4 Week 4

The Lorentz group

We're now going to use all the tools developed over the past three weeks to study the group of transformations that preserve distance in the Minkowski space-time metric of special relativity. The full group of transformations is called the *Poincaré group* and includes translations in space and time. As before, we will start by restricting our analysis to transformations that fix the origin; these transformations form the *Lorentz group*. So the Lorentz group is to the Poincaré group what the Orthogonal group is to the isometry group of $\mathbb{R}^n$.

Recall that Minkowski space-time is a 4-dimensional vector space, $\mathbb{R}^4$, with components $x_0 = ct$ being the distance a photon in a vacuum travels in time t, and x_1, x_2, x_3, being spatial coordinates. The inner product is defined using the matrix

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

with $\langle a, b\rangle = a^T \eta b$, for column vectors $a, b \in \mathbb{R}^4$. A Lorentz transformation, Λ, is a 4×4 invertible matrix that preserves distances in Minkowski space-time, so we must have that for all $x, y \in \mathbb{R}^4$

$$\begin{aligned} \langle (x-y), (x-y)\rangle &= \langle \Lambda(x-y), \Lambda(x-y)\rangle \\ (x-y)^T\eta(x-y) &= (\Lambda(x-y))^T\eta(\Lambda(x-y)) = (x-y)^T\Lambda^T\eta\Lambda(x-y) \\ \implies \eta &= \Lambda^T\eta\Lambda \end{aligned}$$

This is very similar to our condition for the Orthogonal matrices, $O^T O = I$, but because η has one positive element and three negative ones on the diagonal, we say it has *signature* $(1,3)$ and we write the Lorentz group as $O(1,3)$.

Now, the $\det(\eta) = -1$ and $\det(\Lambda^T) = \det(\Lambda)$ as always, so from the preservation of distances we find that $\det(\Lambda) = \pm 1$, just as for orthogonal matrices.

There is another signature of Lorentz transformations though, and this is the sign of its top-left element, Λ_{00}. Our definition $\eta = \Lambda^T\eta\Lambda$ tells us that the top left matrix element on each side is

$$\eta_{00} = 1 = \begin{pmatrix} \Lambda_{00} & \Lambda_{10} & \Lambda_{20} & \Lambda_{30} \end{pmatrix} \begin{pmatrix} \Lambda_{00} \\ -\Lambda_{10} \\ -\Lambda_{20} \\ -\Lambda_{30} \end{pmatrix} = (\Lambda_{00})^2 - (\Lambda_{10})^2 - (\Lambda_{20})^2 - (\Lambda_{30})^2$$

Rearranging this equation we see that

$$\Lambda_{00} = \pm\sqrt{1 + (\Lambda_{10})^2 + (\Lambda_{20})^2 + (\Lambda_{30})^2}$$

The expression in the square root is always positive, so we see that either $\Lambda_{00} \leq -1$, or $\Lambda_{00} \geq +1$.

Together with the determinant condition, we see that the Lorentz matrices fall into four distinct sets (following notation in [S])

$$\begin{aligned} L_+^\uparrow &= \{\Lambda \mid \det(\Lambda) = +1,\ \Lambda_{00} \geq 1\} \\ L_-^\uparrow &= \{\Lambda \mid \det(\Lambda) = -1,\ \Lambda_{00} \geq 1\} \\ L_+^\downarrow &= \{\Lambda \mid \det(\Lambda) = +1,\ \Lambda_{00} \leq -1\} \\ L_-^\downarrow &= \{\Lambda \mid \det(\Lambda) = -1,\ \Lambda_{00} \leq -1\} \end{aligned}$$

The identity matrix $I \in L_+^\uparrow$, so this subgroup is analogous to the special subgroup of the orthogonal group, and it is called the *proper orthochronus Lorentz group*. Another symbol for it is $SO^+(1,3)$. The other parts of the full Lorentz group are like copies of this $L_+^\uparrow$ subgroup, but none are closed under composition, so they are not subgroups. The maps that take a matrix from $L_+^\uparrow$ to its three copies are special transformations called Λ_T, Λ_P and $\Lambda_{TP} = \Lambda_T\Lambda_P = \Lambda_P\Lambda_T$. These are the matrices

$$\Lambda_T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \Lambda_P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \Lambda_{TP} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Physically, these transformations correspond to *time-reversal*, *parity-reversal*, and the combined *time-parity-reversal*. Notice that parity-reversal is the same as spatial inversion, the standard right-hand orientation of axes is mapped to a left-hand orientation.

