim}{x},w) = 0}$$

για κάθε ω λύνουμε

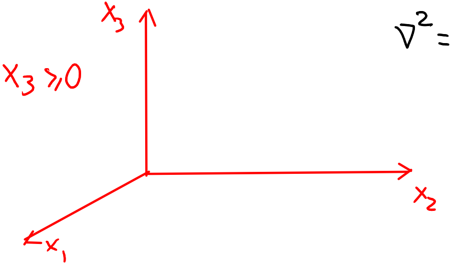
$$\boxed{\nabla^2\phi + \left(\frac{\omega}{\alpha}\right)^2\phi = 0}$$

(Helmholtz)

$$\nabla^2\underset{\sim}{\psi} + \left(\frac{\omega}{\beta}\right)^2\underset{\sim}{\psi} = \underset{\sim}{0} \quad \rightarrow \quad \boxed{\nabla^2\psi_i + \left(\frac{\omega}{\beta}\right)^2\psi_i = 0}$$

$x_3 \geq 0$

Ψάχνουμε λύσεις της μορφής $\phi(\underset{\sim}{x},\omega) = A\exp\{-ik_{x_1}x_1 - ik_{x_2}x_2 - i\nu x_3\}$



$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$$\nabla^2\phi = \underline{A}(-ik_{x_1})^2\underline{\exp\{\dots\}} + A(-ik_{x_2})^2\exp\{\dots\} + A(-i\nu)^2\exp\{\dots\}$$

$$\nabla^2 \phi = \left(-k_{x_1}^2 - k_{x_2}^2 - \nu^2\right)\phi$$

άρα

$$\left(-k_{x_1}^2 - k_{x_2}^2 - \nu^2\right)\phi + \left(\frac{\omega}{\alpha}\right)^2 \phi = 0 \ , \qquad k_\alpha = \frac{\omega}{\alpha}$$

$$\boxed{\nu^2 = \left(\frac{\omega}{\alpha}\right)^2 - k_{x_1}^2 - k_{x_2}^2} \Rightarrow \quad \nu = \left(k_\alpha^2 - k_{x_1}^2 - k_{x_2}^2\right)^{1/2}$$

$$\nu \in \mathbb{C} \Rightarrow \nu = \nu_R + i\nu_I$$

$$\phi(x_1, x_2, x_3, \omega) = A \exp\left\{-ik_{x_1}x_1 - ik_{x_2}x_2 - i\overbrace{\left(k_\alpha^2 - k_{x_1}^2 - k_{x_2}^2\right)^{1/2}}\, x_3\right\}$$

Θέλουμε.

$$\lim_{x_3 \to \infty} \phi(x_1, x_2, x_3, \omega) = 0$$

$$\phi(\underset{\sim}{x}, \omega) \propto e^{-i\nu x_3} = e^{-i\nu_R x_3} e^{-i \cdot i \nu_I x_3} =$$

$$= e^{\nu_I x_3} e^{-i\nu_R x_3}$$

$$e^{\nu_I x_3} \xrightarrow{x_3 \to \infty} 0 \quad \text{ανν} \quad \nu_I < 0$$

$x_3 \leq 0 \qquad \phi(\underset{\sim}{x}, \omega) = A' \exp\{-ik_{x_1}x_1 - ik_{x_2}x_2 + i\nu x_3\}$

$$\nu = \left(k_\alpha^2 - k_{x_1}^2 - k_{x_2}^2\right)^{1/2}, \quad \nu_I < 0$$

$$\lim_{x_3 \to -\infty} \phi(x_1, x_2, x_3, \omega) = 0$$

$x_3 \geq 0 \qquad -i\nu x_3$

$x_3 \leq 0 \qquad i\nu x_3$

$\Gamma_{1\alpha}$

$x_3 \in \mathbb{R} \qquad \phi = A' \exp\{-ik_{x_1}x_1 - ik_{x_2}x_2 - i\nu|x_3|\}$

όμοια $\qquad \psi_i = B_i \exp\{-ik_{x_1}x_1 - ik_{x_2}x_2 - i\gamma|x_3|\}$

$$\gamma = \left(k_\beta^2 - k_{x_1}^2 - k_{x_2}^2\right)^{1/2}, \quad \gamma_I < 0$$

