3 Week 3

Equivalent representations

As we saw above, two n-dimensional representations of a group G, say ρ_1, ρ_2, are equivalent or similar if there is a (single) linear transformation $S \in GL(n, \mathbb{C})$ so that $\rho_2(g) = S^{-1}\rho_1(g)S$ for every $g \in G$.

Reducible and irreducible representations

An n-dimensional representation of a group G is *reducible* if it is equivalent to a representation ρ for which every linear transformation $\rho(g) \in GL(n, \mathbb{C})$ has a fixed upper-triangular block form, e.g.,

$$\rho(g) = \begin{pmatrix} R_{11}(g) & R_{12}(g) \\ 0 & R_{22}(g) \end{pmatrix}$$

where $R_{11}(g)$ are $k \times k$ matrices, $R_{22}(g)$ are $(n-k) \times (n-k)$ matrices and $R_{12}(g)$ are $k \times (n-k)$. This block form means that the first k coordinates and the last $(n-k)$ coordinates form invariant subspaces, so R_{11}, R_{22} are k- and $(n-k)$-dimensional representations for G respectively.

An n-dimensional representation of a group G is *completely reducible* if it is equivalent to a representation ρ for which every linear transformation $\rho(g) \in GL(n, \mathbb{C})$ has a fixed *diagonal* block form, e.g.,

$$\rho(g) = \begin{pmatrix} R_{11}(g) & 0 & \cdots & 0 \\ 0 & R_{22}(g) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & R_{rr}(g) \end{pmatrix}$$

Where each R_{kk} is an *irreducible* representation for the group G. This means the coordinates belonging to each block form invariant and independent subspaces.

Aside: It can be shown that if ρ is a finite-dimensional *unitary* representation of a matrix Lie group then ρ is completely reducible.

For example, the group $SO(2)$ has a 3-dimensional representation as the group of rotations fixing the z-axis:

$$\rho(R_\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These matrices already have a block form of $(2 \times 2) \oplus (1 \times 1)$. The 1 block is just the identity; the third coordinate of any point is unchanged by the action of this matrix. The 2×2 block acts only on the first and second coordinates, but mixes them up. However the 2×2 block can be further reduced to two 1×1 blocks by a similarity $S \in GL(3, \mathbb{C})$ built from the eigenvectors for $\rho(R_\theta)$ because they are independent of θ.

$$S^{-1}\rho(R_\theta)S = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In this basis of eigenvectors, each (complex) coordinate is transformed separately; we write $\rho = \rho_{1,1} \oplus \rho_{1,-1} \oplus \rho_{1,0}$.

An *irreducible* representation is defined as one that is not reducible.

In other words, an irreducible representation has no invariant subspaces other than the trivial one and all of $\mathbb{C}^n$. What this means is that all representations are either irreducible, or they are built up in a block-matrix fashion from smaller irreducible parts. The irreducible representations are the fundamental building blocks in representation theory. Our challenge now is to find all the irreducible representations (shortened to *irreps*) for a given group.

Representations of a Lie Algebra

Definition An n-dimensional representation of a Lie Algebra $\mathfrak{g}$ is a homomorphism into the vector space of $n \times n$ matrices with elements in $\mathbb{C}$ $\rho : \mathfrak{g} \to M(n, \mathbb{C})$. As a homomorphism, ρ preserves addition and multiplication of elements from $\mathfrak{g}$:

$$\rho(aX + bY) = a\rho(X) + b\rho(Y), \quad \rho(XY) = \rho(X).\rho(Y) \text{ (whether or not } XY \in \mathfrak{g}).$$

Representations of an abstract Lie algebra must also map the Lie bracket to the matrix commutator:

$$\rho([A, B]) = [\rho(A), \rho(B)] = \rho(A).\rho(B) - \rho(B).\rho(A)$$

When working with a Lie algebra, we only need to define and compare the representations of a given basis for the algebra. So let $A_1, \ldots, A_d$ be the basis and $[A_p, A_q] = \sum_r c^r_{pq} A_r$ define the structure constants c^r_{pq}.

