

## Tutorial 10

**Main topics:** Adjoint, spectral theorem

1. For each of the following matrices decide if it is: (i) self-adjoint (ii) isometric (iii) normal

$$A = \begin{bmatrix} 2 & i \\ -i & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & i \\ 1 & 2+i \end{bmatrix}$$

2. Find a unitary matrix  $U$  such that  $U^*AU$  is diagonal where  $A = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}$ .

3. Let  $f: V \rightarrow V$  be a linear transformation on an inner product space  $V$ .  
Show that  $\ker(f^*) = \operatorname{im}(f)^\perp$ .

4. Let  $f: V \rightarrow V$ , where  $V$  is a finite-dimensional inner product space.

(a) Show that if  $f$  is self-adjoint, then its eigenvalues are real.

(Hint: consider  $\langle f(v), v \rangle$  where  $v$  is an eigenvector)

(b) Show that if  $f$  is an isometry, then its eigenvalues have absolute value 1.

(Hint: consider  $\langle f(v), f(v) \rangle$ )

5. Show that every normal matrix  $A$  has a square root (i.e., there exists a complex matrix  $B$  such that  $B^2 = A$ ).

6. Show that any complex square matrix  $A$  can be written uniquely as a sum  $A = B + C$  where  $B$  is self-adjoint and  $C$  is skew self-adjoint (i.e.,  $C^* = -C$ ). Further,  $A$  is normal if and only if  $B$  and  $C$  commute (i.e.,  $BC = CB$ ).