MAST20022 Group Theory and Linear Algebra

Assignment 3

Due: 4pm Friday October 18

- ▻ Submission is by file upload on the LMS. Scans or photos must be of good quality. You are responsible for checking that your file(s) has uploaded correctly.
- ▻ Include your name, student number and tutorial time at the top of every page.
- ▻ **All answers should be fully justified.**
- ▻ Soliciting answers to assignment questions from internet forums (or elsewhere) is strictly forbidden.

1. For each of the following pairs determine whether or not the two groups are isomorphic. In each case you should either give an isomorphism or explain why none exist.

   (a) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$

   (b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/8\mathbb{Z}$

   (c) $\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$ and $\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})$

   (d) $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$

2. Let G be a group and $X \subseteq G$ a non-empty subset. Define

$$H = \{x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n} \mid n \in \mathbb{N}, x_i \in X, \epsilon_i = \pm 1\} \subseteq G$$

   (a) Show that H is a subgroup of G.

   (b) Show that $H = \langle X \rangle$ (the subgroup of G generated by X).

3. Let $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 3 \end{bmatrix} \in M_2(\mathbb{C})$ and let $\mathcal{B} \subset \mathbb{C}^2$ be a basis of $\mathbb{C}^2$.

   (a) Show that the following defines an inner product on $\mathbb{C}^2$:

$$\langle u, v \rangle = [u]_{\mathcal{B}}^t \, A \, \overline{[v]}_{\mathcal{B}}$$

   (b) Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be the linear transformation determined by $[f]_{\mathcal{B}} = \begin{bmatrix} i & 0 \\ 0 & 2i \end{bmatrix}$.
   Find the matrix $[f^*]_{\mathcal{B}}$.
   (Note that the adjoint is defined using the inner product from part (a))

4. Let $f : V \to V$ be a normal linear transformation on an inner product space.

   (a) Prove that any eigenvector of f is also an eigenvector of f^*.

   (b) Suppose that $u, v \in V$ are eignevectors of f having different eigenvalues. Show that $\langle u, v \rangle = 0$