Tutorial 8

Main topics: Normal subgroups, Lagrange's theorem, quotient groups

1. (a) Write down the left cosets of $H = \langle (1,0) \rangle$ in $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Find the order of each element in the quotient group G/H. Hence identify this quotient group. (Is it isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$?)

   (b) Repeat (a) for $H = \langle (0,2) \rangle$ in $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

2. (a) A group G has fewer than 100 elements and has subgroups of orders 10 and 25. What is the order of G?

   (b) (i) If H and K are subgroups of a finite group G, prove that $|(H \cap K)|$ is a common divisor of $|H|$ and $|K|$.

   (ii) Deduce that if $|H| = 7$ and $|K| = 29$, then $H \cap K = \{e\}$.

3. Prove that if G is a cyclic group, then any quotient group G/N is also cyclic.

4. Let G be a group and let H be a subgroup such that the index $[G : H] = 2$. Prove that H is normal.

5. Let B be the subgroup of $\mathrm{GL}(2, \mathbb{R})$ consisting of upper triangular matrices, and T the subgroup of $\mathrm{GL}(2, \mathbb{R})$ consisting of diagonal matrices.

   (a) Prove that T is isomorphic to $\mathbb{R}^\times \times \mathbb{R}^\times$.

   (b) Show that $f \colon B \to T$ defined by

   $$f\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$$

   is a homomorphism and find its kernel U.

   (c) Use the first isomorphism theorem to identify (i.e., give a simple description of) the quotient group B/U.

   *Try to generalise this result to $\mathrm{GL}(n, \mathbb{R})$.

6. Determine all subgroups of the dihedral group D_5 (which has order 10).