Assignment 2, Group Theory and Linear Algebra, Sam Lloyd 994940

Q1:

a) Characteristic polymial:

$c(x) = (x-i)^2(x+)$

$\Rightarrow$ Candidates for minimal polymial:

$(x-i)(x+1)$, $(x-i)^2(x+1)$

as eigenvalues are $i, -1$ ~~and~~ $\Rightarrow (x-i), (x+1)$ are linear factors of $m(x)$ and $m(x) \mid c(x)$ (Lemma 2.22, 2.24)

try $(x-i)(x+i)$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -1-i & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i+1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & i+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \neq 0$$

Therefore $(x-i)^2(x+1)$ is the minimal polynomial.

b) ~~M is a 3×3 matrix, yet there are only two distinct eigenvalues.~~ Therefore the matrix is not diagonalisable.

$m(x)$ is not the product of distinct linear factors Therefore is not diagonalisable. (2.35)

c)

~~the~~ $m(x) = (x-i)^2(x+1)$, and $c(x) = (x-i)^2(x+1)$

$\Rightarrow$ ① Size of largest i-Jordan block is two,

② Sum of sizes of i-Jordan blocks is 2

③ Size of largest -1-Jordan block is 1

$\therefore J(i,2) \oplus J(-1,1)$

Q2:

i) $A^2 = A^3$, pre-multiply by A

$\Rightarrow A^3 = A^4$

$\Rightarrow A^4 = A^2$

$\Rightarrow A^4 - A^2 = 0$

$\Rightarrow (A^2)(A^2 - I) = 0$

We have ~~expressed~~ found a polynomial p such that $p(A^2) = 0$ and this polynomial has only linear factors in A^2. Thus the minimal polynomial must have the same property.

As the minimal polynomial of A^2 has distinct linear factors, A^2 is diagonalisable.

ii) nilpotency of $A^2 - A$ implies $\exists n \in \mathbb{N}$:

$(A^2 - A)^n = 0$

try $n = 2$

$(A^2 - A)^2 = A^4 - 2A^2 \cdot A + A^2$

$= A^4 - 2A^3 \cdot A + A^4$, as $A^2 = A^3 = A^4$

$= A^4 - 2A^4 + A^4$

$= 0$

$\therefore A^2 - A$ is nilpotent.

/ /

Q3:

a) consider the polynomials $\left(\frac{1}{2}\right)$ ~~$(-x-1)$~~ $\frac{1}{2}(-x-1)\,\frac{1}{2} \in \mathbb{R}[X]$

notice $\frac{1}{2}(-x-1)(x-1) + \frac{1}{2}(x^2+1)$

~~$= \frac{1}{2}(-x^2-1)$~~

$\frac{1}{2}(-x-1)(x-1) + \frac{1}{2}(x^2+1)$

$= \frac{1}{2}[-x^2+1] + \frac{1}{2}[x^2+1]$

$= 1$, $\therefore$ x^2+1 and $x-1$

~~are~~ have gcd 1

ie are co-prime.

b)

the minimal polynomial $m(x)$ divides ~~f~~ $(x-1)(x^2+1)$ but must have $(x-1)$ and (x^2+1) as factors,

therefore $m(x) = (x-1)(x^2+1)$

c)

as $\mathbb{R}^3$ is a finite dimensional vector space with pairwise relatively prime factors in A. we can use proposition 2.27.

- find a basis for $\ker(A-I)$:

$$A-I = \begin{bmatrix} 4 & 5 & -3 \\ -2 & -4 & 2 \\ 4 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 4 & 5 & -3 \\ 0 & -3 & 1 \\ 0 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & -4 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \{(1,1,3)\}$ forms a basis

- find a basis for $\ker(A^2+I)$ (B_1)

$$A^2+I = \begin{bmatrix} 5 & 5 & -3 \\ -2 & -3 & 2 \\ 4 & 2 & -1 \end{bmatrix}^2 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ 12 & 12 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$, equivalently find basis vectors that span the plane $2x+2y-z=0$ in $\mathbb{R}^3$

$\rightarrow$

$2x + 2y - z = 0$

fix $x=0$, ~~...~~ : $(0, 1, \cancel{-}2)$

fix $y=0$ ~~...~~ : $(1, 0, \cancel{-}2)$

$\therefore \{(0,1,\cancel{-}2), (1,0,\cancel{-}2)\}$ forms our basis (B_2)

