UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS: Part III Physics
MASTER OF ADVANCED STUDY IN PHYSICS

Monday 20th January 2025 10:00 to 12:00

MAJOR TOPICS
Paper 1/PP (Particle Physics)

*Answer **two** questions only. The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate. The paper has content on 17 sides, including this one, and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*

*You should use a **separate Answer Book** for each question.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2x20-page answer books	Mathematical Formulae Handbook
Rough workpad	Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

The information in this box may be used in any question.

The Pauli-matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Dirac representation of the gamma matrices is:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

The Dirac representation of the gamma matrices has the following properties:

$$(\gamma^0)^* = \gamma^0,\ \ (\gamma^1)^* = \gamma^1,\ \ (\gamma^2)^* = -\gamma^2,\ \ (\gamma^3)^* = \gamma^3 \ \ \textit{and}\ \ \gamma^2(\gamma^\mu)^* = -\gamma^\mu\gamma^2.$$

Using the above representation, the Part III Particles lecture course defined the following particle and anti-particle spinors:

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix}, \qquad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix},$$

$$v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix}, \qquad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}$$

for objects whose three-momentum $\vec{p}$ is given by $|\vec{p}|(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ where $c = \cos\frac{\theta}{2}$ and $s = \sin\frac{\theta}{2}$. The normalising constant is $N = \sqrt{E+m}$.

$\hbar \approx 1.05 \times 10^{-34}$ kg m^2/s, $\qquad c \approx 3.00 \times 10^8$ m/s, $\qquad e \approx 1.60 \times 10^{-19}$ C.

$m_e = 5.11 \times 10^{-4}$ GeV. $\qquad m_p = m_n = 1.67 \times 10^{-27}$ kg.

1 In electron-proton scattering, the Lorentz invariant quantities:

$$s = (p_1 + p_2)^2\,, \quad Q^2 = -q^2 = -(p_1 - p_3)^2\,, \quad x = \frac{Q^2}{2p_2 \cdot q} \text{ and } y = \frac{p_2 \cdot q}{p_1 \cdot p_2}$$

are defined in terms of p_1 and p_2, the four-momenta of the initial-state electron and proton respectively, and p_3, the four-momentum of the scattered electron. Neglecting the Q^2 dependence of the structure functions, F_1^{ep} and F_2^{ep}, the differential cross section for electron-proton deep inelastic scattering can be written as

$$\frac{\mathrm{d}^2\sigma^{ep}}{\mathrm{d}x\,\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2^{ep}(x)}{x} + y^2 F_1^{ep}(x)\right].$$

(a) For the case where the proton is at rest, express s, Q^2, x and y in terms of the proton mass, m_p, the electron scattering angle θ in the lab frame, and the energies of the incoming and scattered electron, E_1 and E_3. [4]

BOOKWORK[These variables were all defined and evaluated in the lecture notes, so a student could simply drop in values they recall here. Alternatively, they could derive them very quickly using little more than recall of how the Lorentz-invariant dot-product is defined. The results are:

$$s = \left(\begin{pmatrix} E_1 \\ 0 \\ 0 \\ E_1 \end{pmatrix} + \begin{pmatrix} m_p \\ 0 \\ 0 \\ 0 \end{pmatrix}\right)^2 = (E_1 + m_p)^2 - E_1^2 = 2E_1 m_p + m_p^2.$$

$$Q^2 = -\left(\begin{pmatrix} E_1 \\ 0 \\ 0 \\ E_1 \end{pmatrix} - \begin{pmatrix} E_3 \\ E_3\sin\theta \\ 0 \\ E_3\cos\theta \end{pmatrix}\right)^2 = -(0^2 - 2E_1E_3(1-\cos\theta) + 0^2) = 2E_1E_3(1-\cos\theta).$$

$$x = \frac{Q^2}{2p_2 \cdot q} = \frac{2E_1E_3(1-\cos\theta)}{2\begin{pmatrix} m_p \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \left(\begin{pmatrix} E_1 \\ 0 \\ 0 \\ E_1 \end{pmatrix} - \begin{pmatrix} E_3 \\ E_3\sin\theta \\ 0 \\ E_3\cos\theta \end{pmatrix}\right)} = \frac{E_1E_3(1-\cos\theta)}{m_p(E_1 - E_3)}.$$

$$y = 1 - \frac{E_3}{E_1}.$$

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(b) In the parton model, show that x can be interpreted as the fraction of the proton's momentum carried by the struck quark in a frame where the proton has infinite momentum. Explain any assumptions made. [4]

BOOKWORK[This answer is largely bookwork as it involves re-capitulating the content of slides 187 the lecture notes which is copied below. It will be necessary here to neglect the mass of the struck parton in comparison to the energy of the proton, and to regard the struck parton having negligible transverse momentum.

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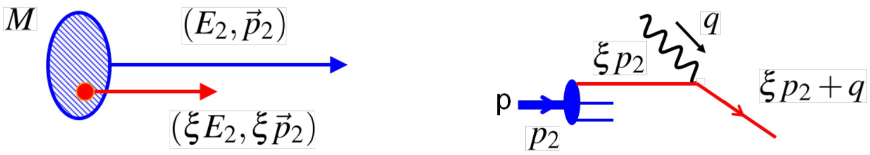
- In the parton model the basic interaction is ELASTIC scattering from a "quasi-free" spin-$\frac{1}{2}$ quark in the proton, i.e. treat the quark as a free particle!
- The parton model is most easily formulated in a frame where the proton has very high energy, often referred to as the "infinite momentum frame", where we can neglect the proton mass, and $p_2 = (E_2, 0, 0, E_2)$.
- In this frame can also neglect the mass of the quark and any momentum transverse to the direction of the proton.
- Let the quark carry a fraction ξ of the proton's four-momentum. $(\xi^2 p_2^2 = m_q^2 \approx 0)$



- After the interaction the struck quark's four-momentum is $\xi p_2 + q$ and it is still a quark, therefore: $m_q^2 = (\xi p_2 + q)^2 = (\xi p_2)^2 + 2\xi p_2 \cdot q + q^2 = m_q^2 + 2\xi p_2 \cdot q + q^2$ and so comparing the first and last expressions we see that $\xi = -q^2/(2p_2 \cdot q)$ and thus

$$\boxed{\xi = \frac{Q^2}{2p_2.q} = x}$$

... therefore Bjorken x can be identified as the fraction of the proton momentum carried by the struck quark in the infinite momentum frame.

