A splitting deep Ritz method for option pricing in Lévy models
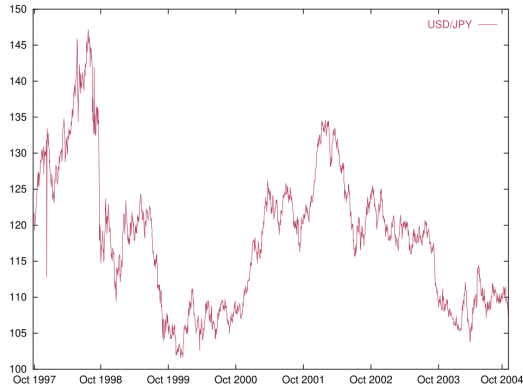
Stochastic Methods in Finance and Physics (SMFP 2023)

Costas Smaragdakis, Antonis Papapantoleon, Emmanuil Georgoulis

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► Pricing options in scenarios where the underlying stock values exhibit discontinuities.

► Our focus is on options involving a large number of underlying assets.

We assume d correlated stocks with values modelled by the stochastic processes

$$S_t^{(i)} = S_0^{(i)} \exp\left(\mu_i t + \sigma_i W_t^{(i)} + \sum_{k=1}^{N_t} Z_k^{(i)} \right), \ t \in \mathbb{T} = [0, T], \ i = 1, \ldots, d$$
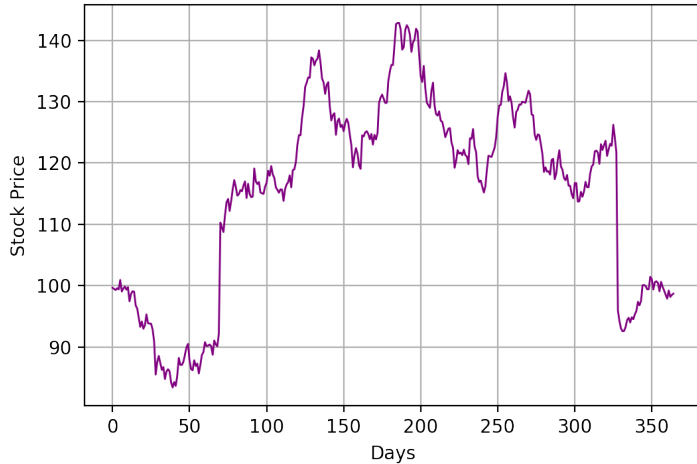
$W_t^{(i)}$: Brownian motions

$Z_k^{(i)}$: Normal random variables

N_t : Poisson process

$$\mathsf{Corr}[W_t^{(i)}, W_t^{(j)}] = \rho_{ij} \in [-1, 1], \quad \mathsf{Corr}[Z_k^{(i)}, Z_k^{(j)}] = \rho_{Jij} \in [-1, 1]$$

▶ Synthetic path

- $\{\alpha_i\}_{i=1,\ldots,d}$: weights on underlyings ($\sum_i \alpha_i = 1$)
- K : the strike price

Payoff function

$$\text{Payoff}(s) = \left(\sum_{i=1}^{d} \alpha_i s_i - K\right)^+$$

Moneynesses : $x_i = s_i/K$

$$\text{Payoff}(x) = \left(\sum_{i=1}^{d} \alpha_i x_i - 1\right)^+$$

▶ $\mathbb{E}^{\mathbb{Q}}[e^{-rt}\mathsf{Payoff}(S_t)|S_0]$, where t expresses the time of maturity.

