

Tutorial 4

Main topics: Minimal polynomials, diagonalisation

1. Show that the matrix

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

has minimal polynomial $X^2 - 2X - 8$. Use this to determine the inverse of A .

2. Find the minimal polynomials of the matrices:

$$A = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

[Hint: Use the following results from the lectures:

- (i) the minimal polynomial divides the characteristic polynomial,
- (ii) every eigenvalue is a root of the minimal polynomial.]

3. Let $f: V \rightarrow V$ be a linear transformation on an n -dimensional vector space with minimal polynomial $m(X) = X^n$.

- (a) Show that there is a vector $v \in V$ such that $f^{n-1}(v) \neq 0$.
- (b) Show that $\mathcal{B} = \{f^{n-1}(v), f^{n-2}(v), \dots, f^2(v), f(v), v\}$ is a basis for V .
- (c) Find the matrix of f with respect to the basis \mathcal{B} .

4. Let K be an algebraically closed field, let $V = M_2(K)$ and consider the linear transformation $f: V \rightarrow V$ given by $f(A) = A^T$ (the transpose of A).

- (a) Find the minimal polynomial of f .
- (b) Find the eigenvalues and corresponding eigenspaces of f .

5. Let $f: V \rightarrow V$ be a linear transformation on an n -dimensional complex vector space satisfying $f^4 = \text{id}_V$. Show that f is diagonalisable. What can you say about the eigenvalues of f ?

6. Let $f: V \rightarrow V$ be a linear transformation on a finite-dimensional vector space V and suppose that $f^2 = f$. Explain how to find a diagonal matrix representing f .

7. Let $f, g: V \rightarrow V$ be commuting linear transformations: $fg = gf$.

- (a) Show that any eigenspace of f is g -invariant.
- (b) If K is algebraically closed and V is finite-dimensional, prove that f and g have a common eigenvector.

8. (Challenging!) Find a linear transformation $f: V \rightarrow V$ on an infinite dimensional vector space V that satisfies no monic polynomial equation $p(f) = 0$.