Notes on Lie Groups and Lie Algebras for PHYS 4301 - Mathematical Methods.

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April 24, 2024

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Introduction

A symmetry is a transformation that leaves something unchanged. This concept is at the heart of one of the foundational concepts in physics: that physics is the same in all inertial frames of reference. The transformation that changes your reference point and coordinate system to another with constant velocity compared to the first will not change the observed physical interactions. By requiring that a physical law, i.e., an equation that relates physical quantities, is invariant under some type of coordinate transformation, we can immediately derive constraints on the form of the equation.

The mathematical description and quantification of symmetry is achieved using group theory. Group theory is a huge field of study and includes everything from abstract finite groups (permutations, for example), to crystallography, to continuously parametrised Lie groups.

The group that captures the invariance of physics in special relativity is called the Poincaré group. This group consists of translations in space and time, rotations, reflections, and inversions of space and time, and other transformations (called boosts) that fix the origin of your coordinate system and preserve the Minkowski scalar product. The group of symmetries of Minkowski space that fix the origin is called the Lorentz group. Since translations and rotations can be arbitrarily small and in arbitrary directions, these groups are examples of continuous groups. They also have a particularly well behaved space of parameters so they are examples of Lie groups.

The remainder of these lectures will focus on the details of how to define certain classes of transformations, i.e., groups, how to get concrete representations of a group, what properties these have, and how this leads to physical consequences.

1 Basic concepts

Groups

A group consists of a set of elements, together with an operation that combines two elements to get another element of the set. A familiar example of a group is the integers, $\mathbb{Z}$, with addition, $+$, as the operation. To satisfy the full definition of a group, the set, G, and the operation, (generally written $*$), must satisfy the following:

1. *Closure.* If $a, b, \in G$, then $a * b \in G$.
2. *Associativity.* $(a * b) * c = a * (b * c)$.
3. *Identity element.* Every group has a special element, I, such that $a * I = I * a = a$ for all $a \in G$.
4. *Inverses.* Every element of the group has an inverse: If $a \in G$, then there is another element denoted $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = I$.

If the group operation also satisfies $a * b = b * a$, the operation is called *commutative* and the group is called *Abelian.*

Examples

- What is the identity in $(\mathbb{Z}, +)$? What is an inverse? Is this group Abelian?
- Do the complex numbers with addition form a group?
- The *trivial group* has just the identity element and nothing else.
- Square, invertible matrices of size $n \times n$ with entries in $\mathbb{R}$ (or $\mathbb{C}$) and standard matrix multiplication form a group called the *general linear group* $GL(n, \mathbb{R})$ (or $GL(n, \mathbb{C})$).

A group can have a finite or infinite number of elements. If the group is finite, the number of elements is called the *order* of the group, written $|G|$. For example, the symmetry group of a square has a finite number of elements. An infinite group could have a countable or uncountable number of elements and be parametrised by any number of variables. Crystallographic groups are examples of infinite groups with a countable number of elements. The general linear group $GL(n, \mathbb{R})$ is an uncountably infinite group parametrised by n^2 real variables. It is a continuous group and an example of a *Lie group.*

Subgroups

Let $H \subset G$. Then if H also satisfies the four properties of a group with the same group operation, we call H a *subgroup* of G.

Example The even integers, $2\mathbb{Z}$, form a subgroup of $(\mathbb{Z}, +)$. Check all the four properties.
Question Do the odd integers form a subgroup of $(\mathbb{Z}, +)$?

Matrices as linear transformations

An invertible $n \times n$ matrix is often interpreted as a linear transformation acting on $\mathbb{R}^n$. In the geometric viewpoint these transformations leave the origin fixed, but can squeeze, stretch, flip, or shear the rest of space. Multiplication of matrices corresponds to composition of transformations.

There are two ways to interpret such a matrix: as a linear transformation mapping a vector space V with basis $\{e_1, \dots, e_n\}$ back onto itself, or as a mapping from one vector space V with basis $\{e_1, \dots, e_n\}$ to a different vector space U with basis $\{f_1, \dots, f_n\}$.