The important consequence of this decomposition is that the pieces are disjoint from one another in parameter space. Just as there was no way to smoothly change from the identity to a reflection in the orthogonal group, there is no way to smoothly change from the identity to any of three matrices above. In other words, the Lie algebra for $O(1,3)$ will only allow us to obtain elements of $L_+^\uparrow = SO^+(1,3)$ under exponentiation.

A representation for the Lie algebra of $SO^+(1,3)$

The defining condition for $O(1,3)$ is $\eta = \Lambda^T\eta\Lambda$. This gives ten independent equations for the $4\times 4 = 16$ real variables of Λ. It follows that the Lie algebra is 6-dimensional and we therefore need to find six matrices for its basis. For now, we will call this Lie algebra $\mathfrak{L}$ and work towards deriving its basis.

Let's start with an infinitesimal transformation, $\Lambda(\epsilon) = I + \epsilon X$, and derive a characterisation of X.

$$\eta = \Lambda^T\eta\Lambda = (I+\epsilon X)^T\eta(I+\epsilon X) = (I+\epsilon X^T)\eta(I+\epsilon X) = \eta + \epsilon(X^T\eta + \eta X) + \epsilon^2(X^T\eta X).$$

To first order in ϵ then, we require $X^T\eta = -\eta X$.

Given that $\eta = \mathrm{diag}(1,-1,-1,-1)$, we see that restricting the above expression to the x_1, x_2, x_3 coordinates gives us the same condition as for the (real) generators of $\mathfrak{so}(3)$: $(X_{ji}) = (-X_{ij})$. This shouldn't be surprising as it amounts to treating transformations of space independently of time, with $SO(3)$ being the natural subgroup of $SO^+(1,3)$. We therefore immediately have three anti-symmetric generators for our real Lie algebra $\mathfrak{L}$ defined by

$$A_k = \begin{pmatrix} 0 & \\ & A_k^{3dim} \end{pmatrix} \qquad \text{or in full}$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The remaining conditions on the matrix X imply that $X_{00} = 0$, and $X_{0l} = X_{l0}$, for $l = 1,2,3$. These give us the three remaining basis matrices for $\mathfrak{L}$ defined by

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We now switch to physics notation for the Lie algebra and write $J_p = iA_p$ and $K_p = iB_p$ for $p = 1, 2, 3$. A general element of $\mathfrak{L}$ is then $L = \alpha K_1 + \beta K_2 + \gamma K_3 + aJ_1 + bJ_2 + cJ_3$, or

$$L = i \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ \alpha & 0 & -c & b \\ \beta & c & 0 & -a \\ \gamma & -b & a & 0 \end{pmatrix}$$

and the corresponding element of $SO^+(1, 3)$ is $\Lambda = e^{iL}$. We already know that $e^{ia_p J_p}$ are rotation matrices, acting on the x_1, x_2, x_3 coordinates and fixing the time-like coordinate. We now use the matrices for K_p to find the associated "elementary" transformations of $SO^+(1, 3)$.

$$\Lambda_p(\alpha) = e^{i\alpha K_p} = I + i\alpha K_p + (i\alpha)^2 K_p^2/2 + \ldots + (i\alpha)^n K_p^n/n! + \ldots$$

Let's start with K_1, and observe that its only non-zero elements are $K_{01} = K_{10} = i$. This means we can restrict to analysing the 2×2 block

$$ik_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \text{with } (ik_1)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Using this in the exponential series we have

$$\begin{aligned} e^{i\alpha k_1} &= I + \alpha(ik_1) + \alpha^2(ik_1)^2/2 + \alpha^3(ik_1)^3/3! + \ldots \\ &= I(1 + \alpha^2/2! + \alpha^4/4! + \ldots) + (ik_1)(\alpha + \alpha^3/3! + \alpha^5/5! + \ldots) \\ &= \begin{pmatrix} \cosh(\alpha) & 0 \\ 0 & \cosh(\alpha) \end{pmatrix} - \begin{pmatrix} 0 & \sinh(\alpha) \\ \sinh(\alpha) & 0 \end{pmatrix} \end{aligned}$$

And so the full matrix for $\Lambda_1(\alpha) = e^{i\alpha K_1}$ is

$$\Lambda_1(\alpha) = \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) & 0 & 0 \\ -\sinh(\alpha) & \cosh(\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And similarly for the other basis generators, K_2, K_3. This type of Lorentz transformation is called a *boost*; it combines the time-like coordinate $x_0 = ct$ with the spatial coordinates.

