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Dynamic Analysis and PID Control for a Quadrotor

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Abstract - In order to analyze the dynamic characteristics and PID controller performance of a quadrotor, this paper firstly describes the architecture of the quadrotor and analyzes the dynamic model of it. Then, based on the classic scheme of PID control, this paper designs a controller, which aims to regulate the posture (position and orientation) of the 6 d.o.f. quadrotor. Thirdly, the dynamic model is implemented in Matlab/Simulink simulation, and the PID control parameters are obtained according to the simulation results. Finally, a quadrotor with PID controllers is designed and made. In order to do the experiment, a flying experiment for the quadrotor has been done. The results of flying experiment show that the PID controllers robustly stabilize the quadrotor.

Keywords- Quadrotor; PID controller; Dynamic model

I. INTRODUCTION

A quadrotor is a kind of non-coaxial multi-rotor aircraft which can achieve vertical take-off and landing (VTOL). The flight attitude control of the quadrotor can be achieved only by adjusting the speed of the four butterfly-distribution rotors. Compared with the conventional rotor-type aircraft, as no tail, quadrotors have a more compact structure. four rotors' lifting force is more uniform than a single rotor, and thus the flight attitude is more stable. Compared with fixed-wing aircraft, the take-off requirements of quadrotor is less, it has other advantages for example, it can hover, has a better environmental adaptability, and so on [1].

As an important representative of multi-rotor aircrafts, the quadrotor has became a new aviation research frontier in the field of aviation and aircraft [2]. In the past few years many research effort has been done in this field. Mesicopter [3] was an ambitious project, which explored the ways to fabricate centimeter-sized vehicles. Such vehicles can be used for gathering planetary atmospheric and meteorological data. The X4 Flyer project of Australian National University [5], aims at developing a quadrotor for indoor and outdoor applications. The control system of the vehicle is based on classical control methodology. Gabe Hoffmann of Stanford University, developed a quadrotor that has achieved autonomous flight. Scott Hanford, designed HMX4141 quadrotor, which is controlled by the flight control computer in ground station [4]. Researchers of Massachusetts Institute of Technology use Draganflyer combining with the computer tracking system have achieved the vision-based hovering, takeoff and landing [6].

The movement of the quadrotor is generated by the lifting force which is provided by the motor driven propeller. According to the quadrotor model diagram (Fig.2), motor (1,3) and motor (2,4) are two parts which are symmetrical with each other. By controlling the speed of the four motor, vertical takeoff and landing, hovering and other movement can be achieved. For example, tumbling movement can be done if (2,4) get reverse control signal, pitch movement can be done if (1,3) get reverse control signal. If we control (1,3) and(2,4) at the same time, yaw movement can be achieved.

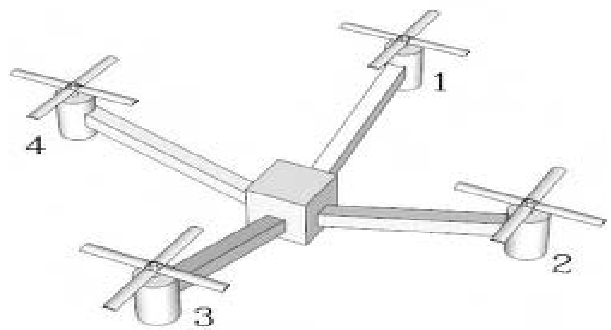

Figure 1. Schematic Diagram of a Quadrotor

II. DYNAMIC MODELLING FOR A QUADROTOR

In order to get the mathematical modeling for the quadrotor, this paper firstly establish , shown in Figure 2.

Of which: the origin of inertial coordinate system E is the initial position of the quadrotor. The positive direction of OX axis is the designated heading of the quadrotor. The positive direction of OX axis is up perpendicular to the horizontal plane. OY axis is perpendicular to the plane OXZ. This coordinate system is used to study the relative movement of ground and quadrotor. The quadrotor's spatial coordinates (X, Y, Z) can be obtained through inertial coordinate system, and thus we can study the position, heading and attitude of the quadrotor.