Then a representation $\rho : \mathfrak{g} \to M(n, \mathbb{C})$ is defined by specifying the matrices for $\rho(A_p)$, because linearity implies that the representation of a general element $X = (a_1 A_1 + \ldots + a_d A_d) \in \mathfrak{g}$ will be given by $\rho(X) = a_1\rho(A_1) + \ldots + a_d\rho(A_d)$. Similarly, the commutator in $\mathfrak{g}$ becomes

$$\rho([A_p, A_q]) = [\rho(A_p), \rho(A_q)] = \sum_r c^r_{pq}\rho(A_r)\,.$$

Definitions for faithful, trivial, equivalent, reducible and irreducible representations of Lie Algebras are directly analogous to those for groups. Note that the trivial representation of a Lie algebra is one where every element is mapped to the zero matrix.

Adjoint representation of a Lie algebra

Now, there is a special representation of the Lie algebra called the *adjoint* representation, written ad, that maps $\mathfrak{g}$ into $M(d, \mathbb{C})$, where d is the dimension of $\mathfrak{g}$. This representation is defined to be the matrix $\mathrm{ad}(X)$ that makes this equation hold for each $q = 1, \ldots, d$:

$$[X, A_q] = \sum_{r=1}^{d} (\mathrm{ad}(X))_{rq} A_r$$

Since we really only need to specify the representation of the basis for the algebra, we have that for $p = 1, \ldots, d$:

$$(\mathrm{ad}(A_p))_{rq} = c^r_{pq}$$

Example. Our Lie algebra $\mathfrak{su}(2)$ has Hermitian basis $\{J_1, J_2, J_3\}$ and structure constants $c^r_{pq} = i\epsilon_{pqr}$, so the adjoint representation is 3-dimensional with $(\mathrm{ad}(J_p))_{rq} = i\epsilon_{pqr}$ giving us

$$\mathrm{ad}(J_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \mathrm{ad}(J_2) = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \mathrm{ad}(J_3) = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice that these matrices are simply i times those found as the anti-Hermitian basis $\{A_1, A_2, A_3\}$ for $\mathfrak{so}(3)$.

As you might have noticed, there are going to be many representations of an abstract Lie algebra; they may have different dimensions; may be reducible or irreducible; and a simple change of basis for our linear space $\mathbb{C}^n$ changes the form of the matrices in our representation in a way that might reveal or obscure this reducibility.

We need a systematic way to build up a basis for the linear space, and to test if a given representation is reducible or not. This systematic method uses something called the Cartan subalgebra. The basis of eigenvectors for this subalgebra will give us a standardised basis for the linear space of each representation.

Cartan subalgebra

A Cartan subalgebra, $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ with the following two properties:

· $\mathfrak{h}$ is a maximal Abelian subalgebra of $\mathfrak{g}$.

· The adjoint representation for $\mathfrak{g}$ restricted to $\mathfrak{h}$ is completely reducible.

The first point means that $[H_1, H_2] = 0$ for all $H_1, H_2 \in \mathfrak{h}$ and if we try to add any other element from $\mathfrak{g}$ to $\mathfrak{h}$ this property no longer holds. So for any $X \in \mathfrak{g}$ with $X \notin \mathfrak{h}$, $[X, H] \neq 0$ for some $H \in \mathfrak{h}$.

The second point means that the set of matrices $\mathrm{ad}(H)$, for $H \in \mathfrak{h}$ can be *simultaneously (block) diagonalised*. The basis for $\mathbb{C}^d$ in which the matrices $\mathrm{ad}(H)$ are diagonalised are eigenvectors for every $H \in \mathfrak{h}$. The eigenvalues of these particular vectors will be used as labels of corresponding physical states.

Example. Returning to our Lie algebra $\mathfrak{su}(2)$, we discover that any Cartan subalgebra can have just a single element of the basis set $\{J_1, J_2, J_3\}$. It is typical to use $\mathfrak{h} = \mathrm{span}\{J_3\}$. To diagonalise the matrix $\mathrm{ad}(J_3)$ we apply the similarity transformation built from its eigenvectors. The eigenvalues of $\mathrm{ad}(J_3)$ are $1, 0, -1$, and we will call the associated eigenvectors v_1, v_0, v_{-1}. Then the change-of-basis matrix has the eigenvectors as its columns: $S = (v_1|v_0|v_{-1})$, and we find the diagonalised version of J_3 is

$$
\begin{aligned}
\tilde{J}_3 = S^{-1} J_3 S &= \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{-i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\end{aligned}
$$

For completeness, the corresponding matrices for J_1 and J_2 are

$$
\tilde{J}_1 = S^{-1} J_1 S = \begin{pmatrix} 0 & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & \frac{-i}{\sqrt{2}} & 0 \end{pmatrix} \quad \tilde{J}_2 = S^{-1} J_2 S = \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}
$$

To recap, we started with the 3-dimensional representation ad for $\mathfrak{su}(2)$, found its Cartan subalgebra, $\mathfrak{h}$ and created a basis for $\mathbb{C}^3$ using the eigenvectors for $J_3 \in \mathfrak{h}$. The eigenvalues for these eigenvectors form labels for these basis vectors.