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(c) The differential cross section for electron-quark scattering can be written as

$$\frac{d\sigma^{eq}}{dQ^2} = \frac{4\pi\alpha^2 e_q^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right]$$

where e_q is the charge of the quark. Using the parton model, including contributions from the light quarks (u, d, s) only, show that

$$\frac{F_2^{ep}(x)}{x} = \frac{4}{9}[u(x) + \bar{u}(x)] + \frac{1}{9}[d(x) + \bar{d}(x) + s(x) + \bar{s}(x)]$$

where $u(x)$, $d(x)$ and $s(x)$ are the up-, down- and strange-quark parton distribution functions for the proton. Obtain a similar expression for the electron-neutron structure function $F_2^{en}(x)$. [6]

BOOKWORK[This is very close to bookwork. The notes make much use of the Callan-Gross relation $F_2(x) = 2xF_1(x)$. With that in mind, it is a trivial exercise to

spot that the difference between the doubly differential ep scattering formula given in the rubric of the question differs from that given in this local part by the sum of the relevant pdfs, each weighted by the charge of the relevant quark squared – as expected. The expression for the neutron should be the same as that of the proton, except that it needs neutron rather than proton structure functions.]

(d) Stating clearly any assumptions made, show that

$$\int_0^1 \frac{\left[F_2^{ep}(x) - F_2^{en}(x)\right]}{x} \mathrm{d}x = \frac{1}{3} + \frac{2}{3}\int_0^1 [\bar{u}(x) - \bar{d}(x)]\mathrm{d}x$$

and comment on the consequences of the observed value being 0.24 ± 0.03. [6]

$$\begin{aligned}
\int_0^1 \frac{\left[F_2^{ep} - F_2^{en}\right]}{x}\mathrm{d}x &= \int_0^1 \frac{4}{9}[u(x) + \bar{u}(x)] + \frac{1}{9}[d(x) + \bar{d}(x) + s(x) + \bar{s}(x)]dx \\
&- \int_0^1 \frac{4}{9}[u^n(x) + \bar{u}^n(x)] + \frac{1}{9}\left[d^n(x) + \bar{d}^n(x) + s(x) + \bar{s}(x)\right]dx \\
&= \int_0^1 \frac{4}{9}[u(x) + \bar{u}(x)] + \frac{1}{9}[d(x) + \bar{d}(x))\Big]dx \\
&- \int_0^1 \frac{4}{9}[u^n(x) + \bar{u}^n(x)] + \frac{1}{9}\left[d^n(x) + \bar{d}^n(x)\right]dx \\
&= \int_0^1 \frac{4}{9}[u(x)_v + 2\bar{u}(x)] + \frac{1}{9}\left[d_v(x) + 2\bar{d}(x))\right]dx \\
&- \int_0^1 \frac{4}{9}[u_v^n(x) + 2\bar{u}^n(x)] + \frac{1}{9}\left[d_v^n(x) + 2\bar{d}^n(x)\right]dx \\
&= \int_0^1 \frac{4}{9}[u(x)_v + 2\bar{u}(x)] + \frac{1}{9}\left[d_v(x) + 2\bar{d}(x))\right]dx \\
&- \int_0^1 \frac{4}{9}\left[d_v(x) + 2\bar{d}(x)\right] + \frac{1}{9}[u_v(x) + 2\bar{u}(x)]dx \\
&= \int_0^1 \frac{1}{3}[u(x)_v - d_v(x)]\mathrm{d}x + \int_0^1 \frac{2}{3}[\bar{u}(x) - \bar{d}(x)]\mathrm{d}x \\
&= \frac{1}{3}(2-1) + \frac{2}{3}\int_0^1 [\bar{u}(x) - \bar{d}(x)]\mathrm{d}x
\end{aligned}$$

wherein the first step we have assumed that $s(s)$ and $\bar{s}(x)$ are identical for neutron and proton, and wherein the third step we have split the quark PDFs into valence and sea parts $u(x) = u_v(x) + u_s(x), d(x) = d_v(x) + d_s(x)$ and simultanesouly assumed that the sea u distribution is the same as the $\bar{u}$ disribution etc, i.e. have replaced $u_s(x)$ with $\bar{u}(x)$ etc. In the fourth step we assumed isospin symmetry $u \leftrightarrow d^n, d \leftrightarrow u^n$ (both for sea and for valence quarks). In the sixth step we assumed two valence up quarks in the proton $\left(\int_0^1 u_v(x)\mathrm{d} = 2\right)$ and one valence down quark in the proton $\left(\int_0^1 d_v(x)\mathrm{d} = 1\right)$.

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Since the measured value is significantly less than $1/3$, and since the prediction is $1/3$ plus an integral of a difference between the number of sea anti-ups and sea anti-downs, we can interpret this result as telling us that there are more anti-downs in the proton than anti-ups, at least on average (i.e. when integrated over x). This is likely to mean the same thing for the sea-component of the ups and downs too, since they should be created most frequently by gluon splitting.

(e) In the parton model for neutrino-nucleon scattering the structure functions are

$$F_2^{\nu p}(x) = 2x[d(x) + s(x) + \bar{u}(x)] \quad \text{and} \quad F_2^{\nu n}(x) = 2x[u(x) + \bar{s}(x) + \bar{d}(x)].$$

Assuming $s(x) = \bar{s}(x)$, obtain an expression for $xs(x)$ in terms of the structure functions for neutrino- and electron-nucleon scattering,
$F_2^{\nu N}(x) = \frac{1}{2}\left(F_2^{\nu p}(x) + F_2^{\nu n}(x)\right)$ and $F_2^{eN}(x) = \frac{1}{2}\left(F_2^{ep}(x) + F_2^{en}(x)\right)$. [7]

$$\begin{aligned}
F_2^{\nu N}(x) &= \frac{1}{2}\left(F_2^{\nu p}(x) + F_2^{\nu n}(x)\right) \\
&= \frac{1}{2}(2x[d(x) + s(x) + \bar{u}(x)] + 2x[u(x) + \bar{s}(x) + \bar{d}(x)]) \\
&= x[d(x) + s(x) + \bar{u}(x) + u(x) + \bar{s}(x) + \bar{d}(x)] \\
&= x[u(x) + \bar{u}(x) + d(x) + \bar{d}(x)] + 2xs(x). \qquad \text{(assuming } s(x) = \bar{s}(x))
\end{aligned}$$

