The University of Melbourne

Department of Mathematics and Statistics

Semester 2 Assessment 2009

620-297 Group Theory and Linear Algebra

Reading time: Fifteen minutes
Writing time: Three hours
Identical Examination Papers: none
Common Content Papers: none

This paper has five pages (including this page).

Total marks 120

Authorized materials:
No materials are authorized. Students entering the examination venue with notes or printed material related to the subject, calculators or computers, or mobile phones, should stand in their place immediately and surrender these to an invigilator before the instruction to commence writing is given.

Instructions to Invigilators:
Script books only are required. Candidates are permitted to take this question paper with them at the end of the examination. No written or printed material related to the subject may be brought into the examination.

Instructions to Students:
The examination paper is in two sections. The questions in Section A are shorter and more routine than those in Section B. It is recommended that candidates attempt the questions in Section A before trying those in Section B. It is possible to pass the examination on marks from Section A alone. All questions may be attempted.

Please give complete explanations in all questions, and give careful statements of any results from the notes or lectures that you use.

Section A

1. (a) Use the Euclidean algorithm to find $d = \gcd(469, 959)$.

   (b) Find integers x, y such that $469x + 959y = d$.

(6 marks)

2. Let $V = \mathcal{P}_2(\mathbb{R})$ denote the vector space of all polynomials in x of degree ≤ 2 with real coefficients. Let $T : V \to V$ be the linear transformation defined by

$$T(p(x)) = p(x+1) + 3p'(x).$$

   (a) Find the matrix of T with respect to the basis $\{1, x, x^2\}$ for V.

   (b) Is T diagonalizable? Give a brief explanation.

(6 marks)

3. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix}.$$

   (a) Find the minimal polynomial of A.

   (b) Find the Jordan normal form for A.

   Give brief reasons for your answers.

(6 marks)

4. The complex vector space $\mathbb{C}^4$ has an inner product defined by:

$$\langle a, b \rangle = a_1\bar{b}_1 + a_2\bar{b}_2 + a_3\bar{b}_3 + a_4\bar{b}_4$$

   for $a = (a_1, a_2, a_3, a_4),\ b = (b_1, b_2, b_3, b_4) \in \mathbb{C}^4$. Let W be the subspace of $\mathbb{C}^4$ spanned by the vectors $(1, 0, -1, 0)$ and $(0, 1, 0, i)$.

   Find a basis for the orthogonal complement $W^{\perp}$ of W.

(6 marks)

5. Determine whether the matrix $A = \begin{bmatrix} 3 & 4i \\ 4i & 3 \end{bmatrix}$ is
   (i) Hermitian, (ii) unitary, (iii) normal, (iv) diagonalizable.

   Give brief explanations.

(6 marks)

6. The sets $G_1 = \{1, 3, 9, 11\}$ and $G_2 = \{1, 7, 9, 15\}$ form groups under multiplication modulo 16. (You don't need to prove this.)

   (a) Find the order of each element in G_1 and each element in G_2.

   (b) Are the groups G_1 and G_2 isomorphic? Explain your answer.

(6 marks)

7. (a) Express the following permutation as a product of disjoint cycles:
$$(234)(56) * (1354)(26)$$

   (b) Find the order of the permutation $(12)(34567)$ in S_7.

   (c) Find all conjugates of $(13)(24)$ in the group S_4.

(6 marks)

8. Let G be a group of order 35.

   (a) What does Lagrange's theorem tell you about the orders of subgroups of G?

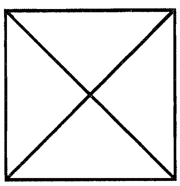
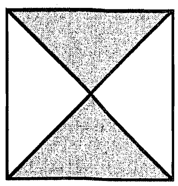
   (b) If H is a subgroup of G with $H \neq G$, explain why H is cyclic.

(6 marks)

9. Consider the set of matrices
$$G = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{R}, a^2 - b^2 = 1 \right\}.$$
Prove that G is a group using matrix multiplication as the operation.