Casimir elements

The next piece of the representation theory puzzle is to find representations of different dimensions. There is a special object called a quadratic Casimir element that will help us with this.

A (quadratic) Casimir element C is a special combination of basis elements for the Lie algebra, that is known to commute with all elements of $\mathfrak{g}$. Note that C is NOT an element of $\mathfrak{g}$, but of its "universal enveloping algebra". When working with matrix Lie algebras this just means that C is a matrix, but not necessarily belonging to the Lie Algebra.

The reason we require C to commute with elements of $\mathfrak{g}$ is due to a collection of results referred to as *Schur's Lemma*. The version we use is

Schur's Lemma: Suppose $\rho : \mathfrak{g} \to M(n, \mathbb{C})$ is an irreducible representation of a Lie algebra and B is an $n \times n$ matrix satisfying $B\rho(X) = \rho(X)B$ for all $X \in \mathfrak{g}$. Then $B = bI_n$ is a multiple of the identity.

The scalar value b above, associated with $\rho(C)$ gives us a way to label or name the irreps for $\mathfrak{g}$.

Example. The Casimir element for $\mathfrak{su}(2)$ is given by $C = J^2 = J_1^2 + J_2^2 + J_3^2$. The notation J^2 for C reflects the physical interpretation of this operator as the magnitude of the orbital angular momentum. Although $C \notin \mathfrak{su}(2)$, we can still obtain its representation with respect to the matrices $\mathrm{ad}(J_k)$. Doing this we have that,

$$\mathrm{ad}(C) = [\mathrm{ad}(J_1)]^2 + [\mathrm{ad}(J_2)]^2 + [\mathrm{ad}(J_3)]^2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

So for this 3-dimensional rep we have the 'magic number' 2. Whatever basis for $\mathbb{C}^3$ we choose, $\mathrm{ad}(C)$ will have exactly the same form. But for a general irrep of dimension n, we have a different value $\rho(C) = bI_n$, and so ρ will be labelled by the value b.

Exercise: Show that $C = J^2$ commutes with the basis elements $\{J_1, J_2, J_3\}$.

Aside: Casimir elements seem to be plucked out of a hat, but are in fact related to the Laplacian operator on a Riemannian manifold.

The irreducible representations of $\mathfrak{su}(2)$

The lectures will follow p. 56–62 of [S]. In summary:

Start by assuming we have an n-dimensional irreducible representation with $J^2 = bI_n$ and use the eigenvectors of J_3 as a basis for the vector space $\mathbb{C}^n$. This means $J_3|b,m\rangle = m|b,m\rangle$ and $J^2|b,m\rangle = b|b,m\rangle$. The notation $|b,m\rangle$ is just a physics-y way to write down an eigenvector that lets us keep track of its labels. We could just as well have used subscripts as in the earlier example, $J_3 v_{b,m} = m v_{b,m}$.

Derivation of the eigenvalues for J_3.
Define $J_+ = J_1 + iJ_2$ and $J_- = J_1 - iJ_2$. (Note that the coefficient i means we are now working with the *complexified* Lie algebra $\mathfrak{su}(2)_{\mathbb{C}}$.) We can compute the commutators and find that $[J_3, J_+] = J_+$, $[J_3, J_-] = -J_-$ and $[J_+, J_-] = 2J_3$. The new terms are called the raising and lowering or ladder operators because if we start with an eigenvector of J_3, we find that the vector $J_+|b,m\rangle$ must also be an eigenvector for J_3, but with eigenvalue $(m+1)$.

$$\begin{aligned} [J_3, J_+]|b,m\rangle &= J_+|b,m\rangle \quad \text{so} \\ J_3J_+|b,m\rangle - J_+J_3|b,m\rangle &= J_+|b,m\rangle \quad \text{rearranging we get} \\ J_3(J_+|b,m\rangle) &= (m+1)J_+|b,m\rangle \end{aligned}$$

Similarly for J_-, the vector $J_-|b,m\rangle$ is an eigenvector for J_3 with eigenvalue $(m-1)$.