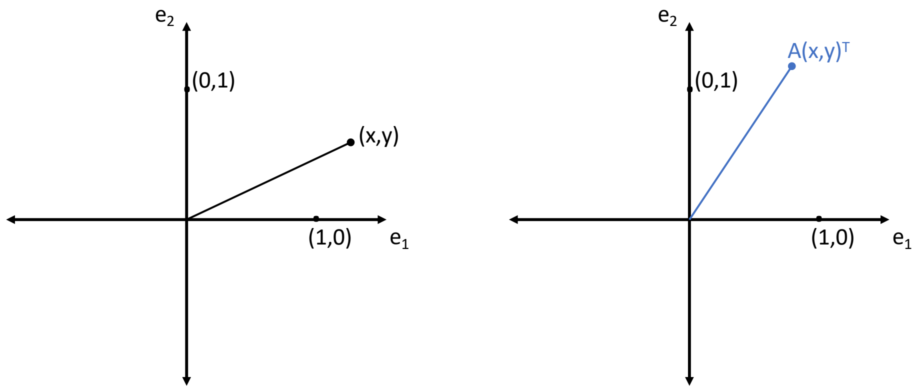


Figure 1: The first interpretation of a matrix as transforming (moving) points in a vector space to new locations.

In the first interpretation, we think of column vectors x as the coordinates of points with respect to the $\{e_1, \ldots, e_n\}$ basis, then the product Ax gives the coordinates of a new point with respect to the $\{e_1, \ldots, e_n\}$ basis; see Figure 1.

In the second interpretation, we start with a column vector x expressed with respect to the $\{e_1, \ldots, e_n\}$ basis, then Ax gives coordinates of the same point, but with respect to the new basis $\{f_1, \ldots, f_n\}$; see Figure 2.

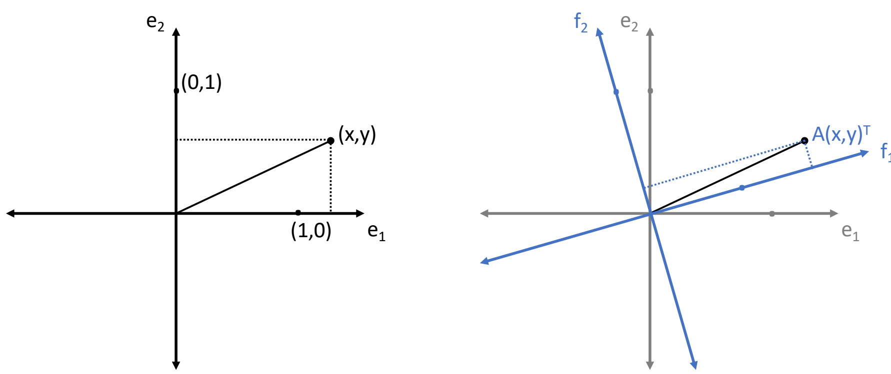


Figure 2: The second interpretation of a matrix as expressing point coordinates with respect to two different bases.

Example Suppose we restrict to studying transformations that leave the distance between any two points, x, y, unchanged. Let A be the matrix of such a transformation, then it must be the case that

$$\|y - x\|^2 = \|Ay - Ax\|^2 \quad \text{distance is preserved,} \tag{1.1}$$

$$(y - x) \cdot (y - x) = (Ay - Ax) \cdot (Ay - Ax) \quad \text{in vector notation,} \tag{1.2}$$

$$(y - x)^T (y - x) = (A(y - x))^T (A(y - x)) = (y - x)^T A^T A(y - x) \quad \text{in matrix notation.} \tag{1.3}$$

As this must hold for any x, y, we see that the matrix A must satisfy $A^T A = I$, or equivalently that $A^T = A^{-1}$. This defines what is known as the *Orthogonal group* $O(n)$. It is a subgroup of $GL(n, \mathbb{R})$

because:

1. *Closure.* If $A, B \in O(n)$, then $(AB)^T = B^T A^T = B^{-1} A^{-1} = (AB)^{-1} \in O(n)$.
2. *Associativity.* Follows because matrix multiplication is associative.
3. *Identity element.* $I^T = I^{-1} = I \in O(n)$.
4. *Inverses.* Check that $(A^{-1})^T = (A^{-1})^{-1}$. We have that $A^T = A^{-1}$, so the LHS of our expression is $(A^T)^T = A$. And the RHS of the expression is also A, so we are done.