- now we find $[M|_{\ker(q_i(M))}]_{B_i}$

where q_i are our polynomials and B_i are our bases

$$M \times \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad M \times \begin{bmatrix} 0 \\ 1 \\ \cancel{-}2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$M \times \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = 2 \times \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$B_{\cancel{1}} =$

$\Rightarrow [1] \in M_1(\mathbb{R})$ is $[M|_{\ker(M-I)}]_{B_1}$

$\Rightarrow C_{\cancel{B}} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \in M_2(\mathbb{R})$ is $[M|_{\ker(M-I)}]_{B_2}$

now we must find P, P^{-1}

we are ~~usin~~ changing ~~M into~~ basis $\{b_1, b_2\}$, then applying $A \oplus B$ and returning to the standard basis.

~~$A \oplus B =$~~

$P(B \oplus C)P^{-1} = A$, $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 2 & 2 \end{bmatrix}$, to find P^{-1}: $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & -1 \\ 0 & 1 & 0 & -2 & -1 & 1 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right]$

$\Rightarrow B \oplus C = P^{-1} A P$, where

$B = [1]$, $C = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 2 & 2 \end{bmatrix}$

$\therefore P^{-1} = \begin{bmatrix} 2 & 2 & -1 \\ -2 & -1 & 1 \\ -1 & -2 & 1 \end{bmatrix}$

P is invertible.

Q4 $\rightarrow f^5 = f^4 \Rightarrow f^5 - f^4 = 0 \Rightarrow x^5 - x^4$ is a "multiple" of the minimal polynomial; $m(x)$ (i.e. $m(x) \mid x^5 - x^4$) (proposition 2.22).

$\Rightarrow$ the only possible candidates for eigenvalues are 0, 1 ~~dim~~ as $x^5 - x^4 = x^4(x-1)$, Lemma 2.24.

$\rightarrow \dim(\ker(f)) = 3 \Rightarrow$ precisely 3 Jordan blocks with ~~0~~ zero on the diagonal in the JNF of $[f]$ in any basis.

~~$\rightarrow$ in the case where one is an eigenvector we have~~

~~$\rightarrow$ from $f^5 - f^4 = 0$ we know the sum of the sizes of 0-Jordan blocks is at most 4. NO!~~

$\rightarrow$ as we have a 5x5 matrix, all Jordan block sizes must add to 5.

$\rightarrow$ in the case where 1 is an eigenvalue we know that the size of the largest ~~sum of the sizes of~~ 1-Jordan blocks is at most 1.

i.e. JNF of $[f]_B = J(1,1) \oplus J(0,1) \oplus J(0,1) \oplus J(0,2)$ up to reordering of blocks. or $J(1,1) \oplus J(1,1) \oplus J(0,1) \oplus J(0,1) \oplus J(0,1)$

$\rightarrow$ in the case where 1 is not an eigenvalue we use the facts that ~~① the sum of sizes of 0-Jordan blocks is at most 4~~ there are three 0-Jordan blocks, and there are two ways of partitioning 5: 1+1+3, 2+2+1.

$\therefore \quad [f]_B = J(0,1) \oplus J(0,1) \oplus J(0,3)$

or $= J(0,1) \oplus J(0,2) \oplus J(0,2)$

ie our options are (up to re-ordering of blocks)

$$\begin{bmatrix} 1&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} \rightarrow$$

if $\dim(\ker(f^2)) = 5$, this means

~~our~~ the dimension of the 0-eigenspace is 5.

$\rightarrow$ there are 5 0-Jordan blocks.

i.e ~~the zer~~ $[f]_B$ is the zero matrix.

if ~~any~~ a matrix M ~~with~~ has a 1 in its diagonal ~~with~~
then M^2 will also have a 1 in its diagonal.
(using Lemma 2.5).

~~the~~ a matrix M cannot have $\dim(\ker(M^2)) = 5$
if it has a one in its diagonal.
Therefore only the matricies from the case
where $\lambda = 0$ only apply.

$\therefore$ $J(0,1) \oplus J(0,1) \oplus J(0,3)$ and
$J(0,1) \oplus J(0,2) \oplus J(0,2)$
(up to reordering...).