Topology I

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Abstract

The following lecture notes are my (inofficial) notes for the 'Topology I' course, taught in the winter semester 2021/2022 at the university of Bonn. I cannot guarantee neither for completeness nor correctness of these notes. Corrections are welcome at any time, albeit just typos. Just write me a Mail, or preferably use the issues feature on GitHub directly. For more information regarding these notes have a look at the GitHub repository or the course page.

Contents

Summary of lectures

Lecture 1
Mi 13 Okt 2021

Organisational stuff 0.0.1.

- All lectures will be recorded and uploaded to eCampus afterwards.
- The same holds for all written lecture notes (from the iPad).
- Daniel is happy if you turn on your camera, these will not be recorded
- If you have questions, feel free to ask them at any time. Beware that your voice / question will be recorded
- It is not yet know whether the exam at the end of the semester will be in person or online.
- You can also ask questions in the chat (these will not be recorded) if you prefer so.
- There is an question session on Monday, 10. You can ask questions there, but there will also be questions asked to the students that are to be solved 'live'.
- On next Monday, we will quickly summarize the basics of category theory
- You need 50% of the points obtainable from the exercise sheet to be admissible to the exam.
- For everything taking place at the university (question session or exercise groups) you are required to fulfill the 3G rule: be vaccinated against, recovered from or testet for the coronavirus.
- If you did not attend the course on 'Geometrie und Topologie' the last semester, you can have a look at the lecture notes [Kas21]. These are in German, though. You can also visit the corresponding eCampus course or the webpage.
- If you want to write a bachelor thesis in topology, during one of the lecture times in January, all the topology group members will be present and present a number of topics for a Bachelor thesis. So don't ask yet for possible topics.
- Exercise sheets can be handed in groups of up to 3 people. Everyone is required to understand all the solutions

As literature, [Lüc05] (in german) and [Hat02] are recommended.

0 Motivation for this lecture

Last semester we had a look at

- basic properties of topological spaces (e.g. Hausdorff or compact spaces)
- The fundamental group of a topological space, coverings to compute it

- Fundamental results we proved were:

$$\pi_1(S^1, 1) \cong \mathbb{Z} \qquad \pi_1(S^n, s_0) = 0 \qquad \forall n \geqslant 2$$

We aim to kind of generalize this to higher n, one can in fact define

$$\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)]$$

but these are very hard to compute and do not behave as one would possibly expect from a generalization, e.g. it holds

$$\pi_3(S^2, s_0) \cong \mathbb{Z}$$

We will consider **homology** instead. Homology is much harder to define than the higher homotopy groups, but will be much easier to compute.

Warning . We will consider R-modules in this course. If you are not familiar with them, just tell Daniel or ask questions at any time.

For now, just think of modules as a sort of vector space, but over a ring R, not a field K.

At the start we will only consider $\mathbb{Z}$-modules, i.e. abelian groups.

1 Δ-complexes & semi-simplicial sets

1.1 Δ-complexes

Definition 1.1 (simplex)**.** The **n-simplex** is the space

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geqslant 0 \right\}$$

That is, Δ^n is the convex subspace of $\mathbb{R}^{n+1}$ spanned by

$$e_0 = (1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 1)$$

Example 1.2 (low dimensional simplices)**.** Have a look at Figure 1:

Δ^0 is just a point.

Δ^1 is the unit interval via the embedding

$$\begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R}^2 \\ t & \longmapsto & te_0 + (1-t)e_1 \end{array}$$

Δ^2 is a triangle

Δ^3 is a tetrahedron

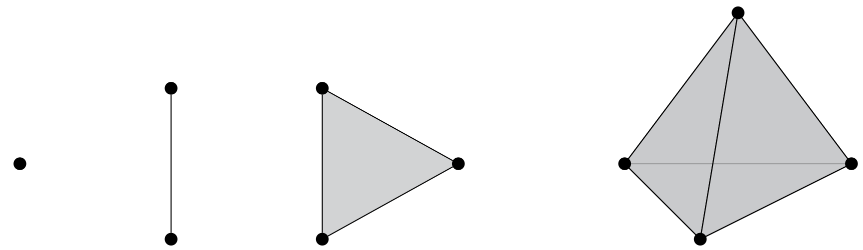


Figure 1: some first simplices

Definition 1.3 (face)**.** For a subset $I \subseteq \{0, \ldots, n\}$ consider the subspace

$$\{(t_0, \ldots, t_n) \in \Delta^n \mid t_i = 0 \ \forall i \in I\}$$

This is called a **face** of or also $(n - |I|)$-face of Δ^n.

Definition 1.3.1 (inclusion)**.** For $0 \leqslant i \leqslant n$ define

$$\delta^i = \delta^{n,i} : \left| \begin{array}{ccc} \Delta^{n-1} & \longrightarrow & \Delta^n \\ (t_0, \ldots, t_{n-1}) & \longmapsto & (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}) \end{array} \right.$$

δ^i is the **inclusion** of the ith face of Δ^n

Definition 1.3.2 (boundary)**.** The **boundary** $\partial\Delta^n$ of Δ^n is the union of the highest dimensional faces, i.e.

$$\partial\Delta^n := \bigcup_{i=0}^{n} \operatorname{im} \delta^i = \{(t_0, \ldots, t_n) \in \Delta^n \mid \exists i \colon t_i = 0\}$$

Definition 1.3.3 (interior)**.** The **interior** of Δ^n is defined as

$$\mathring{\Delta}^n := \Delta^n \backslash \partial\Delta^n$$

Remark* 1.3.4. Note that the boundary of the 0-simplex is empty, since there are no -1-dimensional faces of the 0-simplex. This also means that $\mathring{\Delta}^0 = \Delta^0$.

Definition 1.4 (Δ-complex)**.** A **Δ-complex** is a topological space X together with the following structure:

For all $n \in \mathbb{N}_0$, there is a set $A_n = \{\alpha \colon \Delta^n \to X\}$ such that

1) $\forall \alpha \in A_n$, the restriction $\alpha \,|_{\mathring{\Delta}^n}$ is injective
2) $\forall x \in X$, there is a unique pair (n, α) with $n \in \mathbb{N}_0$ and $\alpha \in A_n$ such that $x \in \alpha \,|_{\mathring{\Delta}^n}$
3) $\forall \alpha \in \Delta^n$ and $i \in \{0, \ldots, n\}$ we have $\alpha \circ \delta^i \in A_{n-1}$.
4) A subset $U \subseteq X$ is open iff

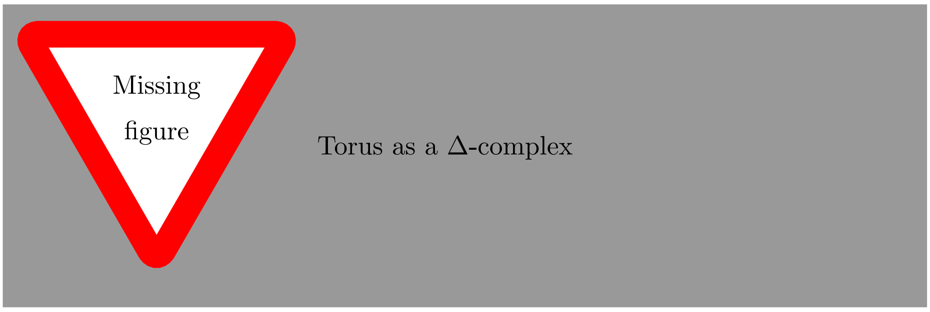
$$\alpha^{-1}(U) \subseteq \Delta^n \text{ is open } \forall n \in \mathbb{N}_0, \alpha \in A_n$$

Oral remark 1.4.1. The last property essentially tells us that you can recover the topology on X from the sets of maps A_n.