Using FTAP and Feynman-Kac the option price is provided by:

$$\partial_t u(t,x) + \mathcal{A}u(t,x) = 0, \ t > 0, \ x \in [0,\infty)^d$$

$$u(0,x) = \mathsf{Payoff}(x), \ x \in [0,\infty)^d$$

▶ $\mathcal{A}$: PIDE operator

$$\mathcal{A}u = -\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + ru + g[u]$$

where $a_{ij}(x) = a_{ji}(x), \ i,j = 1,\ldots,d$ and $g[u]$ is an <u>integral operator</u>.

$$\mathcal{A}u = -\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + ru + g[u]$$

We rewrite the operator as follows

$$\mathcal{A}u = -\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d}\left(b_i(x) + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}a_{ij}(x)\right)\frac{\partial u}{\partial x_i} + ru + g[u]$$

$$\mathcal{A}u = \mathcal{L}u + f[u]$$

$$\mathcal{L}u = -\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + ru \quad \text{(symmetric)}$$

$$f[u] = \sum_{i=1}^{d}\left(b_i(x) + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}a_{ij}(x)\right)\frac{\partial u}{\partial x_i} + g[u] \quad \text{(remainder)}$$

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$$\mathcal{A}u = \mathcal{L}u + f[u], \quad \mathcal{E}[u] = \int_\Omega L(x, u, \nabla u)dx$$

$$\frac{\delta\mathcal{E}[u]}{\delta u} = \frac{d}{d\epsilon}\mathcal{E}[u + \epsilon v]\Big|_{\epsilon=0} = \int_\Omega \sum_{k=1}^d \frac{\partial L}{\partial(\nabla u)_k}\frac{\partial v}{\partial x_k} + \frac{\partial L}{\partial u}v \, dx$$

$$= \int_\Omega \left(-\sum_{k=1}^d \frac{\partial}{\partial x_k}\left(\frac{\partial L}{\partial(\nabla u)_k}\right) + \frac{\partial L}{\partial u}\right)v \, dx.$$

$$\frac{\delta\mathcal{E}[u]}{\delta u} = 0 \Leftrightarrow -\sum_{k=1}^d \frac{\partial}{\partial x_k}\left(\frac{\partial L}{\partial(\nabla u)_k}\right) + \frac{\partial L}{\partial u} = 0 \quad \text{(Euler-Lagrange)}$$

$$L_\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \frac{1}{2}ru^2 \quad \text{(Lagrangian)} \Rightarrow -\sum_{k=1}^d \frac{\partial}{\partial x_k}\left(\frac{\partial L_\mathcal{L}}{\partial(\nabla u)_k}\right) + \frac{\partial L_\mathcal{L}}{\partial u} = \mathcal{L}u = 0$$

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Time deep Ritz method (TDRM)

Suppose we would like to estimate the solution of the following equation:

$$\partial_t u(t,x) + \mathcal{L}u(t,x) + f[u(t,x)] = 0, \quad x \in [0, x_{\max}]^d = \Omega, \quad t \in [0, T]$$
$$u(0,x) = \text{Payoff}(x)$$

We consider a time subdivision of the time interval $[0, T]$

$$\tau = T/n, \quad t_k = k\tau, \quad k = 0, \dots, n$$

$$u^k(x) \doteq u(t_k, x)$$

BDF-2 method (implicit - explicit)

$$\frac{u^k - 4/3u^{k-1} + 1/3u^{k-2}}{\tau} + \frac{2}{3}\mathcal{L}u^k + \frac{2}{3}(2f[u^{k-1}] - f[u^{k-2}]) = 0$$

BDF-2 method (implicit - explicit)

▶ Approximate u^1, u^2 using the implicit-explicit Euler method

$$u - \frac{4}{3}u^{k-1} + \frac{1}{3}u^{k-2} + \frac{2}{3}\tau\mathcal{L}u + \frac{2}{3}\tau(2f[u^{k-1}] - f[u^{k-2}])) = 0, \quad u^{k-2}, \ u^{k-1} \text{ given}, \ k \geq 3$$