$$\begin{aligned}
F_2^{eN}(x) &= \frac{1}{2}\left(F_2^{ep}(x) + F_2^{en}(x)\right) \\
&= \frac{4}{18}x[u(x) + \bar{u}(x)] + \frac{1}{18}x[d(x) + \bar{d}(x) + s(x) + \bar{s}(x)] \\
&+ \frac{4}{18}x\left[u^n(x) + \bar{u}^n(x)\right] + \frac{1}{18}x\left[d^n(x) + \bar{d}^n(x) + s(x) + \bar{s}(x)\right] \\
&= \frac{4}{18}x[u(x) + \bar{u}(x)] + \frac{1}{18}x[d(x) + \bar{d}(x) + 2s(x)] \\
&+ \frac{4}{18}x[d(x) + \bar{d}(x)] + \frac{1}{18}x[u(x) + \bar{u}(x) + 2s(x)] \\
&= \frac{5}{18}x[u(x) + \bar{u}(x) + d(x) + \bar{d}(x)] + \frac{2}{9}xs(x)
\end{aligned}$$

where again we assumed isospin symmetry $d^n \sim u$, $u^n \sim d$. Putting these two results together:

$$\frac{5}{18}F_2^{\nu N}(x) - F_2^{eN}(x) = \left(\frac{5}{9} - \frac{2}{9}\right)xs(x) = \frac{1}{3}xs(x)$$

and so

$$\begin{aligned}
xs(x) &= 3\left(\frac{5}{18}F_2^{\nu N}(x) - F_2^{eN}(x)\right) \\
&= \frac{5}{6}F_2^{\nu N}(x) - 3F_2^{eN}(x).
\end{aligned}$$

(f) Provide possible physical explanations for why $\bar{d}(x) > \bar{u}(x) > \bar{s}(x)$. [3]

The strange quark has by most methods of determination a substantially higher mass than the up or down, and so should be considerably kinematically suppressed in the sea with respect to the other two. It is harder to say why the sea contribution of the d should be higher than that of the u. One cannot rely again on mass since the up and down are very similar in mass and almost massless relative to the proton (having 0.2% and 0.5% of the proton's mass), unlike the strange quark (which has about 10% of the proton mass, a significant fraction - given that only half of the momentum of a proton is typically carried by quarks!). In the lecture course the supplied plot of proton parton distribution functions (with $Q^2 = 10\ \mathrm{GeV}^2$) indicated the sea d-quark contribution was a 20 to 30% higher than that of the u, noting that this effect was 'not understood – exclusion principle?'. This would be acceptable as an answer if backed up by some additional explanation. E.g. "Whenever a sea $\bar{u}$ is created by a gluon, there must be an associated u. This u would then find itself competing against the valence up quarks in the proton for phase space. Given that there are already two valence up quarks in the proton but only one valence down quark, sea up quarks (being fermions and therefore being subject to Fermi's exclusion principle) might find it harder than down quarks to find states that are not already occupied by other similar quarks. This might disfavour $\bar{u}(x)$ with respect to $\bar{d}(x)$."

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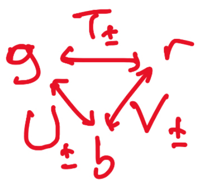
2 This question concerns the colour wave functions of hadrons (baryons, mesons, tetraquarks, pentaquarks, *etc*.). Within this question the basis colour states r, g and b are sometimes written as c_1, c_2 and c_3, respectively, to allow them to be used in indexed sums. Similarly $\bar{c}_1$, $\bar{c}_2$, and $\bar{c}_3$ refer to $\bar{r}$, $\bar{g}$ and $\bar{b}$, respectively.

(a) Write down the action of the SU(3)-colour ladder operators $T^c_\pm$, $U^c_\pm$ and $V^c_\pm$:

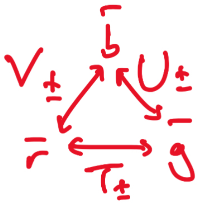
(i) on each of the colour basis states r, g and b; and

(ii) on each of the anti-colour basis states $\bar{r}$, $\bar{g}$ and $\bar{b}$. [2]

BOOKWORK[



$T_+ g = r$
$T_- r = g$
$V_+ b = r$
$V_- r = b$
$U_+ b = g$
$U_- g = b$
others zero!



$T_+ \bar{r} = -\bar{g}$
$T_- \bar{g} = -\bar{r}$
$V_+ \bar{r} = -\bar{b}$
$V_- \bar{b} = -\bar{r}$
$U_+ \bar{g} = -\bar{b}$
$U_- \bar{b} = -\bar{g}$
others zero!

]

(b) What is the defining property of a singlet state S under a general SU(n) symmetry? [Note that this question specifies general SU(n), not SU(3)-colour!] [1]

BOOKWORK[The defining property of S is that it is annihilated by the action of any of the ladder operators of the symmetry in question.]

(c) What special role do colour-singlet states have within QCD? [1]

BOOKWORK[The "Colour Confinement Hypothesis" (lectured) says that the colour wavefunction of any hadron must be a colour singlet.]

(d) Which of the following states A, B, C and D are colour singlet states? (Justify your answers!) [2]

$$A = rgb.$$

$$B = \frac{1}{\sqrt{6}}(rgb - rbg + brg - bgr + gbr - grb)\,.$$

$$C = \frac{1}{\sqrt{3}}\left(r\bar{r} + g\bar{g} + b\bar{b}\right).$$

$$D = \frac{1}{\sqrt{3}}\left(r\bar{r}r\bar{r} + g\bar{g}g\bar{g} + b\bar{b}b\bar{b}\right).$$

Part bookwork part unseen.

The bookwork part is that B and C should be recognised as the colour singlet wavefunctions of (respectively) the baryons and the mesons. I would accept such a statement as a valid justification, although candidates could also demonstrate that B and/or C are both annihilated by (w.l.o.g.) T_+^c.

