A splitting deep Ritz method for option pricing in Lévy models

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- Pricing options in scenarios where the underlying stock values exhibit discontinuities.
- Our focus is on options involving a large number of underlying assets.

- $\{\alpha_i\}_{i=1,\ldots,d}$: weights on underlyings ($\sum_i \alpha_i = 1$)
- K : the strike price

Payoff function

$$\text{Payoff}(s) = \left(\sum_{i=1}^{d} \alpha_i s_i - K\right)^+$$

Moneynesses : $x_i = s_i / K$

$$\text{Payoff}(x) = \left(\sum_{i=1}^{d} \alpha_i x_i - 1\right)^+$$

Using FTAP and Feynman-Kac the option price is provided by:

$$\partial_t u(t,x) + \mathcal{A}u(t,x) = 0,\ t>0,\ x\in[0,\infty)^d$$

$$u(0,x) = \mathsf{Payoff}(x),\ x\in[0,\infty)^d$$

- $\mathcal{A}$: PIDE operator

$$\mathcal{A}u = -\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + ru + g[u]$$

where $a_{ij}(x) = a_{ji}(x),\ i,j=1,\dots,d$ and $g[u]$ is an integral operator.

$$\mathcal{A}u = -\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + ru + g[u]$$

We rewrite the operator as follows

$$\mathcal{A}u = -\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d}\left(b_i(x) + \sum_{j=1}^{d}\frac{\partial}{\partial x_j}a_{ij}(x)\right)\frac{\partial u}{\partial x_i} + ru + g[u]$$

$$\mathcal{A}u = \mathcal{L}u + f[u]$$

$$\begin{aligned}
\mathcal{L}u &= -\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + ru \quad \text{(symmetric)} \\
f[u] &= \sum_{i=1}^{d}\left(b_i(x) + \sum_{j=1}^{d}\frac{\partial}{\partial x_j}a_{ij}(x)\right)\frac{\partial u}{\partial x_i} + g[u] \quad \text{(remainder)}
\end{aligned}$$

$$\mathcal{A}u = \mathcal{L}u + f[u], \quad \mathcal{E}[u] = \int_\Omega L(x, u, \nabla u) dx$$

$$\begin{aligned}\frac{\delta \mathcal{E}[u]}{\delta u} &= \frac{d}{d\epsilon}\mathcal{E}[u + \epsilon v]\Big|_{\epsilon=0} = \int_\Omega \sum_{k=1}^{d} \frac{\partial L}{\partial (\nabla u)_k}\frac{\partial v}{\partial x_k} + \frac{\partial L}{\partial u} v \, dx \\ &= \int_\Omega \left(- \sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(\frac{\partial L}{\partial (\nabla u)_k}\right) + \frac{\partial L}{\partial u}\right) v \, dx.\end{aligned}$$

$$\frac{\delta \mathcal{E}[u]}{\delta u} = 0 \Leftrightarrow - \sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(\frac{\partial L}{\partial (\nabla u)_k}\right) + \frac{\partial L}{\partial u} = 0 \quad \text{(Euler-Lagrange)}$$

$$L_{\mathcal{L}} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \frac{1}{2} r u^2 \quad \text{(Lagrangian)} \Rightarrow - \sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(\frac{\partial L_{\mathcal{L}}}{\partial (\nabla u)_k}\right) + \frac{\partial L_{\mathcal{L}}}{\partial u} = \mathcal{L}u = 0$$

Suppose we would like to estimate the solution of the following equation:

$$\begin{aligned} \partial_t u(t,x) + \mathcal{L}u(t,x) + f[u(t,x)] &= 0, \quad x \in [0, x_{\max}]^d = \Omega, \quad t \in [0,T] \\ u(0,x) &= \text{Payoff}(x) \end{aligned}$$

We consider a time subdivision of the time interval $[0,T]$

$$\tau = T/n, \quad t_k = k\tau, \quad k = 0, \ldots, n$$

$$u^k(x) \doteq u(t_k, x)$$

BDF-2 method (implicit - explicit)

$$\frac{u^k - 4/3u^{k-1} + 1/3u^{k-2}}{\tau} + \frac{2}{3}\mathcal{L}u^k + \frac{2}{3}(2f[u^{k-1}] - f[u^{k-2}]) = 0$$

BDF-2 method (implicit - explicit)

- Approximate u^1, u^2 using the implicit-explicit Euler method

$$u - \frac{4}{3}u^{k-1} + \frac{1}{3}u^{k-2} + \frac{2}{3}\tau\mathcal{L}u + \frac{2}{3}\tau(2f[u^{k-1}] - f[u^{k-2}])) = 0, \quad u^{k-2},\ u^{k-1} \text{ given},\ k \geq 3$$