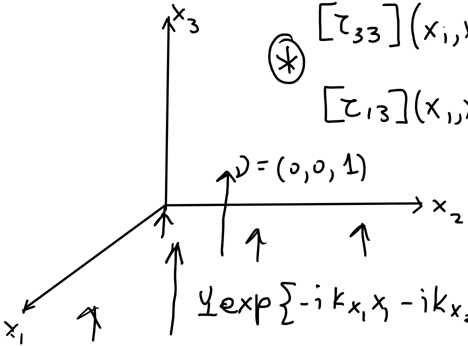
$$u(\underset{\sim}{x}, \omega) = \nabla \phi(\underset{\sim}{x}, \omega) + \nabla \times \underset{\sim}{\psi}(x, \omega)$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \partial x_1 \phi \\ \partial x_2 \phi \\ \partial x_3 \phi \end{bmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x_1 & \partial x_2 & \partial x_3 \\ \psi_1 & \psi_2 & \psi_3 \end{vmatrix}$$

$$\boxed{\begin{aligned} u_1 &= \partial x_1 \phi + \partial x_2 \psi_3 - \partial x_3 \psi_2 \\ u_2 &= \partial x_2 \phi - \partial x_1 \psi_3 + \partial x_3 \psi_1 \\ u_3 &= \partial x_3 \phi + \partial x_1 \psi_2 - \partial x_2 \psi_1 \end{aligned}}$$

$$\phi = A \exp\{-i k_{x_1} x_1 - i k_{x_2} x_2 - i\nu |x_3|\}, \quad \mathrm{Im}\{\nu\} < 0$$

$$\underset{\sim}{\psi} = \underset{\sim}{B} \exp\{-i k_{x_1} x_1 - i k_{x_2} x_2 - i\gamma |x_3|\}, \quad \mathrm{Im}\{\gamma\} < 0$$



$$(*) \quad [\tau_{33}](x_1,x_2,0) = \tau_{33}(x_1,x_2,0^+) - \tau_{33}(x_1,x_2,0^-) = -Y \exp\{-i k_{x_1} x_1 - i k_{x_2} x_2\}$$

$$[\tau_{13}](x_1,x_2,0) = [\tau_{23}](x_1,x_2,0) = 0$$

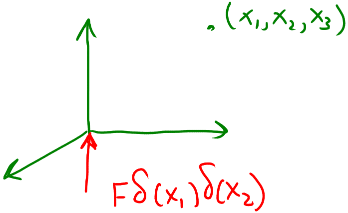
επίπεδο $x_3 = 0$

$$(*) + \text{Νόμος Hooke} \longrightarrow A = \mathrm{sgn}(x_3) \cdot \frac{Y}{2\mu k_\beta^2}, \qquad k_\beta = \frac{\omega}{\beta}$$

$$\underset{\sim}{B}: \quad B_1 = -k_{x_2} A/\gamma, \quad B_2 = k_{x_1} A/\gamma, \quad B_3 = 0$$

$$[\tau_{33}](x_1, x_2, x_3=0) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk_{x_1}\, dk_{x_2}\, [\tau_{33}](k_{x_1}, k_{x_2}, x_3=0) \exp\{-ik_{x_1}x_1 - ik_{x_2}x_2\}$$

$$[\tau_{33}](k_{x_1}, k_{x_2}, x_3=0) = \int_{\mathbb{R}} \int_{\mathbb{R}} [\tau_{33}](x_1', x_2', x_3=0) \exp\{ik_{x_1}x_1' + ik_{x_2}x_2'\}\, dx_1'\, dx_2'$$



Έστω

$$[\tau_{33}](x_1, x_2, x_3=0) = \underline{-F\delta(x_1)\delta(x_2)}$$

$$[\tau_{33}](k_{x_1}, k_{x_2}, 0) = -F$$

$$[\tau_{33}](x_1, x_2, x_3=0) = -\frac{F}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\{-ik_{x_1}x_1 - ik_{x_2}x_2\}\, dk_{x_1}\, dk_{x_2}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(-\frac{F}{4\pi^2}\right) \exp\left\{-ik_{x_1}x_1 - ik_{x_2}x_2\right\} dk_{x_1}\, dk_{x_2}$$