The non-bookwork part is demonstrating that neither A nor D are singlet states. To justify this it is necessary to (w.l.o.g.) demonstrate that neither is annihilated by T_+^c:

$$T_+^c(rgb) = 0gb + rrb + rg0 = rrb \neq 0.$$

$$T_+^c(r\bar{r}r\bar{r} + g\bar{g}g\bar{g} + b\bar{b}b\bar{b}) =$$
$$= -r\bar{g}r\bar{r} - r\bar{r}r\bar{g} + r\bar{g}g\bar{g} + g\bar{g}r\bar{g} \neq 0.$$

It may be shown that if S_1 and S_2 are both colour singlet states, then all of the following states X_1, X_2, X_3 and X_4 are also colour singlet states (when non-zero), albeit not-necessarily normalised.

$X_1 = S_1 S_2$.
$X_2 = \lambda_1 S_1 + \lambda_2 S_2$, for any constants λ_1 and λ_2.
$X_3 =$ [S_1 symmetrized (or antisymmetrized) over any two colour states].
$X_4 =$ [S_1 symmetrized (or antisymmetrized) over any two anti-colour states].

[*In the above, it is intended that the words 'symmetrization over first and third colour states' would describe the linear transformation which would take a term like $c_1c_2c_3$ (within some singlet wave function) into the new expression $c_1c_2c_3 + c_3c_2c_1$. Similarly 'antisymmetrization over anti-colour states' would take a term like $c_1\bar{c}_2c_3\bar{c}_4$ (within some singlet wave function) into the new expression $c_1\bar{c}_2c_3\bar{c}_4 - c_1\bar{c}_4c_3\bar{c}_2$, and so on.*]

(e) Prove **only one** of the last four results just given – i.e. prove either that X_1 is a singlet, or that X_2 is a singlet, or that X_3 is a singlet, or that X_4 is a singlet. [1]

Each candidate needs to prove **only one** of the starred statements below. (In all cases we know that $T_+^c(S_1) = T_+^c(S_2) = 0$.)

(*) A candidate who wished to prove that X_1 is a singlet could use this argument:

$$T_+^c X_1 = T_+^c(S_1S_2) = S_1(T_+^c S_2) + (T_+^c S_1)S_2 = S_1 \cdot 0 + 0 \cdot S_2 = 0.$$

(*) A candidate who wished instead to prove that X_2 is a singlet could (w.l.o.g.) use the linearity of T_+^c in this argument:

$$T_+^c X_2 = T_+^c(\lambda_1 S_1 + \lambda_2 S_2) = \lambda_1 T_+^c(S_1) + \lambda_2 T_+^c(S_2) = \lambda_1 \cdot 0 + \lambda_1 \cdot 0 = 0.$$

(*) A candidate wanting to prove that either X_3 or X_4 was a singlet would be best off answering as follows. Note that this argument clearly longer than the other two just given, so the smart choice might be to choose to prove that X_1 or X_2 is a singlet rather than to work with X_3 or X_4. All are presented, though, as the examiner wishes all candidates to see and think a little about why each might be true as they will all be needed in the later parts of the question.

For any ab (singlet or not) we may trivially write.

$$ab \equiv \tfrac{1}{2}(ab+ba) + \tfrac{1}{2}(ab-ba).$$

So if additionally ab is a singlet then (w.l.o.g)

$$T_+(ab) = 0 \Rightarrow T_+(2ab) = 0$$
$$\Rightarrow T_+(ab+ba) + T_+(ab-ba) = 0$$
$$\Rightarrow T_+(ab+ba) = T_+(ba-ab). \quad (*)$$

But $\left\{\begin{array}{l}\text{LHS } (*) \text{ is even under } a \leftrightarrow b \\ \text{RHS } (*) \text{ is odd under } a \leftrightarrow b\end{array}\right\}$ $\therefore$ Both sides are zero.

$$\therefore T_+(ab+ba) = 0 = T_+(ab-ba)$$

$\therefore$ $ab+ba$ and $ab-ba$ are singlets as both are annihilated by (w.l.o.g) T_+^c.

The rest of this question will focus on colour singlet wave functions that could potentially belong to tetraquark (i.e. $q\bar{q}q\bar{q}$) states. You may assume that these will be linear combinations of terms of the form $c_i\bar{c}_jc_k\bar{c}_l$ for various values i, j, k and l. [We will call these 'four-colour' singlet wave functions even though they contain two colours and two anti-colours.]

(f) Using the results from earlier in this question (or otherwise) write down two linearly-independent four-colour singlet wave functions F_S and F_A. Choose F_S such that it is **symmetric** under exchange of the colour states (positions 1 and 3 in the wave function) while also being symmetric with respect to exchange of the anti-colour states (position 2 and 4) in the wave function. Similarly, choose F_A to be **antisymmetric** with respect to those same interchanges.

[*When answering, avoid of multiplying out long expressions unless it helps you. Of the $3^4 = 81$ colour combinations $c_i\bar{c}_jc_k\bar{c}_l$ there are fifteen which (like $r\bar{g}g\bar{r}$) have $I_3^c = Y^c = 0$, and so you will want to avoid manipulating explicit linear combinations of such terms wherever possible! If you define suitable notation you should mostly be able to avoid multiplying out or manipulating long expressions.*]

[10]

The answers for (f) and (g) are written together at the end of (g).

(g) If your states F_S and F_A were unit normalised (like states A, B, C and D of part (d) above) and were written in the forms:

$$F_S = \frac{1}{\sqrt{24}} \sum_{i,j,k,l=1}^{3} \lambda_{i,j,k,l} \cdot c_i\bar{c}_jc_k\bar{c}_l, \quad \text{and} \quad F_A = \frac{1}{\sqrt{12}} \sum_{i,j,k,l=1}^{3} \mu_{i,j,k,l} \cdot c_i\bar{c}_jc_k\bar{c}_l$$

in terms of 81 indexed constant coefficients $\lambda_{i,j,k,l}$ and 81 indexed constant coefficients $\mu_{i,j,k,l}$, then state the values you would need for:

(i) $(\lambda_{1,1,1,1}, \lambda_{1,1,2,2}, \lambda_{1,2,2,1}, \lambda_{1,2,3,1})$ and

(ii) $(\mu_{1,1,1,1}, \mu_{1,1,2,2}, \mu_{1,2,2,1}, \mu_{1,2,3,1})$.

[4]

UNSEEN (not bookwork):

The examiner's answer prior to the paper being sat was as follows:

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We know from earlier that $C \propto r\bar{r} + g\bar{g} + b\bar{b}$ is a singlet, and so by the X_1 result CC is a singlet too.