Beware that the set of maps A_n is not necessarily the set of all maps, this set is part of the data of the Δ-complex.

Placeholder 1.5. In the lecture, 1.5 was somehow skipped.

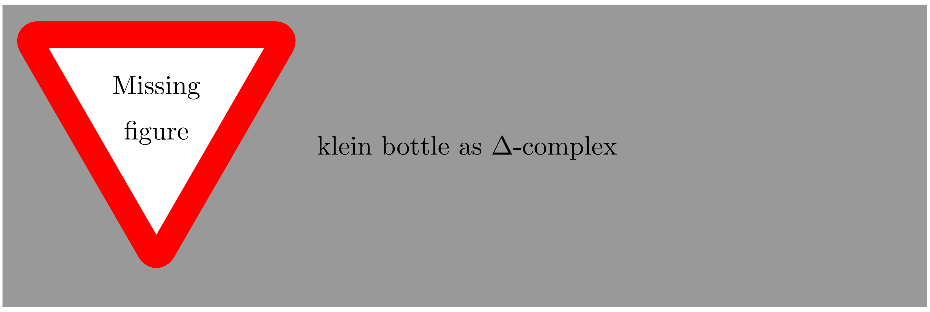
Example 1.6 (Torus as Δ-complex)**.** Considering the torus as a quotient from I^2 (the unit square).



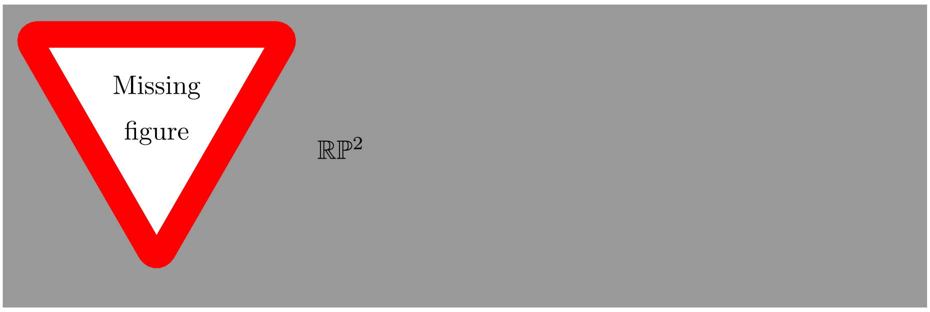
Inserting a diagonal into the unit square realizes the torus as a Δ-complex with two 2-simplices (namely the two triangles), three 1-simplices (the diagonal, and the two pairs of opposite sides of the square) and one 0-simplex (the corners of the square that are glued together)

Note that in this case $A_n = \varnothing$ for $n \geqslant 3$.

Example 1.6.1 (Klein bottle as Δ-complex)**.** We can also realize the **Klein bottle** as a simplicial complex, just as we did with the torus



Example 1.6.2 ($\mathbb{RP}^2$ as simplicial complex)**.** We can do the same for $\mathbb{RP}^2$.



Example 1.6.3 (S^n as simplicial complex)**.** For S^n (the unit sphere), there are a lot of examples:

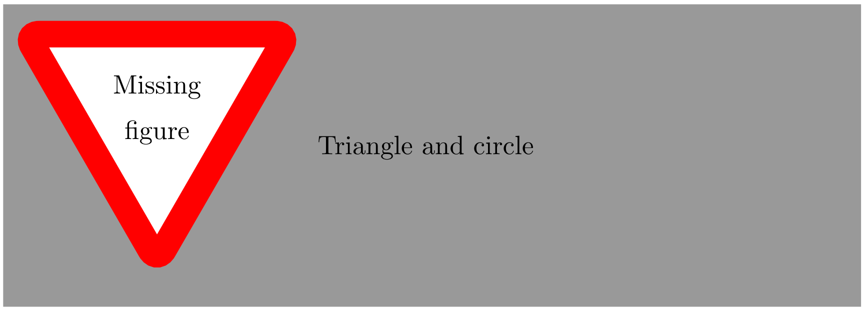
We can also consider S^1 as ${}^{\Delta^1}\!/\!_{\partial\Delta^1}$ as a Δ-complex, but this won't hold for $n \geqslant 2$, since the embeddings of the interiors of the maps won't be injective anymore, so they don't satisfy the requirements of a structure.

Remark 1.7.

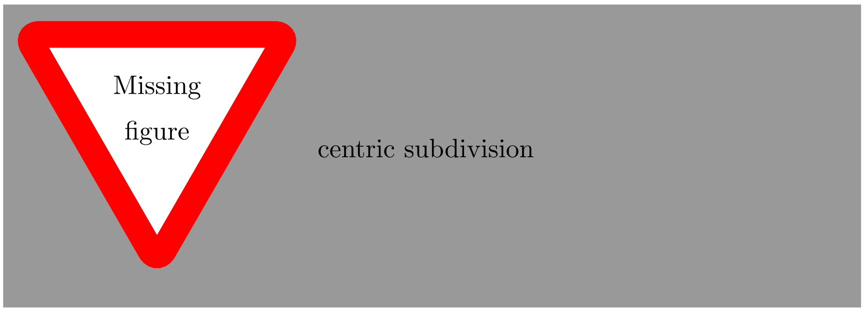
- There is also the notion of (an ordered) simplicial complex. This is slightly more restrictive: We assume that for $\alpha, \beta \in A_n$ it follows $\alpha = \beta$ if

$$\alpha \circ \delta^i = \beta \circ \delta^i \qquad \forall i = 0, \dots, n$$

i.e. if they have the same 0-faces.



- A Δ-complex can be turned into a simplicial complex by **centric subdivision**:



We will not define this for now.

1.2 Simplicial homology

Definition 1.8 (chains, boudary homomorphism)**.** Let X be a Δ-complex.

- The **n-chains** of X are the free abelian groups $\Delta_n(X)$ generated by A_n, i.e.

$$\Delta_n(X) := \mathbb{Z}[A_n] \cong \bigoplus_{A_n} \mathbb{Z}$$

- The **boundary homomorphism** is defined as

$$\partial_n : \left| \begin{array}{ccc} \Delta_n(X) & \longrightarrow & \Delta_{n-1}(X) \\ \alpha & \longmapsto & \sum_{i=0}^{n} (-1)^i (\alpha \circ \delta^i) \end{array} \right.$$

Lemma 1.9. If $i \leqslant j$, then

$$\delta^i \circ \delta^j = \delta^{j+1} \circ \delta^i : \Delta^{n-2} \to \Delta^n$$

Proof. We have

$$(t_0, \ldots, t_{n-2}) \xrightarrow{\delta^j} (t_0, \ldots, t_{j-1}, 0, \ldots, t_j, \ldots, t_{n-2}) \xrightarrow{\delta^i} (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{j-1}, 0, t_j, \ldots, t_{n-2})$$

Note that if $i = j$, $t_i, \dots, t_{j-1}$ expands to nothing, i.e. the two zeros will be adjacent to each other. For the other composition, we have

$$(t_0, \dots, t_{n-2}) \xrightarrow{\delta^i} (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-2}) \xrightarrow{\delta^{j+1}} (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{j-1}, 0, t_j, \dots, t_{n-2})$$

Also for $j = i$ we have the zeros being adjacent.

Now we directly see that these are in fact the same maps. □

Lemma 1.10. It holds $\partial_{n-1} \circ \partial_n = 0$.