Commutation relations, structure constants, complexification

We can now compute the structure constants for the Lie algebra $\mathfrak{L}$ simply by finding the matrix commutators for the basis elements:

$$[J_p, J_q] = i\epsilon_{pqr} J_r, \quad [K_p, K_q] = -i\epsilon_{pqr} J_r, \quad [J_p, K_q] = i\epsilon_{pqr} K_r$$

This tells us the rotations J_p are closed under commutation, but the boosts are not.

Exercise: Write out the matrices for J_1 and K_1 in the adjoint representation of the Lie algebra.

It turns out that the adjoint representation is *reducible* when complex coefficients are allowed. The basis that achieves this decomposition is $N_k^\pm$ for $k = 1, 2, 3$ with

$$N_p^\pm = \tfrac{1}{2}(J_p \pm iK_p) \quad \Longleftrightarrow \quad J_p = N_p^+ + N_p^-, \;\; iK_p = N_p^+ - N_p^-.$$

The commutation relations for these operators are now

$$[N_p^+, N_q^+] = i\epsilon_{pqr}N_r^+, \quad [N_p^-, N_q^-] = i\epsilon_{pqr}N_r^-, \quad [N_p^+, N_q^-] = 0$$

This shows us that the complex coefficient Lie algebra $\mathfrak{L}_{\mathbb{C}}$ is exactly two distinct copies of the complexified Lie algebra $\mathfrak{su}(2)_{\mathbb{C}}$. We write this as $\mathfrak{L}_{\mathbb{C}} = \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$. Every $X \in \mathfrak{L}_{\mathbb{C}}$ can be decomposed uniquely into a pair $X = (X_+, X_-)$, with $X = X_+ + X_-$ and $X_\pm \in \text{span}\{N_p^\pm, \mathbb{C}\}$. The associated Lie group elements satisfy

$$e^{iX} = e^{iX_+ + iX_-} = (e^{iX_+})(e^{iX_-}) \quad \text{because } [N_k^+, N_l^-] = 0.$$

If X_+ and X_- did not commute, we would have to invoke the Baker-Campbell-Hausdorff formula.

Combining representations

We will now be able to obtain representations for the Lorentz Lie algebra and its covering group by combining two different representations for $\mathfrak{su}(2)_{\mathbb{C}}$. In the covering group G we want $\rho_{m,n}^G : G \to GL(d, \mathbb{C})$ to be defined by something like

$$\rho_{m,n}^G(g) = e^{i\rho_{m,n}(X)} = (e^{i\rho_m(X_+)}).(e^{i\rho_n(X_-)})$$

where ρ_m and ρ_n are two representations for $\mathfrak{su}(2)_{\mathbb{C}}$. The difficult part of all this is to figure out what vector space the representation $\rho_{m,n}^G$ acts on: what is its dimension, d; and what is the product on the right hand side? Clearly it cannot be a product of matrices as the representations ρ_m and ρ_n have different dimensions. What's required is a *tensor product of vector spaces* and the tensor product of the linear maps that act on them.

First, we have that $\rho_m : \mathfrak{su}(2) \to M(m, \mathbb{C})$, so $\rho_m(X_+)$ and $e^{i\rho_m(X_+)}$ act on vectors $u \in \mathbb{C}^m$. Similarly, the matrices for second representation $\rho_n(X_-)$ act on vectors $v \in \mathbb{C}^n$. Combining the exponential versions of these representations means that the new representation must act on the tensor product space $\mathbb{C}^m \otimes \mathbb{C}^n$, with dimension $d = mn$. If the basis for $\mathbb{C}^m$ is $\{e_1, \dots e_m\}$ and the one for $\mathbb{C}^n$ is $\{f_1, \dots, f_n\}$, then the basis for $\mathbb{C}^m \otimes \mathbb{C}^n$ is written $e_j \otimes f_k$, for all combinations of j, k.

