The University of Melbourne

Department of Mathematics and Statistics

Semester 2 Assessment 2010

620-297 (MAST20022) Group Theory and Linear Algebra

Reading time: Fifteen minutes
Writing time: Three hours
Identical Examination Papers: none
Common Content Papers: none

This paper has five pages (including this page).

Total marks 120

Authorized materials:
No materials are authorized. Students entering the examination venue with notes or printed material related to the subject, calculators or computers, or mobile phones, should stand in their place immediately and surrender these to an invigilator before the instruction to commence writing is given.

Instructions to Invigilators:
Script books only are required. Candidates are permitted to take this question paper with them at the end of the examination. No written or printed material related to the subject may be brought into the examination.

Instructions to Students:
The examination paper is in two sections. The questions in Section A are shorter and more routine than those in Section B. It is recommended that candidates attempt the questions in Section A before trying those in Section B. It is possible to pass the examination on marks from Section A alone. All questions may be attempted.

Please give complete explanations in all questions, and give careful statements of any results from the notes or lectures that you use.

Section A

1. (a) Use the Euclidean algorithm to find $d = \gcd(2761, 506)$.

   (b) Find integers x, y such that $2761x + 506y = d$.

(6 marks)

2. Consider the polynomials $\{x^3+1, x^3+x^2, x^2+1\}$ in the vector space $V = \mathcal{P}_3(\mathbb{R})$ of all polynomials in x of degree ≤ 3 with real coefficients.

   (a) Are the polynomials linearly independent?

   (b) Do the polynomials form a basis for V?

   Give brief explanations.

(6 marks)

3. Let V be a real vector space with bases $\mathcal{B} = \{v_1, v_2\}$ and $\mathcal{B}' = \{v_1', v_2'\}$ where $v_1' = v_1 + v_2$ and $v_2' = 2v_1 - 3v_2$. Let $f : V \to V$ be a linear transformation satisfying

$$f(v_1) = v_1 + 3v_2, \quad f(v_2) = 2v_1.$$

   (a) Find the matrix of f with respect to the basis $\mathcal{B}$.

   (b) Find the matrix of f with respect to the basis $\mathcal{B}'$.

   (c) Deduce the eigenvalues of f.

(6 marks)

4. Consider the matrix

$$A = \begin{bmatrix} 5 & 4 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

   (a) Find the minimal polynomial of A.

   (b) Find a Jordan normal form for A.

   Give brief reasons for your answers.

(6 marks)

5. Determine whether the matrix $A = \begin{bmatrix} 2i & -1 \\ 1 & 3i \end{bmatrix}$ is

   (a) Hermitian, (b) unitary, (c) normal, (d) diagonalizable.

   Give brief explanations.

(6 marks)

6. (a) Calculate the product $(143)(123)$, and find the order of this element in the permutation group S_4.
   (b) Find the conjugacy class of (34) in S_4.
   (c) How elements are there in the centralizer of (34) in S_4?

   Give brief explanations.

(6 marks)

7. A group G contains subgroups H and K of orders 7 and 9.
   (a) What is the least possible order for G?
   (b) What can you say about the order of $H \cap K$?

   Give brief explanations.

(6 marks)

8. Give example of a group G and a subgroup H that is *not normal* in G. Justify your answer.

(6 marks)

9. Let $GL(2, \mathbb{R})$ denote the group of all invertible 2×2 matrices with real entries. For each $A \in GL(2, \mathbb{R})$ and each column vector $x \in \mathbb{R}^2$ let $A \cdot x$ be the matrix product Ax.
   (a) Show that this defines an action of the group $GL(2, \mathbb{R})$ on $\mathbb{R}^2$.
   (b) Find the orbit and stabilizer of $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
   (c) Find the orbit and stabilizer of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(6 marks)

10. A group G of order 35 acts on a set X of 13 elements. Show that there exists an element $x \in X$ that is *fixed* by all elements in G.

(6 marks)

Section B

11. Let $f : V \to V$ be a linear operator on a finite dimensional complex inner product space V.

    (a) Give definitions of

    (i) the adjoint f^* of f,

    (ii) the orthogonal complement $W^\perp$ of a subspace W of V.

    (b) Prove that the nullspace of f is equal to the orthogonal complement of the range of f^*.

    (c) Deduce that $\dim(\text{range}\, f) = \dim(\text{range}\, f^*)$.

(10 marks)

12. Let A be a 5×5 complex matrix with characteristic polynomial

$$c(X) = (X-1)^2(X-2)^3.$$

    (a) List the possible minimal polynomials for A.

    (b) Hence list the possible Jordan normal forms for A (up to reordering the Jordan blocks).

    (c) Write down the dimensions of the eigenspaces of A corresponding to each of the possible Jordan normal forms. Do the dimensions of these eigenspaces determine the exact Jordan form of A (up to reordering the Jordan blocks)?

    Give brief explanations.

(10 marks)

13. Let V be a finite dimensional complex inner product space with $\dim V = n$, and let $T : V \to V$ be a self-adjoint linear operator.

    (a) Show that $\langle Tx, x \rangle$ is real for all $x \in V$.

    (b) Show that every eigenvalue of T is real.

    (c) If $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are the eigenvalues of T, show that

$$\lambda_1 ||x||^2 \leq \langle Tx, x \rangle \leq \lambda_n ||x||^2$$

    for all $x \in V$. [Hint: use the spectral theorem.]

(10 marks)

14. Consider the set U of upper triangular matrices in $GL(2,\mathbb{R})$:

$$U = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R}, a \neq 0, c \neq 0 \right\}.$$

(a) Show that U is a group using matrix multiplication as the operation.

(b) Show that the function $f : U \to \mathbb{R}^*$ defined by

$$f\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \right) = a$$

is a homomorphism, where $\mathbb{R}^*$ is the group of non-zero real numbers under multiplication.

(c) Find the image and kernel of f.

(d) Find a normal subgroup H of U with $H \neq U$ such that the quotient group U/H is isomorphic to a subgroup of $\mathbb{R}^*$. Give a brief explanation.

(10 marks)

15. The set of eight elements $G = \{\pm 1, \pm 2, \pm 4, \pm 7\}$ forms a group under multiplication modulo 15. (You do not need to prove this.)

(a) What does Lagrange's theorem tell you about the possible orders of elements in G?

(b) Find the orders of all the elements in G.

(c) Decide which of the groups $\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to G. (You do not need to give an isomorphism.)

Give brief explanations.

(10 marks)

16. Let G be a finite group acting on a finite set X. For each $g \in G$, let

$$\mathrm{Fix}(g) = \{x \in X : g \cdot x = x\}$$

denote the set of elements of X fixed by g. For each $x \in X$, let $\mathrm{Stab}(x)$ denote the stabilizer of x. For any set A, let $|A|$ denote the number of elements in A.

(a) Let

$$S = \{(g, x) \in G \times X : g \cdot x = x\}.$$

By counting S in two ways, show that

$$|S| = \sum_{x \in X} |\mathrm{Stab}(x)| = \sum_{g \in G} |\mathrm{Fix}(g)|.$$

(b) Show that if $g \cdot x = y$ then $g\,\mathrm{Stab}(x) g^{-1} = \mathrm{Stab}(y)$, hence

$$|\mathrm{Stab}(x)| = |\mathrm{Stab}(y)|.$$

(c) Prove that the number of distinct orbits is

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|.$$

(10 marks)