(6 marks)

10. Let X be a subset of $\mathbb{R}^2$ consisting of the four edges of a square together with its two diagonals. Let Y be obtained from X by filling in two triangles as shown below.

X  Y 

Let G be the symmetry group of X and H the symmetry group of Y.

   (a) Describe the group G by giving geometric descriptions of the symmetries in G, and writing down a familiar group isomorphic to G.

   (b) Give a similar description of H.

   (c) Explain why H is a normal subgroup of G.

(6 marks)

Section B

11. Let $f : V \to V$ be a self-adjoint linear operator on an inner product space V, (i.e. $f^* = f$).

    (a) Prove that every eigenvalue of f is real.

    (b) Let v_1, v_2 be eigenvectors of f corresponding to eigenvalues λ_1, λ_2 with $\lambda_1 \neq \lambda_2$. Prove that v_1 and v_2 are orthogonal.

(10 marks)

12. Let A be a 6×6 complex matrix with minimal polynomial

$$m(X) = (X+1)^2(X-1).$$

    (a) Describe the possible characteristic polynomials for A.

    (b) List the possible Jordan normal forms for A (up to reordering the Jordan blocks).

    (c) Explain why A is invertible and write A^{-1} as a polynomial in A.

(10 marks)

13. (a) Let $f : V \to V$ be a normal linear operator on a complex inner product space V such that $f^4 = f^3$. Use the spectral theorem to prove that f is self-adjoint and that $f^2 = f$.

    (b) Give an example of a linear operator $g : V \to V$ on a complex inner product space Vsuch that $g^4 = g^3$ but $g^2 \neq g$.

(10 marks)

14. Consider the subgroup $H = \{\pm 1, \pm i\}$ of the multiplicative group $G = \mathbb{C}^*$ of non-zero complex numbers.

    (a) Describe the cosets of H in G. Draw a diagram in the complex plane showing a typical coset.

    (b) Show that the function $f : G \to G$ defined by $f(z) = z^4$ is a homomorphism, and find its kernel and image.

    (c) Explain why H is a normal subgroup of G, and identify the quotient group G/H.

(10 marks)

15. Let $G = < (12)(3456) >$ be the cyclic subgroup of S_7 generated by the permutation $(12)(3456)$. Consider the action of G on $X = \{1, 2, 3, 4, 5, 6, 7\}$.

   (a) Write down all the elements of G.

   (b) Find the orbit and stabilizer of (i) 1, (ii) 3, (iii) 7.
   Check that your answers are consistent with the orbit-stabilizer theorem.

   (c) Prove that if a group H of order 4 acts on a set Y with 7 elements then there must be at least one element of Y fixed by all elements of H.

(10 marks)

16. Let p be a prime number, and let V be the vector space over the field $\mathbb{Z}_p$ consisting of all column vectors in $\mathbb{Z}_p^2$:

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{Z}_p \right\}.$$

Let $G = GL(2, \mathbb{Z}_p)$ be the group of all invertible 2×2 matrices with $\mathbb{Z}_p$ entries using matrix multiplication. This acts on V by matrix multiplication as usual: $A \cdot v = Av$ for all $A \in G$ and all $v \in V$.

   (a) Consider the 1-dimensional subspaces of V. Show that there are exactly $p + 1$ such subspaces: spanned by the vectors

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} p-1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   (b) Explain briefly why G also acts on the set X of 1-dimensional subspaces of V. This gives a homomorphism $\phi : G \longrightarrow S_{p+1}$.

   (c) Show that the kernel of ϕ consists of the scalar matrices

$$K = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{Z}_p^* = \mathbb{Z}_p - \{0\} \right\}.$$

   Deduce that the quotient group G/K is isomorphic to a subgroup of S_{p+1}.

   (d) For the case where $p = 3$, find $|K|$ and $|G|$. Deduce that G/K is isomorphic to S_4.

(10 marks)