So our representation of the covering group is defined by the tensor product

$$\begin{aligned} \rho_{m,n}^G : G \to GL(\mathbb{C}^m \otimes \mathbb{C}^n), &\quad \text{with} \\ \rho_{m,n}^G(g) = (\rho_m^G \otimes \rho_n^G)(g) = (e^{i\rho_m(X_+)}) \otimes (e^{i\rho_n(X_-)}) &\quad \text{and the action} \\ \rho_{m,n}^G(g)(u \otimes v) = (e^{i\rho_m(X_+)})(u) \otimes (e^{i\rho_m(X_+)})(v) & \end{aligned}$$

What does this all mean for the Lie algebra representations? We have to go back to the definition of the algebra elements as infinitessimal generators for a one-parameter subgroup, $iX = \frac{d}{dt}e^{itX}\big|_{t=0}$, and work through the consequences for the representation matrices. This leads to a result which is analogous to the product rule of differentiation:

$$\rho_{m,n}(X) = (\rho_m \otimes \rho_n)(X_+, X_-) = \rho_m(X_+) \otimes I + I \otimes \rho_n(X_-)$$

The expression $\rho_m \otimes \rho_n$ is an abuse of notation here, but used as a mnemonic or shorthand notation for the combination of Lie algebra representations as given on the right.

More details on tensor products of vector spaces and linear transformations

Suppose we have the two vector spaces $U = \mathbb{C}^m$ and $V = \mathbb{C}^n$ with bases $\{u_1, \dots u_m\}$ and $\{v_1, \dots, v_n\}$. The tensor product of the vector spaces is written $U \otimes V$, it has dimension mn, with a basis of vectors of the form $u_j \otimes v_k$. We fix an ordering of these basis vectors, by convention incrementing the second index before the first:

$$\{u_1 \otimes v_1, u_1 \otimes v_2, \dots, u_1 \otimes v_n, u_2 \otimes v_1, \dots, u_2 \otimes v_n, \dots, u_m \otimes v_n\}$$

Suppose we have two linear transformations $A : U \to U$ and $B : V \to V$, given by $m \times m$ and $n \times n$ matrices (a_{ij}) and (b_{kl}) respectively. The tensor product of these transformations is then $A \otimes B : U \otimes V \to U \otimes V$, defined by taking $u \otimes v$ to $A(u) \otimes B(v)$. We can represent this bilinear transformation using an $mn \times mn$ matrix with respect to the above basis as

$$(A \otimes B) = \begin{bmatrix} a_{11} \left[\, B \,\right] & a_{12} \left[\, B \,\right] & \dots & a_{1m} \left[\, B \,\right] \\ a_{21} \left[\, B \,\right] & a_{22} \left[\, B \,\right] & \dots & a_{2m} \left[\, B \,\right] \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} \left[\, B \,\right] & a_{m2} \left[\, B \,\right] & \dots & a_{mm} \left[\, B \,\right] \end{bmatrix}$$

This type of matrix operation is called a *Kronecker product* (and is implemented in Mathematica).

Low-dimensional representations of $\mathfrak{L}$

Recall that rather than use the dimension n of the $\mathfrak{su}(2)$ representation as a label we used j, the maximal eigenvalue of J_3 where $j = (n-1)/2$ was an integer or half-integer. It is the value of j that carries the most important physical significance, so a representation of $\mathfrak{L}$ usually carries the label (j_1, j_2), with the first eigenvalue referring to the $\mathfrak{su}(2)$ irrep for the N^+ matrices, and the second one to that of N^-.

The $(0,0)$ rep is the trivial one. All elements of $\mathfrak{L}$ are mapped to zero and this exponentiates to 1. Objects acted on in this representation are those that do not change under any Lorentz transformations and are called *scalars*.

