Game Theory and Applications (博弈论及其应用)

Chapter 1： Strategy Game and Nash Equilibrium

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Recap

- Game theory: **study of mathematical models of conflict and cooperation between intelligent rational decision-makers**
  - Player
  - Strategy/Decision
  - Payoff
  - Information
  - Rationality

Prisoners' Dilemma

Prisoner 2

Prisoner 1	Confess(c)	Don't confess(d)
Confess(c)	-6 -6	0 -12
Don't confess(d)	-12 0	-1 -1

- Set of players $N = \{1, 2\}$
- Set of strategies $A_1 = A_2 = \{\mathrm{c}, \mathrm{d}\}$
- Payoff matrix

Strategy Games

- How to model the game where each player select his strategy simultaneously with full information, without knowledge of strategy choices by the other players (non-cooperate)

- Strategy game = normal form game

A strategy game consists of

有限玩家集
非空策略集
收益函数

- A finite set N of players
- A non-empty strategy set A_i for each player $i \in N$
- A payoff function $u_i: A_1 \times A_2 \times \cdots \times A_N \rightarrow R$ for $i \in N$

Strategy Games (cont.)

A strategy game consists of

- A finite set N of players
- A non-empty strategy set A_i for each player $i \in N$
- A payoff function $u_i: A_1 \times A_2 \times \cdots \times A_N \to R$ for $i \in N$

- An outcome $a = (a_1, a_2, \ldots, a_N)$ is a collection of strategies, one for each player
- Outcome space:

$$A = A_1 \times A_2 \times \cdots \times A_N$$

- The payoff function can be replaced by a **preference relation(偏好关系)** $\succsim_i$ over A for each player i

其实只需知道大小关系，无需知道具体的值

Preference Relation

A preference relation (偏好关系) $\succsim$ over a set A satisfies

– complete

a $\succsim b$ or $b \succsim a$ for every a $\in A, b \in A$

– reflexive

a $\succsim a$ for every a $\in A$

– transitive

if a $\succsim b$ or $b \succsim c$, then a $\succsim c$ for every a$, b, c \in A$

We write 强偏好关系

$a \succ b$ if $a \succsim b$ but not $b \succsim a$ for a $\in A, b \in A$

Strategy Games (cont.)

A strategy game consists of

- A finite set N of players
- A non-empty strategy set A_i for each player $i \in N$
- A payoff function $u_i\colon A \to R$ for $i \in N$; or a preference relation $\succsim_i$ over A for $i \in N$

$$G=\{N, \{A_i\}_{i=1}^{N}, \{u_i\}_{i=1}^{N}\}$$

$$G=\{N, \{A_i\}_{i=1}^{N}, \{\succsim_i\}_{i=1}^{N}\}$$

Prisoners' Dilemma (Formally)

		Prisoner 2: Confess(c)	Prisoner 2: Don't confess(d)
Prisoner 1	Confess(c)	-6 -6	0 -12
Prisoner 1	Don't confess(d)	-12 0	-1 -1

- Set of players $N = \{1, 2\}$
- Set of strategies $A_1 = A_2 = \{\mathrm{c}, \mathrm{d}\}$
- $u_1(c, c) = -6, u_1(c, d) = 0$, etc.
- $(\mathrm{c}, \mathrm{d}) \gtrsim_1 (c, c), (\mathrm{d}, \mathrm{d}) \gtrsim_1 (c, c)$, etc.

Notations

- For an outcome $a = (a_1, a_2, \dots, a_N)$, we define

$$\boldsymbol{a_{-i}} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_N)$$

the outcome of strategies taken by all players other than i

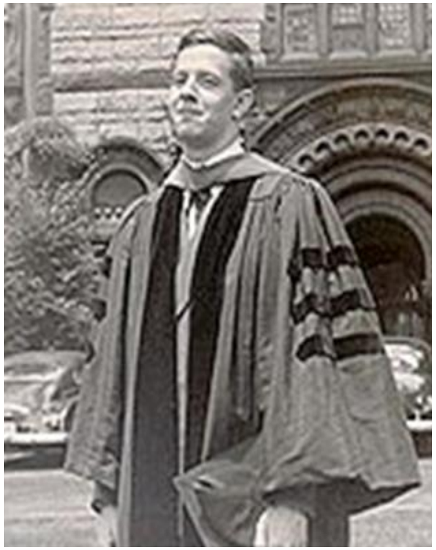
$$a = (a_i, a_{-i})$$

- For a game $G = (N, \{A_i\}, \{u_i\})$, we define $\boldsymbol{A_{-i}}$ the set of all such a_{-i}, i.e.,

$$\boldsymbol{A_{-i}} = A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_N$$