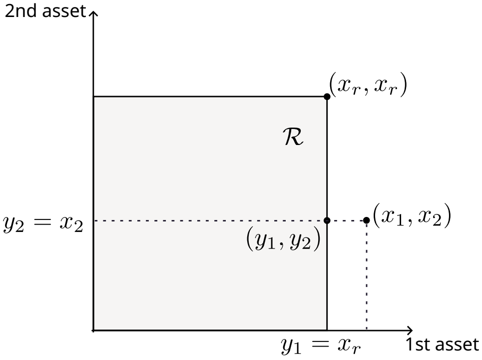
Lagrangian (BDF-2)

$$L = \frac{1}{2}\left(u - \frac{4}{3}u^{k-1} + \frac{1}{3}\right)^2 + \frac{2}{3}\tau L_{\mathcal{L}}(u) + \frac{2}{3}\tau(2f[u^{k-1}] - f[u^{k-2}])u$$

Minimization Problem

$$\mathcal{C}(\theta;\theta^{k-2},\theta^{k-1}) \doteq \frac{1}{2}\left\| u - \frac{4}{3}u^{k-1} + \frac{1}{3}u^{k-2} \right\|^2_{L^2(\Omega)} + \frac{2}{3}\tau\mathcal{E}_{\mathcal{L}}[u] + \frac{2}{3}\tau\int_\Omega (2f[u^{k-1}] - f[u^{k-2}])u\,dx,$$

$$\theta^k \to \min_\theta \mathcal{C}(\theta;\theta^{k-2},\theta^{k-1}),\ k \in \{3,\ldots,n\}$$



Choose $x_r \le x_{\text{max}}$

Let $\mathcal{R}$ be a rectangle of $\mathbb{R}^d$ such that

$$\mathcal{R} \doteq \{y \in \mathbb{R}^d : y_i \in [0, x_r]\}$$

For $x \in \mathbb{R}_+^d$ we define $y \in \mathcal{R}$ such that

$$y_i = \min\{x_i, x_r\}, \; i = 1, 2, \ldots, d$$

The approach : Taking into account the asymptotic behaviour of the solution

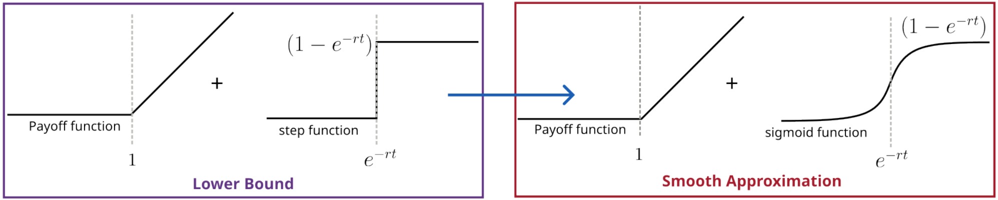
$$u(t,x) \doteq u(t,y) + \sum_{i=1}^{d} \alpha_i (x_i - y_i), \; x \in \mathbb{R}_+^d$$

$$\forall x \in \mathbb{R}_+^d \quad x \to y = y(x) \to u(t,y) \to u(t,x)$$

(Solution at time t_k) = (Lower Bound at time t_k) + (positive function)

$$u(t_k, y) = \tilde{u}_{\mathrm{LB}}(t_k, y) + w^k(y, \theta^k)$$

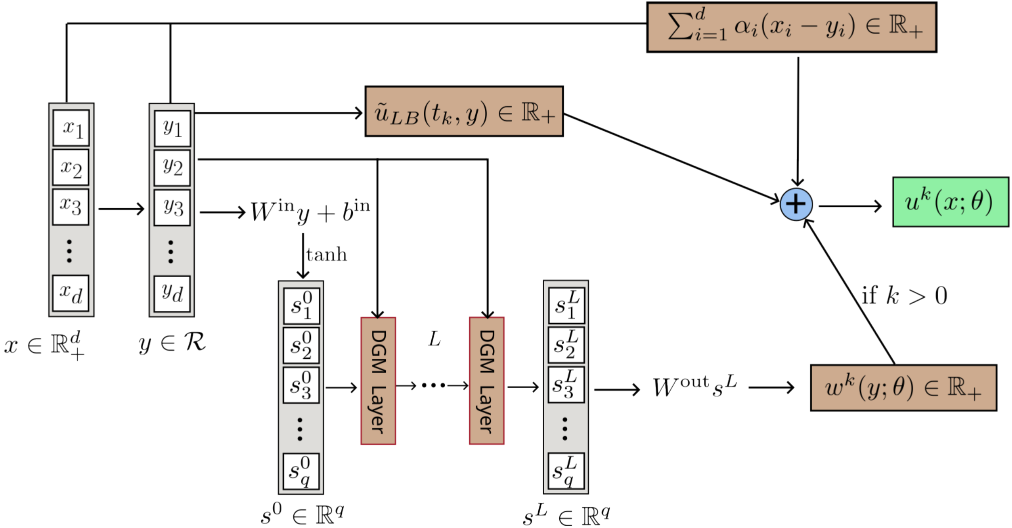
Lower bound at each time-step



$$\tilde{u}_{\mathrm{LB}}(t, x; \gamma) = \mathrm{Payoff}(x) + (1 - e^{-rt})\mathrm{Sigmoid}\Big(\sum_{i=1}^{d} \alpha_i x_i - e^{-rt/2}; \gamma\Big), \ \gamma > 0$$

$$\mathrm{Sigmoid}(x; \gamma) = \frac{1}{1 + e^{-\gamma x}}.$$

Model : Neural Network



$$w^k(y;\theta) = \text{Softplus}(W^{\text{out}} S^L; \delta)$$

$$\text{Output}: u^k(x;\theta) = (1-\delta_{k0})w^k(y;\theta) + \tilde{u}_{\text{LB}}(t_k, y) + \sum_{i=1}^{d} \alpha_i(x_i - y_i),$$