Proof. We compute the composition:

$$\begin{aligned}
\partial_{n-1}(\partial_n(\alpha)) &= \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} (\alpha \circ \delta^i \circ \delta^j) \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} (\alpha \circ \delta^i \circ \delta^j) + \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} (\alpha \circ \delta^i \circ \delta^j) \\
&\overset{\text{Lemma 1.9}}{=} \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} (\alpha \circ \delta^{j+1} \circ \delta^i) + \sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} (\alpha \circ \delta^i \circ \delta^j) \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} (\alpha \circ \delta^{j+1} \circ \delta^i) + \sum_{i=0}^{n-1} \sum_{j=0}^{i} (-1)^{i+j+1} (\alpha \circ \delta^{i+1} \circ \delta^j) \\
&= 0
\end{aligned}$$

□

Definition 1.11 (chain complex)**.** A **chain complex** $(C_\bullet, \partial_\bullet)$ of $\mathbb{Z}$-modules is a sequence of abelian groups:

$$\dots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \dots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

such that $\partial_{n-1} \circ \partial_n = 0$ for all $n \geqslant 1$.

The chain complex is called **exact** if $\ker \partial_n = \operatorname{im} \partial_{n+1}$ for all n.

Remark 1.11.1. Note that the condition $\partial_{n-1} \circ \partial_n = 0$ just imposes the condition $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$

Example 1.12. Let X be a Δ-complex. By Lemma 1.10, $(\Delta_\bullet(X), \partial_\bullet)$ is a chain complex.

Definition 1.13 (homology group)**.** Let $(C_\bullet, \partial_\bullet)$ be a Δ-complex. We define its

***n*-th homology group** by

$$H_n(C_\bullet, \partial_\bullet) := {}^{\ker \partial_n}\!/\!_{\operatorname{im} \partial_{n+1}}$$

Definition 1.14. Let X be a Δ-complex. We define its **nth homology group** as

$$H_n^\Delta(X) = H_n(\Delta_\bullet(X), \partial_\bullet)$$

Example 1.15. Pick S^1 as a Δ-complex as in Example 1.6.3. Then $\Delta_0 \cong \Delta_1 \cong \mathbb{Z}$ and $\Delta_n = 0$ for $n \geqslant 2$. As $a \circ \delta^0 = a \circ \delta^1 = v$ we have

$$\partial_1(a) = a \circ \delta^0 - a \circ \delta^1 = v - v = 0$$

Thus the corresponding chain complex is:

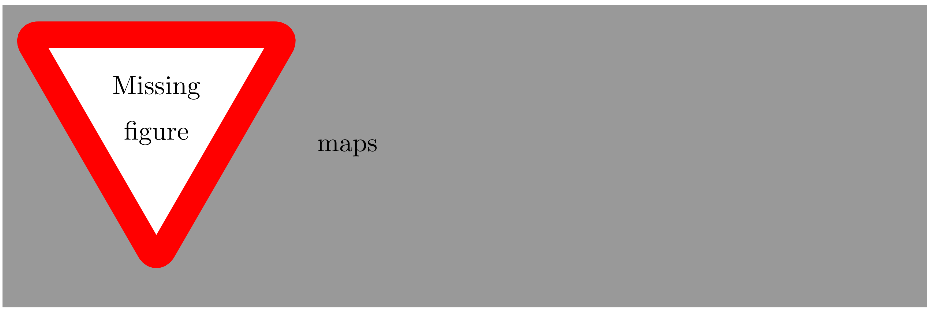
$$\ldots \to 0 \to 0 \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

and thus

$$H_n^\Delta(S^1) \cong \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & n \geqslant 2 \end{cases}$$

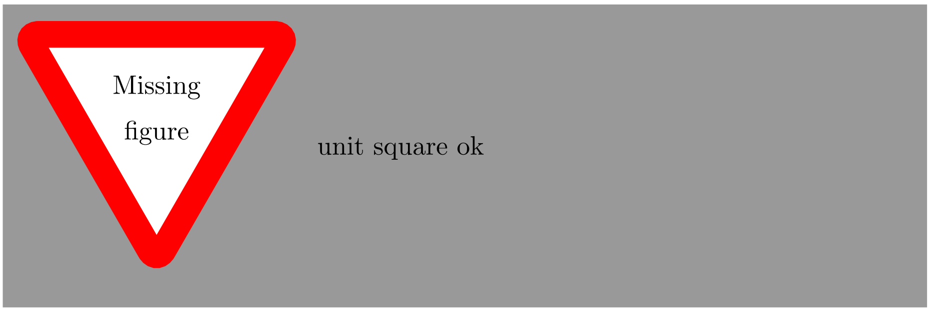
Lecture 2
Th 14 Oct 2021

Remark 1.15.1. The vertices of Δ^n are ordered. Thus also $\Delta^n \to X$ specifies this, i.e. the embeddings of Δ^2 into $\mathbb{R}^2$

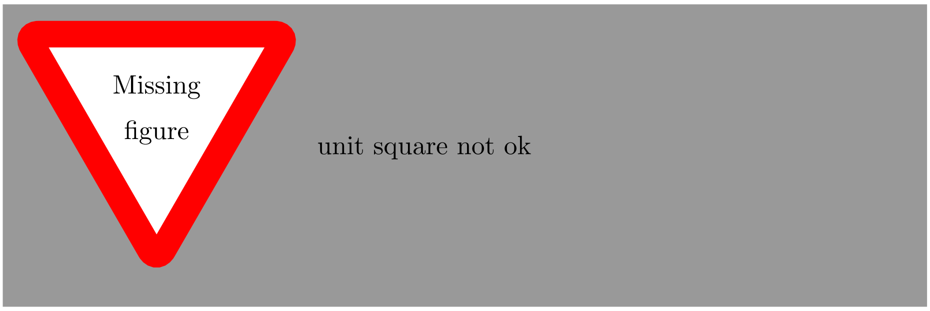


are different maps.

The orders on the boundaries have to agree, e.g. we can view $[0, 1]^2$ as



but not as

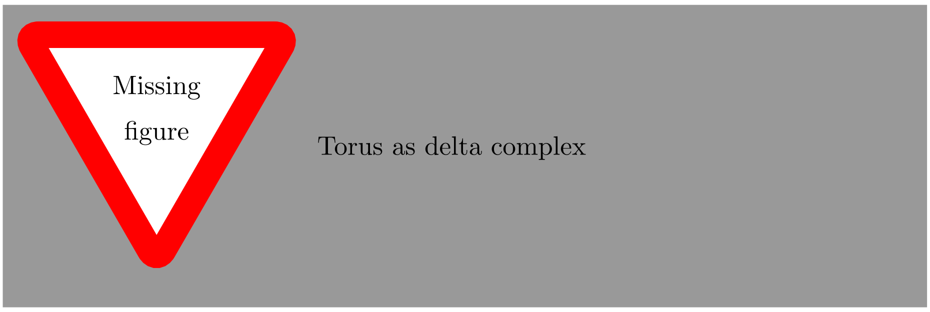


.

This is imposed by the condition that $\alpha \circ \delta^i \in A_{n-1}$: If the order does not agree on the boundary, this implies that we have at least maps from Δ^{n-1} to the boundary to the boundary, one for each direction, which will violate condition 2) in Definition 1.4.

To express the ordering of the simplices, in images we will often draw arrows indicating the direction of the corresponding edges, like in the images above.