Lagrangian (BDF-2)

$$L = \frac{1}{2}\left(u - \frac{4}{3}u^{k-1} + \frac{1}{3}\right)^2 + \frac{2}{3}\tau L_{\mathcal{L}}(u) + \frac{2}{3}\tau(2f[u^{k-1}] - f[u^{k-2}])u$$
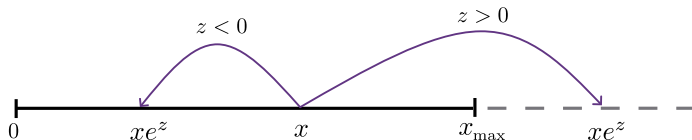
Minimization Problem

$$\mathcal{C}(\theta; \theta^{k-2}, \theta^{k-1}) \doteq \frac{1}{2}\left\|u - \frac{4}{3}u^{k-1} + \frac{1}{3}u^{k-2}\right\|_{L^2(\Omega)}^2 + \frac{2}{3}\tau\mathcal{E}_{\mathcal{L}}[u] + \frac{2}{3}\tau\int_\Omega(2f[u^{k-1}] - f[u^{k-2}])u\,dx,$$

$$\theta^k \to \min_\theta \mathcal{C}(\theta; \theta^{k-2}, \theta^{k-1}), \ k \in \{3, \ldots, n\}$$

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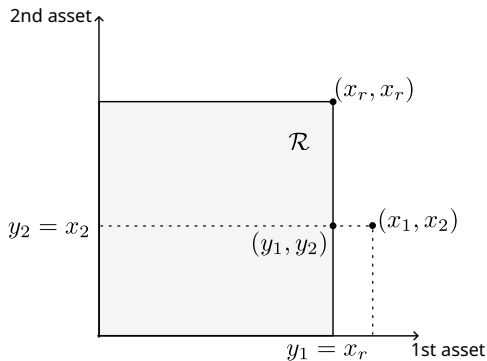
$$\partial_t u(t,x) + \mathcal{L}u(t,x) + f(u(t,x)) = 0, \quad x \in [0, x_{\max}]^d = \Omega, \quad t > 0$$
$$u(0,x) = u^0(x)$$

$$\int_{\mathbb{R}^d} (u(xe^z) - u(x))\nu(dz)$$

where the measure is given as

$$\nu(dz) = \lambda p(z)dz, \; p(z) = (2\pi)^{-d/2}|\Sigma_J|^{-1/2}\exp\left[-\frac{1}{2}(z-\mu_J)^T\Sigma_J^{-1}(z-\mu_J)\right]$$

Choose $x_r \leq x_{\max}$

Let $\mathcal{R}$ be a rectangle of $\mathbb{R}^d$ such that

$$\mathcal{R} \doteq \{y \in \mathbb{R}^d : y_i \in [0, x_r]\}$$

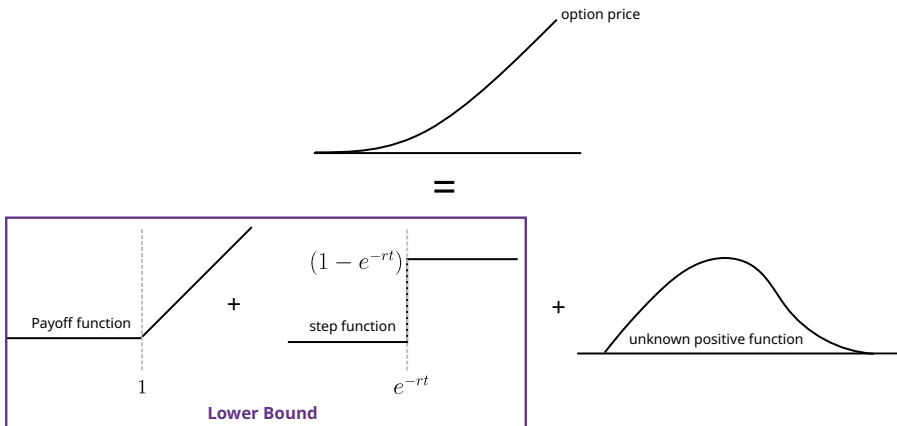For $x \in \mathbb{R}^d_+$ we define $y \in \mathcal{R}$ such that

