The University of Melbourne

Department of Mathematics and Statistics

Semester 2 Assessment 2012

MAST20022 Group Theory and Linear Algebra

Reading time: Fifteen minutes
Writing time: Three hours
Identical Examination Papers: none
Common Content Papers: none

This paper has five pages (including this page).

Total marks 120

Authorized materials:
No materials are authorized. Students entering the examination venue with notes or printed material related to the subject, calculators or computers, or mobile phones, should stand in their place immediately and surrender these to an invigilator before the instruction to commence writing is given.

Instructions to Invigilators:
Script books only are required. Candidates are permitted to take this question paper with them at the end of the examination. No written or printed material related to the subject may be brought into the examination.

Instructions to Students:
The examination paper is in two sections. The questions in Section A are shorter and more routine than those in Section B. It is recommended that candidates attempt the questions in Section A before trying those in Section B. It is possible to pass the examination on marks from Section A alone. All questions may be attempted.

Please give complete explanations in all questions, and give careful statements of any results from the notes or lectures that you use.

Section A

1. (a) Use the Euclidean algorithm to find $d = \gcd(1015, 221)$.
   (b) Find integers x, y such that $1015x + 221y = d$.
   (c) Find all integers n satisfying the equation $1015n \equiv 3 \bmod 221$.

(6 marks)

2. Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

   (a) Find the minimal polynomial of A.
   (b) Is A diagonalizable?

   Give brief reasons for your answers.

(6 marks)

3. A complex matrix has characteristic polynomial $c(X) = (X-1)^4(X-5)^2$ and minimal polynomial $m(X) = (X-1)^2(X-5)^2$.
   (a) What are the possible Jordan normal forms (up to rearranging the Jordan blocks)?
   (b) What additional information would allow you to determine the correct Jordan form?

   Give brief reasons for your answers.

(6 marks)

4. Let $V = \mathcal{P}_2(\mathbb{R})$ be the vector space of all polynomials of degree at most 2 with real coefficients. An inner product on V is defined by

$$\langle p, q \rangle = \int_{-1}^{1} p(t)q(t)\, dt.$$

   Find a basis for the orthogonal complement of the subspace spanned by $\{x, x^2\}$.

(6 marks)

5. Let f, g be linear operators on an inner product space V, with adjoints f^*, g^*. Prove that the composition $fg = f \circ g$ has adjoint $(fg)^* = g^* f^*$.

(6 marks)

6. The set of 6 elements $\{\pm 1, \pm 3, \pm 5\}$ forms a group G under multiplication modulo 14.

   (a) Find the orders of the elements of G.

   (b) Is the group cyclic? Give brief reasons for your answer.

(6 marks)

7. (a) Calculate the permutation $(146) * (13524) * (354)$, writing it as a product of disjoint cycles.

   (b) Find the order of $(12)(4356)$ in S_6.

   (c) Find the conjugacy class and centralizer of (12) in S_3.

   Give brief explanations.

8. The dihedral group D_7 consists of all symmetries of a regular 7-gon.

   (a) State Lagrange's theorem.

   (b) List the possible orders of subgroups of D_7 allowed by Lagrange's theorem.

   (c) Give an example of a subgroup of D_7 of each order listed in (b).

(6 marks)

9. (a) The non-identity isometries of the Euclidean plane $\mathbb{E}^2$ are of four types: rotations, reflections, translations and glide reflections. For each type of isometry describe the set of fixed points.

   (b) Let f and g be reflections of the Euclidean plane in lines L_1 and L_2 respectively. What type of isometry is the composition $g \circ f$ if $L_1 \cap L_2$ is a single point p?

   (c) If f is a rotation about a point p and g is a reflection in a line L, what can you say about the composition $g \circ f$ if $p \in L$?

   Give brief explanations.

(6 marks)

10. Let G be a group of order 27 acting on a set X of 25 elements.

   (a) What are the possible sizes of orbits? Explain your answer.

   (b) Prove that there is at least one point fixed by all elements of G.

(6 marks)

Section B

11. Let $f : V \to V$ be a linear operator on a finite dimensional complex inner product space V such that $f^*f = ff^*$.

    (a) State the spectral theorem.

    (b) Prove that $f^* = f$ if and only if all eigenvalues of f are real.

(10 marks)

12. Let A be a $n \times n$ matrix with complex entries and let $p(x)$ be a polynomial in x with complex coefficients.

    (a) Explain how $p(A)$ is defined.

    (b) Prove that if λ is an eigenvalue of A then $p(\lambda)$ is an eigenvalue of $p(A)$.

    (c) Prove that if μ is an eigenvalue of $p(A)$ then $\mu = p(\lambda)$ for some eigenvalue λ of A. (Hint: Use the Jordan normal form theorem.)

(10 marks)

13. Let $\theta = \sqrt[3]{2}$ be the real cube root of 2. Then

$$V = \{a_0 + a_1\theta + a_2\theta^2 : a_0, a_1, a_2 \in \mathbb{Q}\}$$

is a vector space over the field of rational numbers $\mathbb{Q}$ with basis $\mathcal{B} = \{1, \theta, \theta^2\}$. (You do not need to prove this.) Let $\alpha = 1 + \theta$.

    (a) Show that multiplication by α defines a linear transformation $T : V \to V$ where $T(x) = \alpha x$ for x in V. (Note: first you will need to show that $\alpha x \in V$ for all $x \in V$.)

    (b) Find the matrix of T with respect to the basis $\mathcal{B}$.

    (c) Find the characteristic polynomial of T.

    (d) Find a polynomial $p(x)$ of degree 3 with rational coefficients such that $p(\alpha) = 0$.

(10 marks)

14. Let A_4 be the subgroup of the symmetric group S_4 consisting of the permutations

$$\{(1),(12)(34),(13)(24),(14)(23),(123),(132),(124),(142),(134),(143),(234),(243)\}.$$

(You may assume that these permutations form a group.)

(a) Find the conjugacy class and the centralizer of $(12)(34)$ in A_4.

(b) Show that the $H = \{(1),(12)(34),(13)(24),(14)(23)\}$ is a normal subgroup of A_4.

(c) Identify the quotient group A_4/H.

(10 marks)

15. (a) Show that

$$G = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

is a subgroup of $GL(3, \mathbb{R})$ using matrix multiplication as the operation.

(b) Show that the function $f : G \to \mathbb{R}^2$ defined by

$$f\left(\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \right) = (a, b)$$

is a homomorphism from G to the additive group $\mathbb{R}^2$.

(c) Find the kernel and image of f.

(d) Find a normal subgroup N of G such that G/N is isomorphic to $\mathbb{R}^2$.

(10 marks)

16. Let $G = GL(2, \mathbb{Z}_2)$ be the group of invertible 2×2 matrices with entries in $\mathbb{Z}_2$, and let $X = \{(1,0),(0,1),(1,1)\}$ be the set of non-zero vectors in $\mathbb{Z}_2^2$. For all $A \in G$ and $x \in X$, let $A \cdot x$ denote the matrix product Ax where x is written as a column vector.

(a) Show that this defines an action of G on X.

(b) Find the orbit and stabilizer of $(1, 0)$.

(c) Hence find the order of G.

(d) Explain how this group action defines a homomorphism $\phi : G \to S_3$.

(e) Find the kernel of ϕ.

(f) Is ϕ an isomorphism? Explain your answer.

(10 marks)