More on the orthogonal groups

The orthogonal group $O(n)$ is defined to be the group of invertible linear transformations of $\mathbb{R}^n$ that fix the origin and preserve Euclidean distances. In terms of the associated $n \times n$ matrices: $A \in O(n)$ means $A^T A = I$. Taking the determinant of both sides, we see that $\det(A^T A) = \det(A^T)\det(A) = [\det(A)]^2 = 1$. So there are two types of matrices in $O(n)$, those with $\det(A) = 1$ and those with $\det(A) = -1$. By the multiplicative property of determinants, we see that the matrices with $\det(A) = 1$ form a subgroup of $O(n)$ called the *special orthogonal group* $SO(n)$. These transformations preserve the orientation of space (*parity*). In $\mathbb{R}^2$ this means $SO(2)$ is the group of rotations about the origin. In $\mathbb{R}^3$, we have that $SO(3)$ is the group of rotations about axes that pass through the origin. We will study these groups in much further detail over the next few weeks. For reference, an element of $SO(2)$ corresponding to anticlockwise rotation of points by the angle θ is given by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

See Figure 3 for an illustration of R_θ as a linear transformation of the plane. Observe also that *as matrices*, $R_{\theta+2\pi} = R_\theta$, so the parameter space for $SO(2)$ is the unit circle, S^1.

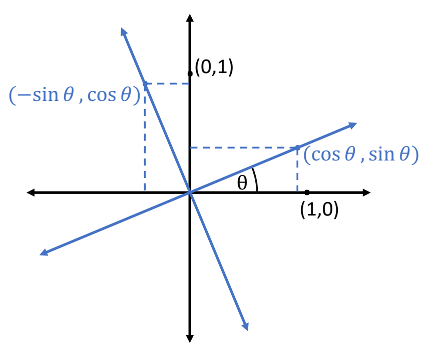


Figure 3: Rotation of the plane by θ in the anticlockwise direction transforms the black axes onto the blue ones.

Remark: The fact that $\det(\cdot)$ is a continuous function of the matrix entries means that the parameter space for $O(n)$ is broken into two disjoint parts. There is no continuous path of orthogonal matrices that can take us from a matrix with $\det(A) = 1$ to another matrix with $\det(B) = -1$. We can join these two parts together by working in a much bigger group: matrices that have complex values. These are the unitary matrices that are defined in the next section.

A matrix that has $\det(A) = -1$ changes the orientation of space by flipping the direction of an odd number of axes. In $O(2)$, these transformations are reflections in a line. For example, the reflection that fixes the x-axis and flips the y-axis is given by ($F_{y=0}$ means 'fix the line $y = 0$', i.e. the x-axis).

$$F_{y=0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In $O(3)$, orientation-reversing transformations can be inversions that reverse all three directions, or reflections that flip just one, e.g.:

$$N = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad F_{z=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The unitary groups

We can enlarge our general linear group to be $n \times n$ invertible matrices with complex elements, denoted $GL(n, \mathbb{C})$. The appropriate version of transpose now requires also taking the complex-conjugate of the elements, and this operation is denoted with a dagger: $A^\dagger = (A^*)^T$. The subgroup of $GL(n, \mathbb{C})$ that preserves the magnitude of vectors in $\mathbb{C}^n$ is called the *unitary group*, denoted $U(n)$. Matrices in $U(n)$ satisfy $A^\dagger A = I$, or $A^\dagger = A^{-1}$. This defining property means that $\det(A^\dagger)\det(A) = \det(A)^*\det(A) = 1$, which implies that $\det(A) = e^{i\alpha}$.