$$y_i = \min\{x_i, x_r\}, \ i = 1, 2, \ldots, d$$

The approach : Taking into account the asymptotic behaviour of the solution

$$u(t, x) \doteq u(t, y) + \sum_{i=1}^{d} \alpha_i(x_i - y_i), \ x \in \mathbb{R}^d_+$$

$$\forall x \in \mathbb{R}^d_+ \quad x \to y = y(x) \to u(t, y) \to u(t, x)$$

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(Solution at time t_k) = (Lower Bound at time t_k) + (positive function)

option price

=

$(1 - e^{-rt})$

Payoff function + step function + unknown positive function

1 e^{-rt}

Lower Bound

$$\begin{aligned}
\mathcal{C}(\theta; \theta^{k-1}, \theta^{k-2}) &\doteq & \frac{1}{2}\|u(x;\theta^k) - \frac{4}{3}u^{k-1} + \frac{1}{3}u^{k-2}\|_{L^2(\Omega)}^2 \\
&+& \frac{2}{3}\tau\mathcal{E}_{\mathcal{L}}[u(x;\theta^k)] + \frac{2}{3}\tau\int_{\Omega}(2f[u^{k-1}] - f[u^{k-2}])u(x;\theta^k)\,dx,
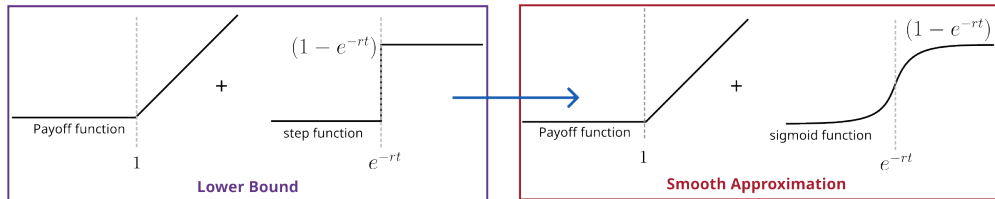\end{aligned}$$

Is the cost functional smooth?

$$\theta^k \approx \theta^{k-1} \approx \theta^{k-2} \Rightarrow u(x;\theta^k) - \frac{4}{3}u^{k-1} + \frac{1}{3}u^{k-2} \approx \text{step function}$$

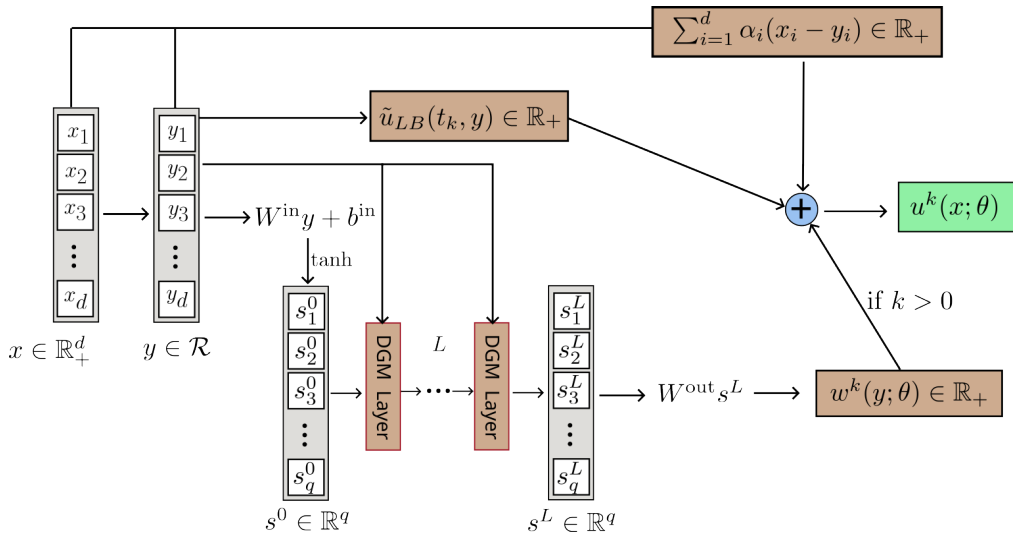(Solution at time t_k) = (Lower Bound at time t_k) + (positive function)

