PHYS4301: TUTORIAL PROBLEM FOR WEEK 8

(1) (a) Show that the unitary group $U(n) \subset GL(n, \mathbb{C})$ can be parametrised by n^2 real numbers. The defining property of matrices in $U(n)$ is that $A^\dagger = A^{-1}$, the conjugate-transpose equals the inverse.
(b) Then show that the *special* unitary group $SU(n) \subset U(n)$ can be parametrised by $n^2 - 1$ real numbers. The defining property of those matrices in $SU(n)$ is that $\det(U) = 1$.

(2) Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be a vector in $\mathbb{R}^3$ and use $\hat{\mathbf{n}}$ to denote a unit vector. We can then define a unit quaternion as $q = \cos(\theta)\mathbf{1} + \sin(\theta)\hat{\mathbf{n}}$, and view its components as elements of $\mathbb{R}^4$. We embed vectors from $\mathbb{R}^3$ into $\mathbb{R}^4$ by setting $v = 0\mathbf{1} + \mathbf{v}$; i.e. by taking $\mathbb{R}^3$ to be the 3-dimensional subspace spanned by $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.
(a) Check that the following product of quaternions $u = T_q(v) = qvq^{-1}$ results in $u \in \operatorname{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. Note that quaternion arithmetic can be done using the Mathematica package invoked with `<< Quaternions‘`, and then defined by `q = Quaternion[a, b, c, d]`.
(b) If we set $\hat{\mathbf{n}} = \mathbf{k}$, show that $T_q(v)$ defines a rotation by angle 2θ about the z-axis by computing its action on $\mathbf{v} = \mathbf{i}, \mathbf{j}$, and $\mathbf{k}$.

(3) What do the eigenvalues and eigenvectors of a matrix in $GL(2, \mathbb{R})$ tell us about the associated transformation of space? What about the case of matrices in $SO(2)$? Discuss this in as much detail as you can. You might like explore example cases using Mathematica.

(4) The definition of the exponential of a matrix is the infinite sum (which converges for any square matrix)

$$\exp(X) = \left(\sum_{n=0}^{\infty} \frac{X^n}{n!} \right) = I + X + \tfrac{1}{2}X^2 + \tfrac{1}{6}X^3 + \cdots$$

Prove the following properties, where A is an $n \times n$ matrix with complex entries.

It may help to recall that every complex matrix is similar to one in *Jordan normal form*. That is, there exists an invertible matrix P, with $J = P^{-1}AP$, such that J is block-diagonal and each block of J is a singleton, or has a repeated eigenvalue on its diagonal and 1 on the first super-diagonal,

$$J_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix}.$$

(a) $(\exp A)^\dagger = \exp(A^\dagger)$.
(b) Given any matrix $S \in GL(n, \mathbb{C})$, $\quad \exp(SAS^{-1}) = S \exp(A) S^{-1}$.
(c) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A, then the eigenvalues of $\exp(A)$ are $e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}$.
(d) $\det(\exp A) = \exp(\operatorname{tr} A)$.