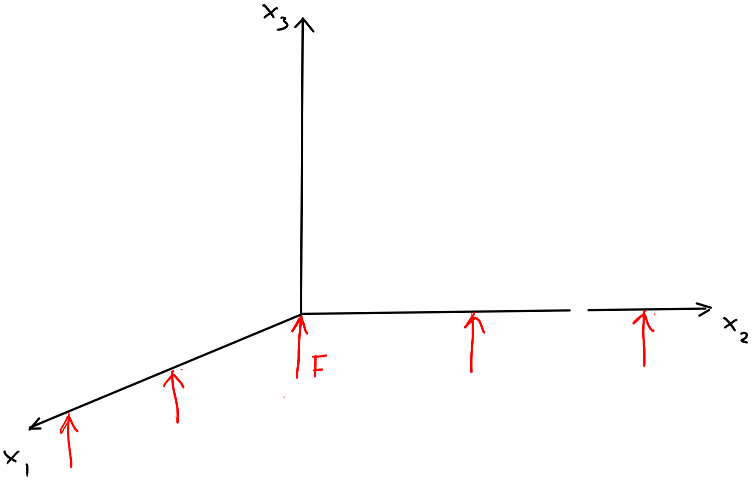
$\nu = (0,0,1) \leftarrow$ κάθετο διάνυσμα στο $x_3 = 0$

$$\phi^{(3)}(x_1, x_2, x_3{=}0, \omega) = \int_{\mathbb{R}}\int_{\mathbb{R}} \operatorname{sgn}(x_3) \overbrace{\frac{F}{4\pi^2 \cdot 2\mu k_{\beta}^2}}^{A} \exp\left\{-ik_{x_1}x_1 - ik_{x_2}x_2 - i\nu|x_3|\right\} dk_{x_1}\, dk_{x_2}$$

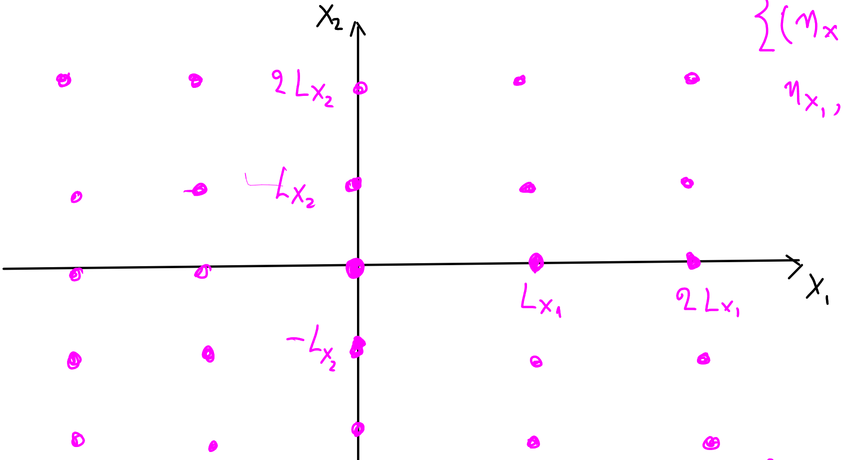
$$\psi_1^{(3)}(x_1, x_2, x_3{=}0, \omega) = -\int_{\mathbb{R}}\int_{\mathbb{R}} k_{x_2} A/\gamma \exp\left\{ \cdots - i\gamma|x_3|\right\} dk_{x_1}\, dk_{x_2}$$

