Tutorial 8: Answers

1. (a) The left cosets of $H = \langle(1,0)\rangle$ in $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ are: $H = \{(0,0),(1,0)\}$, $(0,1)+H = \{(0,1),(1,1)\}$, $(0,2)+H = \{(0,2),(1,2)\}$, $(0,3)+H = \{(0,3),(1,3)\}$. In G/H, these are elements of orders: 1, 4, 2, 4. (For example, for the coset $C = (0,1)+H$, we have $C+C = (0,2)+H$, $C+C+C = (0,3)+H$, $C+C+C+C = (0,0)+H = H = e_{G/H}$. So C has order 4 in G/H.) Since $|G/H| = |G|\,/\,|H| = 8/2 = 4$ and G/H contains an element of order 4, it is cyclic and hence isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

   (b) The left cosets of $H = \langle(0,2)\rangle$ in $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ are: $H = \{(0,0),(0,2)\}$, $(0,1)+H = \{(0,1),(0,3)\}$, $(1,0)+H = \{(1,0),(1,2)\}$, $(1,1)+H = \{(1,1),(1,3)\}$. In G/H, these are elements of orders: 1, 2, 2, 2. Since there is no element of order 4, G/H is not cyclic. It is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

2. (a) By Lagrange's theorem, 10 divides $|G|$ and 25 divides $|G|$, so $50 = \mathrm{lcm}(10,25)$ divides $|G|$. Since $|G| < 100$ we must have $|G| = 50$.

   (b) (a) If H and K are subgroups of a group G, then $H \cap K$ is also a subgroup of G. So $H \cap K$ is a subgroup of H and a subgroup of K. Hence, by Lagrange's theorem, $|(H \cap K)|$ divides both $|H|$ and $|K|$.

   (b) From (a) $|(H \cap K)|$ divides $\gcd(7,29) = 1$. Hence $|(H \cap K)| = 1$ and $H \cap K = \{e\}$.

3. Let G be cyclic with generator a and let N be a subgroup of G. Note that N is normal since G is abelian. If $g \in G$, then $g = a^k$ for some $k \in \mathbb{Z}$, hence $gN = a^kN = (aN)^k$. Thus aN generates G/N, and G/N is cyclic.

4. We want to show that for all $g \in G$, $gH = Hg$. If $g \in H$, then $gH = H = Hg$. So assume that $g \in G \setminus H$. Then $gH \neq H$ and $Hg \neq H$. We also know that the left cosets of H partition G and that the right cosets of H also partition G. That is,

$$\begin{aligned} G &= H \cup gH \quad \text{and} \quad H \cap gH = \emptyset \\ G &= H \cup Hg \quad \text{and} \quad H \cap Hg = \emptyset \end{aligned}$$

   Therefore $gH = G \setminus H = Hg$.

5. (a) Consider the map $\varphi\colon T \to \mathbb{R}^\times \times \mathbb{R}^\times$ defined by

$$\varphi\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = (a,d).$$

   The map is clearly surjective. It is also a homomorphism:

$$\varphi\left(\begin{bmatrix} a_1 & 0 \\ 0 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & 0 \\ 0 & d_2 \end{bmatrix}\right) = \varphi\left(\begin{bmatrix} a_1a_2 & 0 \\ 0 & d_1d_2 \end{bmatrix}\right) = (a_1a_2, d_1d_2) = (a_1,d_1)(a_2,d_2).$$

   Finally, the kernel of φ consists of only the identity matrix.

   (b) For

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \quad A' = \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix} \in B$$

we have

$$f(AA') = f\left(\begin{bmatrix} aa' & ab' + bc' \\ 0 & cc' \end{bmatrix}\right) = \begin{bmatrix} aa' & 0 \\ 0 & cc' \end{bmatrix} = f(A)f(A'),$$

so f is a homomorphism. The kernel of f is

$$U = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \mid b \in \mathbb{R} \right\}.$$

(c) The image of f is T. Hence $B/U \cong \mathrm{im}(f) = T$, by the first isomorphism theorem.

*For $n \times n$ matrices, there is an analogous homomorphism from upper triangular matrices to diagonal matrices, with kernel consisting of the upper triangular matrices with entries 1 on the main diagonal.

6. Recall that D_5 is the group of symmetries of a regular pentagon. Each subgroup of D_5 has order dividing 10, i.e. 1, 2, 5 or 10. The trivial subgroups $\{e\}$ and D_5 have orders 1 and 10. The others have prime order, so are cyclic. There are 5 subgroups of order 2 generated by the 5 reflections in D_5, and one cyclic subgroup of order 5 consisting of all the rotations in D_5.