The $(\frac{1}{2}, 0)$ representation has the definition

$$\rho_{(\frac{1}{2},0)} = \rho_{\frac{1}{2}} \otimes 1 + I \otimes \rho_0 \qquad \Longrightarrow \rho_{(\frac{1}{2},0)}(X) = \rho_{\frac{1}{2}}(X_+).$$

This means the representation acts on 2-dimensional complex vectors and ignores the N^- part of the basis for $\mathcal{L}$. We have the following matrices for the basis $N_k^{\pm}$:

$$\begin{aligned} &\rho_{(\frac{1}{2},0)}(N_1^+) = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_{(\frac{1}{2},0)}(N_2^+) = \tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_{(\frac{1}{2},0)}(N_3^+) = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ &\rho_{(\frac{1}{2},0)}(N_1^-) = \rho_{(\frac{1}{2},0)}(N_2^-) = \rho_{(\frac{1}{2},0)}(N_3^-) = 0. \end{aligned}$$

Notice that this representation is not faithful. In the J_k, K_k basis we find $\rho_{(\frac{1}{2},0)}(N_k^-) = 0$ means $\rho_{(\frac{1}{2},0)}(J_k) = \rho_{(\frac{1}{2},0)}(iK_k)$. Using this in the expressions for J_k and K_k in terms of $N_k^{\pm}$ from p.26, we have that

$$\rho_{(\frac{1}{2},0)}(J_k) = \rho_{(\frac{1}{2},0)}(iK_k) = \rho_{(\frac{1}{2},0)}(N_k^+) = \text{ scaled Pauli matrices, } \tfrac{1}{2}\sigma_k$$

Matrix exponentiation then gives us representations for elements of the Lorentz $(L_+^{\uparrow})$ *covering group*. So for example,

$$\begin{aligned} e^{i\theta\rho(N_1^+)} &= e^{i\theta\sigma_1/2} = \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \\ e^{i\theta\rho(N_2^+)} &= e^{i\theta\sigma_2/2} = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \\ e^{i\theta\rho(N_3^+)} &= e^{i\theta\sigma_3/2} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \end{aligned}$$

It follows that Lorentz spatial rotations, R_θ, and boosts, T_ϕ, are given by

$$\rho_{(\frac{1}{2},0)}(R_\theta) = e^{i\theta\rho(J)} = e^{i(\theta_1\sigma_1+\theta_2\sigma_2+\theta_3\sigma_3)/2}$$
$$\rho_{(\frac{1}{2},0)}(T_\phi) = e^{i\phi\rho(K)} = e^{(\phi_1\sigma_1+\phi_2\sigma_2+\phi_3\sigma_3)/2}$$

For example, a (clockwise) rotation about the x_1-axis becomes

$$\rho_{(\frac{1}{2},0)}(R_{\theta,1}) = e^{i\theta\sigma_1/2} = \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

While an x_1-coordinate boost is

$$\rho_{(\frac{1}{2},0)}(T_{\phi,1}) = e^{\phi\sigma_1/2} = \begin{pmatrix} \cosh(\phi/2) & \sinh(\phi/2) \\ \sinh(\phi/2) & \cosh(\phi/2) \end{pmatrix}$$

Note that just as the 2-dimensional $j = \frac{1}{2}$ representation for $\mathfrak{su}(2)$ fails to give a proper representation for $SO(3)$, but rather represents its covering group $SU(2)$, the $\rho_{(\frac{1}{2},0)}$ representation is also only well-defined for the covering group of $SO^+(1,3)$. Nevertheless, this representation has physical meaning and the objects they act on (2-dimensional complex vectors) are called *left chiral spinors*.

Next we investigate the $(0, \frac{1}{2})$ representation. The derivation works almost exactly as above, but now we have $N_k^+ \mapsto 0$ and $N_k^- \mapsto \frac{1}{2}\sigma_k$. Together these imply that

$$\tfrac{1}{2}\sigma_k = \rho_{(0,\frac{1}{2})}(N_k^-) = \rho_{(0,\frac{1}{2})}(J_k) = \rho_{(0,\frac{1}{2})}(-iK_k)$$

The boosts in this representation differ only by a sign from those in the one above. Using this in the expressions for rotations and boosts we see

$$\rho_{(0,\frac{1}{2})}(R_\theta) = e^{i\theta\rho(J)} = e^{i(\theta_1\sigma_1+\theta_2\sigma_2+\theta_3\sigma_3)/2}$$
$$\rho_{(0,\frac{1}{2})}(T_\phi) = e^{i\phi\rho(K)} = e^{-(\phi_1\sigma_1+\phi_2\sigma_2+\phi_3\sigma_3)/2}$$

And the same two transformations as before are now

$$\rho_{(0,\frac{1}{2})}(R_{\theta,1}) = \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \quad \rho_{(0,\frac{1}{2})}(T_{\phi,1}) = \begin{pmatrix} \cosh(\phi/2) & -\sinh(\phi/2) \\ -\sinh(\phi/2) & \cosh(\phi/2) \end{pmatrix}$$

The rotations are the same, but the boosts have changed. Objects are acted on by this $(0, \frac{1}{2})$ representation are therefore slightly different to those in the $(\frac{1}{2}, 0)$ representation and are called *right chiral spinors*.