$$u(t_k, y) = \tilde{u}_{\mathsf{LB}}(t_k, y) + w^k(y, \theta^k)$$

Lower bound at each time-step

$$\tilde{u}_{\mathsf{LB}}(t, x; \gamma) = \mathsf{Payoff}(x) + (1 - e^{-rt})\mathsf{Sigmoid}(\sum_{i=1}^{d} \alpha_i x_i - e^{-rt/2}; \gamma), \ \gamma > 0$$
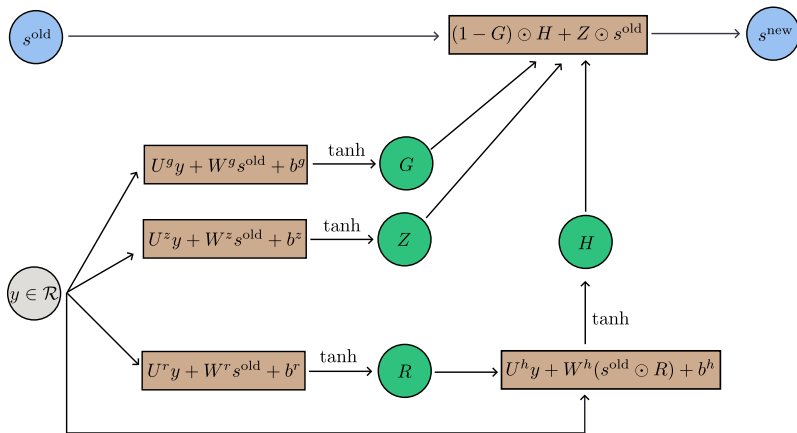
$$\mathsf{Sigmoid}(x; \gamma) = \frac{1}{1 + e^{-\gamma x}}.$$

$$w^k(y;\theta) = \mathsf{Softplus}(W^{\mathrm{out}}S^L; \delta)$$

DGM Layer

$$\text{Output}: u^k(x;\theta) = (1-\delta_{k0})w^k(y;\theta) + \tilde{u}_{\mathsf{LB}}(t_k,y) + \sum_{i=1}^{d}\alpha_i(x_i - y_i),$$

We sample N points in the interval $[0, x_{\mathsf{max}}]^d$. By applying Monte-Carlo to approximate the integrals of the cost functional, we get:

$$
\begin{aligned}
\mathcal{C}(\theta;\theta^{k-2},\theta^{k-1}) &\approx& \frac{(x_{\mathsf{max}})^d}{N}\sum_{i=1}^{N}\Big[(u(x_i;\theta) - \frac{4}{3}u^{k-1}(x_i) + \frac{1}{3}u^{k-2}(x_i))^2 \\
&+& \frac{2}{3}\tau\bigg(L(x_i) + (2f[u^{k-1}(x_i)] - f[u^{k-2}(x_i)])u(x_i;\theta)\bigg)\Big]
\end{aligned}
$$

Merton model

$$\mathcal{A}u = -\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + ru + g[u]$$

$$a_{ij}(x) = \frac{1}{2}\sigma_i\rho_{ij}\sigma_j x_i x_j, \qquad b_i(x) = [-r + \lambda\exp(\mu_{Ji} + \frac{1}{2}\sigma_{Ji}^2) - \lambda]x_i$$

$$g[u] = -\int_{\mathbb{R}^d}(u(xe^z) - u(x))\nu(dz)$$

$\sigma_i = 0.5, \ \rho_{ij} = \delta_{ij} + 0.5(1 - \delta_{ij}), \ i,j = 1, \ldots, d, \ r = 0.05, \text{ time-step} = 0.02$

