Tutorial 7

Main topics: Direct product, homomorphisms and isomorphisms, cosets

1. Find the order of the element $(1, 2)$ in the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.

2. Do any of the following define homomorphisms from $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{GL}(n, \mathbb{C})$ for $n \geqslant 2$? Give brief explanations.

   a) $f(A) = A^T$ b) $f(A) = (A^{-1})^T$ c) $f(A) = A^2$

3. Let $g \in G$, where G is a group. Prove that the function $\varphi\colon G \to G$ given by $\varphi(h) = ghg^{-1}$ is an isomorphism from G to itself (i.e., φ is an ***automorphism*** of G).

4. List all (group) homomorphisms $\varphi\colon \mathbb{Z} \to \mathbb{Z}$. [Hint: if $\varphi(1) = a$, what is $\varphi(n)$?]

5. Explain why the following groups are not isomorphic:

   a) $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and D_4 b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/12\mathbb{Z}$ c) $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Q}$

6. Write down the left cosets of $H = \langle 3 \rangle$ in $G = \mathbb{Z}/6\mathbb{Z}$. What is the index $[G : H]$?

7. Let s be the reflection in D_6 with respect to the vertices 1 and 4 (as shown) and let r be rotation by $2\pi/6$. Let $H = \langle s, r^2 \rangle \leqslant D_6$.

   a) List the elements of H (in terms of r and s).

   b) Write down the left cosets of H.

   c) What is the index $[G\colon H]$?