Example 1.16. Let T be a torus, realized with three 2-simplices, three 1-simplices and a 0-simplex as a Δ-complex as follows:



Computing the chain complex yields:

$$\begin{aligned}
\Delta_2(T) &\cong \mathbb{Z}^2 \quad \text{(generated by } U, V\text{)} \\
\Delta_1(T) &\cong \mathbb{Z}^3 \quad \text{(generated by } a, b, c\text{)} \\
\Delta_0(T) &\cong \mathbb{Z} \quad \text{(generated by } v\text{)}
\end{aligned}$$

For the boundary maps, we get the images:

$$\begin{aligned}
\partial_2(U) &= b - c + a \\
\partial_2(V) &= a - c + b = \partial_2(u) \\
\Longrightarrow H_2^\Delta(T) &\cong \ker \partial_2 \cong \mathbb{Z} \text{ (generated by } U - V\text{)}
\end{aligned}$$

Thus, as $b - c + a = a - c + b \neq 0 \in \Delta_1(T)$, we have:

$$H_2^\Delta(T) = {\ker \partial_2}/{\operatorname{im} \partial_3} = \ker \partial_2 \cong \mathbb{Z} \cong \langle U - V \rangle$$

For the first homology, we get that all boundaries vanish

$$\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0$$

and hence

$$H_1^\Delta(T) \cong {\ker \partial_1}/{\operatorname{im} \partial_2} \cong {\mathbb{Z}^3}/{\operatorname{im} \partial_2} \cong {\mathbb{Z}^3}/{\langle c - a + b \rangle} \overset{c=a+b}{\cong} \mathbb{Z}^2$$

Finally, we also get

$$H_0^\Delta(T) \cong {\ker \partial_1}/{\operatorname{im} \partial_1} = {\mathbb{Z}}/{0} \cong \mathbb{Z}$$

Oral remark 1.16.1. We will observe that $H_0^\Delta \cong \mathbb{Z}$ is quite common, namely for connected spaces.

In the example, the groups are torsion-free, however this does not have to be the case in general (in contrast to the chain groups that are free by definition).

Example 1.17. Consider the **figure eight**, namely $S^1 \vee S^1$. We can realize this as a Δ-complex with two 1-simplices and one 0-simplex as follows:

Again, ∂_1 is trivial to compute:

$$\partial_1(a) = \partial_1(b) = v - v = 0.$$

Thus

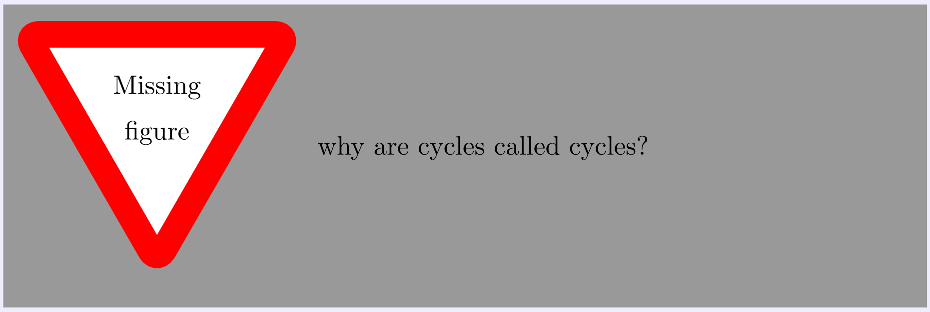
$$H_n^{\Delta}(S^1 \vee S^1) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^2 & \text{if } n = 1 \\ 0 & \text{if } n \geqslant 2 \end{cases}$$

Remark 1.18. Note that for the Torus, we had $\pi_1(T) \cong H_1^{\Delta}(T)$, but for $S^1 \vee S^1$ this did not hold.

However, in all examples above we have $H_1^{\Delta} \cong \pi_1^{\mathrm{ab}}$. We will see that this holds in general.

We will further see that H_n^{Δ} only depends on the underlying space and not on the Δ-structure. For this we will first generalize homology to all spaces.

Definition 1.19 (Cycles, boundaries)**.** Elements in $\ker \partial_n \subseteq C_n$ are called **cycles**. Elements in $\operatorname{im} \partial_{n+1} \subseteq C_n$ are called **boundaries**.



1.3 Semi-simplicial sets

Oral remark 1.19.1. Semi-simplicial sets are somewhat more abstract. If you don't like that, you can mostly ignore them for the rest of the lecture.

Definition 1.20. A **semi-simplicial set** $S_\bullet$ (or Δ-set) is a sequence $(S_n)_{n=0}^{\infty}$ of sets together with maps $d_i : S_{n+1} \to S_n$ for $i \in \{0, \dots, n+1\}$ that satisfy $d_j \circ d_i = d_i \circ d_{j+1}$ $\forall i \leqslant j$.

Oral remark 1.20.1. First note that ther really are $n+2$ (potentially different) maps $S_{n+1} \to S_n$ that are part of the data of a simplicial set.

Remark 1.20.2. A semi-simplicial set can also be viewed as a functor, as we will see in Lemma 1.25.

Example 1.21. Let X be a Δ-complex. Then setting $S_n := A_n$ and d_i as the restriction along δ^i (i.e. $d_i(\alpha) = \alpha \circ \delta^i$) yields a a semi-simplicial set, as we have

$$d_j \circ d_i = d_i \circ d_{j+1} \text{ for } i \leqslant j$$

by Lemma 1.9.

As with the n-chains of a Δ-complex , one can form a chain complex from a semi-simplicial complex as well:

Lemma and Definition 1.22. The **linearization** of $S_\bullet$ is the chain complex $(\mathbb{Z}S_\bullet, \partial)$ given by

$$(\mathbb{Z}S)_n := \mathbb{Z}\left[S_n\right]$$

(the formal finite linear combinations of the elements of S_n) and as boundary maps

$$\partial_n := \sum_{i=0}^{n} (-1)^i d_i$$

. This forms a chain complex, that gives $(\Delta_\bullet, \partial)$ in the special case of a Δ-complex treated as in Example 1.21.

Proof. The chain complex property just follows analoguously to Lemma 1.10, as we just needed the property of Lemma 1.9 for its proof and this is just part of the definition of a semi-simplicial set. It is easy to check that this yields $(\Delta_\bullet, \partial)$ in the case of a Δ-complex. $\square$

Definition 1.23 (category of). The category Δ_{inj} consists of:

- the non-empty, linearly ordered, finite sets as objects
- the injective, order-preserving maps between them

Remark 1.24. We can consider the full subcategory Δ'_{inj} of Δ_{inj} on the objects $[n] = \{0, \ldots, n\}$. The inclusion of this subcategory is an equivalence of categories.

A functor $\mathcal{F}\colon \mathscr{C} \to \mathscr{D}$ is **full**, if for each pair of objects $A, B \in \mathscr{C}$, the map of sets

$$\mathcal{F}\colon \operatorname{Mor}_{\mathscr{C}}(A, B) \to \operatorname{Mor}_{\mathscr{D}}(\mathcal{F}(A), \mathcal{F}(B))$$

is surjective. $\mathcal{F}$ is called **faithful**, if this map is injective. A functor that is both full and faithful (meaning above map is a bijection) is also called a **fully faithful** functor.
$\mathcal{F}$ is **essentially surjective**, if for each $B \in \mathcal{D}$, there is some $A \in \mathscr{C}$ with $\mathcal{F}(A) \cong B$, i.e. it is surjective up to isomorphism (in the category $\mathcal{D}t$).
If $\mathcal{F}$ is fully faithful and essentially surjective, $\mathcal{F}$ is called an **equivalence** of categories, we also say that $\mathbb{C}$ and $\mathscr{D}$ are equivalent.
If you are not familiar with these notions yet, check them in the case of $\Delta'_{\text{inj}} \hookrightarrow \Delta_{inj}$.

Organisational stuff 1.24.1. If you don't know what a category is, come to the Q&A-session on next monday.

Lemma 1.25. There is a bijection between semi-simplicial sets and functors ${\Delta'_{\text{inj}}}^{\text{op}} \to \mathbf{Set}$.