Note $CC \propto r\bar{r}r\bar{r} + r\bar{r}g\bar{g} + \ldots\ldots$

Also, by the X_3 and X_4 results: $F_S \propto (CC)\big|_{\substack{S(1\leftrightarrow 3)\\ S(2\leftrightarrow 4)}}$ and $F_A \propto (CC)\big|_{\substack{A(1\leftrightarrow 3)\\ A(2\leftrightarrow 4)}}$ are singlets too, where

$S(m\leftrightarrow n)$ means Symmetrising in positions m and n,
and
$A(m\leftrightarrow n)$ means Anti Symmetrising in positions m and n.

i.e.
$$F_S \propto (r\bar{r}r\bar{r})\big|_{\substack{S(1\leftrightarrow 3)\\ S(2\leftrightarrow 4)}} + (r\bar{r}g\bar{g})\big|_{\substack{S(1\leftrightarrow 3)\\ S(2\leftrightarrow 4)}} + \ldots\ldots$$
$$\begin{aligned} &= r\bar{r}r\bar{r} + r\bar{r}r\bar{r} + r\bar{r}r\bar{r} + r\bar{r}r\bar{r} \\ &\quad + r\bar{r}g\bar{g} + g\bar{r}r\bar{g} + r\bar{g}g\bar{r} + g\bar{g}r\bar{r} \\ &\quad + \ldots\ldots \\ &= 4(r\bar{r}r\bar{r} + \text{cyclic perms}) \quad \begin{matrix} r\to g \\ g\to b \\ b\to r \end{matrix} \\ &\quad + 2((r\bar{r}g\bar{g} + g\bar{g}r\bar{r}) + \text{cyclic perms}) \\ &\quad + 2((r\bar{g}g\bar{r} + g\bar{r}r\bar{g}) + \text{cyclic perms}) \end{aligned}$$

$$\begin{aligned} \therefore F_S \propto f_S = {} & 2(r\bar{r}r\bar{r} + \text{cyclic perms}) \leftarrow \text{3 terms in bracket} \\ & + ((r\bar{r}g\bar{g} + g\bar{g}r\bar{r}) + \text{cyclic perms}) \leftarrow \text{6 terms in bracket} \\ & + ((r\bar{g}g\bar{r} + g\bar{r}r\bar{g}) + \text{cyclic perms}) \leftarrow \text{6 terms in bracket} \end{aligned}$$

$$\therefore \langle f_S | f_S \rangle = 3(2^2) + 6(1^2) + 6(1^2) = 24$$

$\therefore$ normalised F_S is

$$F_S = \frac{1}{\sqrt{24}}\left[2(r\bar{r}r\bar{r} + g\bar{g}g\bar{g} + b\bar{b}b\bar{b}) \begin{matrix} +(r\bar{r}g\bar{g} + g\bar{g}r\bar{r} + g\bar{g}b\bar{b} + b\bar{b}g\bar{g} + b\bar{b}r\bar{r} + r\bar{r}b\bar{b}) \\ +(r\bar{g}g\bar{r} + g\bar{r}r\bar{g} + g\bar{b}b\bar{g} + b\bar{g}g\bar{b} + b\bar{r}r\bar{b} + r\bar{b}b\bar{r}) \end{matrix} \right]$$

$\underset{(r\bar{r}r\bar{r})}{\lambda_{1111} = 2}$, $\underset{(r\bar{r}g\bar{g})}{\lambda_{1122} = 1}$, $\underset{(r\bar{g}g\bar{r})}{\lambda_{1221} = 1}$, $\underset{(r\bar{g}b\bar{r})}{\lambda_{1231} = 0}$.

and
$$F_A \propto (r\bar{r}r\bar{r})\big|_{\substack{A(1\leftrightarrow 3)\\ A(2\leftrightarrow 4)}} + (r\bar{r}g\bar{g})\big|_{\substack{A(1\leftrightarrow 3)\\ A(2\leftrightarrow 4)}} + \ldots\ldots$$
$$\begin{aligned} &= r\bar{r}r\bar{r} - r\bar{r}r\bar{r} - r\bar{r}r\bar{r} + r\bar{r}r\bar{r} \\ &\quad + r\bar{r}g\bar{g} - g\bar{r}r\bar{g} - r\bar{g}g\bar{r} + g\bar{g}r\bar{r} \\ &\quad + \ldots\ldots \\ &= 2((r\bar{r}g\bar{g} + g\bar{g}r\bar{r}) + \text{cyclic perms}) \quad \begin{matrix} r\to g \\ g\to b \\ b\to r \end{matrix} \\ &\quad - 2((r\bar{g}g\bar{r} + g\bar{r}r\bar{g}) + \text{cyclic perms}) \end{aligned}$$

$$\begin{aligned} \therefore F_A \propto f_a = {} & ((r\bar{r}g\bar{g} + g\bar{g}r\bar{r}) + \text{cyclic perms}) \leftarrow \text{6 terms in bracket} \\ & - ((r\bar{g}g\bar{r} + g\bar{r}r\bar{g}) + \text{cyclic perms}) \leftarrow \text{6 terms in bracket} \end{aligned}$$