We sample N points in the interval $[0, x_{\text{max}}]^d$. By applying Monte-Carlo to approximate the integrals of the cost functional, we get:

$$\begin{aligned}\mathcal{C}(\theta;\theta^{k-2},\theta^{k-1}) &\approx \frac{(x_{\text{max}})^d}{N}\sum_{i=1}^{N}\Big[\big(u(x_i;\theta) - \frac{4}{3}u^{k-1}(x_i) + \frac{1}{3}u^{k-2}(x_i)\big)^2 \\ &+ \frac{2}{3}\tau\Big(L(x_i) + (2f[u^{k-1}(x_i)] - f[u^{k-2}(x_i)])u(x_i;\theta)\Big)\Big]\end{aligned}$$

Merton model

$$\mathcal{A}u = -\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + ru + g[u]$$

$$a_{ij}(x) = \frac{1}{2}\sigma_i\rho_{ij}\sigma_j x_i x_j, \qquad b_i(x) = [-r + \lambda\,\mathsf{exp}(\mu_{Ji} + \frac{1}{2}\sigma_{Ji}^2) - \lambda]x_i$$

$$g[u] \quad = \quad -\int_{\mathbb{R}^d}(u(xe^z) - u(x))\nu(dz)$$

$$\sigma_i = 0.5,\ \rho_{ij} = \delta_{ij} + 0.5(1-\delta_{ij}),\ \ i,j = 1,\ldots,d,\ r = 0.05,\ \text{time-step} = 0.02$$

Morever, the parameters of the jumps are

$$\lambda = 1,\ \mu_{Ji} = 0,\ \sigma_{Ji} = 0.5,\ \rho_{Jij} = \delta_{ij} + 0.2(1-\delta_{ij}),\ i,j = 1,\ldots,d$$

The integral : Gauss-Hermite quadrature

In Merton model, the integral is simply the expected value of the function $h(z) = u(xe^z) - u(x)$ multiplied by the Poisson parameter λ.

- Singular Value Decomposition (SVD) of Σ_J

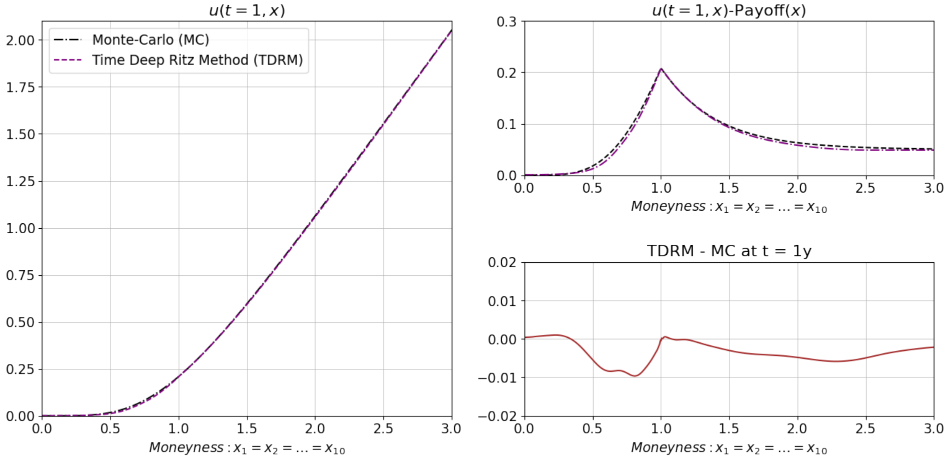
$$\Sigma_J = U\Lambda U^T = U\Lambda^{1/2}\Lambda^{1/2}U^T = VV^T,$$

where $V = U\Lambda^{1/2}$.

- change of variable $Z - \mu = \sqrt{2}VY$

$$\begin{aligned} \int_{\mathbb{R}^d} h(z)\nu(dz) &= \lambda\pi^{-d/2}\int_{\mathbb{R}^d} \exp(y^T y)h(\mu + \sqrt{2}Vy)dy \\ &\approx \lambda\pi^{-d/2}\sum_{\mathsf{i}\in\Theta_p} h(\mu + \sqrt{2}Vy^{\mathsf{i}})W^{\mathsf{i}}, \end{aligned}$$

10 Assets - Maturity : 1 year



- We proposed a modified splitting deep Ritz method for option pricing.
- Our method is capable of accurately approximating the solutions of the partial integro-differential equations (PIDEs) that arise in European basket call options.
- To evaluate the effectiveness of our method, we compared its results with those obtained using Monte-Carlo simulations.

- We are working on the multivariate variance Gamma model (infinite jump activity)

Thank you for your attention