Proof (sketched). Every morphism $[k] \to [n]$ in Δ'_{inj} is a composition of morphisms of the form $\delta^i = \delta^{ni} : [n-1] \to [n]$ that send j to j for $j < i$. j to $j+1$ for $j \geqslant i$.

Roughly, one sees this by iteratively embedding $[k] \hookrightarrow [k+1] \hookrightarrow \ldots \hookrightarrow [n]$ whilst skipping the elements that do not lie in the image.

This decomposition (into maps δ^i) is unique up to the relation $\delta^i \circ \delta^j = \delta^{j+1} \circ \delta^i \ \forall i \leqslant j$.

Hence a contravariant functor from Δ'_{inj} is determined by a set S_n for each $[n]$ and functions $(\delta^i)^\star = d_i : S_n \to S_{n-1} \ \forall i \in \{0, \ldots, n\}$ with $d_j \circ d_i = d_i \circ d_{j+1} \forall i \leqslant j$.

But this is precisely the datum of a semi-simplicial set, so the two notions agree. □

Remark 1.25.1. For a quick motivation, consider the map

$$\begin{array}{ccc} [1] & \longrightarrow & [3] \\ 0 & \longmapsto & 1 \\ 1 & \longmapsto & 3 \end{array}$$

Then there are two ways of decomposing this, namely

$$\begin{array}{ccccc} [1] & \longrightarrow & [2] & \longrightarrow & [3] \\ 0 & \longmapsto & 0 & \longmapsto & 1 \\ 1 & \longmapsto & 2 & \longmapsto & 3 \\ & & 1 & \longmapsto & 2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} [1] & \longrightarrow & [2] & \longrightarrow & [3] \\ 0 & \longmapsto & 1 & \longmapsto & 1 \\ 1 & \longmapsto & 2 & \longmapsto & 3 \\ & & 0 & \longmapsto & 0 \end{array}$$

To see that this is in fact (and in general) unique up to the given relation, one would have to do some combinatorics, but this is not to be part of our course.

Remark 1.26. $\mathrm{Fun}({\Delta'_{\mathrm{inj}}}^{\mathrm{op}}, \mathbf{Set})$ is equivalent to $\mathrm{Fun}(\Delta_{\mathrm{inj}}^{\mathrm{op}}, \mathbf{Set})$.

Definition 1.27 (Geometric realization of a semi-simplicial set)**.** The **geometric realization** of a semi-simplicial set $S_\bullet$ is the (quotient) space

$$|S| = \coprod_{n \in \mathbb{N}_0} S_n \times \Delta^n \big/ \sim$$

where the relation $\sim$ is generated by

$$(\sigma, \delta^i t) \sim (d_i \sigma, t) \qquad \forall \sigma \in S_n, t \in \Delta^{n-1}$$

Oral remark 1.27.1. Essentially, this takes a simplex Δ^n for each element of each S_n, identifying those parts of the simplices that correspond via the boundary maps d_i.

Also, this is quite abstract now, if you are confused about it, do not worry too much.

Lemma 1.28. For every semi-simplicial set $S_\bullet$ the inclusion induces a bijection (not homeomorphism!)

$$\coprod_{n \in \mathbb{N}_0} S_n \times \Delta^{\circ n} \xrightarrow{\simeq} |S|$$

Proof. For every $(s,t) \in S_n \times \Delta^n$ t is contained in the interior of a unique k-face of Δ^n (potentially $k = n$). Let $f : \Delta^k \to \Delta^n$ be the inclusion of this face and let $y \in \Delta^k$ such that $f(y) = t$. For a decomposition $f = \partial^{i_1} \circ \ldots \circ \partial^{i_k}$ denote $f^* \coloneqq d_{i_k} \circ \ldots \circ d_{i_1} : S_n \to S_k$ as the 'dual' map on the S_n's.

By inductively applying the relation from Definition 1.27, we see that $(s,t) \sim (f^* s, y)$. Thus the map is surjective, as $f^\star s$ lies in the interior of y by construction.

It remains to show injectivity: For a given (s,t) f and y already are unique. If $(s,t) = (d_i s', t) \sim (s', \delta^i t)$ then

$$(s', \delta^i t) = s', (\delta^i \circ f)(y)) \cong ((f^* \circ d_i)(s'), y) = (f^* s, y)$$

Similarly, if $(s,t) = s, \delta^i t') \sim (d_i s, t')$, then there exists

$$f' : \Delta^k \to \Delta^{n-1} \text{ with } \delta^i \circ f' = f$$

and thus $(d_i s, t') = (d_i s, f'(y)) \sim (f'^* d_i s, y) = (f^* s, y)$ □

Corollary 1.29. The realization of a semi-simplicial set is a Δ-complex.

Proof. Have a look at Definition 1.4.

1) & 2) follow from the bijectivity in Lemma 1.28.
3) follows from $(s, \delta^i t) \sim (d_i s, t)$
4) follows from the definition of the quotient topology.

□

Remark 1.30. If the semi-simplicial set comes from a Δ-complex, its realization is homeomorphic to the Δ-complex. Thus, Δ-complex and semi-simplicial sets are really just the same thing as one can switch between each other.

2 Singular homology

Definition 2.1 (Singular sets)**.** Let X be a space.

1) We define the **singular set** $\mathrm{sing}(X)$ as the semi-simplicial set given by

$$\mathrm{sing}_n(X) := \{\sigma \colon \Delta^n \to X \mid \delta \text{ continuous}\}$$

where
$$d_i \colon \mathrm{sing}_n(X) \to \mathrm{sing}_{n-1}(X)$$
are given by the natural restrictions along δ^i.

2) Denote $C_n^{\mathrm{sing}}(X) := \mathbb{Z}\left[\mathrm{sing}_n(X)\right]$ as the chains.

Definition 2.2 (Singular homology)**.** The ***n*th singular homology** group of X is defined as
$$H_n(X) := H_n(C_\bullet^{\mathrm{sing}}(X), \partial)$$

Oral remark 2.2.1. The motivation for this comes from the semi-simplicial sets: For an arbitrary space, we are not (directly) able to define chains as we did for Δ-complexes, so the singular sets are just there to have such a structure we can compute homlogy on.

Remark 2.3. Note that $\text{sing}_n(X)$ might be really large, in fact most of the time have uncountably many generators (e.g. as soon as X is uncountable). But $H_n(X)$ is often finitely generated. We will in fact see that for Δ-complexes it holds

$$H_\star^\Delta(X) \cong H_\star(X)$$

Example 2.4. Let $X = \{\star\}$ be the one-point-space. Then

$$\text{sing}_n(X) = \{\Delta^n \to \star\} = \{\star\}$$

thus $\mathbb{Z}[\text{sing}_n(X)] \cong \mathbb{Z}$ for all n. Since all the d_i agree, we have

$$\partial_n = \begin{cases} \text{id} = n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

(remember that ∂_n was an alternating sum of d_i). Thus as a chain complex, we get

$$\ldots \to \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

Thus

$$H_n(\{\star\}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \geqslant 1 \end{cases}$$

Lecture 3
Mi 20 Okt 2021

Definition 2.5 (Chain map)**.** A **chain map** $f\colon (C_\bullet, \partial^C) \to (D_\bullet, \partial^D)$ is a sequence of maps (group homomorphisms)

$$f_n\colon C_n \to D_n$$

such that

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "\partial^C"'] & D_n \arrow[d, "\partial^D"] \\ C_{n-1} \arrow[r, "f_{n-1}"'] & D_{n-1} \end{tikzcd}$$

commutes for all n.

