PHYS4301: TUTORIAL PROBLEMS FOR WEEK 11

(1) (a) Consider the $\rho_{(\frac{1}{2},\frac{1}{2})}$ representation of the $\mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$ Lie algebra basis $\{N_k^{\pm},\ k = 1,2,3\}$. Construct matrices for this representation with respect to the tensor basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$ built from the eigenvectors for $N_3^{\pm}$. We derived the matrices for N_p^+ in class, and they are in the lecture notes. Now write out the matrices for N_p^-.

(b) Using the matrices you found for $N_k^{\pm}$, find the matrices for

$$J_k = (N_k^+ + N_k^-) \quad \text{and} \quad K_k = -i(N_k^+ - N_k^-)$$

Solution: (a) You must calculate the action of the representations on the tensorial basis to get:

$$\rho_{(\frac{1}{2},\frac{1}{2})}(N_1^+) = \begin{pmatrix} & & \frac{1}{2} & \\ & & & \frac{1}{2} \\ \frac{1}{2} & & & \\ & \frac{1}{2} & & \end{pmatrix} \qquad \rho_{(\frac{1}{2},\frac{1}{2})}(N_1^-) = \begin{pmatrix} & \frac{1}{2} & & \\ \frac{1}{2} & & & \\ & & & \frac{1}{2} \\ & & \frac{1}{2} & \end{pmatrix}$$

$$\rho_{(\frac{1}{2},\frac{1}{2})}(N_2^+) = \begin{pmatrix} & & -\frac{i}{2} & \\ & & & -\frac{i}{2} \\ \frac{i}{2} & & & \\ & \frac{i}{2} & & \end{pmatrix} \qquad \rho_{(\frac{1}{2},\frac{1}{2})}(N_2^-) = \begin{pmatrix} & -\frac{i}{2} & & \\ \frac{i}{2} & & & \\ & & & -\frac{i}{2} \\ & & \frac{i}{2} & \end{pmatrix}$$

$$\rho_{(\frac{1}{2},\frac{1}{2})}(N_3^+) = \begin{pmatrix} \frac{1}{2} & & & \\ & \frac{1}{2} & & \\ & & -\frac{1}{2} & \\ & & & -\frac{1}{2} \end{pmatrix} \qquad \rho_{(\frac{1}{2},\frac{1}{2})}(N_3^-) = \begin{pmatrix} \frac{1}{2} & & & \\ & -\frac{1}{2} & & \\ & & \frac{1}{2} & \\ & & & -\frac{1}{2} \end{pmatrix}$$

(b)

$$\rho_{(\frac{1}{2},\frac{1}{2})}(J_1) = \frac{1}{2}\begin{pmatrix} & 1 & 1 & \\ 1 & & & 1 \\ 1 & & & 1 \\ & 1 & 1 & \end{pmatrix} \qquad \rho_{(\frac{1}{2},\frac{1}{2})}(K_1) = \frac{i}{2}\begin{pmatrix} & 1 & -1 & \\ 1 & & & -1 \\ -1 & & & 1 \\ & -1 & 1 & \end{pmatrix}$$

$$\rho_{(\frac{1}{2},\frac{1}{2})}(J_2) = \frac{i}{2}\begin{pmatrix} & -1 & -1 & \\ 1 & & & -1 \\ 1 & & & -1 \\ & 1 & 1 & \end{pmatrix} \qquad \rho_{(\frac{1}{2},\frac{1}{2})}(K_2) = \frac{1}{2}\begin{pmatrix} & 1 & -1 & \\ -1 & & & -1 \\ 1 & & & 1 \\ & 1 & -1 & \end{pmatrix}$$

$$\rho_{(\frac{1}{2},\frac{1}{2})}(J_3) = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix} \qquad \rho_{(\frac{1}{2},\frac{1}{2})}(K_3) = \begin{pmatrix} 0 & & & \\ & -i & & \\ & & i & \\ & & & 0 \end{pmatrix}$$

(2) Show that the 2-dimensional version of the special orthochronous Lorentz group $SO^+(1,1)$ is a subgroup of $O(1,1)$ i.e., that the product of two matrices with determinants $= 1$ and $\Lambda_{00} \geq 1$ has the same properties.

Hint: First derive expressions relating the matrix elements a, b, c, d, with

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Solution: The conditions for $SO^+(1,1)$ imply that $ad - bc = 1$, and $a = \sqrt{1+c^2}$, and further,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a^2 - c^2 & ab - cd \\ ab - cd & b^2 - d^2 \end{pmatrix}$$

$$\implies d^2 = 1 + b^2, \;\; ab = cd \quad \implies \ldots\; b = c \;\text{ and } d = \sqrt{1+c^2}$$

So we can parametrise these matrices with hyperbolic sine and cosine:

$$\begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}$$

Using the hyperbolic angle formulae we see that

$$\begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} \cosh(\beta) & \sinh(\beta) \\ \sinh(\beta) & \cosh(\beta) \end{pmatrix} = \begin{pmatrix} \cosh(\alpha+\beta) & \sinh(\alpha+\beta) \\ \sinh(\alpha+\beta) & \cosh(\alpha+\beta) \end{pmatrix}$$

The function $\cosh(t) \geq 1$ for all $t \in \mathbb{R}$, so we see that $SO^+(1,1)$ is closed under multiplication.

(3) The Poincaré group adds spatial translations to the Lorentz group, so let's consider an operator $T(a)$ that acts on a function $f(x)$ and makes it $f(x+a)$. That is:

$$\tilde{f}(x) = T(a)f(x) = f(x+a)$$

Show that the infinitessimal operator corresponding to the Lie group of translation operators is given by

$$p_x = -i\frac{\partial}{\partial x}, \qquad \text{with } T(a) = e^{iap_x}$$

That is, quantum mechanical linear momentum is the Lie algebra generator associated to translation.

Solution: Use the taylor expansion:

$$f(x+a) = f(x) + a\frac{d}{dx}f(x) + \frac{a^2}{2!}\frac{d^2}{dx^2}f(x) + \ldots + \frac{a^n}{n!}\frac{d^n}{dx^n}f(x) + \ldots$$

(4) Consider the natural or defining representation for the Lorentz group and its Lie algebra. Suppose we change our coordinate system to reverse time. Show that the transformed versions of the matrices of J_p and K_p are

$$\widetilde{J_p} = \Lambda_T J_p \Lambda_T^{-1} = J_p, \qquad \widetilde{K_p} = \Lambda_T K_p \Lambda_T^{-1} = -K_p.$$

Explain how this transforms the left-chiral spinor representation into the right-chiral spinor representation.

Solution: The first part is a straightforward computation. The second part uses the observation that $\rho_{(1/2,0)}(J_p) = \rho_{(0,1/2)}(J_p) = \frac{1}{2}\sigma_p$ while $\rho_{(1/2,0)}(iK_p) = -\rho_{(0,1/2)}(iK_p) = \frac{1}{2}\sigma_p$.