όμοια $\psi_2^{(3)}, \psi_3^{(3)} = 0$

Θέλουμε $\iint \rightarrow \sum\sum$



x_3
x_2
x_1
F



$$\{(n_{x_1}L_{x_1},\, n_{x_2}L_{x_2})\}$$

$$n_{x_1}, n_{x_2} \in \mathbb{Z}$$

Η λύση που αντιστοιχεί σε δύναμη στο σημείο $(n_{x_1}L_{x_1}, n_{x_2}L_{x_2})$

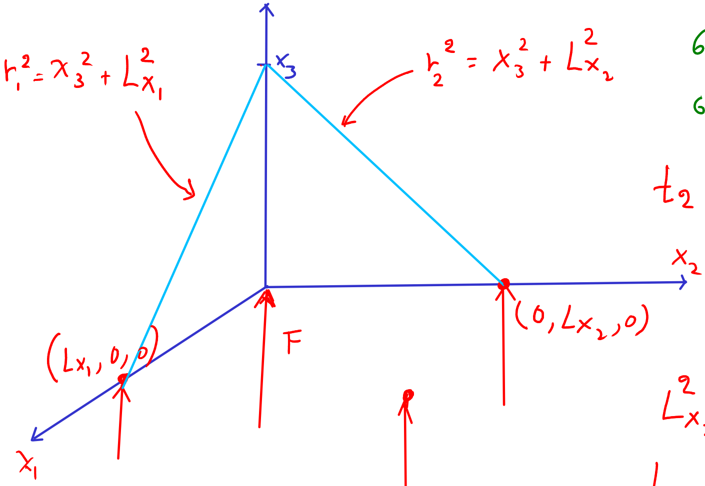
$$\phi^z \propto \int_{\mathbb{R}}\int_{\mathbb{R}} \exp\{-ik_{x_1}(x_1 - n_{x_1}L_{x_1}) - ik_{x_2}(x_2 - n_{x_2}L_{x_2}) - i\nu|x_3|\}\, dk_{x_1}dk_{x_2}$$

$$\sum_{n_{x_1}=-\infty}^{+\infty} \exp\{i k_{x_1} n_{x_1} L_{x_1}\} = 2\pi \sum_{n_{x_1}=-\infty}^{\infty} \delta(k_{x_1} L_{x_1} - 2\pi n_{x_1}) \rightarrow k_{x_1} = \frac{2\pi}{L_{x_1}} n_{x_1}$$

$$\phi^z \propto \sum_{n_{x_1}=-\infty}^{\infty} \sum_{n_{x_2}=-\infty}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\{-i k_{x_1} x_1 - i k_{x_2} x_2 - i\nu |x_3|\} \cdot \exp\{i k_{x_1} n_{x_1} L_{x_1}\} \cdot \exp\{i k_{x_2} n_{x_2} L_{x_2}\} \cdot dk_{x_1} dk_{x_2}$$

$$\phi^z \propto \sum_{n_{x_1}=-\infty}^{\infty} \sum_{n_{x_2}=-\infty}^{+\infty} \exp\left\{-i \frac{2\pi}{L_{x_1}} n_{x_1} x_1 - i \frac{2\pi}{L_{x_2}} n_{x_2} x_2 - i\nu |x_3|\right\}$$

$$\phi^z \propto \sum_{n_{x_1}=-N_1}^{N_1} \sum_{n_{x_2}=-N_2}^{N_2} \exp\left\{-i \frac{2\pi}{L_{x_1}} n_{x_1} x_1 - i \frac{2\pi}{L_{x_2}} n_{x_2} x_2 - i\nu |x_3|\right\}$$



Θέλουμε την επίδραση στις μετατοπίσεις της $F\delta(x_1)\delta(x_2)$ στους χρόνους $[0, T]$, $T > 0$

$$t_2 = \frac{r_2}{\alpha} \geq T \Rightarrow \frac{r_2^2}{\alpha^2} \geq T^2 \Rightarrow$$

$$\Rightarrow \frac{x_3^2}{\alpha^2} + \frac{L_{x_2}^2}{\alpha^2} \geq T^2$$

$$L_{x_2}^2 \geq \alpha^2 T^2 - x_3^2$$

ή

$$L_{x_2} \geq \sqrt{\alpha^2 T^2 - x_3^2}$$

$$L_{x_1} \geq \sqrt{\alpha^2 T^2 - x_3^2}$$

Discrete Wavenumber Representation Method