In general, both left and right chiral spinors are referred to as *Weyl spinors*.

The *Dirac spinor* representation combines the left and right chiral spinors into a 4-dimensional representation $\rho_D = \rho_{(\frac{1}{2},0)} \oplus \rho_{(0,\frac{1}{2})}$. This representation is a reducible one with the two components combining in block-diagonal fashion: e.g.,

$$\rho_D(X) = \begin{pmatrix} \rho_{(\frac{1}{2},0)}(X) & 0 \\ 0 & \rho_{(0,\frac{1}{2})}(X) \end{pmatrix} = \begin{pmatrix} \rho_{\frac{1}{2}}(X_+) & 0 \\ 0 & \rho_{\frac{1}{2}}(X_-) \end{pmatrix}$$

Finally, we study the $(\frac{1}{2}, \frac{1}{2})$ representation. This is defined to be

$$\rho_{(\frac{1}{2},\frac{1}{2})}(X) = \rho_{\frac{1}{2}}(X_+) \otimes I + I \otimes \rho_{\frac{1}{2}}(X_-)$$

The operator $\rho_{(\frac{1}{2},\frac{1}{2})}(X)$ acts on the 4-dimensional tensor product space $\mathbb{C}^2 \otimes \mathbb{C}^2$ with

$$\rho_{(\frac{1}{2},\frac{1}{2})}(X)(u \otimes v) = \rho_{\frac{1}{2}}(X_+)(u) \otimes v + u \otimes \rho_{\frac{1}{2}}(X_-)(v)$$

Working through this for the $N_k^{\pm}$ basis, recall that $\rho_{\frac{1}{2}}(N_k^{\pm}) = \frac{1}{2}\sigma_k$, so using the Kronecker product of matrices described earlier we find that

$$\rho_{(\frac{1}{2},\frac{1}{2})}(N_1^+) = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$\rho_{(\frac{1}{2},\frac{1}{2})}(N_2^+) = \begin{pmatrix} 0 & 0 & \frac{-i}{2} & 0 \\ 0 & 0 & 0 & \frac{-i}{2} \\ \frac{i}{2} & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 \end{pmatrix}$$

$$\rho_{(\frac{1}{2},\frac{1}{2})}(N_3^+) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & \frac{-1}{2} \end{pmatrix}$$

Exercise: Write out the matrices for $\rho_{(\frac{1}{2},\frac{1}{2})}(N_k^-)$.

Now $J_k = N_k^+ + N_k^-$ and $K_k = -i(N_k^+ - N_k^-)$ so we have

$$\rho_{(\frac{1}{2},\frac{1}{2})}(J_k) = \frac{1}{2}(\sigma_k \otimes I + I \otimes \sigma_k)$$
$$\rho_{(\frac{1}{2},\frac{1}{2})}(K_k) = \frac{-i}{2}(\sigma_k \otimes I - I \otimes \sigma_k)$$

And, for example, the matrices for J_1 and K_1 are

$$\rho_{(\frac{1}{2},\frac{1}{2})}(J_1) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$\rho_{(\frac{1}{2},\frac{1}{2})}(K_1) = \begin{pmatrix} 0 & \frac{i}{2} & \frac{-i}{2} & 0 \\ \frac{i}{2} & 0 & 0 & \frac{-i}{2} \\ \frac{-i}{2} & 0 & 0 & \frac{i}{2} \\ 0 & \frac{-i}{2} & \frac{i}{2} & 0 \end{pmatrix}$$

Comparing representations

We have now constructed three apparently different 4-dimensional representations for the (complexified) Lorentz Lie algebra, and by exponentiation, the Lorentz group $SO^+(1,3)$. These are the natural or standard representation, the Dirac representation, ρ_D, and the $\rho_{(\frac{1}{2},\frac{1}{2})}$ representation.

How do we determine whether these representations are equivalent or not? One approach is to compute quantities from the matrices that we know do not change under similarity transformations (i.e., a unitary change of basis). Such quantities are called *invariants* or *characters* for a group representation.