Morever, the parameters of the jumps are

$$\lambda = 1, \ \mu_{Ji} = 0, \ \sigma_{Ji} = 0.5, \ \rho_{Jij} = \delta_{ij} + 0.2(1 - \delta_{ij}), \ i,j = 1, \ldots, d$$

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The integral : Gauss-Hermite quadrature

In Merton model, the integral is simply the expected value of the function $h(z) = u(xe^z) - u(x)$ multiplied by the Poisson parameter λ.
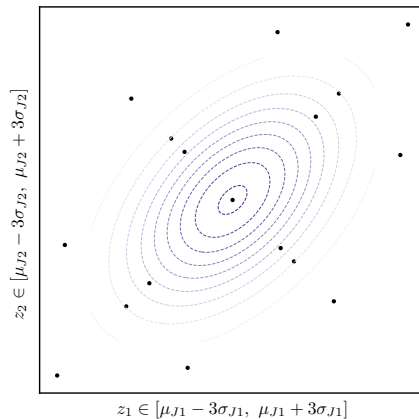
▶ Singular Value Decomposition (SVD) of Σ_J

$$\Sigma_J = U\Lambda U^T = U\Lambda^{1/2}\Lambda^{1/2}U^T = VV^T,$$
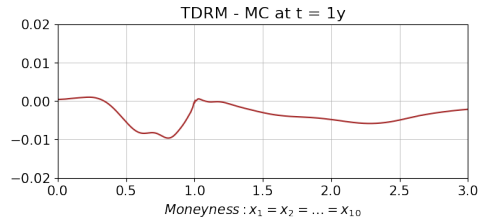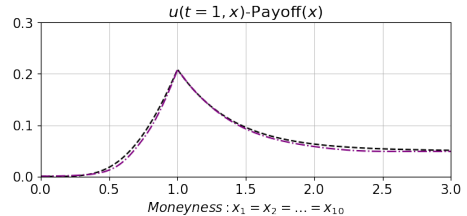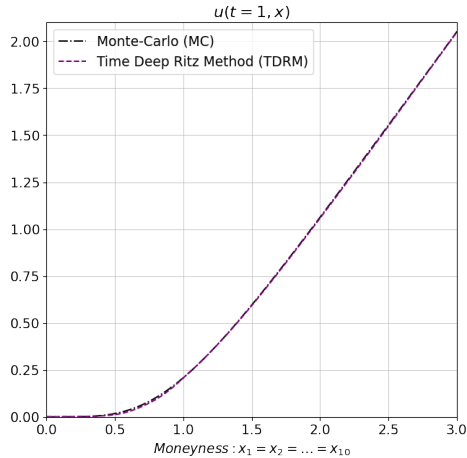
where $V = U\Lambda^{1/2}$.

▶ change of variable $Z - \mu = \sqrt{2}VY$

$$\begin{aligned}
\int_{\mathbb{R}^d} h(z)\nu(dz) &= \lambda\pi^{-d/2}\int_{\mathbb{R}^d}\exp(y^T y)h(\mu + \sqrt{2}Vy)dy \\
&\approx \lambda\pi^{-d/2}\sum_{\mathsf{i}\in\Theta_p}h(\mu + \sqrt{2}Vy^{\mathsf{i}})W^{\mathsf{i}},
\end{aligned}$$

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The integral : Gauss-Hermite quadrature

$z_1 \in [\mu_{J_1} - 3\sigma_{J_1}, \ \mu_{J_1} + 3\sigma_{J_1}]$

$z_2 \in [\mu_{J_2} - 3\sigma_{J_2}, \ \mu_{J_2} + 3\sigma_{J_2}]$

10 Assets - Maturity : 1 year

- ▶ We proposed a modified splitting deep Ritz method for option pricing.
- ▶ Our method is capable of accurately approximating the solutions of the partial integro-differential equations (PIDEs) that arise in European basket call options.
- ▶ To evaluate the effectiveness of our method, we compared its results with those obtained using Monte-Carlo simulations.

- ▶ We are working on the multivariate variance Gamma model (infinite jump activity)

Thank you for your attention