We are working in a finite-dimensional representation, so the number of eigenvectors is finite and equals n, the dimension of the representation. So we have a maximum eigenvalue $m = j$ and $J_+|b,j\rangle = 0$. Then J_- takes us down through the n eigenvectors to one with $m = j-n+1 = k$, say. So $J_-|b,k\rangle = 0$.

Now consider the action J_-J_+ on the maximal eigenvector $|b,j\rangle$. First we expand the ladder operators in terms of J_1, J_2, and use the Casimir operator to write $J_-J_+ = J^2 - J_3^2 - J_3$ Then:

$$J_-J_+|b,j\rangle = 0 \implies (J^2 - J_3^2 - J_3)|b,j\rangle = (b - j^2 - j)|b,j\rangle = 0$$

And we conclude $b = j(j+1)$.

Similarly, the action of J_+J_- on the minimal eigenvector $|b,k\rangle$ is found to be

$$J_+J_-|b,k\rangle = 0 \implies (J^2 - J_3^2 + J_3)|b,k\rangle = (b - k^2 + k)|b,k\rangle = 0 \implies b = -k(-k+1)$$

Comparing the two expressions for b, we see that $k = -j$. But the integer decrements of eigenvalues tell us that $j - n + 1 = k = -j$, so that $j = (n-1)/2$, where n, the dimension of the representations, is an integer.

To conclude, an irreducible n-dimensional representation of $\mathfrak{su}(2)_{\mathbb{C}}$ has a basis of eigenvectors for J_3, with eigenvalues $m = j, j-1, \ldots, -j$, the Casimir element $J^2 = bI_n$ with $b = j(j+1) = (n^2-1)/4$, where $j = (n-1)/2$. It is commonplace to name these irreps by their value for j, so the possibilities are $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

Derivation of the matrix representations of J_1, J_2, J_3.

The matrix for J_3 is simply the diagonal one with its eigenvalues listed in decreasing order:

$$J_3 = \begin{pmatrix} j & 0 & \ldots & 0 \\ 0 & (j-1) & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & -j \end{pmatrix}$$

To find the matrices for J_1 and J_2 we must find their action on each of the eigenvectors $|b, m\rangle$. We know that $J_\pm |b, m\rangle$ is an eigenvector of J_3 with eigenvalue $(m \pm 1)$, but all this tells us is that $J_+|b, m\rangle = C_+^m |b, m+1\rangle$, and similarly $J_-|b, m\rangle = C_-^m |b, m-1\rangle$. We find these constants by imposing the condition that each eigenvector $|b, m\rangle$ has unit norm:

$$(J_+|b, m\rangle)^\dagger (J_+|b, m\rangle) = |C_+|^2 |b, m+1\rangle^\dagger |b, m+1\rangle$$

Next use the fact that $J_+^\dagger = J_-$ and $J_- J_+ = J^2 - J_3^2 - J_3$ to find that

$$C_+^m = \sqrt{b - m^2 - m} = \sqrt{j^2 - m^2 + j - m}$$

A similar derivation shows that

$$C_-^m = \sqrt{b - m^2 + m} = \sqrt{j^2 - m^2 + j + m}$$

The matrices for J_+ and J_- then have a single non-zero upper diagonal and lower diagonal form respectively:

$$J_+ = \begin{pmatrix} 0 & C_+^{j-1} & 0 & \ldots & 0 \\ 0 & 0 & C_+^{j-2} & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & C_+^{-j} \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ C_-^j & 0 & \ldots & 0 & 0 \\ 0 & C_-^{j-1} & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & C_-^{-j+1} & 0 \end{pmatrix}$$

Then $J_1 = \frac{1}{2}(J_+ + J_-)$, and $J_2 = -\frac{i}{2}(J_+ - J_-)$ will give their respective matrices.

