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# Exam: MAST20022 Group Theory and Linear Algebra

## The University of Melbourne

## Department of Mathematics and Statistics

## Examination duration: 3 hours

## Reading time allowed: 15 minutes

This paper has 3 pages, including the coversheet.

### Authorised materials:

No authorised material. In particular, no written or printed material may be brought into the examination room. Calculators and computers and phones are not permitted.

If you have any such material in your possession you should immediately surrender it to an invigilator.

### Instructions to invigilators:

Initially each student is to receive two 14 page script booklets.

No special materials are to be supplied.

No written or printed material related to the subject may be brought into the examination.

Students should NOT include the examination paper with their script booklets.

### Instructions to students:

Write all your solutions in the booklet(s) provided. Number the questions and the parts clearly. The quality of the mathematical writing in solutions to the problems is an important factor.

There are 16 questions in this paper worth 10 marks each for 160 marks total. All questions may be attempted.

1. Show that the group $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the group $D_4$ are not isomorphic.

2. Describe all group homomorphisms $f: \mathbb{Z} \to \mathbb{Z}$.

3. Let $G$ be a group and let $g, x, y \in G$. Show that if $gx = gy$ then $x = y$.

4. Let $f: V \to V$ be a linear transformation on a finite dimensional inner product space $V$. Show that the adjoint $f^*$ exists and is unique.

5. Let $a, b, c \in \mathbb{C}$. Find the possible Jordan normal forms (up to reordering the Jordan blocks) of matrices that have characteristic polynomial $(x-a)(x-b)(x-c)$.

6. Let $\mathbb{F}$ be a field and let $d, a \in \mathbb{F}[t]$. Define the ideal generated by $d$ and "$d$ divides $a$" and give some illustrative examples.

7. Let $f: G \to H$ be a group homomorphism. Show that $f$ is injective if and only if $\ker f = \{1\}$.

8. Let $\mathbb{F}$ be a field. Define $\mathbb{F}[t]$ and $\mathbb{F}(t)$ and give some illustrative examples.

9. Find the multiplicative inverse of 71 in $\mathbb{Z}/131\mathbb{Z}$.

10. Define $\mathbb{R}^2$ and $\mathbb{E}^2$ and give some illustrative examples.

11. Let $G$ be the group of symmetries of the rectangle $X$ with vertices $(2,1),(2,-1),(-2,1)$, $(-2,-1)$.

(a) Give geometric descriptions of the symmetries in $G$.

(b) Find the orbit and stabilizer of the point $Q=(2,0)$ under the action of $G$ on $X$.

(c) Check that your answers to parts (a) and (b) are consistent with the orbit-stabiliser theorem.

12. Let $V$ be the subspace of $\mathbb{R}^3$ spanned by the vectors $(1,1,0)$, $(0,1,2)$. Find the orthogonal complement of $V$, using the dot product as inner product on $\mathbb{R}^3$.

13. Let $\mathcal{J}$ be the group of isometries of $\mathbb{E}^2$. Let $P$ be a point of $\mathbb{E}^2$. Show that every element of $\mathcal{J}$ can be uniquely expressed as an isometry fixing $P$ followed by a translation.

14. Define the dihedral group $D_n$ and give some illustrative examples.

15. Let $A$ be an $n\times n$ complex Hermitian matrix. Define a product on $\mathbb{C}^n$ by $(X,Y)=XAY^*$, where $X,Y\in\mathbb{C}^n$ are written as row vectors. Show that this is an inner product if all the eigenvalues of $A$ are positive real numbers.

16. Show that if $A=B^*B$, where $B$ is any invertible $n\times n$ complex matrix, then $A$ is a Hermitian matrix and all the eigenvalues of $A$ are real and positive.