THE UNIVERSITY OF
MELBOURNE

Semester 2 Assessment, 2014

Department of Mathematics and Statistics

MAST20022 Group theory and linear algebra

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 4 pages (including this page)

Authorised Materials:

- No materials are authorised.

Instructions to Students:

- This examination consists of 14 questions. The total number of marks is 100.
- The examination paper is in two sections. The questions in Section A are shorter and more routine than those in Section B. It is recommended that students attempt the questions in Section A before trying those in Section B. It is possible to pass the examination on marks from Section A alone. All questions should be attempted.
- Please give complete explanations in all questions and show all your calculations and working. Give careful statements of any results from the notes or lectures that you use.
- You may remove this question paper at the conclusion of the examination.

Instructions to Invigilators:

- Students may remove the examination paper at the conclusion of the examination.

Section A: 50 marks total

1. (a) Use the Euclidean algorithm to compute the greatest common divisor $\gcd(78, 28)$.

(b) Find integers x and y such that $78x + 28y = 2$.

(c) Does $[28]_{78}$ have a multiplicative inverse modulo 78?

[5 marks]

2. (a) Define "algebraically closed field".

(b) Prove that the field $\mathbb{F}_{11}$ is not algebraically closed.

[5 marks]

3. Let $f\colon \mathbb{C}^6 \to \mathbb{C}^6$ be a linear transformation. You are told that its characteristic polynomial is

$$c(x) = (x+2)(x-1)^5$$

and its minimal polynomial is

$$m(x) = (x+2)(x-1)^3.$$

What can you say about the Jordan normal form J of f? [5 marks]

4. Let $V = \mathbb{R}^5$ with the standard inner product (that is, the dot product); let $(e_1, e_2, e_3, e_4, e_5)$ denote the standard basis of V. Let $f\colon V \to V$ be an orthogonal linear transformation. You are told that the 1-eigenspace of f is

$$V_1 = \mathrm{Span}(e_2)$$

and the (-1)-eigenspace of f is

$$V_{-1} = \mathrm{Span}(e_1 + e_3, e_4).$$

Let $W = V_1 \oplus V_{-1}$.

(a) Find a basis for the orthogonal complement $W^\perp$ of W.

(b) Give a geometric interpretation for the restricted linear transformation $f|_{W^\perp}$.

[5 marks]

5. Let V be a finite-dimensional complex inner product space.

(a) Define "isometry on V".

(b) Let f be an isometry on V. Show that the eigenvalues of f have absolute value 1.

[5 marks]

6. Let G be a group with identity element e such that $g^2 = e$ for all $g \in G$. Prove that G is abelian. [5 marks]

7. (a) Write the following product of permutations in S_6

$$(1\,2\,4)(3\,6) \circ (1\,5\,6)(2\,4\,3)$$

as a product of disjoint cycles.

(b) What is the order of the following element of S_4:

$$(1\,2\,3) \circ (1\,4\,3\,2)?$$

(c) What is the order of the following element of S_{100}:

$$(1\,2\,\ldots\,50) \circ (51\,52\,\ldots\,100)?$$

[5 marks]

8. (a) Explain why the size of the group $\mathrm{GL}_2(\mathbb{F}_5)$ is

$$(5^2 - 1)(5^2 - 5) = 480.$$

(b) Is there a subgroup of $\mathrm{GL}_2(\mathbb{F}_5)$ that is isomorphic to the dihedral group D_{25}?

[5 marks]

9. Consider the group of orthogonal 2×2 matrices

$$\mathrm{O}_2 = \{M \in \mathrm{GL}_2(\mathbb{R}) \mid M^T M = I\}$$

and its subset of matrices of determinant 1

$$\mathrm{SO}_2 = \{M \in \mathrm{O}_2 \mid \det(M) = 1\}.$$

(a) Prove that SO_2 is a normal subgroup of O_2.

(b) Prove that the quotient $\mathrm{O}_2 / \mathrm{SO}_2$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

[5 marks]

10. (a) State the orbit-stabiliser theorem.

(b) Compute the number of rotational symmetries of a rectangular box with dimensions $1 \times 1 \times 3$.

[5 marks]

Section B: 50 marks total

11. Let $V = \mathbb{R}[x]_{\leq 3}$ be the vector space of real polynomials of degree at most 3. Consider the linear transformation given by differentiation

$$\delta \colon V \to V, \qquad \delta(f) = \frac{df}{dx}.$$

Find the Jordan normal form J of δ, and a basis $\mathcal{B}$ of V such that J is the matrix of δ with respect to $\mathcal{B}$. [10 marks]

12. Let V be a finite dimensional complex inner product space. Recall that a self-adjoint linear transformation $f \colon V \to V$ is *positive* if all its eigenvalues are nonnegative.

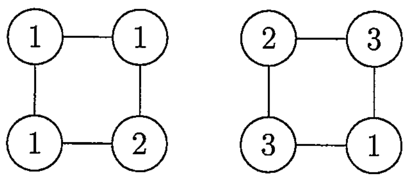
(a) Prove that a self-adjoint f is positive if and only if $\langle f(v), v \rangle \geq 0$ for all $v \in V$. (Hint: for one direction, use the spectral theorem; for the other direction, consider an eigenvector v.)

(b) Prove that the sum of two positive linear transformations is positive.

(c) Is the composition of two positive linear transformations always positive? Give a proof or a counterexample.

[12 marks]

13. Consider the set X of all decorated squares, where each vertex is assigned a label from $\{1, 2, 3\}$, and the same label may be used for more than one vertex. Here are some examples of elements of X:



The dihedral group D_4 acts on X by symmetries of the square.

(a) Prove that if $x \in X$, the size of the orbit of x is one of $1, 2, 4, 8$.

(b) For each of the numbers $1, 2, 4, 8$, find an element $x \in X$ whose orbit has that size.

(c) If we only allow $\{1, 2\}$ as possible labels, can we still get orbits of sizes $1, 2, 4, 8$?

[13 marks]

14. (a) State Lagrange's theorem about the order of elements in a finite group.

(b) According to Lagrange's theorem, which **prime** numbers are possible orders of elements in the symmetric group S_5?

(c) Which of the primes in part (b) actually occur as orders of elements in S_5? Justify your answer by exhibiting an element for each of these orders.

(d) What is the largest order (prime or otherwise) of any element in S_5?

(e) How many Sylow 5-subgroups does S_5 have?

[15 marks]

End of exam