PHYS4301: TUTORIAL PROBLEMS FOR WEEK 10

(1) The following three matrices are a 4-dimensional representation of $\mathfrak{su}(2)$.

$$\rho(J_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \rho(J_2) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \qquad \rho(J_3) = \begin{pmatrix} 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Show that this representation is reducible.

- Examine the eigenvalues and eigenvectors of these matrices (use Mathematica, or similar tool).
- Find a common invariant subspace.
- Find a similarity transform that brings these matrices to block-diagonal form.

(2) Consider the Lie algebra $\mathfrak{su}(2)$, with its Hermitian generators J_1, J_2, J_3 such that

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2.$$

Define the Casimir element, $J^2 = J_1^2 + J_2^2 + J_3^2$. Show that J^2 commutes with J_3, i.e., that

$$[J^2, J_3] = J^2 J_3 - J_3 J^2 = 0.$$

(3) The Heisenberg algebra has commutation relations $[X, Y] = Z \neq 0$, $[X, Z] = 0$, $[Y, Z] = 0$. Construct its adjoint representation. Is this a faithful representation?

(4) Construct the irreducible 4-dimensional representations of $\mathfrak{su}(2)$. That is, find the matrices for J_3, then J_1 and J_2.

(5) Show that an irreducible matrix representation of an Abelian group must be one-dimensional.