John F. Nash

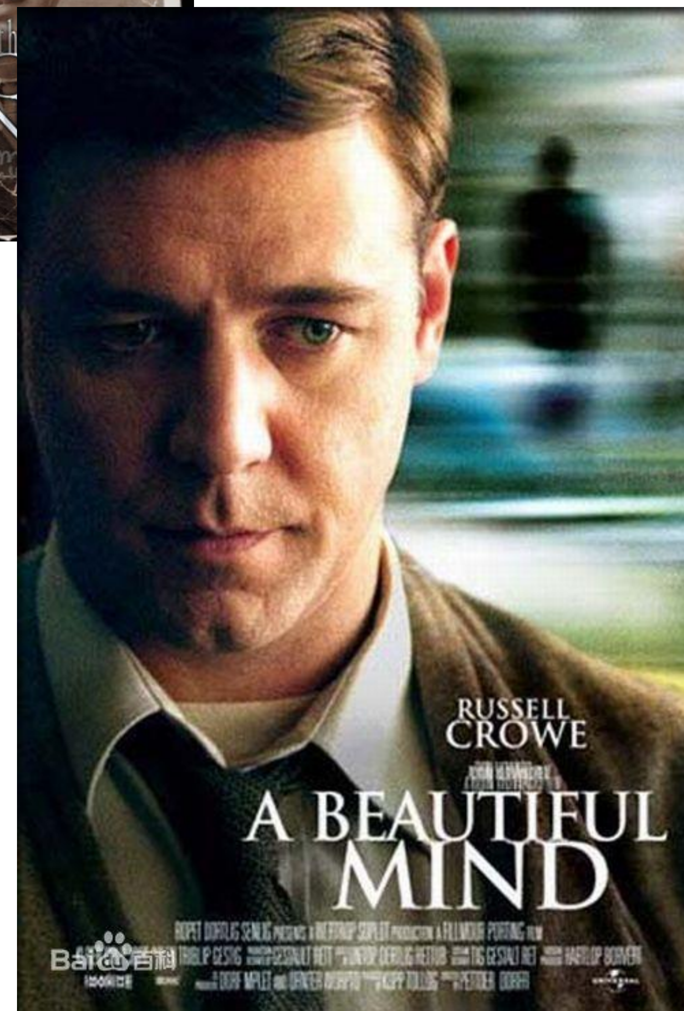
- 1928 Born in American
- 1950 PhD (Princeton)
  - Non-cooperative Games
  - 28 pages
- 1951 Teacher MIT



- 1957 Married

John F. Nash (cont.)

- 1959 Paranoia (精神分裂症)
- 1980s Health



- 1994 Nobel Prize
- 1999 Steel Prize
- 2015 Abel Prize
- 2015 Accident

Nash Equilibrium (NE)

Nash equilibrium is a strategy outcome (a collection of strategies, one for each player) such that each strategy is a best response (maximizes payoff) to all other strategies

An outcome $a^* = (a_1^*, a_2^*, \dots, a_N^*)$ is a **Nash equilibrium (NE)** if for each players i