$$\therefore \langle f_a | f_a \rangle = 6(1^2) + 6(1^2) = 12$$

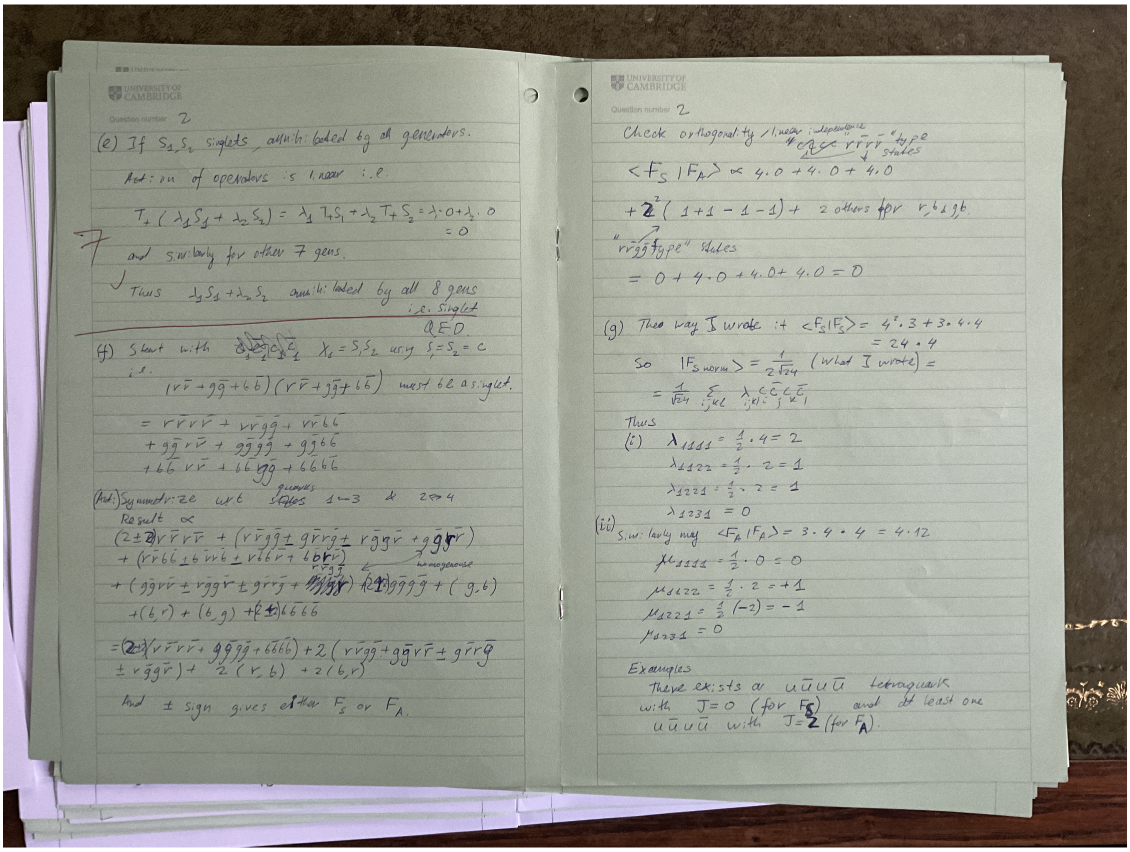
$\therefore$ normalised F_A is

$$F_A = \frac{1}{\sqrt{12}}\left[\begin{matrix} +(r\bar{r}g\bar{g} + g\bar{g}r\bar{r} + g\bar{g}b\bar{b} + b\bar{b}g\bar{g} + b\bar{b}r\bar{r} + r\bar{r}b\bar{b}) \\ -(r\bar{g}g\bar{r} + g\bar{r}r\bar{g} + g\bar{b}b\bar{g} + b\bar{g}g\bar{b} + b\bar{r}r\bar{b} + r\bar{b}b\bar{r}) \end{matrix} \right]$$

$\underset{(r\bar{r}r\bar{r})}{\mu_{1111} = 0}$, $\underset{(r\bar{r}g\bar{g})}{\mu_{1122} = 1}$, $\underset{(r\bar{g}g\bar{r})}{\mu_{1221} = -1}$,

$\underset{(r\bar{g}b\bar{r})}{\mu_{1231} = 0}$.

Here is a similar answer which was offered by one of the candidates in the actual examination. This answer is reproduced here as it cleverly does both F_S and F_A in one calculation.



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Question number 2

(e) If S_1, S_2 singlets, annihilated by all generators.

Action of operators is linear i.e.

$T_+(\lambda_1 S_1 + \lambda_2 S_2) = \lambda_1 T_+ S_1 + \lambda_2 T_+ S_2 = \lambda_1 \cdot 0 + \lambda_2 \cdot 0 = 0$

and similarly for other 7 gens.

Thus $\lambda_1 S_1 + \lambda_2 S_2$ annihilated by all 8 gens i.e. singlet

QED.

7

(f) Start with $\chi_1 = S_1 S_2$ using $S_1 = S_2 = c$

i.e.

$(r\bar{r} + g\bar{g} + b\bar{b})(r\bar{r} + g\bar{g} + b\bar{b})$ must be a singlet.

$= r\bar{r}r\bar{r} + r\bar{r}g\bar{g} + r\bar{r}b\bar{b}$
$+ g\bar{g}r\bar{r} + g\bar{g}g\bar{g} + g\bar{g}b\bar{b}$
$+ b\bar{b}r\bar{r} + b\bar{b}g\bar{g} + b\bar{b}b\bar{b}$

(Anti:)Symmetrize wrt quarks 1 ↔ 3 & 2 ↔ 4

Result ∝

$(2\pm 2)r\bar{r}r\bar{r} + (r\bar{r}g\bar{g} \pm g\bar{r}r\bar{g} \pm r\bar{g}g\bar{r} + g\bar{g}r\bar{r})$
$+ (r\bar{r}b\bar{b} \pm b\bar{r}r\bar{b} \pm r\bar{b}b\bar{r} + b\bar{b}r\bar{r})$ homogenouse
$+ (g\bar{g}r\bar{r} \pm r\bar{g}g\bar{r} \pm g\bar{r}r\bar{g} + r\bar{r}g\bar{g}) + (2\pm 2)g\bar{g}g\bar{g} + (g, b)$
$+ (b, r) + (b, g) + (2\pm 2)b\bar{b}b\bar{b}$

$= (2\pm 2)(r\bar{r}r\bar{r} + g\bar{g}g\bar{g} + b\bar{b}b\bar{b}) + 2(r\bar{r}g\bar{g} + g\bar{g}r\bar{r} \pm g\bar{r}r\bar{g}$
$\pm r\bar{g}g\bar{r}) + 2(r, b) + 2(b, r)$

And ± sign gives either F_S or F_A.

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Question number 2

Check orthogonality / linear independence

4 "$r\bar{r}r\bar{r}$" type states

$\langle F_S | F_A \rangle \propto 4 \cdot 0 + 4 \cdot 0 + 4 \cdot 0$

$+ 2^2(1 + 1 - 1 - 1) +$ 2 others for r, b & g, b.

