

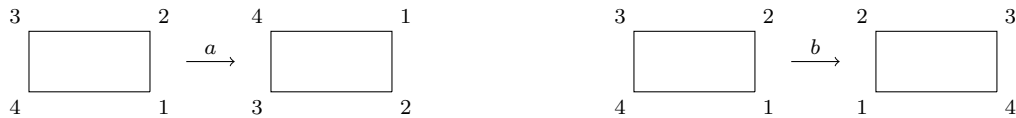
## Tutorial 11

**Main topics:** group actions, orbit-stabiliser theorem, Sylow theorems.

1. Let  $G$  be a group. Show that the following gives an action of  $G$  on  $X = G$ :

$$g \cdot x = xg^{-1} \text{ for } g \in G, x \in X$$

2. Let  $G = \{e, a, b, ab\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  act as the symmetries of a rectangle, with  $a$  and  $b$  as shown below.



What is the stabiliser and orbit of: (a) a vertex (b) the midpoint of an edge?

3. Let  $\text{GL}(2, \mathbb{R})$  act on  $\mathbb{R}^2$  in the usual way:  $A \cdot x = Ax$  for  $A \in \text{GL}(2, \mathbb{R})$  and  $x \in M_{2 \times 1}(\mathbb{R})$ .

Describe the stabiliser and orbit of: (a)  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  (b)  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

4. A group  $G$  of order 9 acts on a set  $X$  having 16 elements. Show that there must be at least one point in  $X$  that is fixed by all elements of  $G$ .

5. Find the conjugacy class and centraliser of:

(a)  $(12) \in S_3$

(b)  $(123) \in S_3$

Check that  $|\text{conjugacy class}| \times |\text{centraliser}| = |S_3|$  in each case.

6. Let  $G$  be a group of order  $84 = 2^2 \times 3 \times 7$ . What can you say about the number of

- (a) Sylow 2-subgroups? (b) Sylow 3-subgroups? (c) Sylow 7-subgroups?

Explain why  $G$  must have a normal subgroup of order 7.

7. Let  $G$  be an abelian group of order  $n$ . Prove that  $G$  has a unique Sylow  $p$ -subgroup for each prime  $p \mid n$ .

8. Let  $G$  be a group of order  $30 = 2 \times 3 \times 5$ , and let  $n_p$  denote the number of Sylow  $p$ -subgroups of  $G$ .

(a) Prove that  $n_3 = 1$  or  $n_5 = 1$ . Hence  $G$  must have a normal subgroup of order 3 or 5.

(b) Prove that if  $n_2 = 15$  then  $n_3 = n_5 = 1$ .