THE UNIVERSITY OF MELBOURNE

Semester 2 Assessment, 2015

School of Mathematics and Statistics

MAST20022 Group theory and linear algebra

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam.

This paper consists of 4 pages (NOT including this page).

Authorised Materials

- No materials are authorised. No written or printed material may be brought into the examination room.
- Mobile phones are not permitted in the examination room.
- Calculators, mathematical tables and computers are not permitted in the examination room.

Instructions to Students

- You may remove this question paper at the conclusion of the examination.
- The paper is in two sections. The questions in Section A are shorter and more routine than those in Section B. It is recommended that students attempt the questions in Section A before trying those in Section B.
- All questions should be attempted.
- Please give complete explanations in all questions and show all your calculations and working. Give careful statements of any results from the notes or lectures that you use.
- Number the questions and question parts clearly. Start each question on a new page.
- Use the left pages for rough working. Write material that you wish to be marked on the right pages only.
- This examination consists of 14 questions. The total number of marks available is 100.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination.
- Each candidate should be issued with an examination booklet, and with further booklets as needed.

Section A: 50 marks total

Question 1 (5 marks)

(a) Are there integers x and y such that $48x + 18y = 5$? If yes, find such integers x and y. If no, explain why.

(b) Are there integers x and y such that $48x + 18y = 12$? If yes, find such integers x and y. If no, explain why.

Question 2 (5 marks)

(a) Show that the field $\mathbb{F}_{13}$ is not algebraically closed.

(b) Find a 2×2 matrix A with entries in $\mathbb{F}_{13}$ such that the eigenvalues of A are not in $\mathbb{F}_{13}$.

Question 3 (5 marks)

Let $f\colon \mathbb{C}^4 \to \mathbb{C}^4$ be a linear transformation. You are told that its characteristic polynomial is

$$c(x) = (x - i)^4$$

and its minimal polynomial is

$$m(x) = (x - i)^2.$$

Describe as precisely as you can the possible Jordan normal forms J of f.

What additional information would allow you to narrow the list of possibilities down to one Jordan normal form?

Question 4 (5 marks)

Let $V = \mathbb{R}^4$ with the standard inner product (that is, the dot product). Let

$$W = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(a) Find an orthonormal basis for W.

(b) Find the distance from the vector $v = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ to W.

Question 5 (5 marks)

Let V be a complex inner product space.

(a) Define “self-adjoint linear transformation on V”.

(b) Let f be a self-adjoint linear transformation on V. Show that the eigenvalues of f are real.

Question 6 (5 marks)

Prove, or find a counterexample for, each of the following statements:

(a) In any group G, $xyz = e$ is equivalent to $yzx = e$.

(b) In any group G, $xyz = e$ is equivalent to $yxz = e$.

Question 7 (5 marks)

Let G be the group $\mathbb{Z}/4\mathbb{Z} \times S_3$.

(a) What is the order of the element $g = ([2]_4, (1\,3))$ in G?

(b) Is G isomorphic to S_4?

Question 8 (5 marks)

(a) Let $f\colon G \to H$ be an injective group homomorphism. Prove that for any $g \in G$, the order of g in G is equal to the order of $f(g)$ in H.

(b) Count the number of group homomorphisms from a group of order 24 to a group of order 25.

Question 9 (5 marks)

Consider the subgroup of $\mathrm{GL}_2(\mathbb{R})$ generated by the two elements below:

$$H = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle.$$

(a) Show that $\det(h) = 1$ for all $h \in H$.

(b) Find the order of each of the two generators given above.

(c) Find the centre of H.

Question 10 (5 marks)

(a) State Burnside's Lemma about the number of orbits under a group action.

(b) We colour the vertices of an equilateral triangle with 3 colours: Red, Green and Blue. We say that two coloured triangles T_1 and T_2 are *equivalent* if there exists an element of the dihedral group D_3 that takes T_1 to T_2. How many non-equivalent coloured triangles are there?

Section B: 50 marks total

Question 11 (10 marks)

Consider the inner product space $V = M_{2\times 2}(\mathbb{C})$ with the usual inner product

$$(A, B) = \mathrm{Tr}(AB^*),$$

where B^* is the conjugate transpose of B.

Let $f \colon V \to V$ be the orthogonal projection onto the subspace

$$W = \mathrm{Span}\{E_{11}, E_{12} + iE_{21}\},$$

where

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is the standard orthonormal basis of V.

Find the minimal polynomial and the Jordan normal form of f.

Question 12 (15 marks)

Let p be prime, let $V = \mathbb{F}_p^2$ and let $G = \mathrm{GL}_2(\mathbb{F}_p)$. Consider the set of $p+1$ vectors

$$\mathcal{L} = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ldots, v_p = \begin{bmatrix} 1 \\ p-1 \end{bmatrix}, v_{p+1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

(a) Prove that any one-dimensional subspace of V is of the form $\mathrm{Span}\{v_j\}$ where v_j is one of the vectors in $\mathcal{L}$.

(b) Let $g \in G$. Prove that left multiplication by g gives a permutation of the set $\mathcal{L}$. Let $f \colon G \to S_{p+1}$ be the resulting group homomorphism, i.e. $f(i) = j$ if $gv_i \in \mathrm{Span}\{v_j\}$.

(c) Find the kernel of f. (Hint: v_1, v_{p+1} and v_2 are of particular interest here.)

(d) Show that f is surjective for $p = 2$ and $p = 3$ but not for $p > 3$.

Question 13 (13 marks)

A 3×3 *Latin square* is a 3×3 box filled with the symbols $\{A, B, C\}$ in such a way that each symbol appears only once in each row and in each column. For example, the following is a Latin square:

$$x = \begin{array}{|c|c|c|} \hline B & A & C \\ \hline C & B & A \\ \hline A & C & B \\ \hline \end{array}$$

Let X denote the set of all 3×3 Latin squares.

The group $G = S_3 \times S_2$ acts on X as follows: if $\sigma \in S_3$, $\tau \in S_2$ and $x \in X$, then $(\sigma, \tau) \cdot x$ is obtained by first applying the permutation σ to the rows of x, and then applying the permutation τ to the columns 2 and 3 of the result. For example,

$$((1\,3\,2), (1\,2)) \cdot \begin{array}{|c|c|c|} \hline B & A & C \\ \hline C & B & A \\ \hline A & C & B \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline C & A & B \\ \hline A & B & C \\ \hline B & C & A \\ \hline \end{array}$$

(a) Prove that there is only one orbit for this action of G on X.

(b) Find the stabiliser of the Latin square x given above.

(c) Using the previous parts (or otherwise), find the number of 3×3 Latin squares.

Question 14 (12 marks)

(a) State the Sylow theorems about p-Sylow subgroups of finite groups.

(b) Recall that the group $G = \mathrm{GL}_2(\mathbb{F}_3)$ has $2^4 \cdot 3 = 48$ elements. Prove that G has exactly four 3-Sylow subgroups. (Hint: you might find it useful to notice that $\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$ is an element of order 6 in G.)

(c) How many elements of order 3 does G have?

End of Exam—Total Available Marks = 100