"$r\bar{r}g\bar{g}$ type" states

$= 0 + 4 \cdot 0 + 4 \cdot 0 + 4 \cdot 0 = 0$

(g) The way I wrote it $\langle F_S | F_S \rangle = 4^2 \cdot 3 + 3 \cdot 4 \cdot 4$
$= 24 \cdot 4$

So $|F_{S\,norm}\rangle = \frac{1}{2\sqrt{24}}$ (What I wrote) $=$

$= \frac{1}{\sqrt{24}} \sum_{ijkl} \lambda_{ijkl} c_i \bar{c}_j c_k \bar{c}_l$

Thus

(i) $\lambda_{1111} = \frac{1}{2} \cdot 4 = 2$

$\lambda_{1122} = \frac{1}{2} \cdot 2 = 1$

$\lambda_{1221} = \frac{1}{2} \cdot 2 = 1$

$\lambda_{1231} = 0$

(ii) Similarly may $\langle F_A | F_A \rangle = 3 \cdot 4 \cdot 4 = 4 \cdot 12$

$\mu_{1111} = \frac{1}{2} \cdot 0 = 0$

$\mu_{1122} = \frac{1}{2} \cdot 2 = +1$

$\mu_{1221} = \frac{1}{2}(-2) = -1$

$\mu_{1231} = 0$

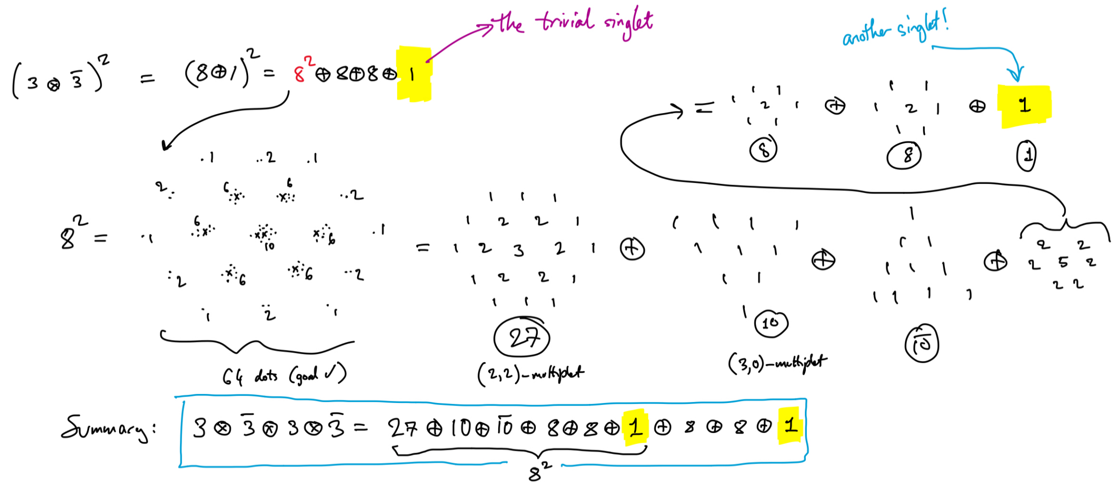
Examples

There exists a $u\bar{u}u\bar{u}$ tetraquark with $J = 0$ (for F_S) and at least one $u\bar{u}u\bar{u}$ with $J = 2$ (for F_A).

Recall that $3 \otimes \bar{3} = 8 \oplus 1$ when an SU(3) colour is combined with an anti-colour.

(h) By multiplying multiplets (or otherwise) determine how many singlets appear in the product $3 \otimes \bar{3} \otimes 3 \otimes \bar{3}$, and thereby state whether or not there exist other tetraquark colour wave functions linearly independent of those already found in part (f). [7]

This part of the question is formally UNSEEN (i.e. this specific calculation was not lectured in lectures and so it is not bookwork). However: (i) lectures did cover how to do this *sort* of calculation, and (ii) this *exact* calculation appeared as question 2b of the 2014 Tripos paper. In 2014 this question had the same number of marks as it has here (seven) and it was well answered in 2014 so I expect it to be answered this year too – or perhaps even answered better due to the foreshadowing!



The working in the diagram above shows that there are only two singlets in $3 \otimes \bar{3} \otimes 3 \otimes \bar{3}$ and so the two singlets we have found are all there is to find!

(i) State two physical consequences you might predict for the spectrum of tetraquark states given your results in this question. You need not consider more than the two lightest quarks. [2]

[ASIDE: There are only a small number of marks available here because only two physical consequences have been requested. Thus **the candidates do not have to produce the whole of the answer given here to get full marks!** On the contrary, they only need to mention two of the consequences. The answer supplied here is long and full only so that the kinds of things that any number of different candidates could say can be seen. Candidates may be able to think of valid consequences which are not given here, and they will be rewarded too.]

We have found that there are two independent colour singlets wave functions available to tetraquarks. This could **potentially** mean that there are two different 'kinds' of tetraquark for a given flavour and spin content (e.g. one perhaps heavier than the other). However whether this is actually the case or not will depend on whether we can find spin wave functions to go with the colour wave functions. We know that the Pauli exclusion principle means that we need the **total hadron wave function** should be antisymmetric under the exchange of any two identical quarks (or any two identical antiquarks). Since F_S and F_A are (respectively) symmetric and antisymmetric under those exchanges (and assuming that we have symmetric spatial wave functions) then F_S and F_A need to be paired with spin and flavour configurations that are (respectively) antisymmetric or symmetric under the two exchanges. We can think of examples of both in the two-flavour limit.

Regarding F_S:

We could pair the quarks in F_S with this $S = 0$ flavour-triplet:

$$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)uu \quad \text{or} \quad \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\frac{1}{\sqrt{2}}(ud + du) \quad \text{or} \quad \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)dd$$

or with this $S = 1$ flavour-singlet:

$$\uparrow\uparrow \frac{1}{\sqrt{2}}(ud - du) \quad \text{or} \quad \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)\frac{1}{\sqrt{2}}(ud - du) \quad \text{or} \quad \downarrow\downarrow \frac{1}{\sqrt{2}}(ud - du)$$

and we could do the same for the anti-quark pair. Putting the anti-quarks and quarks together, we thus expect F_S to be associated with:

- •an $S = 0$ flavour nonet (this is the product of the two $S = 0$ flavour-triplets)
- •two $S = 1$ flavour triplets (each is a product of the one $S = 0$ flavour-triplet and one $S = 1$ flavour-singlet)
- •an $S = 2$ flavour-singlet, an $S = 1$ flavour-singlet and an $S = 0$ flavour-singlet (all coming from the product of the two $S = 1$ flavour-singlets).

This makes a total of 9 + (3+3) + (1+1+1) = 18 ground state tetraquarks attached to the F_S colour wave function.