We also define the *special unitary group* $SU(n)$ to be the subgroup of $U(n)$ defined by matrices with $\det(A) = 1$.

There are many relationships between the unitary and orthogonal groups. Firstly, $O(n)$ is naturally a subgroup of $U(n)$, found by restricting the matrix entries from $\mathbb{C}$ to $\mathbb{R}$. But where $O(n)$ has two components, $U(n)$ has just one, because it is possible to smoothly change a matrix from the identity I to $F_{z_n=0}$, the matrix that fixes the complex plane $z_n = 0$ and swaps the sign of the z_n coordinate.

Rotations of the plane

We can model the plane either as $\mathbb{R}^2$ or $\mathbb{C}$. A transformation, R_θ that fixes the origin and rigidly rotates points in the plane by θ in an anticlockwise direction can then either be represented by a matrix in $SO(2)$, or simply by multiplying complex points by a phase factor $e^{i\theta}$. This shows us that $SO(2)$ is the same group as $U(1)$.

Formal definitions

So far we have described Lie groups just as some type of continuous group. The modern mathematical definition is quite involved.

Definition: A *Lie group* is a group, G, with product $\circ$. Elements of the group form a differentiable manifold M, and the product induces a differentiable map f_g of the manifold onto itself

$$f_g(h) = g \circ h \quad \text{is differentiable with respect to } h.$$

If we have coordinates for M, then this means the coordinates of $g \circ h$ are differentiable functions of the coordinates for h, and this must hold for all $g \in G \simeq M$. The map of M taking a group element to its inverse must also be differentiable, $v : M \to M$ with $v(h) = h^{-1}$.

Example: The group $SO(2)$ has the unit circle S^1 as its manifold, and the product map

$$R(\phi)R(\theta) = \begin{pmatrix} \cos(\phi + \theta) & -\sin(\phi + \theta) \\ \sin(\phi + \theta) & \cos(\phi + \theta) \end{pmatrix} = R(\phi + \theta)$$

is then $f_\phi(\theta) = \phi + \theta$, which clearly maps S^1 back onto itself and is a differentiable function of θ. The inverse map is

$$R(\phi)^{-1} = R(\phi)^T = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} = \begin{pmatrix} \cos(-\phi) & -\sin(-\phi) \\ \sin(-\phi) & \cos(-\phi) \end{pmatrix} = R(-\phi)$$

So $v(\phi) = -\phi$ is clearly a continuous mapping of S^1.

Rotations of three-dimensional space

Rotations of $\mathbb{R}^3$ fix a linear axis through the origin with direction vector $\hat{\mathbf{n}}$, and move points anti-clockwise by angle θ uniformly about this axis. So these transformations require a specification of the direction vector and angle: $R_{n,\theta}$. These are difficult to write out in full generality, but when $\hat{\mathbf{n}}$ is an axis direction we have that:

$$R_{x,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \; R_{y,\theta} = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \text{ and } R_{z,\theta} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The *inverse* of a rotation $R_{n,\theta}$, is a clockwise rotation about the same direction $\hat{\mathbf{n}}$, $R^{-1}_{n,\theta} = R_{n,-\theta}$, e.g.,

$$R^{-1}_{z,\theta} = R_{z,-\theta} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = R^T_{z,\theta}$$

Another observation about three-dimensional rotations is that anti-clockwise rotation about $\hat{\mathbf{n}}$ is identical to clockwise rotation about $-\hat{\mathbf{n}}$, so that $R_{-n,-\theta} = R_{n,\theta}$.