Suppose ρ_1 and ρ_2 are equivalent representations of a group $G \to GL(n,\mathbb{C})$. This means that the two matrices $\rho_1(g)$ and $\rho_2(g) = S^{-1}\rho_1(g)S$ for a particular group element g, must have the same invariant. Consequently, if we compute the invariant for $\rho_1(g)$ and $\rho_2(g)$ and get different values, we have shown that the representations are different.

The simplest of these invariants is the matrix trace, $\mathrm{tr}(A) = \sum_k a_{kk}$, the sum of the matrix diagonal entries, which is also equal to the sum of the matrix eigenvalues.

You may recall that one of the defining properties for the Lie algebras of the "special" Lie groups is that $\mathrm{tr}(X) = 0$. [[Why?]] So the trace invariant is not much help in determining equivalence in the Lie algebra, and we must instead look at the matrices for the group.

Let's start by looking at the matrices for the Lorentz rotation $R_{t,1} = e^{itJ_1}$,

$$R_{t,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & \cos(t) \end{pmatrix} \quad \text{trace } = 2 + 2\cos t$$

$$\rho_D(R_{t,1}) = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & i\sin\left(\frac{t}{2}\right) & 0 & 0 \\ i\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) & 0 & 0 \\ 0 & 0 & \cos\left(\frac{t}{2}\right) & i\sin\left(\frac{t}{2}\right) \\ 0 & 0 & i\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) \end{pmatrix} \quad \text{trace } = 4\cos\left(\frac{t}{2}\right)$$

$$\rho_{(\frac{1}{2},\frac{1}{2})}(R_{t,1}) = \begin{pmatrix} \cos^2\left(\frac{t}{2}\right) & \frac{1}{2}i\sin(t) & \frac{1}{2}i\sin(t) & -\sin^2\left(\frac{t}{2}\right) \\ \frac{1}{2}i\sin(t) & \cos^2\left(\frac{t}{2}\right) & -\sin^2\left(\frac{t}{2}\right) & \frac{1}{2}i\sin(t) \\ \frac{1}{2}i\sin(t) & -\sin^2\left(\frac{t}{2}\right) & \cos^2\left(\frac{t}{2}\right) & \frac{1}{2}i\sin(t) \\ -\sin^2\left(\frac{t}{2}\right) & \frac{1}{2}i\sin(t) & \frac{1}{2}i\sin(t) & \cos^2\left(\frac{t}{2}\right) \end{pmatrix} \quad \text{trace } = 4\cos^2\left(\frac{t}{2}\right) = 2 + 2\cos t$$

This immediately tells us that the Dirac representation is different from the other two, and that the natural and $\rho_{(\frac{1}{2},\frac{1}{2})}$ reps may be equivalent.

The matrices for a Lorentz boost $T_{\alpha,1} = e^{i\alpha K_1}$ are

$$T_{t,1} = \begin{pmatrix} \cosh(t) & -\sinh(t) & 0 & 0 \\ -\sinh(t) & \cosh(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{trace } = 2 + 2\cosh(t)$$

$$\rho_{(\frac{1}{2},\frac{1}{2})}(T_{t,1}) = \begin{pmatrix} \cosh^2\left(\frac{t}{2}\right) & -\frac{\sinh(t)}{2} & \frac{\sinh(t)}{2} & -\sinh^2\left(\frac{t}{2}\right) \\ -\frac{\sinh(t)}{2} & \cosh^2\left(\frac{t}{2}\right) & -\sinh^2\left(\frac{t}{2}\right) & \frac{\sinh(t)}{2} \\ \frac{\sinh(t)}{2} & -\sinh^2\left(\frac{t}{2}\right) & \cosh^2\left(\frac{t}{2}\right) & -\frac{\sinh(t)}{2} \\ -\sinh^2\left(\frac{t}{2}\right) & \frac{\sinh(t)}{2} & -\frac{\sinh(t)}{2} & \cosh^2\left(\frac{t}{2}\right) \end{pmatrix} \quad \text{trace } = 4\cosh^2\left(\frac{t}{2}\right) = 2 + 2\cosh t$$

This further supports our hunch that the representations are equivalent. In fact the trace for all six Lie algebra basis generators agree and the natural and $\rho_{(\frac{1}{2},\frac{1}{2})}$ representations are equivalent. This representation is called the *vector representation*.

