PHYS4301 - REVISION GROUP-THEORY QUESTIONS

(1) (a) The Lie group $SL(2,\mathbb{R})$ consists of 2×2 real matrices with $\det(A) = 1$. What are the possible cases for the form of eigenvalues of a matrix in this group?
(b) What geometric transformations do these cases represent?
(c) The real-coefficient Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ consists of 2×2 real matrices with $\mathrm{tr}(X) = 0$. What are the possible cases for eigenvalues of a matrix in the Lie algebra?
(d) Find a matrix $A \in SL(2,\mathbb{R})$ that is not in the image of the exponential map: $A \neq e^X$ for any $X \in \mathfrak{sl}(2,\mathbb{R})$.

(2) *Matrix group question*
Consider G to be the set of 2×2 real matrices that preserve the following inner product:

$$\langle x_1, x_2 \rangle = x_1^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x_2.$$

That is, $M \in G$, if and only if $\langle Mx_1, Mx_2 \rangle = \langle x_1, x_2 \rangle$ for any choice of $x_1, x_2 \in R^2$.
Derive one or more condition(s) on the entries of M that guarantee $M \in G$.
Show that G is a group.

(3) *Lie algebra/ Lie group question*
Define the two matrix groups $SO(3) \subset GL(3,\mathbb{R})$ and $SU(2) \subset GL(2,\mathbb{C})$.
Characterise the matrices in their respective Lie algebras, $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$.
Write down a basis and commutator relations for both algebras.
Explain how the two Lie algebras are related. (1-2 sentences)
Explain how the two Lie groups are related. (short paragraph)

(4) *SU(3) Lie algebra (extension question)*
$SU(3)$ is a local symmetry of the Lagrangian for three fermion fields. It is a simply-connected and compact Lie group. $U \in SU(3)$ satisfies $U^\dagger U = I$ and $\det U = 1$. The Lie algebra defined by $U = e^{itH}$ is $H^\dagger = H$ and $\mathrm{tr}\, H = 0$.
(a) Show that the Lie algebra has 8 elements in its basis.
(b) The structure constants derived from the standard basis for $\mathfrak{su}(3)$ (the Gell-Mann) matrices are defined as follows, using $[T_a, T_b] = i f_{abc} T_c$,

$$f_{123} = 1;\ f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \tfrac{1}{2};\ \text{ and } f_{458} = f_{678} = \tfrac{\sqrt{3}}{2}.$$

All permutations of the indices take ± 1 of these values as appropriate and all other combinations of indices are 0. Show that T_3 and T_8 form a maximal commuting set for $\mathfrak{su}(3)$, and hence a Cartan subalgebra.
(c) Show that the following are eigenvectors for the adjoint representations of T_3 and T_8, and find these eigenvalues. In other words show that $[T_3, v_k] = \lambda_k v_k$ and $[T_8, v_k] = \mu_k v_k$ for $k = 1, \ldots, 8$, and give the values of λ_k, μ_k.

$$\begin{aligned} v_1 &= T_1 + iT_2, \quad v_2 = T_1 - iT_2, \quad v_3 = T_3, \\ v_4 &= T_4 + iT_5, \quad v_5 = T_4 - iT_5, \\ v_6 &= T_6 + iT_7, \quad v_7 = T_6 - iT_7, \quad v_8 = T_8. \end{aligned}$$

The vectors $v_1, v_2, v_4, v_5, v_6, v_7$ form a *root system* for $\mathfrak{su}(3)_{\mathbb{C}}$.