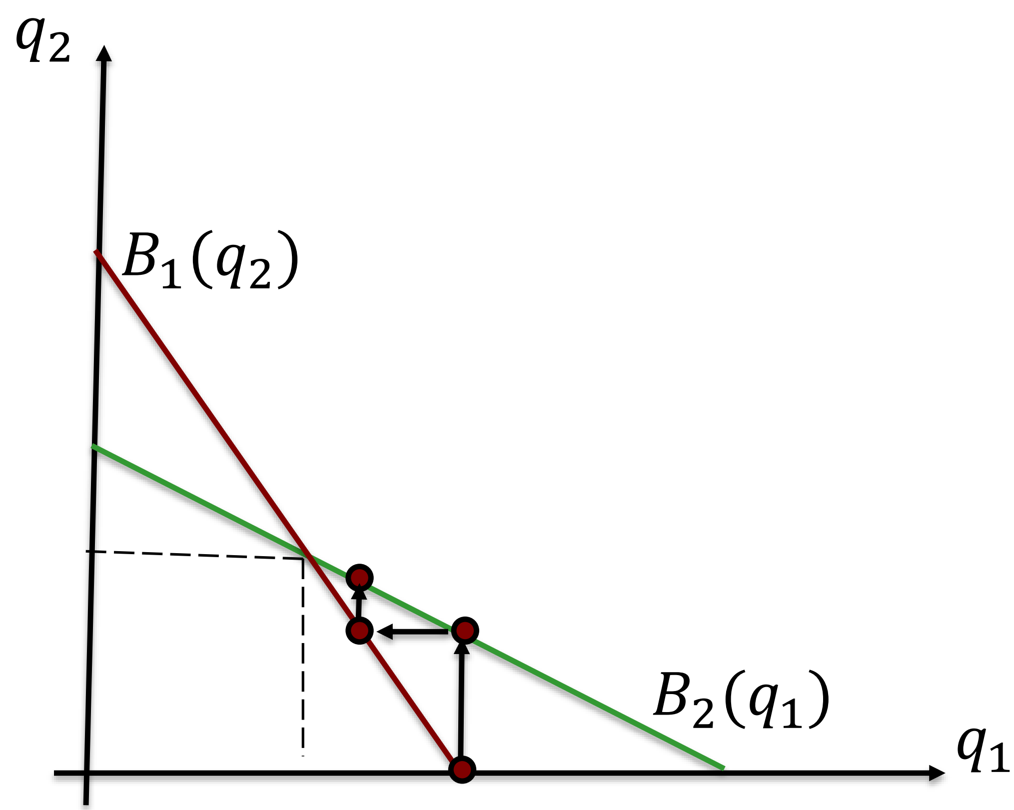
$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \text{ for all } a_i \in A_i.$$

本身具有强制力的

Nash equilibrium is self-enforcing: no player has an incentive to alter his strategy unilaterally (单方面).

Nash Equilibrium (cont.)

- Let $(a_1^*, a_2^*, \dots, a_N^*)$ be an Nash equilibrium
  - Player i makes the best strategies a_i^* with respect to a_{-i}^*



Nash Equilibrium

Player 2

Player 1

	d		e		f	
a	-6	-6	0	8	0	-2
b	-12	0	1	3	1	1
c	-10	2	4	0	1	4

Best Response Correspondence 最佳反应函数

- The best response correspondence of player i is given by

$$B_i(a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(b_i, a_{-i}) \text{ for all } b_i \in A_i\}$$

$B_i(a_{-i})$ may be a set or a singleton

Prisoner 1 \ Prisoner 2	Confess(c)	Don't confess(d)
Confess(c)	-6 -6	0 -12
Don't confess(d)	-12 0	-1 -1

$B_1(c) = \{c\}$ $B_1(d) = \{c\}$

$B_2(c) = \{c\}$ $B_2(d) = \{c\}$ P2在对方选d时最好选c

How to Find Nash Equilibria

- One way of finding Nash equilibria for payoff matrix:

  (1) Find the best response correspondence for each player

  Best response correspondence gives the set of payoff maximizing strategies for each strategy profile of the other players

  (2) Find where they intersect

  Find all outcomes $(a_1^*, a_2^*, \dots, a_N^*)$ such that

$$a_i^* \in B_i(a_{-i}^*)$$

Nash Equilibrium: Example 1

Primitive hunting

		Hunter 2	
		Rabbit (r)	Deer (d)
Hunter 1	Rabbit (r)	3 3	3 0
	Deer (d)	0 3	9 9

Nash Equilibrium: Example 2

Prisoners' Dilemma

Prisoner 2

Prisoner 1

	Confess(c)	Don't confess(d)
Confess(c)	-6 -6	0 -12
Don't confess(d)	-12 0	-1 -1

Nash Equilibrium: Example 3

Rock-Paper-Scissors

Player 2

Player 1

		Rock	Paper	Scissors
Player 1	Rock	0, 0	-1, 1	1, -1
	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

An Exercise

- Find all Nash equilibria

P2

		h	i	j	k	l	m
	a	7, 5	8, 6	2, 2	2, 3	6, 9	6, 5
	b	6, 5	9, 6	5, 8	6, 7	8, 8	7, 4
P1	c	9, 7	1, 1	7, 9	3, 2	9, 6	9, 2
	d	2, 14	10, 12	6, 5	6, 3	7, 2	9, 12
	e	8, 6	5, 9	3, 9	7, 5	13, 15	8, 9