Lemma 2.6. A chain map $f\colon (C_\bullet, \partial^C) \to (D_\bullet, \partial^D)$ induces a map

$$f_\star : \left| \begin{array}{ccc} H_n(C_\star, \partial^C) & \longrightarrow & H_n(D_\star, \partial^D) \\ [x] & \longmapsto & [f_n(x)] \end{array} \right.$$

Proof. First note that if $x \in \ker \partial_n^C$, $[x]$ is its image in ${}^{\ker \partial_n}\!/\!_{\operatorname{im} \partial_{n+1}} = H_n$ Consider now the commutative diagram

$$\begin{tikzcd}
C_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\partial^C_{n+1}"'] & D_{n+1} \arrow[d, "\partial^D_{n+1}"] \\
C_n \arrow[r, "f_n"'] \arrow[d, "\partial^C_n"'] & D_n \arrow[d, "\partial^D_n"] \\
C_{n-1} \arrow[r, "f_{n-1}"] & D_{n-1}
\end{tikzcd}$$

Given $x \in \ker \partial^C_n$, then $\partial^D_n(f_n(x)) = f_{n-1}(\partial^C_n(x)) = f_{n-1}(0) = 0$. Thus also $f_n(x) \in \ker \partial^D_n$.

It remains to show that $f_\star$ is well-defined on $H_n = {\ker \partial^C_n}/{\operatorname{im} \partial^C_{n+1}}$. A computation yields that

$$\begin{aligned}
f_n(x + \partial^C_{n+1}(y)) &= f_n(x) + f_n(\partial^C_{n+1}(y)) \\
&= f_n(x) + \underbrace{\partial^D_{n+1}(f_{n+1}(y))}_{\in \partial^D_{n+1}}
\end{aligned}$$

□

Oral remark 2.6.1. We now do the same proof again using a technique called **diagram chasing**. We will be doing this kind of proof quite a lot in the future.

Lemma 2.7. A map $f\colon X \to Y$ of topological spaces induces a chain map

$$C^{\text{sing}}_\star(X) \xrightarrow{f_\star} C^{\text{sing}}_\star(Y)$$

via

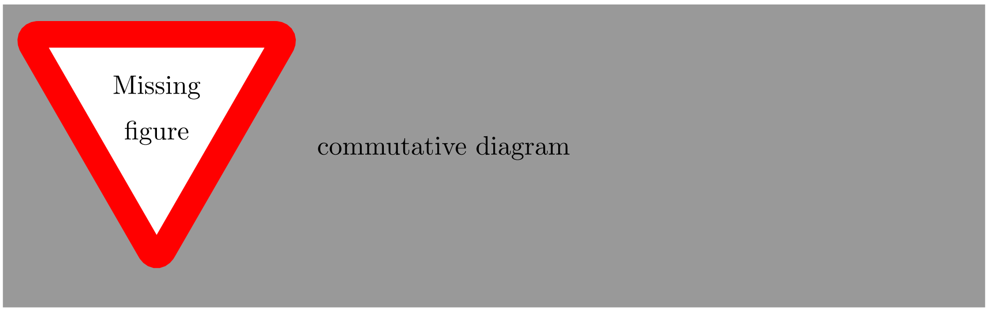
$$(\Delta^n \xrightarrow{\alpha} X) \longmapsto (\Delta^n \xrightarrow{A} X \xrightarrow{f} Y)$$

Proof. Note that it suffices to understand the map on generators, since the images of the generators determine the group homomorphism uniquely.

Consider again the diagram

$$\begin{tikzcd}
C^{\text{sing}}_n(X) \arrow[r, "f_n"] \arrow[d, "\partial^X"'] & C^{\text{sing}}_n(Y) \arrow[d, "\partial^Y"] \\
C^{\text{sing}}_{n-1}(X) \arrow[r, "f_{n-1}"'] & C^{\text{sing}}_{n-1}(Y)
\end{tikzcd}$$

One easily checks that on the level of generators, we just get the commutative diagram



□

Definition† 2.7.1 (Pair of spaces)**.** A **pair of spaces** (X, A) is a pair of topological spaces such that $A \subseteq X$ is a subspace (endowed with the subspace topology).

Note. We do not require that A is open / closed / anything.

Definition† 2.7.2 (Map of pairs)**.** A map of a pair of spaces $f\colon (X, A) \to (Y, B)$ is a continuous map $f\colon X \to Y$ such that $f\mid_A (A) \subseteq B$, i.e. we can restrict

$$f|_A\colon A \to B$$

Lemma and Definition 2.8 (Relative chain group)**.** Let (X, A) be a pair of topological spaces. Let $i\colon A \to X$ be the inclusion. Then we define the **relative chain group** $C_n^{\text{sing}}(X, A)$ as

$$C_n(X, A) := C_n^{\text{sing}}(X) \big/ i_n(C_n^{\text{sing}}(A))$$

∂^X induces a differential $\partial^{(X,A)}$ for $C_n^{\text{sing}}(X, A)$. The nth homology group of the pair (X, A) is then defined as

$$H_n(X, A) := H_n(C_\bullet^{\text{sing}}(X, A), \partial^{(X,A)})$$

Proof. Since

$$\partial_n^X(i_\star(\Delta^n \xrightarrow{\alpha} X)) = i_\star(\partial_n^A(\Delta^n \xrightarrow{\alpha} A))$$

, ∂^X induces a differential $\partial^{(X,A)}$ for $C_n^{\text{sing}}(X, A)$. There is nothing more to proof. □

Lemma 2.9. A map of pairs $f\colon (X, A) \to (Y, B)$ induces a chain map

$$f_\star\colon C_\star^{\text{sing}}(X, A) \to C_\star^{\text{sing}}(Y, B)$$

Proof. Let $i\colon A \to X$ and $j\colon B \to Y$ be the inclusions of the subspaces. Let $\alpha \in C_n^{\text{sing}}(X)$ and $\Delta^n \xrightarrow{\beta} A$ be a generator. We have to see that

$$C_n^{\text{sing}}(X, A) \xrightarrow{f_n} C_n^{\text{sing}}(Y, B)$$

is well-defined. But now

$$[\alpha] = [\alpha + i_\star\beta] \longmapsto [f_\star\alpha + \underbrace{f_\star i_\star\beta}_{f\circ i\circ\beta \ = j\circ f\circ\beta}] = [f_\star\alpha]$$

As before

$$f_{n-1}\partial^{(X,A)}[\alpha] = [f_\star\partial^X\alpha] \overset{\text{Lemma 2.7}}{=} [\partial^Y f_\star\alpha] = \partial^{(Y,B)} f_n[\alpha]$$

□

Notation† 2.9.1. We denote by $\mathbf{Top}^2$ the category of pairs of topological spaces.

Corollary 2.10. For all $n \in \mathbb{N}$, H_n is a functor

$$H_n \colon \mathbf{Top}^2 \to \mathbf{Ab}$$

Proof. Just follows from Lemma 2.6 and Lemma 2.9 noting that $\mathrm{id}_\star = \mathrm{id}$ and $(f\circ g)_\star = f_\star \circ g_\star$.

In more detail, we have

$$\mathrm{id}_n[\alpha] = [\mathrm{id}\circ a] = [\alpha]$$

and

$$(f\circ g)_n[\alpha] = [(f\circ g)\circ\alpha] = [f\circ(g\circ\alpha)] = f_n[g\circ\alpha] = f_n(g_n([\alpha]))$$

□

Remark 2.11. We denote $H_n(X) \coloneqq H_n(X, \varnothing)$. This gives the previous definition of $H_n(X)$ for a single space.