To find a matrix representation for rotation by θ about a general axis $\hat{\mathbf{n}}$, requires some conventions for specifying direction vectors, and the concept of *conjugacy.* Basically, we find a transformation, T, that changes the coordinate system so that the direction $\hat{\mathbf{n}}$ becomes the z-axis (say), apply the rotation matrix $R_{z,\theta}$, to this new set of coordinates, and then transform the result back into the original coordinate system using T^{-1}:

$$R_{n,\theta} = T^{-1} R_{z,\theta} T = T^T R_{z,\theta} T$$

because $T \in O(3)$.

Remark: The definition of conjugacy applies in any group. Two elements, $g_1, g_2 \in G$ are said to be conjugate if there exists an element $h \in G$ such that $g_2 = h^{-1} g_1 h$. This means g_1 and g_2 will have similar properties. In $O(2)$ for example, all rotations are conjugate to one another, and no rotation is conjugate to a flip.

A unit direction vector in $\mathbb{R}^3$ is specified by two angles (ϕ, ψ), where, for example, $\phi \in [0, 2\pi)$ corresponds to a longitude and $\psi \in [0, \pi)$ to a latitude, see Figure 4. This implies *Euler's Theorem* that any rotation of 3-dimensional space requires just three angles to define it. With this method we see that $R_{z,-\phi}$ transforms the rotation axis $\hat{\mathbf{n}}$ to sit above the x-axis, and then $R_{y,-\psi}$ moves it to the z-axis, so that $R_{n,\theta} = R_{z,\phi} R_{y,\psi} R_{z,\theta} R_{y,-\psi} R_{z,-\phi}$.

Another method, commonly used in engineering applications, requires just three rotation matrices (rather than five) to find the transformation that maps one set of axes (x, y, z), onto another set (X, Y, Z). The angles for these three rotation matrices are called *Euler angles.* There are many different conventions for defining these, but a common one is to use the z-x-z convention; see Figure 5. Then we have the following transformation that gives the image of the implied rotation of (x, y, z) onto (X, Y, Z), given w.r.t. to the initial coordinate system

$$R_{\alpha,\beta,\gamma} = R_{z,\alpha} R_{x,\beta} R_{z,\gamma}.$$

Notes:

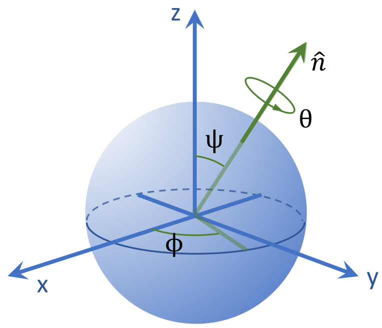
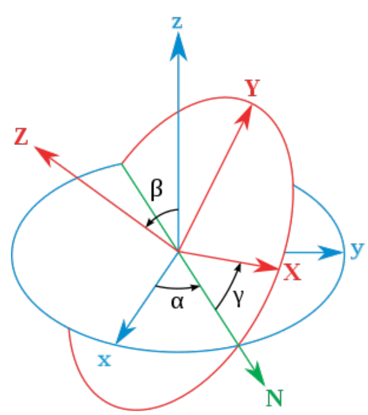


Figure 4: A rotation in three dimensions is defined by a unit direction vector $\hat{\mathbf{n}}$ and the angle, θ, to be rotated about this axis.

Figure 5: Euler angle definition. The original axes (x, y, z) are in blue, the transformed frame (X, Y, Z) is in red. The green axis N is called the "line of nodes"; it is the line of intersection between the xy-plane and the XY-plane. The angle α is that between x and N, β is the angle between z and Z, and γ is the angle between N and X. Image from wikipedia. `https://en.wikipedia.org/wiki/Euler_angles/media/File:Eulerangles.svg`



- The Euler angles definition for a rotation doesn't identify the invariant axis or angle rotated directly.
- The choice of angles for α and γ becomes ill-defined when $\beta = 0$ or π, an issue called "gimbal lock".
- Mathematica uses a z-y-z convention by default in its EulerMatrix command, but can be set to other combinations.

Quaternions

A third method for working with rotations of 3-dimensional space uses unit quaternions. The unit quaternions provide a continuous parametrisation, that doesn't suffer from the angle degeneracies of the previous two methods.