3- Players Game

$$G = \{\{1,2,3\}, \{\{a,b,c\},\{x,y,z\},\{L,R\}\}, \{u_i\}_{i=1}^{3}\}$$

P3 chooses *L*

P2

P1		x			y			z		
	a	8	7	4	2	9	1	4	1	8
	b	4	6	5	7	2	6	1	3	7
	c	6	2	2	5	1	7	4	4	2

P3 chooses *R*

P2

P1		x			y			z		
	a	5	3	2	6	5	4	1	2	4
	b	8	6	2	2	8	10	5	2	6
	c	6	9	4	1	1	3	9	7	8

How to Find Nash Equilibria

- One way of finding Nash equilibrium for continuous strategies A_i:

(1) Find the best response correspondence for each player

Best response correspondence

$$B_i(a_{-i}) = \{a_i \in \text{argmax}_{a_i \in A_i} u_i(a_i, a_{-i})\}$$

(2) Find all Nash Equilibria $(a_1^*, a_2^*, \ldots, a_N^*)$ such that

$$a_i^* \in B_i(a_{-i}^*) \text{ for each player}$$

Optimization

可微

- Let $f\colon R \to R$ be a continuous and differential function.
  The optimization problem: $\max_{x\in[a,b]} f(x)$

This problems can be solved by following procedures:

- Find all critical points: $x \in [a,b], f'(x) = 0$
- Evaluate f at all critical point and boundaries a and b
- Find the highest f

Example

$$\max_{x\in[-2,5]} x^3 - 3x^2 - 9x + 6$$

Cournot Competition(古偌竞争, 1838)

- Two firms compete by choosing how much to produce

$$G = \{\{1,2\},\{q_1,q_2\},\{u_1,u_2\}\}$$

  - Price

$$p(q_1+q_2) = \max(0, a - b(q_1+q_2))$$

  - Costs $(i = 1,2)$

$$c_i(q_i) = cq_i$$

  - Payoffs $(i = 1,2)$

$$u_i(q_1,q_2) = (\max\big(0, a - b(q_1+q_2)\big) - c)q_i$$

  - Condition $a > \mathrm{b}, c > 0, q_1 \geq 0, q_2 \geq 0$

Cournot: Best Response Correspondence

Best Response Correspondence is given by

$$B_i(q_{-i}) = \max(0, (a - c - bq_{-i})/2b)$$

Proof. We will prove for i=1 (similarly for i=2)

If $q_2 \geq (a-c)/b$, then $u_1(q_1, q_2) \leq 0$ for any $q_1 > 0$. $q_1 = 0$.

If $q_2 < (a-c)/b$, then

$$u_i(q_1, q_2) = (a - c - b(q_1 + q_2))q_i$$

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = a - c - bq_2 - 2bq_1 = 0$$

$$q_1 = (a - c - bq_2)/2b$$

Cournot: Nash Equilibrium

The Nash equilibria is give by

$$\left\{\left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)\right\}$$

Proof. Assume that (q_1^*, q_2^*) is a Nash equilibrium.