The origin of quadrotor coordinate system B (Oxyz) is the center of the quadrotor, Ox parallels to the center connection of the front rotor and the rear rotor, and the positive direction points to the front. Oz parallels to the center connection of the left rotor and the right rotor, and the positive direction points to the right. Oy axis perpendicular to the plane Oxz, the positive direction is the direction conform to the right hand rule. This two coordinate can be converted to each other through transition matrix R.

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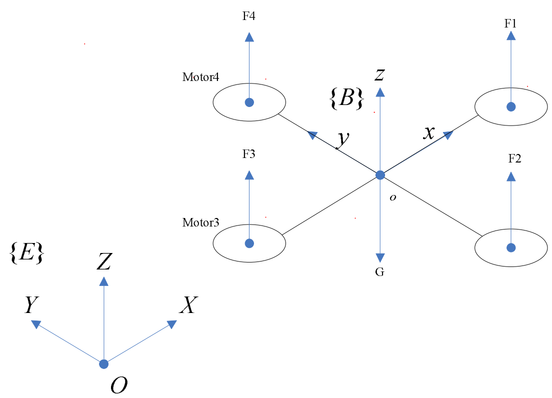


Figure 2. Structure model of the quadrotor

In figure3, the Euler angles are defined as follows:

Yaw angle ψ—angle between the X-axis and the projection of Ox in the OXY plane.

Pitch angle θ—angle between the Z-axis and the projection of Oz in the OXY plane.

Roll angle ϕ—angle between the Y-axis and the projection of Oy in the OXY plane.

So, we can get the transition matrix R which is from the quadrotor coordinate system B (Oxyz) to the inertial frame E (OXYZ).

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{1}$$

$$R_y = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{2}$$

$$R_z = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

$$R = R_x \cdot R_y \cdot R_z = \begin{pmatrix} \cos\psi\cos\phi & \cos\psi\sin\theta\sin\phi & \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\phi & \sin\psi\sin\theta\cos\phi - \sin\phi\cos\psi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix} \tag{4}$$

The transition matrixes from every axis of quadrotor coordinate system B (Oxyz) to the inertial frame E (OXYZ).

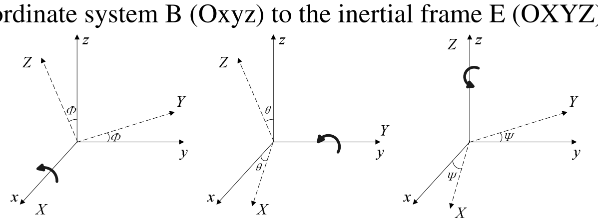


Figure 3. Digram of Euler angle.

In order to establish the dynamic model of the quadrotor, without loss of generality, we make the following assumptions to the quadrotor:

- The quadrotor is a symmetrical rigid body;
- The origin of inertial coordinate system is in the same position with the geometric center and centroid of the quadrotor;
- Resistance and gravity of the quadrotor do not affected by flight altitude and other factors;
- Tensions in all directions are proportional to the square of the propeller speed.

Define Fx, Fy, Fz are components of $\vec{F}$ on the three coordinate axes of quadrotor coordinate system; p, q, r are components of $\vec{\omega}$ on the three coordinate axes of quadrotor coordinate system.

Stress analysis of the quadrotor is shown in Figure 2, where:

Gravity: $G = mg$

Resistance: $D_i = \frac{1}{2}\rho C_d \omega_i^2 = k_d \omega_i^2$

Lift of a single rotor: $T_i = \frac{1}{2}\rho C_t \omega_i^2 = k_t \omega_i^2$

Newton's second law and dynamics equation of the quadrotor can be described in vector forms as following:

$$\vec{F} = m\frac{d\vec{V}}{dt} \tag{5}$$

$$\vec{M} = \frac{d\vec{H}}{dt} \tag{6}$$

Where, $\vec{F}$ is the external forces acting on the aircraft, m is the quality of the quadrotor. $\vec{V}$ is the speed of the quadrotor, $\vec{M}$ is the moment the quadrotor suffered. $\vec{H}$ is the angular momentum the quadrotor relative to the ground inertial frame E.

