THE UNIVERSITY OF MELBOURNE

Student Number

Semester 2 Assessment, 2018

School of Mathematics and Statistics

# MAST20022 Group Theory and Linear Algebra

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 5 pages (including this page)



**Authorised Materials**

- Mobile phones, smart watches and internet or communication devices are forbidden.
- Calculators, tablet devices or computers must not be used.
- No handwritten or print materials may be brought into the exam venue.

**Instructions to Students**

- You must NOT remove this question paper at the conclusion of the examination.
- The paper is in two sections. The questions in section A are shorter and more routine than those in Section B. It is recommended that students attempt the questions in Section A first. You should attempt all questions.
- Number the questions and question parts clearly. Start each question on a new page.
- Please give complete explanations in all questions and show all your calculations and working. Give careful statements of any results from the notes or lectures that you use.
- There are 14 questions with marks as shown. The total number of marks available is 95.

**Instructions to Invigilators**

- Students must NOT remove this question paper at the conclusion of the examination.

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## Section A: 50 marks total

### Question 1 (5 marks)

(a) Find the multiplicative inverse of $[9]_{14}$ in $\mathbb{Z}/14\mathbb{Z}$.

(b) What can be said about the multiplicative inverse of $[6]_{14}$ in $\mathbb{Z}/14\mathbb{Z}$?

(c) Using the Euclidean Algorithm, find $\gcd(299, 377)$.

### Question 2 (5 marks)

(a) Show that $\mathbb{Q}(i) = \{a + bi \mid a, b, \in \mathbb{Q}\} \subset \mathbb{C}$ is a field (using the usual operations on $\mathbb{C}$). (Hint: You may use that $\mathbb{C}$ is a field.)

(b) (i) Give the definition of what it means to say that a field is *algebraically closed*.

(ii) Show that the field $\mathbb{Q}(i)$ is *not* algebraically closed.

### Question 3 (5 marks)

Consider the matrix $M = \begin{bmatrix} i & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & i \end{bmatrix} \in M_3(\mathbb{C})$.

(a) Find the minimal polynomial of $M$.

(b) Use your answer for part (a) to determine whether $M$ is diagonalizable. (Be sure to give a justification.)

### Question 4 (5 marks)

Find all possible Jordan normal forms (up to permutation of Jordan blocks) for a matrix whose characteristic polynomial is $(X + 2)^2(X - 5)^3$

### Question 5 (5 marks)

Let $V$ be an inner product space and $f : V \to V$ a linear transformation.

(a) Give the definition of the adjoint $f^*$.

(b) Show that if $f = f^*$, then all eigenvalues of $f$ are real.

**Question 6 (5 marks)**

Let $V$ be a vector space and $U, W \leq V$ two subspaces of $V$.

(a) Give the definition of what it means to say that $V = U \oplus W$.

(b) Suppose that $V = U \oplus W$. Show that for all $v \in V$ there exist unique vectors $u \in U$ and $w \in W$ such that $v = u + w$.

**Question 7 (5 marks)**

Let $G$ be a group and $H, K \leq G$ two subgroups.

(a) Prove that $H \cap K$ is a subgroup of $G$.

(b) Give and example to show that $H \cup K$ need not be a subgroup of $G$.

**Question 8 (5 marks)**

For each of the following pairs decide whether or not the two groups are isomorphic.
You should justify your answers.

(a) $(\mathbb{F}_5^\times, \times)$ and $(\mathbb{Z}/5\mathbb{Z}, +)$

(b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$

(c) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ and $D_5$

(d) $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$ and $\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})$

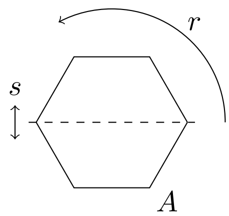
**Question 9 (5 marks)**

(a) Let $G$ be a finite group and $\varphi : G \to H$ a homomorphism.
Prove that the order of $\varphi(G)$ divides the order of $G$.

(b) How many homomorphisms are there from $\mathbb{Z}/3\mathbb{Z}$ to $S_3 \times \mathbb{Z}$?
Be careful to justify your answer.

**Question 10 (5 marks)**

Let $s, r \in D_6$ be the symmetries of a regular hexagon corresponding to reflection across the line shown and rotation by $2\pi/3$ respectively. Consider the subgroup $H \leq D_6$ generated by $\{s, r\}$.




(a) Find the orbit and stabilizer of the vertex $A$ under the action of $H$.
(Label the vertices of the hexagon $A, B, C, D, E, F$ clockwise from the vertex $A$ shown.)

(b) Is the action of $H$ transitive?

## Section B: 45 marks total

### Question 11 (10 marks)

(a) State the Orbit-Stabilizer relation.

(b) Let $G$ be a group of size 16 and $X$ a set having 25 elements. Show that every action of $G$ on $X$ has a fixed point.

(c) Suppose that a finite group $G$ acts non-trivially on a finite set $X$. Let $n = |G|$ and $r = |X|$. Prove that if $n > r!$ then $G$ has a normal subgroup $N \lhd G$ satisfying both $N \neq \{e\}$ and $N \neq G$.

### Question 12 (10 marks)

Let $V$ be a $K$-vector space and let $f : V \to V$ be a linear transformation.

(a) Give the definition of an *$f$-invariant subspace* of $V$.

(b) Let $p(X) \in K[X]$. Prove that $W = \ker(p(f))$ is an $f$-invariant subspace of $V$.

(c) Suppose now that $p(f) = 0$ and that $p(X) = q_1(X)q_2(X)$ for some relatively prime $q_1(X), q_2(X) \in K[X]$. Show that $V = (\ker(q_1(f))) \oplus (\ker(q_2(f)))$.

### Question 13 (10 marks)

(a) Let $V$ be an inner product space. Show that if $f : V \to V$ is a normal linear transformation, then $f(v) = 0$ if and only if $f^*(v) = 0$.

(b) State the Spectral Theorem for linear transformations on a finite dimensional inner product space.

(c) Let $A = \begin{bmatrix} 2 & i \\ i & 2 \end{bmatrix}$.

  (i) Show that $A$ is normal.

  (ii) Find a unitary matrix $U$ such that $U^*AU$ is diagonal.

  (iii) Show that there exists a matrix $B$ such that $B^2 = A$.

**Question 14 (15 marks)**
The set

$$Q = \{1, -1, i, -i, j, -j, k, -k\}$$

has the structure of a group in which 1 is the identity element and the multiplication satisfies:

$$\begin{gathered} i^2 = j^2 = k^2 = -1, \qquad (-1)^2 = 1 \\ ij = k, \qquad jk = i, \qquad ki = j \\ -i = (-1)i, \qquad -j = (-1)j, \qquad -k = (-1)k \end{gathered}$$

(You do not have to prove that this is a group.)

(a) Show that $ji = -k$ in $Q$.

(b) Find the order of each element in $Q$.

(c) Is $Q$ isomorphic to $D_4$? Justify your answer.

(d) Calculate the centre $Z(Q)$ of $Q$.

(e) Determine which of the following groups is isomorphic to the quotient $Q/Z(Q)$:

(i) $\mathbb{Z}/2\mathbb{Z}$

(ii) $\mathbb{Z}/4\mathbb{Z}$

(iii) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(iv) $\mathbb{Z}/8\mathbb{Z}$



**End of Exam—Total Available Marks = 95**