The *quaternions*, $\mathbb{Q}$, are a 4-dimensional version of the complex numbers. There are now three complex 'units' $\mathbf{i}, \mathbf{j}, \mathbf{k}$, and one real unit $\mathbf{1}$, that satisfy the following:

$$\begin{aligned} &\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 = \mathbf{ijk} \\ \Rightarrow \quad &\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j} \end{aligned}$$

A general quaternion is given by $q = a\mathbf{1}+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$, and its conjugate-transpose by $q^\dagger = a\mathbf{1}-b\mathbf{i}-c\mathbf{j}-d\mathbf{k}$, so that the squared-magnitude of q is given by

$$|q|^2 = q^\dagger q = (a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k})(a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = a^2 + b^2 + c^2 + d^2$$

A *unit* quaternion is then defined by $q^\dagger q = 1$ and these can be viewed as points on the unit 3-sphere in $\mathbb{R}^4$, in the same way that unit complex numbers are points on the unit circle in $\mathbb{R}^2$. The unit quaternions form a group under quaternion multiplication, $U(1, \mathbb{Q})$. This multiplication is defined as follows:

$$\begin{aligned} &(a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(w\mathbf{1} + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \\ &\quad (aw - bx - cy - dz)\mathbf{1} + (bw + ax - dy + cz)\mathbf{i} + (cw + dx + ay - bz)\mathbf{j} + (dw - cx + by + az)\mathbf{k} \end{aligned}$$

[[You can use Mathematica to do the multiplying for you with its Quaternion package.]] The inverse of q is defined similarly to that for the complex numbers,

$$\begin{aligned} q^{-1} &= \frac{q^\dagger}{q^\dagger q} = (a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k})/(a^2 + b^2 + c^2 + d^2) \\ &= q^\dagger \quad \text{when } |q|^2 = 1. \end{aligned}$$

The correspondence with a rotation transformation is made as follows. Let $\hat{\mathbf{n}}$ be the direction of the rotation axis and suppose θ is the angle of anti-clockwise rotation about this axis. A vector from $\mathbb{R}^3$ is written in $(\mathbf{i}, \mathbf{j}, \mathbf{k})$-notation as $v = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ and viewed naturally as a quaternion. Now set

$$q = \cos(\theta/2)\mathbf{1} + \sin(\theta/2)\hat{\mathbf{n}}.$$

Then the function T_q that represents the rotation transformation is defined to act on the $\mathbb{R}^3$ subspace as

$$u = T_q(v) = qvq^{-1} = qvq^\dagger \quad \text{because } q \in U(1, \mathbb{Q}).$$

The tutorial worksheet will guide you through the steps to see that this transformation is indeed a rotation by θ about $\hat{\mathbf{n}}$.

So what we get is that given any unit quaternion, q, there is a transformation T_q acting on $\mathbb{R}^3$ as a rotation from $SO(3)$. A simple calculation shows that in fact the two unit quaternions

$$\begin{aligned} q_1 &= \cos(\theta/2)\mathbf{1} + \sin(\theta/2)\hat{\mathbf{n}} \quad \text{and} \\ q_2 &= \cos(\theta/2 + \pi)\mathbf{1} + \sin(\theta/2 + \pi)\hat{\mathbf{n}} \\ &= -q_1 \end{aligned}$$

both define the *same* rotation by the angle θ anticlockwise about the axis $\hat{\mathbf{n}}$.

Geometrically, this means antipodal points of the unit 3-sphere are mapped to the same rotation matrix in $SO(3)$. This means $SO(3)$ is the space (manifold) called RP^3. It has the curious property that there are closed paths (loops) through the space that can't be shrunk to a point (like a string wrapped around a tyre), but when you follow this path twice around in the same direction, you can now 'pull the string free'. When the same type of structure and reasoning is viewed within the context of the Lorentz group $O(1, 3)$, we are led to the origin of the property of *spin*.