According to mechanical analysis, Newton's second law and dynamic equation of the quadrotor, the line motion equation can be obtained, described as follows:

$$\begin{cases} \ddot{x} = (F_x - K_1 \cdot \dot{x})\big/ m = \left(k_t \sum_{i=1}^{4} \omega_i^2 (\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi) - K_1 \cdot \dot{x}\right)\Big/ m \\ \ddot{y} = (F_y - K_2 \cdot \dot{y})\big/ m = \left(k_t \sum_{i=1}^{4} \omega_i^2 (\sin\psi\sin\phi\cos\phi - \cos\psi\sin\phi) - K_2 \cdot \dot{y}\right)\Big/ m \\ \ddot{z} = (F_z - K_3 \cdot \dot{z} - mg)\big/ m = \left(k_t \sum_{i=1}^{4} \omega_i^2 (\cos\phi\cos\phi) - K_3 \cdot \dot{z}\right)\Big/ m - g \end{cases} \tag{7}$$

According to the relationship between Euler angle and angular velocity of the quadrotor, the following result can be obtained:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\phi} - \dot{\psi}\sin\theta \\ \dot{\theta}\cos\phi + \dot{\psi}\sin\phi\cos\theta \\ -\dot{\theta}\sin\phi + \dot{\psi}\cos\phi\cos\theta \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} (p\cos\theta + q\sin\phi\sin\theta + r\cos\phi\sin\theta)/\cos\theta \\ q\cos\phi + r\sin\phi \\ (q\sin\phi + r\cos\phi)/\cos\theta \end{bmatrix} \tag{9}$$

Previously the quadrotor has been assumed to be symmetrical in quality and structure, so the inertia matrix I can be defined as a diagonal matrix:

$$I=\begin{bmatrix} I_x & & \\ & I_y & \\ & & I_z \end{bmatrix} \tag{10}$$

And according to $\vec{M}=\frac{d\vec{H}}{dt}$ (11)

We can get

$$\sum M = I\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} \tag{12}$$

By calculating the angular momentum, we can obtain the three axial components angular motion equations of M in the quadrotor coordinate system, which is Mx, My, Mz.

$$\begin{bmatrix} M_X \\ M_Y \\ M_Z \end{bmatrix} = \begin{bmatrix} \dot{p}I_x - \dot{r}I_{xz} + qr(I_z - I_y) - pqI_{xz} \\ \dot{q}I_y + pr(I_x - I_z) + \left(p^2 - r^2\right)I_{xz} \\ \dot{r}I_z - \dot{p}I_{xz} + pq(I_y - I_z) + qrI_{xz} \end{bmatrix} \tag{13}$$

After simplification, the formula is

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \left[M_x + (I_x - I_z)qr\right]/I_x \\ \left[M_y + (I_z - I_x)rp\right]/I_y \\ \left[M_z + (I_x - I_y)pq\right]/I_z \end{bmatrix} \tag{14}$$

nonlinear motion equations of the quadrotor can be obtained by combining the line motion equation and angular motion equation:

$$\begin{cases} \ddot{x} = (F_x - K_1\cdot\dot{x})\big/m = \left(k_t\sum_{i=1}^{4}\omega_i^2\left(\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi\right) - K_1\cdot\dot{x}\right)\Big/m \\ \ddot{y} = (F_y - K_2\cdot\dot{y})\big/m = \left(k_t\sum_{i=1}^{4}\omega_i^2\left(\sin\psi\sin\phi\cos\phi - \cos\psi\sin\phi\right) - K_2\cdot\dot{y}\right)\Big/m \\ \ddot{z} = (F_z - K_3\cdot\dot{z} - mg)\big/m = \left(k_t\sum_{i=1}^{4}\omega_i^2\left(\cos\phi\cos\theta\right) - K_3\cdot\dot{z}\right)\Big/m - g \\ \dot{p} = \left[M_x + (I_y - I_z)qr\right]/I_x \\ \dot{q} = \left[M_y + (I_z - I_x)rp\right]/I_y \\ \dot{r} = \left[M_z + (I_x - I_y)pq\right]/I_z \\ \dot{\phi} = \left(p\cos\theta + q\sin\phi\sin\theta + r\cos\phi\sin\theta\right)/\cos\theta \\ \dot{\theta} = q\cos\phi + r\sin\phi \\ \dot{\psi} = \left(q\sin\phi + r\cos\phi\right)/\cos\theta \end{cases} \tag{15}$$

Where, ***l*** is the distance between the center of the rotor and the origin of the coordinate system. ***Ki*** is drag coefficient.

