PHYS4301: SOLUTIONS TO THE TUTORIAL PROBLEMS FOR WEEK 9

(1) *$SO(2)$ and its generator.* Recall that a 2D rotation matrix is

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ and } X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that $R(\theta) = e^{\theta X}$ by evaluating the expression for each entry in the series expansion for the matrix $e^{\theta X} = \sum_{n=0}^{\infty} \frac{(\theta X)^n}{n!}$.

Solution: First show that

$$X^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then turn the matrix series into a series expansion for each element in the matrix. Observe that the top left and bottom right elements (i.e. diagonal terms) only involve even powers of θ. And the other two off-diagonal matrix elements only contain odd powers of θ. You should recognise these as the series expansions for $\cos\theta$ and $\sin\theta$ as required.

(2) Consider the Lie algebra associated with both $SO(3)$ and $SU(2)$. For $SO(3)$ use the physics convention $R(\vec{n}, \theta) = e^{i\theta J}$ and the basis

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \quad J_y = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad J_z = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

subscripts x, y, z refer to rotation about x,y,z-axes respectively. For $SU(2)$ use $U = e^{iH}$ and the basis

$$s_1 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

Make the correspondence $J_x \to s_1$, $J_y \to s_2$, and $J_z \to s_3$.

The matrix exponential e^{itJ_z} for $t \in [0, 2\pi]$ defines a closed path of elements in $SO(3)$ from the identity back to itself.

$$e^{itJ_z} = R_z(t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What is the corresponding path through $SU(2)$? Write down the parametrised matrix for this path (i.e. e^{its_3}). Is the path closed?

Solution:

$$U(t) = e^{its_3} = \exp\left[\begin{pmatrix} it/2 & 0 \\ 0 & -it/2 \end{pmatrix}\right] = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}$$

This path is not closed in $SU(2)$ because $U(0) = I$, but $U(2\pi) = -I$. This is the origin of the concept that "you have to turn through an angle of 4π to get back to where you started".

(3) The Heisenberg Lie algebra consists of 3×3 matrices of the form

$$H = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \quad \text{with } a, b, c \in \mathbb{R}.$$

· Write out the basis matrices $\{X, Y, Z\}$ for this Lie algebra so that $H = aX + bY + cZ$.

· Compute the commutation relations for this basis.
· Determine the Lie group associated with this algebra, using $g = e^H$.

Solution:

$$[X,Y] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = Z$$
$$[Y,Z] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0$$
$$[Z,X] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

The matrix exponential is $e^H = I + H + \frac{1}{2}H^2 + \ldots$ so we start by computing the powers of H:

$$H^2 = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$H^3 = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & ab \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This means the series expansion for $e^H = I + H + \frac{1}{2}H^2$ exactly.

$$e^H = \begin{pmatrix} 1 & a & z \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } z = c + \tfrac{1}{2}ab$$

The Heisenberg algebra appears in quantum mechanics as the canonical commutation relations with X being position, Y momentum and Z the identity times Planck's constant $i\hbar I$: $[\hat{x}, \hat{p}] = i\hbar I$, and (trivially) $[\hat{x}, i\hbar I] = 0, [\hat{p}, i\hbar I] = 0$. It is also an example of a *nilpotent* Lie algebra.

(4) Show that the Baker-Campbell-Hausdorff formula holds up to second order terms. Try higher order terms if you're feeling adventurous.

$$\begin{aligned} e^X e^Y &= \left(\sum_{n=0}^{\infty} \frac{X^n}{n!}\right) \left(\sum_{n=0}^{\infty} \frac{Y^n}{n!}\right) \\ &= \exp\left[X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) + \cdots\right] \end{aligned}$$

Solution: Expand the product of series to third order:

$$\begin{aligned} LHS &= \left(I + X + \tfrac{1}{2}X^2 + \tfrac{1}{6}X^3\right)\left(I + Y + \tfrac{1}{2}Y^2 + \tfrac{1}{6}Y^3\right) \\ &= I + X + Y + \tfrac{1}{2}(X^2 + Y^2) + XY + \tfrac{1}{6}(X^3 + Y^3) + \tfrac{1}{2}(X^2Y + XY^2) \end{aligned}$$

And similarly for the claimed expression $Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) + \cdots$. We must compute terms to third order from the series for e^Z:

$$
\begin{aligned}
RHS &= \sum_{n=0}^{\infty} \frac{Z^n}{n!} \\
&= I + \big(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}([X,[X,Y]] + [Y,[Y,X]]) + \cdots\big) \\
&\quad + \tfrac{1}{2}\left(X + Y + \tfrac{1}{2}[X,Y] + \cdots\right)^2 \\
&\quad + \tfrac{1}{6}\left(X + Y + \cdots\right)^3 + \text{ higher order terms.} \\
&= I + \big(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}([X,[X,Y]] - [Y,[X,Y]]) + \cdots\big) \\
&\quad + \tfrac{1}{2}(X^2 + Y^2 + XY + YX) + \tfrac{1}{4}\left(X[X,Y] + Y[X,Y] + [X,Y]X + [X,Y]Y\right) + \tfrac{1}{8}([X,Y])^2 \\
&\quad + \tfrac{1}{6}(X+Y)(X^2 + XY + YX + Y^2) \\
&= I + X + Y + \tfrac{1}{2}(XY - YX) + \tfrac{1}{12}(X[X,Y] - [X,Y]X - Y[X,Y] + [X,Y]Y) \\
&\quad + \tfrac{1}{2}(X^2 + Y^2 + XY + YX) + \tfrac{1}{4}\left(X[X,Y] + Y[X,Y] + [X,Y]X + [X,Y]Y\right) \\
&\quad + \tfrac{1}{6}\left(X^3 + XY^2 + X^2Y + XYX + YX^2 + Y^3 + YXY + Y^2X\right) \\
&= I + X + Y + \tfrac{1}{2}(X^2 + Y^2) + XY + \text{ terms of order 3.}
\end{aligned}
$$

This agrees with the LHS up to terms of order 2. To simplify the terms of order 3, we have to further expand the commutators.

$$
\begin{aligned}
\text{order-3 terms } &= (\tfrac{1}{12} + \tfrac{1}{4})X[X,Y] + (\tfrac{1}{4} - \tfrac{1}{12})[X,Y]X + (\tfrac{1}{4} - \tfrac{1}{12})Y[X,Y] + (\tfrac{1}{12} + \tfrac{1}{4})[X,Y]Y \\
&\quad + \tfrac{1}{6}\left(X^3 + XY^2 + X^2Y + XYX + YX^2 + Y^3 + YXY + Y^2X\right) \\
&= \tfrac{1}{3}(X^2Y - XYX) + \tfrac{1}{6}(XYX - YX^2) + \tfrac{1}{6}(YXY - Y^2X) + \tfrac{1}{3}(XY^2 - YXY) \\
&\quad + \tfrac{1}{6}\left(X^3 + XY^2 + X^2Y + XYX + YX^2 + Y^3 + YXY + Y^2X\right) \\
&= \tfrac{1}{6}(X^3 + Y^3) + \tfrac{1}{2}X^2Y + \tfrac{1}{2}XY^2
\end{aligned}
$$

This expression is the same as the LHS terms of order 3 so we are finished.