Lemma 2.12. For all $n \in \mathbb{N}$, there is a natural transformation

$$H_n(X,A) \xrightarrow{\partial} H_{n-1}(A)$$

More precisely, this transformation is between teh functors

$$\begin{array}{ccccc} \mathbf{Top}^2 & \longrightarrow & \mathbf{Top}^2 & \xrightarrow{H_{n-1}} & \mathbf{Ab} \\ (X,A) & \longmapsto & (A,\varnothing) & \longmapsto & H_{n-1}(A) \end{array}$$

Proof. The following commutative diagram has exact rows:

$$0 \longrightarrow C_n^{\text{sing}}(A) \xrightarrow{i_n} C_n^{\text{sing}}(X) \longrightarrow C_n^{\text{sing}}(X,A) \longrightarrow 0$$

$$0 \longrightarrow C_{n-1}^{\text{sing}}(A) \xrightarrow{i_{n-1}} C_{n-1}^{\text{sing}}(X) \longrightarrow C_{n-1}^{\text{sing}}(X,A) \longrightarrow 0$$

The exactness of the right half follows from the definition

$$C_n^{\text{sing}}(X,A) := C_n^{\text{sing}}(X)\big/C_n^{\text{sing}}(A)$$

and i_n, i_{n-1} are injective, because $i\colon A \to X$ is injective, so that

$$\Delta^n \xrightarrow{\alpha} A \xrightarrow{i} X = \Delta^n \xrightarrow{\beta} A \xrightarrow{i} X \qquad \Longleftrightarrow \qquad \alpha = \beta$$

Consider $[\alpha] \in H_n(X,A)$ with $\alpha \in C_n^{\text{sing}}(X)$. Then $\partial_n^X(\alpha) \in \operatorname{im} i_{n-1}$. Let $\beta \in C_{n-1}^{\text{sing}}(A)$ be a preimage. Then

$$i_{n-2}(\partial_{n-1}^A(\beta)) = \partial_{n-1}^X(i_{n-1}(\beta)) = \partial_{n-1}(X)(\partial_n^X(\alpha)) = 0$$

Thus as i_{n-2} is injective, we have $\partial_{n-1}^A(\beta) = 0$. Then we can define the natural transformation as

$$\begin{array}{ccc} H_n(X,A) & \longrightarrow & H_{n-1}(A) \\ [\alpha] & \longmapsto & [\beta] \end{array}$$

We have yet to see that this definition is independent of the choice of α we made. To see this, we have to see that α and $\alpha + i_n\alpha' + \partial_{n+1}^X\alpha''$ represent have the same image. But then

$$\begin{aligned} \partial_n^X(\alpha + i_n\alpha' + \partial_{n+1}^X\alpha'') &= \partial_n^X\alpha + \underbrace{\partial_n^X i_n}_{=i_{n-1}\partial_n^A}\alpha' + \underbrace{\partial_n^X\partial_{n+1}^X}_{=0}\alpha'' \\ &= \partial_n^X\alpha + i_{n-1}\partial_n^A\alpha' \end{aligned}$$

This has preimage $\beta + \partial_n^A\alpha'$, which defines the same element in $H_{n-1}(A)$, since $\partial_n^A\alpha'$ is just divided out so that

$$[\beta] = [\beta + \partial_n^A\alpha'] \in H_{n-1}(A)$$

□

Aside 2.12.1. A short exact sequence is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \twoheadrightarrow 0$$

This means that f is injective, g is surjective, and $\ker g = \operatorname{im} f$.

Definition 2.13 (Homology theory)**.** A **homology theory** H is a sequence of functors

$$H_n\colon \mathbf{Top}^2 \to \mathbf{Ab}$$

for all $n \in \mathbb{Z}$ together with a natural transformation

$$H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) := H_{n-1}(A,\varnothing)$$

called the **connecting** homomorphisms) such that

3) (**Homotopy invariance**) If $f,g\colon (X,A) \to (X,B)$ are maps of pairs and $f \simeq g$, then $H_n(f) = H_n(g)$.

3) (Long exact sequence of pairs). $\forall (X, A) \in \mathbf{Top}^2$, we have a long exact sequence

$$\dots H_n(A) \to H_n(X) \to H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \dots$$

(i.e. the image of one map is the kernel of the next).

3) (Excision) Let $A \subseteq B \subseteq X$ be subspaces such that $\overline{A} \subseteq \mathring{B}$. Then the inclusion

$$(X \backslash A, B \backslash A) \to (X, B)$$

induces isomorphisms on the homology, i.e.

$$\forall n \in \mathbb{Z}\colon H_n(X \backslash A, B \backslash A) \xrightarrow{\simeq} H_n(X, B)$$

Remark† 2.13.1.
- The maps $H_n(A) \to H_n(X)$ and $H_n(X) \to H_n(X, A)$ are induced by the corresponding inclusions of spaces.
- The long exact sequence of pairs might be infinite in both directions

Definition† 2.13.2 (Further axioms)**.** Consider a homology theory H_n. We define further properties as follows:

4) (Dimension axiom) We have

$$H_n(\{\star\}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

5) (Disjoint union). Let $\{X_i\}_{i \in I}$ be a family of spaces. Then the inclusions $j_i : X_i \hookrightarrow \coprod_I X_i$ induce an isomorphism

$$\bigoplus_{i \in I} H_n(X_i) \to H_n\left(\coprod_{i \in I} X_i\right)$$

for all $n \in \mathbb{Z}$.

Oral remark 2.13.3. This quite abstract machinery is not totally unmotivated: Actually, the singular homology of a space is an example of a homology theory and this is also why we denote a general homology theory by H_n as well.

Theorem 2.14 (Singular homology theory is a homology theory)**.** Singular homology, with $H_n = 0$ for $n < 0$ is a homology theory that satisfies 4) and 5).

Proof. Later. □

Theorem 2.15. Let H be a homology theory satisfying the dimension axiom. Then

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ \mathbb{Z}^2 & n = k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.16 (Reduced homology)**.** The **reduced homology** $\tilde{H}_n(X)$ of X (for $X \neq \varnothing$) is the kernel of $H_n(X) \to H_n(\{\star\})$.

Lemma 2.17. For all $X \neq \varnothing$, we have

1) $H_n(X) \cong \tilde{H}_n(X) \oplus H_n([\star])$
2) $\forall x \in X$, $\tilde{H}_n(X) \cong H_n(X, \{\star\})$

Proof. exercise. □

Proof of Theorem 2.15 . To show

$$\tilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z} & n = k \\ 0 & \text{otherwise} \end{cases}$$

we use induction. For $n = 0$, we get that

$$\tilde{H}_n(S^0) \overset{\textit{Lemma 2.17}}{\cong} H_n(\{-1, 1\}, \{1\}) \overset{\text{excision}}{\cong} H_n(\{-1\}) \overset{\text{dim. axiom}}{\cong} \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

For the induction step, cover S^n by $U := S^n \backslash \{N\}$ and $V := S^n \backslash \{S\}$. Then $U \simeq \star$ and

$$\begin{aligned} \tilde{H}_k(S^n) &\overset{\textit{Lemma 2.17}}{\cong} H_k(S^n, \{S\}) \\ &\overset{\text{Homotopy invariance}}{\cong} H_k(S^n, U) H_k(S^n \backslash \{S\}, U \backslash \{S\}) \end{aligned}$$

Pari seg and homotopy invariance now give us a long exact sequence

$$\ldots \to H_k(\star) \to \tilde{H}_k(S^n) \to H_{k-1}(S^{n-1}) \to H_{k-1}(\star) \to \ldots$$