Define$\boldsymbol{U_1}$, $\boldsymbol{U_2}$, $\boldsymbol{U_3}$, $\boldsymbol{U_4}$, as the inputs of the four independent control channels:

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} F_1 + F_2 + F_3 + F_4 \\ F_4 - F_2 \\ F_3 - F_1 \\ F_2 + F_4 - F_3 - F_1 \end{bmatrix} = \begin{bmatrix} k_t\sum_{i=1}^{4}\omega_i^2 \\ k_t\left(\omega_4^2 - \omega_2^2\right) \\ k_t\left(\omega_3^2 - \omega_1^2\right) \\ k_d\left(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2\right) \end{bmatrix} \tag{16}$$

Where, $\boldsymbol{U_1}$ is vertical speed control input ; $\boldsymbol{U_2}$ is roll control input; $\boldsymbol{U_3}$ is pitch control input; $\boldsymbol{U_4}$ is yaw control input; ω is rotor speed; F is the tension rotor suffered.

In the circumstance of no wind and low wind, drag coefficient can be ignored, so we can get the following model:

$$\begin{cases} \ddot{x} = \left(\cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi\right)U_1/m \\ \ddot{y} = \left(\sin\psi\sin\phi\cos\phi - \cos\psi\sin\phi\right)U_1/m \\ \ddot{z} = \left(\cos\phi\cos\theta\right)U_1/m - g \\ \ddot{\phi} = \left[lU_2 + \dot{\theta}\dot{\psi}\left(I_y - I_z\right)\right]/I_x \\ \ddot{\theta} = \left[lU_3 + \dot{\phi}\dot{\psi}\left(I_z - I_x\right)\right]/I_y \\ \ddot{\psi} = \left[U_4 + \dot{\phi}\dot{\theta}\left(I_x - I_y\right)\right]/I_z \end{cases} \tag{17}$$

Ⅲ. Design of PID Algorithm

By define $\boldsymbol{U_1}$, $\boldsymbol{U_2}$, $\boldsymbol{U_3}$, $\boldsymbol{U_4}$, the complex nonlinear coupled model are decomposed to the four independent control channels. So, the model can be viewed as two independent subsystems, which are line movement subsystem and angular movement subsystem.

We can see from the above, line motion does not affect the angular motion, but the angular motion affects line motion.

Based on this, we use small perturbation method, after ignoring the small perturbation, the motion equation of the quadrotor is:

$$m\dot{x} = Ax + Bu$$

Where, state variable $x=\left[\dot{x},\dot{y},\dot{z},p,q,r,\theta,\phi,\psi\right]^T$ and controlled variable $u=\left[u_1,u_2,u_3,u_4\right]^T$ .Because there is a quasi-integral relation between the attitude angle and angular velocity, in order to simplify the control system, suppose there is simple integral relation:

$$\dot{\phi} = p\,,\quad \dot{\theta} = q\,,\quad \dot{\psi} = r$$

When straight flying, the longitudinal motion equation of quadrotor is $\dot{x}_g = A_g x_g + B_g u_g$, where, state variable $x_g=\left[\dot{x},\dot{z},q,\theta\right]^T$, controlled variable $u_g=\left[u_1,u_4\right]^T$.

Coefficient matrix is as follows:

$$A_g=\begin{bmatrix} x_{\dot{x}} & x_{\dot{z}} & x_q & x_\theta \\ z_{\dot{x}} & z_{\dot{z}} & z_q & z_\theta \\ m_{\dot{x}} & m_{\dot{z}} & m_q & m_\theta \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B_g=\begin{bmatrix} x_t & x_\delta \\ 0 & z_\delta \\ 0 & m_\delta \\ 0 & 0 \end{bmatrix} \tag{18}$$

When horizontal flying, the motion equation of quadrotor is $\dot{x}_h = A_h x_h + B_h u_h$, where, state variable $x_h=\left[v,p,r,\phi,\psi\right]^T$, controlled variable $u_s = u_2$.

