Combining representations

We have seen two methods for combining representations so far. These are the direct sum of representations (p.18); and one version of the tensor product of representations (p.28). The tensor product we used to build representations of the Lorentz group really combines two different groups into a larger one, in general we would have $\rho_m : G \to GL(m, \mathbb{C})$ and $\rho_n : H \to GL(n, \mathbb{C})$ with $\rho_m \otimes \rho_n : G \times H \to GL(mn, \mathbb{C})$ being defined by $\rho_m \otimes \rho_n(g, h) = \rho_m(g) \otimes \rho_n(h)$.

It is also possible to combine two representations for a group into a new tensor-product representation for that group, as follows.

· Given two irreducible representations ρ_m, ρ_n for a Lie group G, we can use a tensor product to obtain an mn-dimensional representation $\rho_{mn}(g) = \rho_m(g) \otimes \rho_n(g)$.

· At the Lie algebra level we have product rule behaviour again with $\tilde{\rho}_{mn}(X) = \tilde{\rho}_m(X) \otimes I_n + I_m \otimes \tilde{\rho}_n(X)$.

· This new representation will, in general, be *reducible*, and if G is compact, or ρ is unitary, we know that it is *completely reducible* and would like to find its irreducible parts.

· This procedure is "finding the Clebsch-Gordan coefficients" or "multiplying ladders". It amounts to finding dimensions of the distinct invariant subspaces $V_{n_r} \subset \mathbb{C}^{mn}$ with $\sum n_r = mn$.

The above method of combining representations is subtly different to what we did with the Lorentz group. With the Lorentz Lie algebra, we saw that $\mathfrak{so}^+(1,3) \simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$ which meant each element of the algebra broke down into two pieces: $X = X^+ + X^-$. So in this setting, we are thinking of the two pieces as distinct parts of the algebra/group, and the representations we combine act on these two different parts.

For the second type of tensor product representations, we take the representations to act on the same element of the algebra/group.

Clebsch-Gordan for $SU(2)$

Given two irreducible representations for $SU(2)$ with $j = (m-1)/2$ and $k = (n-1)/2$, assume $j \geq k$. The tensor product space for the representation $D_m \otimes D_n$ decomposes as

$$\mathbb{C}^m \otimes \mathbb{C}^n \sim \mathbb{C}^{mn} = V_{j+k} \oplus V_{j+k-1} \oplus \cdots \oplus V_{j-k}$$

where the dimension of $V_{n_r} = 2n_r + 1$.
The representation on each V_{n_r} is the unique irreducible representation for $SU(2)$ of that dimension.

Exercise: Check the vector space dimensions for the decomposition add up appropriately for some choice of j, k.

Clebsch-Gordan decompositions exist for other Lie groups, and are different in each case.