Challenge Exercise: find a matrix S that makes the $\rho_{(\frac{1}{2},\frac{1}{2})}$ and natural group representations equivalent.

Back to the full Lorentz group

Last week we found that linear transformations, Λ, in the Lorentz group $O(1,3)$ preserved Minkowski space-time metric:

$$\|(x-y)\|^2 = (x-y)^T\eta(x-y) = (x-y)^T\Lambda^T\eta\Lambda(x-y)$$

and this implied a condition on the matrices $\Lambda^T\eta\Lambda = \eta$.

We then saw that the group $O(1,3)$ has four classes of matrix according to the sign of the determinant, and the sign of the Λ_{00}. Only matrices Λ with $\det(\Lambda) = 1$ and $\Lambda_{00} \geq 1$ form a subgroup, $SO^+(1,3)$, the proper orthochronous Lorentz group. The other parts of the group are "copies" of $SO^+(1,3)$ with a flip in time, Λ_T, flip in space, Λ_P, or both, $\Lambda_T\Lambda_P = -I$. So every transformation in

$O(1,3)$ is one of the following: $\Lambda^+ \in SO^+(1,3)$, or it can be written as $\Lambda = \Lambda_T \Lambda^+$, or $\Lambda = \Lambda_P \Lambda^+$, or $\Lambda = -\Lambda^+$.

Changing coordinates

Now we're going to change perspective a little bit and think of the elements of $SO^+(1,3)$ as linear transformations acting on space-time, and consider what happens to their matrix representation under certain changes of coordinate system.

For example, suppose we consider a proper, space rotation, R_θ, and the change of coordinates that reverses time: $X = \Lambda_T x$. If we have the matrix M for R_θ in the x-coordinate system, then the matrix $\widetilde{M}$ in the X-coordinate system is given by

$$\widetilde{M} = \Lambda_T M \Lambda_T^{-1}.$$

We know that the matrix for a proper rotation will fix the x_0 coordinate, and mix together the other three coordinates. So the above expression has block-diagonal form:

$$\widetilde{M} = \begin{pmatrix} 1 & \\ & -I \end{pmatrix} \begin{pmatrix} 1 & \\ & R^{3dim} \end{pmatrix} \begin{pmatrix} 1 & \\ & -I \end{pmatrix} = \begin{pmatrix} 1 & \\ & R^{3dim} \end{pmatrix} = M$$

The matrix is exactly the same.

Now consider a boost transformation B_α that involves just the x_0, x_1 coordinates. We now have a block diagonal form as

$$\widetilde{M} = \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & -I \end{pmatrix} \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) & \\ \sinh(\alpha) & \cosh(\alpha) & \\ & & I \end{pmatrix} \begin{pmatrix} 1 & 0 & \\ 0 & -1 & \\ & & -I \end{pmatrix} = \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) & \\ -\sinh(\alpha) & \cosh(\alpha) & \\ & & I \end{pmatrix} = M^{-1}$$

So the coordinate system with time reversed sees the inverse of the boost transformation matrix.

Similar expressions hold for the matrices of a general boost T_ϕ and for changes of coordinate that reverse parity or both time and parity. Rotation matrices stay the same in a parity-reversed coordinate system but boosts again become their inverses. A time- and parity-reversed coordinate system sees the same matrices for rotation and boost transformations, because $\Lambda_T \Lambda_P = -I$.

Recall that the coordinate system for the matrix of R_θ is exactly the same coordinate system that its exponential generator has. So the change of coordinates applies to the Lie algebra basis matrices J_k, K_k as well. In this context the matrices for the J_k stay the same under time and parity reversals of coordinate systems, while the matrices for the boost generators K_k change sign,

$$\Lambda_T J_k \Lambda_T^{-1} = J_k, \qquad \Lambda_T K_k \Lambda_T^{-1} = -K_k$$

$$\Lambda_P J_k \Lambda_P^{-1} = J_k, \qquad \Lambda_P K_k \Lambda_P^{-1} = -K_k$$

Pushing this through to the low-dimensional representations for $SO^+(1,3)$, we then find that changing coordinates with a time or a parity reversal converts the $\rho_{(\frac{1}{2},0)}$ representation into the $\rho_{(0,\frac{1}{2})}$ This is why they were called *left chiral* and *right chiral spinors*.