Coefficient matrix is as follows:

$$A_g=\begin{bmatrix} y_v & y_p & y_r & y_\phi & 0 \\ l_v & l_p & l_r & l_\phi & 0 \\ n_v & n_p & n_r & n_\phi & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{19}$$

$$B_g = \begin{bmatrix} y_\delta \\ l_\delta \\ n_\delta \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

According to the data of the quadrotor, as well as relative references, we can get the parameter table of the quadrotor:

TABLE I. THE PARAMETER TABLE OF THE QUADROTOR

Parameter	Unit	Value
M	kg	1.2
L	m	0.2
Kt	Ns^2	$3.13e^{-5}$
Kd	Nms^2	$7.5e^{-7}$
Ix	kgm^2	$2.353e^{-3}$
Iy	kgm^2	$2.353e^{-3}$
Iz	kgm^2	$5.262e^{-2}$

According to the transfer function $G(s)=(sI-A)^{-1}B$ and parameter table of the quadrotor, we can get the transfer function of each channel:

Pitch channel:

$$G_1 = \frac{\theta}{u_1} = \frac{56.95s+4391}{s^3+105s^2+870s+4430} \tag{21}$$

Roll channel:

$$G_2 = \frac{\phi}{u_2} = \frac{65s+4560}{s^3+109s^2+1023s+2935} \tag{22}$$

Yaw channel:

$$G_3 = \frac{\psi}{u_3} = \frac{105}{s^2+413s} \tag{23}$$

X-axis direction and pitch angle:

$$G_4 = \frac{x}{\theta} = \frac{\dot{x}}{s\theta} = \frac{-190s+567}{s(57.95s+4400)} \tag{24}$$

Z-axis:

$$G_6 = \frac{z}{u_4} = \frac{\dot{z}}{su_4} = \frac{1.63}{s(s+5)} \tag{25}$$

Z-axis direction and roll angle:

$$G_5 = \frac{y}{\phi} = \frac{\dot{y}}{s\phi} = \frac{-276.4s+743.5}{s(61s+4463)} \tag{26}$$

Ⅳ. SIMULATION AND EXPERIMENT

Based on PID system structure, simulink model is built(figure.4) and simulation is done, through commissioning, values is obtained and shown in Table 2, the step response of position and attitude of the designed controller are shown in Figure5, followed by the yaw angle, pitch angle, roll angle, X axis, Z axis and Y axis.

TABLE II. PARAMETER OF SIMULATION

N	Kpn	Kin	Kdn
1	2	0.01	1
2	2	8	0.1
3	3	0.01	1
4	3	10	0.1
5	5	5	1
6	30	1	1

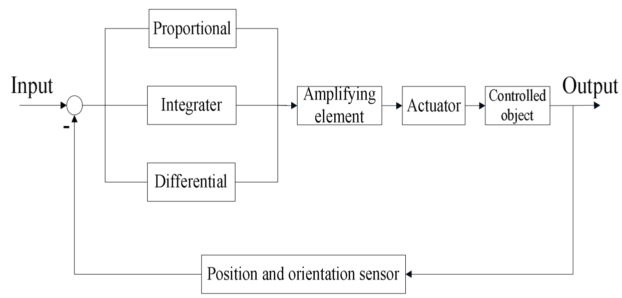
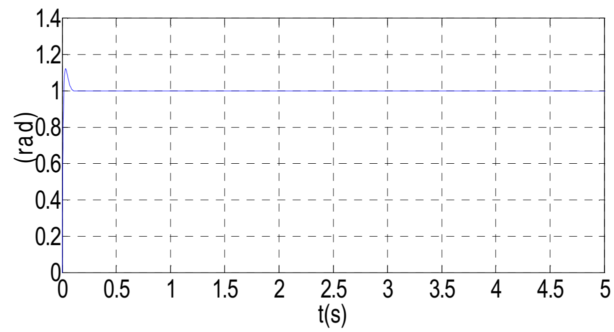
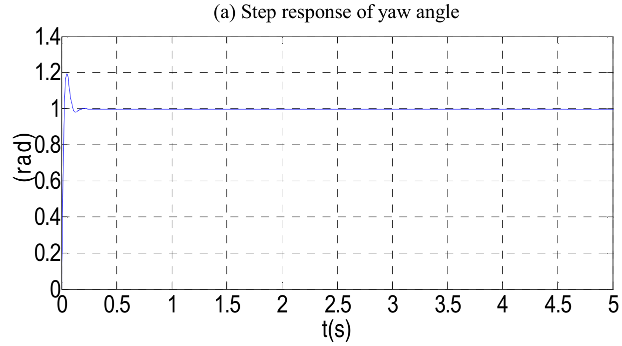


