

## Tutorial 6

**Main topics:** Properties and examples of groups; subgroups, cyclic groups, orders of elements

1. Let  $K$  be a field.
  - (a) Does the set  $M_n(K)$  of  $n \times n$  matrices form a group under matrix addition?
  - (b) Does  $M_n(K)$  form a group under matrix multiplication?
  - (c) The set  $\text{GL}_n(K)$  of invertible  $n \times n$  matrices forms a group under matrix multiplication. Does  $\text{GL}_n(K)$  form a group under matrix addition?
2. Let  $G$  be a group and  $x, y, z, w \in G$ .
  - (a) Given that  $xyz^{-1}w = e$ , solve for  $y$  (in terms of  $x, z$  and  $w$ ).
  - (b) Assume  $xyz = e$ . Does it follow that  $yzx = e$ ? Does it follow that  $yxz = e$ ?
3. Let  $n \in \mathbb{N}$ . Show that the set of all complex  $n$ -th roots of unity  $\mu_n = \{z \in \mathbb{C} \mid z^n = 1\}$  forms a group under multiplication. (It's useful to notice that  $\mu_n$  is a subset of the group  $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ , so it is enough to show that  $\mu_n$  is a subgroup.)
4. Compute the following products of permutations in the symmetric group  $S_6$ :
  - (a)  $(123)(456) \times (12)(34)(56)$
  - (b)  $(12) \times (246) \times (123654)$
5. (a) For each of the following  $n \in \mathbb{N}$  write down an element of  $S_5$  that has order  $n$ .
  - (i) 1      (ii) 2      (iii) 3      (iv) 4      (v) 5      (vi) 6(b) Find the orders of the following elements of the group  $\mathbb{C}^\times = (\mathbb{C} \setminus \{0\}, \times)$ .
  - (i) 1      (ii)  $-1$       (iii) 3      (iv)  $i$
6. List all the cyclic subgroups of  $S_3$ . How many are there?
7. Show that if  $g^2 = e$  for all  $g$  in a group  $G$ , then  $G$  is abelian.