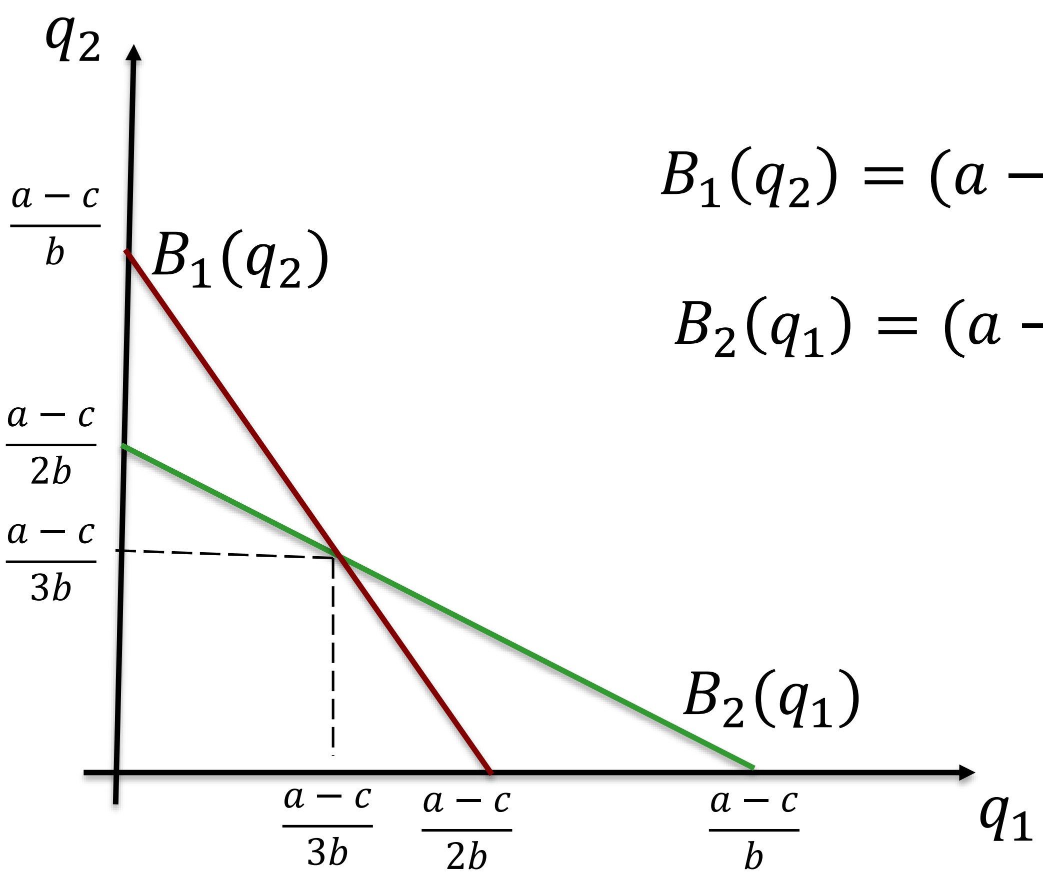
1) Prove $q_1^* > 0$ and $q_2^* > 0$ by contradiction

2) (q_1^*, q_2^*) is such that $q_1^* > 0, q_2^* > 0$

$$q_1^* = B_1(q_2^*) = (a - c - bq_2^*)/2b$$

$$q_2^* = B_2(q_1^*) = (a - c - bq_1^*)/2b$$

Cournot: Nash Equilibrium (cont.)



$$B_1(q_2) = (a - c - bq_2)/2b$$

$$B_2(q_1) = (a - c - bq_1)/2b$$

An Exercise: n-Cournot Competition

- n firms compete by choosing how much to produce

$$G = \{\{1, \dots, n\}, \{q_1, \dots, q_n\}, \{u_1, \dots, u_n\}\}$$

  - Price

$$p(q_1 + \cdots + q_n) = a - b(q_1 + \cdots + q_n)$$

  - Costs $(i = 1, \dots, n)$

$$c_i(q_i) = cq_i$$

  - Payoffs $(i = 1, \dots, n)$

$$u_i(q_1, \dots, q_n) = (a - b(q_1 + \cdots + q_n) - c)q_i$$

  - Condition $a > c > 0, b > 0 \ q_i \geq 0$

n-Cournot: Best Response Correspondence

Best Response Correspondence is given by

$$B_i(q_{-i}) = \max\left(0, \frac{a - c - b\sum_{k=1,k\neq i}^{n} q_k}{2b}\right)$$

Proof. Solve

$$\max_{q_i \geq 0} u_i(q_1, \dots, q_n) = \max_{q_i \geq 0} (a - b(q_1 + \cdots + q_n) - c)q_i$$

$$\frac{\partial u_i(q_1, \dots, q_n)}{\partial q_i} = a - c - b\sum_{k=1,k\neq i}^{n} q_k - 2bq_i = 0$$

$$q_i = \frac{\left(a - c - b\sum_{k=1,k\neq i}^{n} q_k\right)}{2b}$$

n-Cournot: Nash Equilibrium

The Nash equilibria is give by

$$\left\{\left(\frac{a-c}{(n+1)b}, \dots, \frac{a-c}{(n+1)b}\right)\right\}$$

Proof. Assume that $(q_1^*, \dots, q_n^*)$ is a Nash equilibrium.

1) Prove $q_i^* > 0$ by contradiction

2) $(q_1^*, \dots, q_n^*)$ is such that

$$q_i^* = B_i(q_{-i}^*) = \frac{a - c - b\sum_{k=1,k\neq i}^{n} q_k}{2b}$$

Summary

- Formulations of Game
- Nash Equilibrium
- How to find Nash Equilibrium for payoff matrices
- How to find Nash Equilibrium for continuous strategies