Figure 4. Simulink model of PID controller

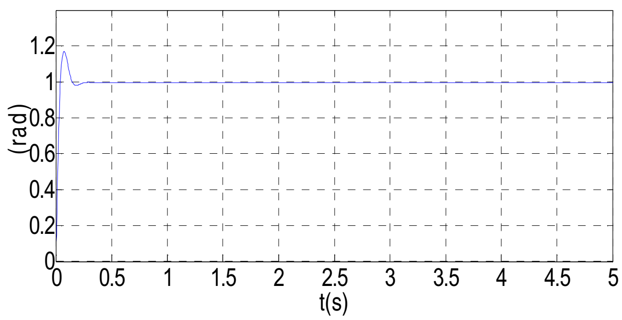
The simulation results show that the system overshoot is small, steady-state error is almost zero, the system response is fast and the performance can be improved. System simulations verify the effectiveness of the design of the control method.



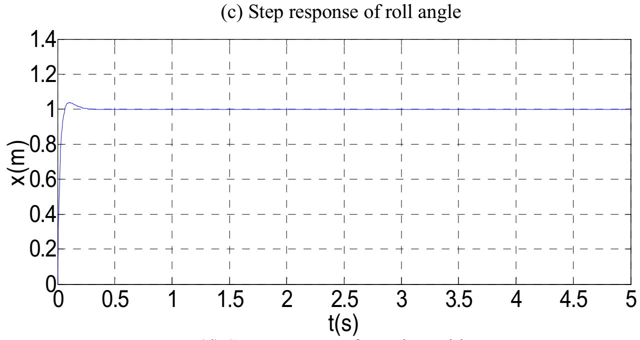
(a) Step response of yaw angle



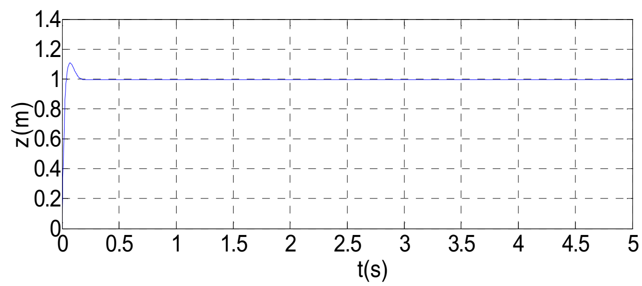
(b) Step response of pitch angle



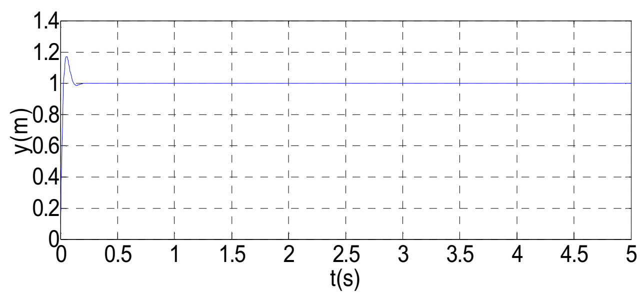
(c) Step response of roll angle



(d) Step response of X axis position



(e) Step response of Z axis position



(f) Step response of Y axis position

Figure 5. Step Response of Position and Attitude

The simulation results only can be a reference, can not be directly used for the actual flight. Therefore, this paper design a outdoor test, through the test, parameters can be obtained eventually.