Now $H_k(S^{n-1}) \to H_k(\star)$ is surjetive for all k and we get short exact sequences

$$0 \to \tilde{H}_k(S^n) \to H_{k-1}(S^{n-1}) \to H_{k-1}(\star) \to 0$$

It follows that

$$\tilde{H}_k(S^n) \cong \tilde{H}_{k-1}\left(S^{n-1}\right) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise} \end{cases}$$

□

Part I

Appendix

A Exercise sheets

1. Exercise Sheet

Exercise 1.1. *Let* $\mathbf{Top}_*$ *denote the category of pointed spaces and based continuous maps, and write* $\mathbf{Top}$ *for the category of (unpointed) topological spaces and continuous maps.*

1. *(*$1+1+1$ *points) We have a 'forgetful' functor* $U\colon \mathbf{Top}_* \to \mathbf{Top}$ *that sends a pointed space* $(X,*)$ *to the underlying space* X *and a based map* $f\colon (X,*) \to (Y,*)$ *to the underlying map* $f\colon X \to Y$*. Is* U *faithful? Is* U *full? Is* U *essentially surjective?*

Let I *be a set and let* $(X_i)_{i\in I}$ *be a family of objects in some category* $\mathscr{C}$*. A* product *of* $(X_i)_{i\in I}$ *consists of an object* X *together with a map* $p_i\colon X \to X_i$ *for each* $i \in I$ *such that these data have the following 'universal property': if* Y *is any other object of* $\mathscr{C}$ *together with maps* $f_i\colon Y \to X_i$ *for all* $i \in I$*, then there exists a* unique *map* $f\colon Y \to X$ *such that* $p_i \circ f = f_i$ *for all* $i \in I$*, i.e. for each* $i \in I$ *the following diagram commutes:*

$$\begin{tikzcd} & X \arrow[d, "p_i"] \\ Y \arrow[ur, bend left, "f"] \arrow[r, "f_i"'] & X_i. \end{tikzcd}$$

If such a product exists, then we will say that the family $(X_i)_{i\in I}$ *'admits a product.'*

2. *(0 points) If you haven't seen this before, convince yourself that products are 'unique up to unique isomorphism' (if they exist). More precisely: if we have a product of* $(X_i)_{i\in I}$ *given by an object* X *together with maps* $p_i\colon X \to X_i$ *as well as a product given by an object* X' *together with maps* $p'_i\colon X' \to X_i$*, then there is a unique map* $f\colon X \to X'$ *with* $p'_i \circ f = p_i$ *for all* $i \in I$*, and this map is an isomorphism. We will therefore often simply say 'the product' instead of 'a product' of* $(X_i)_{i\in I}$ *and denote any fixed choice of a product by* $\prod_{i\in I} X_i$*.*

3. *(*$1+1$ *points) Prove that every family of objects in* $\mathbf{Top}$ *or* $\mathbf{Top}_*$ *admits a product by explicitly constructing one.*

4. *(*$1+2$ *points) Let* $(X_i)_{i\in I}$ *be a family of path-connected spaces. Show that* $\prod_{i\in I} X_i$ *is path-connected. Are products of connected spaces always connected?*

5. *(*$0+1+1$ *points) Coproducts are defined dually to products, i.e. they are products in the opposite category* $\mathscr{C}^{\mathrm{op}}$*. Spell out the definition of a coproduct, and construct coproducts in* $\mathbf{Top}$ *and* $\mathbf{Top}_*$*.*

Exercise 1.2. 1. *(*$2+2$ *points) Let* X, Y *be topological spaces, and let* $A \subseteq X, B \subseteq Y$ *be closed subsets. Show that the boundary* $\partial(A\times B)$ *inside* $X \times Y$ *agrees with the union* $(\partial A \times B) \cup (A \times \partial B)$*. Use this to show that* S^3 *is homeomorphic to* *the space $Z = T_1 \cup_{\mathrm{id}} T_2$ obtained by gluing the two solid tori $T_1 = D^2 \times S^1, T_2 = S^1 \times D^2$ (note that different order of the factors!) along the identity of $S^1 \times S^1$.*

2. *(1 point) Let $L \subseteq Z$ be the union of the 'center curves' $\{0\} \times S^1 \subseteq T_1$ and $S^1 \times \{0\} \subseteq T_2$, and set $U := \{p, q\} \times S^1 \subseteq T_1$ for some distinct points $p, q \in (D^2)^\circ$. Draw* schematic *pictures of the images of L and U under the above homeomorphism $Z \cong S^3$.*

3. *(2+2+1 points) Compute the fundamental groups $\pi_1(Z\backslash L, *)$ and $\pi_1(Z\backslash U, *)$ for your favourite choice of basepoints, and in particular give explicit sets of generators for these groups (e.g. by drawing representatives into your picture from the previous subtask). Are $Z\backslash L$ and $Z\backslash U$ homeomorphic?*

Exercise 1.3. *For any group G, we write $[G, G]$ for the* commutator *of G, i.e. the subgroup generated by all elements of the form $ghg^{-1}h^{-1}$ with $g, h \in G$.*

1. *(0 points) If you haven't seen this in another lecture before, convince yourself that $[G, G]$ is a normal subgroup and that the quotient map $p\colon G \to G/[G, G] =: G^{ab}$ has the following universal property: G^{ab} is abelian, and for every homomorphism $f\colon G \to A$ to an abelian group A there exists a unique homomorphism $\bar{f}\colon G^{ab} \to A$ making the following diagram commute:*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] & A. \\ G^{ab} \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

We call $p\colon G \to G^{ab}$ (and, although somewhat imprecise, also just the group G^{ab} itself) the abelianization *of G. You can convince yourself that the above universal property characterizes $p\colon G \to G^{ab}$ up to unique isomorphism.*

2. *(2 points) Let $n > 0$ and write $\mathfrak{F}_n$ for the free group on n letters. Show that the homomorphism $\mathfrak{F}_n \to \mathbb{Z}^n$ that sends the k-th standard generator to the k-th standard generator for $k = 1, \dots, n$ factors through an isomorphism $(\mathfrak{F}_n)^{ab} \cong \mathbb{Z}^n$.*

   Hint. *You can do this by just using the universal property of abelianization.*

3. *(4 points) Let $X = S^1 \vee S^1$ be the 'figure eight,' i.e. the space obtained by gluing two copies of the unit circle along their basepoint $*$, and recall that the two inclusions of S^1 induce an isomorphism $\mathfrak{F}_2 \cong \pi_1(X, *)$. Give an explicit description of 'the' cover $p\colon X' \to X$ whose characteristic subgroup $\mathrm{im}\big(\pi_1(p)\colon \pi_1(X', *) \to \pi_1(X, *)\big)$ is the commutator $[\pi_1(X, *), \pi_1(X, *)]$.*

4. *(2 + 2 points) Conclude that $\mathfrak{F}_2$ contains copies of all finitely generated free groups, i.e. for every $n > 0$ there exists an injective homomorphism $\mathfrak{F}_n \to \mathfrak{F}_2$. Construct an explicit family of such homomorphisms.*

$*$ 5. *(3 + 7 bonus points) Denote the standard generators of $\mathfrak{F}_2$ by a and b. Prove that for every $n > 0$ the homomorphism $\mathfrak{F}_n \to \mathfrak{F}_2$ sending the k-th standard generator to $a^k b^k$ for each $k = 1, \dots, n$ is injective, but not surjective. Can you give a general procedure to decide whether a homomorphism $\mathfrak{F}_n \to \mathfrak{F}_m$, specified in terms of the images of the standard generators, is injective and/or surjective?*

Exercise 1.4. *(5 + 5 points) Compute the simplicial homology of the Klein bottle and of $\mathbb{RP}^2$ using the Δ-complex structures from Example 1.6 and Example 1.6.1.*