PHYS4301: TUTORIAL PROBLEMS FOR WEEK 10

(1) The following three matrices are a 4-dimensional representation of $\mathfrak{su}(2)$.

$$\rho(J_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \rho(J_2) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \quad \rho(J_3) = \begin{pmatrix} 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Show that this representation is reducible.

- Examine the eigenvalues and eigenvectors of these matrices (use Mathematica, or similar tool).
- Find a common invariant subspace.
- Find a similarity transform that brings these matrices to block-diagonal form.

Solution: To see that the representation is reducible we must first find the eigenvalues and eigenvectors of the three matrices. All three matrices have the eigenvalues $\{i, -i, 0, 0\}$ with their corresponding eigenvectors (not normalised) given as the columns of the following matrices:

$$E[\rho(J_1)] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ i & -i & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad E[\rho(J_2)] = \begin{pmatrix} -i & i & 0 & -1 \\ -i & i & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \sqrt{2} & \sqrt{2} & 0 & 0 \end{pmatrix} \quad E[\rho(J_3)] = \begin{pmatrix} i & -i & 0 & -1 \\ i & -i & 0 & 1 \\ \sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

It is clear that the matrices $\rho(J_2), \rho(J_3)$ share the eigenvector $v = (-1, 1, 0, 0)^T$. Closer inspection of the eigensystem for $\rho(J_1)$ reveals that this vector is also in its 0-eigenspace, as a linear combination of the two eigenvectors with eigenvalue 0. Since all three matrices share an eigenvector they form a reducible set. The eigenvalue and eigenvectors for the invariant subspace are all real, so these matrices are reducible over the reals.

We find a similarity transform (i.e. change of basis) to bring the matrices to block-diagonal form by making a new orthonormal basis for $\mathbb{R}^4$ containing $v/\|v\|$. One possible basis is $v_1 = v/\|v\| = (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)^T, v_2 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)^T, v_3 = e_3, v_4 = e_4$, so the similarity matrix is

$$S = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with } S^{-1} = S^T = S.$$

Finally, the block diagonal form has a 1×1 block in the top left corner and a 3×3 block for the remainder, and we see that these 3×3 blocks are ($-i$ times) the standard representation for $\mathfrak{so}(3)$, which we know is irreducible.

$$S^T\rho(J_1)S = \begin{pmatrix} 0 & . & . & . \\ . & 0 & 0 & 0 \\ . & 0 & 0 & -1 \\ . & 0 & 1 & 0 \end{pmatrix} \quad S^T\rho(J_2)S = \begin{pmatrix} 0 & . & . & . \\ . & 0 & 0 & 1 \\ . & 0 & 0 & 0 \\ . & -1 & 0 & 0 \end{pmatrix} \quad S^T\rho(J_3)S = \begin{pmatrix} 0 & . & . & . \\ . & 0 & -1 & 0 \\ . & 1 & 0 & 0 \\ . & 0 & 0 & 0 \end{pmatrix}$$

(2) Consider the Lie algebra $\mathfrak{su}(2)$, with its Hermitian generators J_1, J_2, J_3 such that

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2.$$

Define the Casimir element, $J^2 = J_1^2 + J_2^2 + J_3^2$. Show that J^2 commutes with J_3, i.e., that

$$[J^2, J_3] = J^2J_3 - J_3J^2 = 0.$$

Solution:

$$\begin{aligned}[J^2, J_3] &= [J_1^2 + J_2^2 + J_3^2, J_3] \\ &= [J_1^2, J_3] + [J_2^2, J_3] + [J_3^2, J_3]\end{aligned}$$

Clearly the third term is zero. We now expand the first term and use the commutator relation $[J_3, J_1] = J_3 J_1 - J_1 J_3 = iJ_2$.

$$\begin{aligned}[J_1^2, J_3] &= J_1 J_1 J_3 - J_3 J_1 J_1 \\ &= J_1(J_3 J_1 - iJ_2) - (iJ_2 + J_1 J_3)J_1 \\ &= J_1 J_3 J_1 - iJ_1 J_2 - iJ_2 J_1 - J_1 J_3 J_1 \\ &= -i(J_1 J_2 + J_2 J_1).\end{aligned}$$

Similarly for the second term of the J^2 commutator we have

$$\begin{aligned}[J_2^2, J_3] &= J_2 J_2 J_3 - J_3 J_2 J_2 \\ &= J_2(iJ_1 + J_3 J_2) - (J_2 J_3 - iJ_1)J_2 \\ &= iJ_2 J_1 + J_2 J_3 J_2 - J_2 J_3 J_2 + iJ_1 J_2 \\ &= i(J_2 J_1 + J_1 J_2).\end{aligned}$$

The expressions for these two terms therefore cancel and we have shown that $[J^2, J_3] = 0$.

(3) The Heisenberg algebra has commutation relations $[X, Y] = Z \neq 0$, $[X, Z] = 0$, $[Y, Z] = 0$. Construct its adjoint representation. Is this a faithful representation?

Solution:

$$\mathrm{ad}(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{ad}(Y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}(Z) = 0.$$

This representation is not faithful because $Z \neq 0$, but $\mathrm{ad}(Z) = 0$.

(4) Construct the irreducible 4-dimensional representations of $\mathfrak{su}(2)$. That is, find the matrices for J_3, then J_1 and J_2.

Solution: The four-dimensional irrep for $\mathfrak{su}(2)$ has Casimir value $b = j(j+1) = 15/4$, and eigenvalues $m = 3/2, 1/2, -1/2, -3/2$. So we have immediately that

$$J_3 = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & \frac{-3}{2} \end{pmatrix}$$

Next we must find the matrices for the raising and lowering elements J_+, J_- w.r.t. the standard basis. Recall that the constant for J_+ is $C^+ = \sqrt{b - m^2 - m}$, $m = 1/2, -1/2, -3/2$ and those for J_- are $C^- = \sqrt{b - m^2 + m}$ for $m = 3/2, 1/2, -1/2$. So

$$J_+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

Now since $J_1 = \frac{1}{2}(J_- + J_+)$ and $J_2 = \frac{i}{2}(J_- - J_+)$ we have that

$$J_1 = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \quad J_2 = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

(5) Show that an irreducible matrix representation of an Abelian group must be one-dimensional.

Solution: An abelian group is one for which $g_1 g_2 = g_2 g_1$. So in a matrix representation, we must have the corresponding matrices commute: $\rho(g_1)\rho(g_2) = \rho(g_2)\rho(g_1)$, for all $g_1, g_2 \in G$. Fix g_1, say, and write $\rho(g_1) = M$, a square matrix. Then we have that $M\rho(g_2) = \rho(g_2)M$ for all group elements g_2. Since we assume ρ is irreducible, Schur's lemma implies $M = \lambda I$ for some λ. But this says that $\rho(g_1) = \lambda I$, for each g_1, and we deduce that ρ must be one-dimensional.