THE UNIVERSITY OF MELBOURNE

Semester 2 Assessment, 2016

School of Mathematics and Statistics

MAST20022 Group theory and linear algebra

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam.

This paper consists of 4 pages (NOT including this page).

Authorised Materials

- No materials are authorised. No written or printed material may be brought into the examination room.
- Mobile phones are not permitted in the examination room.
- Calculators, mathematical tables and computers are not permitted in the examination room.

Instructions to Students

- You may NOT remove this question paper at the conclusion of the examination.
- The paper is in two sections. The questions in Section A are shorter and more routine than those in Section B. It is recommended that students attempt the questions in Section A before trying those in Section B.
- All questions should be attempted.
- Please give complete explanations in all questions and show all your calculations and working. Give careful statements of any results from the notes or lectures that you use.
- Number the questions and question parts clearly. Start each question on a new page.
- Use the left pages for rough working. Write material that you wish to be marked on the right pages only.
- This examination consists of 14 questions. The total number of marks available is 100.

Instructions to Invigilators

- Students may NOT remove this question paper at the conclusion of the examination.
- Each candidate should be issued with an examination booklet, and with further booklets as needed.

Dihedral group D_4

Recall that the dihedral group D_4 is the group of symmetries of a square. It has presentation

$$D_4 = \langle r, s \mid r^4 = e, s^2 = e, sr = r^3 s \rangle,$$

where r is the anticlockwise rotation by $\pi/2$ and s is any of the four reflections.

Section A: 50 marks total

Question 1 (5 marks)

(a) Let a, b, and c be integers. If $a \mid b$ and $a \mid c$, prove that $a^2 \mid b^2 + 3c^2$.

(b)

i. Use Euclid's algorithm to find $d = \gcd(323, 377)$.

ii. Find integers x, y such that $323x + 377y = d$.

Question 2 (5 marks)

Consider the set

$$\mathbb{Q}[i] = \{a + bi \mid a, b \in \mathbb{Q}\}, \qquad \text{where } i^2 = 1.$$

Show that $\mathbb{Q}[i]$ forms a field under the usual operations of addition and multiplication of complex numbers.

Question 3 (5 marks)

Let $f \colon V \to V$ be a linear transformation on an n-dimensional vector space with minimal polynomial $m(X) = X^n$.

(a) Show that there is a vector $v \in V$ such that $f^{n-1}(v) \neq 0$.

(b) Show that $\mathcal{B} = (f^{n-1}(v), f^{n-2}(v), \dots, f^2(v), f(v), v)$ is a basis for V.

(c) Find the matrix of f with respect to the basis $\mathcal{B}$.

Question 4 (5 marks)

Find the minimal polynomials and Jordan normal forms of the matrices:

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Question 5 (5 marks)

Which of the following pairs of matrices (over $\mathbb{C}$) are similar?

(a) $\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix};$

(b) $\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 5 \\ 0 & -1 \end{bmatrix};$

(c) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$

Question 6 (5 marks)

(a) Find the order of each element in $\mathbb{Z}/10\mathbb{Z}$.

(b) Hence find all subgroups of $\mathbb{Z}/10\mathbb{Z}$.

Question 7 (5 marks)

For each pair of groups, determine whether they are isomorphic or not and briefly justify your answer.

(a) $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and D_4

(b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and $\mathbb{Z}/12\mathbb{Z}$

(c) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$.

Question 8 (5 marks)

Consider the subgroup $H = \langle (0, 2) \rangle$ of $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

(a) Write down the left cosets of H in G.

(b) Find the order of each element in the quotient group G/H.

(c) Identify the quotient group G/H. (Is it isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$?)

Question 9 (5 marks)

Let V be an inner product space and let W be a subspace of V.

(a) Define the orthogonal complement $W^\perp$ of W.

(b) Show that $W \subset (W^\perp)^\perp$.

(c) Suppose now that V is finite-dimensional. Show that $W = (W^\perp)^\perp$.

Question 10 (5 marks)

Let $\mathrm{GL}_2(\mathbb{R})$ act on $\mathbb{R}^2$ in the usual way: $A \cdot v = Av$ for $A \in \mathrm{GL}(2, \mathbb{R})$ and $v \in \mathbb{R}^2$. Describe the stabiliser and orbit of:

(a) $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (b) $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Section B: 50 marks total

Question 11 (8 marks)

(a) State the Jordan normal form theorem.

(b) Give an explicit matrix over $\mathbb{F}_5$ that **has** a Jordan normal form. Justify your answer.

(c) Give an explicit matrix over $\mathbb{F}_5$ that **does not have** a Jordan normal form. Justify your answer.

(d) Let $V = \mathbb{C}[x]_{\leq 2}$ and consider a **non-diagonalisable** linear transformation $f\colon V \to V$ satisfying the conditions

$$\begin{aligned}(f - 2\,\mathrm{id}_V)^3 &= 0\\ f(x-1) &= 2x - 2\\ f(x^2+1) &= 2x^2 + 2\end{aligned}$$

Find the Jordan normal form of f. Justify your answer.

Question 12 (17 marks)

Let p be a prime number. Recall the groups of 2×2 matrices

$$\begin{aligned}\mathrm{GL}_2(\mathbb{F}_p) &= \{A \in M_2(\mathbb{F}_p) \mid \det(A) \neq 0\}\\ \mathrm{SL}_2(\mathbb{F}_p) &= \{A \in M_2(\mathbb{F}_p) \mid \det(A) = 1\}\end{aligned}$$

(a) Prove that $\#\,\mathrm{GL}_2(\mathbb{F}_p) = (p^2-1)(p^2-p)$.

(b) Use the determinant group homomorphism to find the cardinality $\#\,\mathrm{SL}_2(\mathbb{F}_p)$.

(c) Consider the subset

$$H = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in \mathbb{F}_p \right\}.$$

Prove that H is a subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$.

(d) Is H a normal subgroup? Justify your answer.

(e) Prove that H is isomorphic to the group $(\mathbb{F}_p, +)$.

(f) Write down an explicit element of order p of $\mathrm{SL}_2(\mathbb{F}_p)$. Justify your answer.

(g) How many p-Sylow subgroups does $\mathrm{SL}_2(\mathbb{F}_p)$ have? Justify your answer.

Question 13 (12 marks)

(a) State the Spectral Theorem for complex matrices.

(b) Show that the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is normal.

(c) Find a complex matrix square root of A, i.e. a complex matrix B such that $B^2 = A$.

Question 14 (13 marks)

(a) Consider the action of the group D_4 on $\mathbb{R}^2$ defined by

$$r \cdot v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} v$$

$$s \cdot v = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} v$$

i. What is the vector

$$(sr^3) \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix}?$$

ii. What cardinalities can the orbits of this action of D_4 have? Give an explicit example for each cardinality.

(b)

i. State Burnside's Lemma for the number of orbits of the action of a finite group on a finite set.

ii. Find the number of 3×3 squares containing only 0's and 1's, up to D_4 symmetry.

End of Exam—Total Available Marks = 100