Final recap: We've seen that by simply choosing the dimension of the representation, n, we can construct the matrix representations for each of J_1, J_2, J_3 in a basis for $\mathbb{C}^n$ built from eigenvectors for J_3. This was achieved using two elements of the Lie algebra: J_+ and J_- called the raising and lowering operators, and a matrix J^2 (the Casimir element) that we know must be a multiple of the identity, $J^2 = bI_n$ in any irreducible representation. The value of $b = (n^2-1)/4$.

Relationship between Lie group and Lie algebra representations

Suppose we have a finite representation for a (matrix) Lie group $\rho : G \to GL(n, \mathbb{C})$, then we can define a representation of the Lie algebra $\mathfrak{g}$ via the exponential map as follows. Let $g = e^X$, with $\rho(g) = \rho(e^X) = A \in GL(n, \mathbb{C})$. Then $\tilde{\rho}(X) := Y$ is defined to be the matrix $Y \in M(n, \mathbb{C})$ such that $e^{\tilde{\rho}(X)} = e^Y = A = \rho(e^X)$.

The converse result holds when G is a simply-connected matrix Lie group and we start with a representation for its Lie algebra.

Another useful result is that equivalent representations of a Lie group give equivalent representations of its Lie algebra, and conversely when G is connected. More formally: let ρ_1, ρ_2 be representations of a connected matrix Lie group G, with $\tilde{\rho_1}$ and $\tilde{\rho_2}$ the associated representations of its Lie algebra. If $\tilde{\rho_1}$ and $\tilde{\rho_2}$ are equivalent representations of $\mathfrak{g}$, then ρ_1 and ρ_2 are equivalent representations of G. This follows from the property of matrix exponential that $e^{S^{-1}AS} = S^{-1}e^{A}S$.

Example We know that $SU(2)$ and $SO(3)$ have the same Lie algebra, and that $SU(2)$ is the simply connected Lie group. We have derived the irreps for $\mathfrak{su}(2)_{\mathbb{C}}$, and these are indexed by the dimension n, or alternatively by the half-integer $j = (n-1)/2$. It can be shown that only the *integer* values of j (odd dimensions n) correspond to representations of $SO(3)$. These are the "integer spin" representations. The half-integer values of j (even dimensions) give valid representations of $SU(2)$ via the exponential map, but not $SO(3)$.

Consider the $j = \frac{1}{2}$ representation, and the matrix for J_3 in particular:

$$\rho_{\frac{1}{2}}(J_3) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Upon exponentiation we have that

$$u(\omega) = \exp\{i\omega\rho_{\frac{1}{2}}(J_3)\} = \begin{pmatrix} e^{i\omega/2} & 0 \\ 0 & e^{-i\omega/2} \end{pmatrix}$$

We saw on page 13 that such a matrix is equivalent to a unit quaternion $q = \cos(\omega/2)\mathbf{1} + \sin(\omega/2)\mathbf{k}$, and that the unit quaternions are isomorphic to $SU(2)$ but are a double cover of $SO(3)$. In particular,

$$u(\omega + 2\pi) = \begin{pmatrix} e^{i\pi}e^{i\omega/2} & 0 \\ 0 & e^{-i\pi}e^{-i\omega/2} \end{pmatrix} = \begin{pmatrix} -e^{i\omega/2} & 0 \\ 0 & -e^{-i\omega/2} \end{pmatrix}$$

which is a distinct element of $SU(2)$, but *should* map to the same element of $SO(3)$. Here we have $e^{\rho(A)} \neq e^{\rho(A')}$ when we want $g_A = g_{A'} = R_z(\omega/2) \in SO(3)$. The 'representation' is mapping one group element to two matrices, which is not allowed.

On the other hand, the $j = 1$ representation has

$$\rho_1(J_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Upon exponentiation we have that

$$R(\omega) = \exp\{i\omega\rho_1(J_3)\} = \begin{pmatrix} e^{i\omega} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\omega} \end{pmatrix}.$$

We now have that $R(\omega+2\pi) = R(\omega)$ as required for $SO(3)$. Note that the exponentiated ρ_1 is no longer a *faithful* representation of $SU(2)$ because $R(0) = I_3 = R(2\pi)$ whereas $u(0) = \mathbf{1}$ but $u(2\pi) = -\mathbf{1}$. Here we have $e^{\rho(A)} = e^{\rho(A')} = R_z(\omega) \in SO(3)$, but since $g_A \neq g_{A'} \in SU(2)$ the representation here is mapping two elements of $SU(2)$ into one matrix.