PHYS4301: TUTORIAL PROBLEMS FOR WEEK 9

(1) *$SO(2)$ and its generator.* Recall that a 2D rotation matrix is

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ and } X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that $R(\theta) = e^{\theta X}$ by evaluating the expression for each entry in the series expansion for the matrix $e^{\theta X} = \sum_{n=0}^{\infty} \frac{(\theta X)^n}{n!}$.

(2) Consider the Lie algebra associated with both $SO(3)$ and $SU(2)$. For $SO(3)$ use the physics convention $R(\vec{n}, \theta) = e^{i\theta J}$ and the basis

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad J_y = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad J_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

subscripts x, y, z refer to rotation about x,y,z-axes respectively. For $SU(2)$ use $U = e^{iH}$ and the basis

$$s_1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

Make the correspondence $J_x \to s_1$, $J_y \to s_2$, and $J_z \to s_3$.

The matrix exponential e^{itJ_z} for $t \in [0, 2\pi]$ defines a closed path of elements in $SO(3)$ from the identity back to itself.

$$e^{itJ_z} = R_z(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What is the corresponding path through $SU(2)$? Write down the parametrised matrix for this path (i.e. e^{its_3}). Is the path closed?

(3) The Heisenberg Lie algebra consists of 3×3 matrices of the form

$$H = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{with } a, b, c \in \mathbb{R}.$$

- Write out the basis matrices $\{X, Y, Z\}$ for this Lie algebra so that $H = aX + bY + cZ$.
- Compute the commutation relations for this basis.
- Determine the Lie group associated with this algebra, using $g = e^H$.

(4) Show that the Baker-Campbell-Hausdorff formula holds up to second order terms. Try higher order terms if you're feeling adventurous.

$$\begin{aligned} e^X e^Y &= \left(\sum_{n=0}^{\infty} \frac{X^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{Y^n}{n!} \right) \\ &= \exp\left[X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \cdots \right] \end{aligned}$$