Student Number: ____________________

The University of Melbourne

Semester 2 Assessment 2013

Department of Mathematics and Statistics

MAST20022 Group Theory and Linear Algebra

Reading Time: 15 minutes

Writing Time: 3 hours

Open Book Status: Closed book

This paper has 5 pages (including this page).

Authorised Materials:

☐ The following items are authorised:

OR

☐ Students may have unrestricted access to all materials.

OR

☒ No materials are authorised.

Paper to be held by Baillieu Library:

☒ Yes ☐ No

Instructions to Invigilators:

Each candidate should be issued with an examination booklet, and with further booklets as needed. The students may remove the examination paper at the conclusion of the examination. No calculators, computers or mobile phones may be used.

Instructions to Students:

This examination consists of 16 questions. The total number of marks is 120.

The examination paper is in two sections. The questions in Section A are shorter and more routine than those in Section B. It is recommended that candidates attempt the questions in Section A before trying those in Section B. It is possible to pass the examination on marks from Section A alone. All questions may be attempted.

Please give complete explanations in all questions and show all of your calculations and working. Give careful statements of any results from the notes or lectures that you use.

Extra materials required:

☐ Graph Paper ☐ Multiple Choice Form ☐ Other

Section A

1. (a) Use the Euclidean algorithm to find $d = \gcd(2639, 301)$.
   (b) Find integers x, y such that $2639x + 301y = d$.
   (c) Find the multiplicative inverse of $[301]_{2639}$ in $\mathbb{Z}_{2639}$ if it exists.

(6 marks)

2. Let $V = \mathcal{P}_3$ be the real vector space of polynomials of degree ≤ 3 with real coefficients and $T : V \to V$ be the linear map defined by

$$T(p(x)) = p(x+1) - p(x).$$

   (a) Find the matrix of T with respect to the ordered basis $\mathcal{B} = \{1, x, x^2, x^3\}$ for V.
   (b) Is T diagonalizable? Justify your answer.

(6 marks)

3. Consider the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

   (a) Find the minimal polynomial of A.
   (b) Hence find the Jordan normal form for A.

(6 marks)

4. Determine whether the matrix

$$B = \begin{bmatrix} 2+i & 2i \\ -2i & -3+i \end{bmatrix}$$

is (i) Hermitian, (ii) unitary, (iii) normal, (iv) diagonalizable.
Give brief explanations.

(6 marks)

5. Find an orthonormal basis for the subspace of $\mathbb{C}^3$ spanned by $(1, i, 0), (1, 1, 1)$ using the complex dot product

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1\overline{b_1} + a_2\overline{b_2} + a_3\overline{b_3}$$

as inner product.

(6 marks)

6. (a) Calculate the permutation $(12)(345) * (246)$, writing it as a product of disjoint cycles. Hence find the order of this permutation.
   (b) Find the conjugacy class of $(12)(34)$ in S_4.

(6 marks)

7. Let G be a group of order 10.

   (a) What does Lagrange's theorem tell you about the possible orders of subgroups of G?

   (b) If $G = \mathbb{Z}_2 \times \mathbb{Z}_5$ explain why G is a cyclic group of order 10.

   (c) Given an example of a group of order 10 that is not cyclic.

(6 marks)

8. Let $f : G \to H$ be a homomorphism between groups.

   (a) Define the kernel of f.

   (b) Prove that the kernel of f is a subgroup of G.

   (c) Prove that the kernel of f is a *normal* subgroup of G.

(6 marks)

9. The set $G = \{\pm 1, \pm 2, \pm 4, \pm 7\}$ forms an abelian group under multiplication modulo 15.

   (a) Find the order of each element in G.

   (b) Decide which of the additive groups $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to G. Give brief reasons for your answer

(6 marks)

10. Let X be the parallelogram in $\mathbb{R}^2$ with vertices $(2, 0), (-2, 0), (0, 4), (0, -4)$, and let G be the group of symmetries of X.

   (a) Sketch X and describe all the elements of G.

   (b) For the action of G on X, find the orbit and stabilizer of the points (i) $P = (2, 0)$, (ii) $Q = (1, 2)$.

   (c) Verify that your answers in (b) are consistent with the orbit-stabilizer theorem.

(6 marks)

Section B

11. Let $S : V \to V$ be a linear operator on a finite dimensional inner product space V such that $S^* = -S$.

   (a) Prove that $||v + Sv||^2 = ||v||^2 + ||Sv||^2$ for all v in V.

   (b) Prove that the nullspace of $I + S$ is $\{\mathbf{0}\}$.

   (c) Deduce that $I + S$ is invertible.

   (d) Prove that S is diagonalizable if V is a *complex* inner product space.

   (e) Give an example to show that S need not be diagonalizable if V is a *real* inner product space.

(10 marks)

12. Let A be a 4×4 complex matrix with characteristic polynomial

$$c(x) = (x-1)^2(x+1)^2.$$

   (a) Describe the possible minimal polynomials for A.

   (b) List the possible Jordan normal forms for A (up to reordering the Jordan blocks). Give a brief explanation.

   (c) Prove that A has a square root B, i.e. a 4×4 complex matrix such that $B^2 = A$. (Hint: first show that each Jordan block has a square root.)

(10 marks)

13. Let $T : V \to V$ be a linear operator on a finite dimensional inner product space V.

   (a) Let W be a subspace of V. Define the orthogonal complement $W^\perp$ of W, and state the relationship between the dimensions of W and $W^\perp$.

   (b) Prove that the nullspace of T is the orthogonal complement of the range of T^*.

   (c) Deduce that the rank of T^* is equal to the rank of T.

(10 marks)

14. (a) Show that the set

$$G = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in \mathbb{R}, a^2 - b^2 \neq 0 \right\}$$

is a group under matrix multiplication.

(b) Show that the function $f : G \to \mathbb{R}^*$ given by

$$f\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) = a + b$$

is a homomorphism from G to the multiplicative group $\mathbb{R}^*$ of non-zero real numbers.

(c) Find the image and kernel of f.

(10 marks)

15. Let G be a group. A *commutator* in G is a product

$$[a, b] = aba^{-1}b^{-1} \text{ where } a, b, \in G.$$

(a) Show that $ab = ba$ if and only if $[a, b] = e$ is the identity in G.

(b) Show that the inverse of $[a, b]$ is a commutator.

(c) Show that the conjugate $g[a, b]g^{-1}$ is a commutator for all $g \in G$.

(d) Show that the set H of all finite products of commutators is a normal subgroup of G.

(e) Prove that the quotient group G/H is abelian.

(10 marks)

16. Let H_1 and H_2 be subgroups of finite index in a group G. Let G/H_i denote the set of left cosets of H_i in G for $i = 1, 2$. (Note that we are not assuming that H_1, H_2 are *normal* subgroups of G.)

(a) Show that there is an action of G on $G/H_1 \times G/H_2$ defined by

$$g \cdot (g_1H_1, g_2H_2) = (gg_1H_1, gg_2H_2) \text{ for } g, g_1, g_2 \in G.$$

(b) Find the stabiliser of (H_1, H_2) under this action.

(c) Prove that the subgroup $H_1 \cap H_2$ is of finite index in G and that

$$|G : H_1 \cap H_2| \leq |G : H_1| \cdot |G : H_2|.$$

(10 marks)