Regarding F_A:

We could pair the quarks in F_A with this $S = 1$ flavour-triplet (in which only $S_z = 1$ is shown to save space):

$$\uparrow\uparrow uu \quad \text{or} \quad \uparrow\uparrow \frac{1}{\sqrt{2}}(ud + du) \quad \text{or} \quad \uparrow\uparrow dd$$

or with this $S = 0$ flavour-singlet:

$$\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\frac{1}{\sqrt{2}}(ud - du)$$

and we could do the same for the anti-quark pair. Putting the anti-quarks and quarks together, we thus expect F_A to be associated with:

- •an $S = 2$ flavour-nonet, an $S = 1$ flavour-nonet, and an $S = 0$ flavour-nonet (these are all from the product of the two $S = 1$ flavour-triplets)
- •two $S = 1$ flavour triplets (each is a product of the one $S = 1$ flavour-triplet and one $S = 0$ flavour-singlet)
- •an $S = 0$ flavour-singlet, (from the product of the two $S = 0$ flavour-singlets).

This makes a total of (9+9+9) + (3+3) + 1 = 34 ground state tetraquarks attached to the F_A colour wave function.

In short, we should not be surprised if there are a lot of tetraquarks to find, 3 Although some of the questions below are discursive rather than mathematical, long essays are not required. Indeed, succinct answers which include all the relevant points will be rewarded more highly than lengthy answers which digress into areas of tangential relevance. You are therefore strongly encouraged to keep your answers focused narrowly on what has been asked.

(a) What can electron deep inelastic scattering tell us about nucleon structure that neutrino deep inelastic scattering cannot, and why? [6]

BOOKWORK[The main message an answer should convey is that whereas electron DIS uses the photons, neutrino DIS uses the W-boson, and consequently the former constrains the Bjorken-x-distributions of things of given ELECTRIC CHARGE MAGNITUDE whereas the latter constrains the Bjorken-x-distributions of things having given "third component of weak isospin". This means that the electron DIS allows us to separate (say) up-type-quark x-distributions from down-type ones since the former has charge magnitude 2/3 and the latter 1/3 .... yet electron DIS does not let us distinguish up from anti-up, or down from anti-down. Conversely, neutrino DIS cannot distinguish up from down, but does distinguish x-distributions of "up and anti-down" from "down and anti-up". A good answer could/should provide evidence in the form of Feynman diagrams illustrating why the above is so, and then should make the further point that what is then MEASURED is not the charges (which are initially assumed or inferred from the quark model itself and hadron charges, etc!) but is the parton distribution functions (or structure functions) totals in the categories mentioned previously.]

(b) Why is one of the central members of the $L = 0$, $S = 0$, $J = 0$, $P = -1$ psuedoscalar meson nonet (the η') considerably heavier than all the other members of that nonet? [6]

(Largely bookwork) The mesons don't contain any identical quarks or antiquarks (since they contain just one of each) and so they don't have to satisfy any special symmetries or antisymmetries on account of Fermi-Dirac statistics. Hence, although $3 \otimes \bar{3}$ decomposes to $8 \oplus 1$ for both colour and flavour, one can legitimately use either the 8 or the 1 (and indeed both) of these multiplets to build mesons. The singlet state has the potential to have a very different mass to the eight members of the octet because it is not forced to be like the others. [Put another way, the 8 states in the octet would have to have identical masses to each other if SU(3)-flavour symmetry was exact. As SU(3)-flavour is not exact their masses vary a bit – but are still held together by the symmetry more than the singlet state.]

(c) Outline the most important parts of the mechanism whereby charged leptons and quarks acquire mass in The Standard Model. [6]

BOOKWORK[Candidates will probably want to show that they know that this is via Yukawa interactions between the Higgs Field and the fermions, and that this is

necessary because direct mass terms in the Dirac Lagrangian are incompatible with SU(2) gauge invariance. Specifically

$$m_e\bar{e}e = m_e\bar{e}(P_L + P_R)e = m_e\bar{e}(P_L^2 + P_R^2)e = m_e\bar{e}P_L^2e + m_e\bar{e}P_R^2e = m_e\bar{e}_Re_L + m_e\bar{e}_Le_R$$

and so under an SU(2) gauge transformation we would have

$m_e\bar{e}_Re_L + m_e\bar{e}_Le_R \to m_e\bar{e}_R($**a lin comb of** e_L **and** ν_L **states**$) + m_e($**a lin comb of** $\bar{e}_L$ **and** $\bar{\nu}_L$ **states**$)e_R$

which is clearly not invariant. The Higgs mechanism avoids this since the Higgs field is a complex doublet that transforms under $\mathrm{SU}(2)_L$ in the opposite direction to the left lepton fields. It is, however, necessary to force the Higgs field to have a VeV (or else particles would still be massless even with Yukawa interactions) and so quadratic and quartic terms are added to the Higgs potential to create a VeV. Candidates might mention other things for which they could also get credit if they are clearly relevant.]

(d) How did the Super-Kamiokande discriminate between electron and muon neutrinos and determine their direction? Why was detection of solar tau neutrinos historically difficult in many neutrino detectors? Describe the innovative features of the experiment that first measured the total solar flux of all three neutrino types. [12]

BOOKWORK[There is a lot that one can describe about neutrino detectors, so there are double the marks for this section. For the first part candidates should explain that Super-K observed Cerenkov rings from super-luminal electrons or muons (from neutrino interactions) and that the electron rings were fuzzier than the muon rings due to the greater ease with which the electrons (being lighter) are scattered. Direction was obtained by looking at the shapes of the rings where they impinged on the walls of the detector — and with the important extra feature that timing-information from the photo-multiplier tubes could determine the tilt of the cone – essentially oblique incidence leads to Cerenkov photons on one side of the cone arriving sooner than from the other side. The difficulty of seeing tau neutrinos is that you need to make a tau lepton (if interacting with the charged current) and since the tau is heavy that sets a kinematic threshold that is above the maximum that solar (non-beam, non-cosmic) neutrinos are emitted with. This means that the only viable way of seeing solar tau neutrinos needs to use the neutral current weak interaction (Z-boson). The Sudbury Neutrino Observatory (SNO) found a way of observing Z-boson-mediated neutrino interactions by using large quantities of heavy water (D_2O). The neutral current reaction in heavy water liberates a neutron which is thermalized in the heavy water and observed following capture on other nuclei within the detector.]

END OF PAPER