In order to do the test, we design a experimental quadrotor. It has three main tasks. Firstly, it achieves stable flight so that we can do further research for the flight control system. Secondly, through the quadrotor, we want to design a new mechanical structure and search for the new materials which are suitable for making quadrotor. Thirdly, equipped with high-definition camera, it will do fixed-point aerial photography supported by GPS.

With respect to the mechanical structure, the frame of the quadrotor is made by carbon fiber structure, which the chassis of the aircraft and four motor carriers are made by high-strength 3K carbon fiber board and the cross-type frame is made of carbon fiber rectangular intersections pipe. Carbon fiber is of light weight, high strength, corrosion resistance, long life and other excellent characteristics. The landing gear of the quadrotor is the commercial one U380. The cross-type frame and chassis, as well as the chassis and landing gear is connected by m5 hex bolts. The overall mechanical structure is simple and practical, which meet the requirements of the design of quadrotor.

Figure 6. Picture of the Quadrotor

Experiment sites are selected in low-speed wind and obstacle-free places. When doing the test, firstly we increase the throttle, do not control the attitude of the quadrotor, so that the quadrotor will vertical launch. The test shows that the PID controller can adjust the posture changes caused by wind or other disturbances. The attitude data can be collected by angular speed sensors carried by the quadrotor itself, and transfer to the lower level computer by bluetooth equipment.The results shown in Figure 5, where the abscissa units is 10ms, the vertical axis unit is 0.1 °. The quadrotor lifted-off in 44s; in 44 ~ 54s, it was in the air; as is shown in figure5, the pitch angle, the roll angle and yaw angle all change little (5 ° or less).

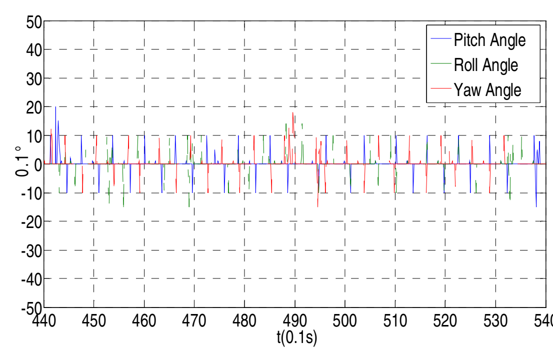


Figure 7. Test Results of the Flying Experiment

V. Conclusion

This paper describes the architecture of a quadrotor and analyzes the dynamic model of it. A quadrotor is designed and made for the experiment. Based on the classic scheme of PID control, this paper designs a controller, which aims to regulate the posture (position and orientation) of the 6 d.o.f. quadrotor. Simulation and experimental studies had been done to the control system.

The simulation results show that the system overshoot is small, at the same time the steady-state error is almost zero, and the system response is fast, which is to say that the performance can be improved by PID controller. So the system simulations of verify the effectiveness of the design of the control method.

The experiment results show that the quadrotor can achieve attitude stabilization if the PID parameters are appropriate.

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References

[1] Ly Dat Minh; Cheolkeun Ha; , "Modeling and control of quadrotor MAV using vision-based measurement," *Strategic Technology (IFOST), 2010 International Forum on* , vol., no., pp.70-75, 13-15 Oct. 2010

[2] Santos, M.; López, V.; Morata, F.; , "Intelligent fuzzy controller of a quadrotor," *Intelligent Systems and Knowledge Engineering (ISKE), 2010 International Conference on* , vol., no., pp.141-146, 15-16 Nov. 2010.

[3] Jun Wu; Hui Peng; Qing Chen; , "RBF-ARX model-based modeling and control of quadrotor," *Control Applications (CCA), 2010 IEEE International Conference on* , vol., no., pp.1731-1736, 8-10 Sept. 2010.

[4] Mian, A.A.; Wang Daobo; , "Nonlinear Flight Control Strategy for an Underactuated Quadrotor Aerial Robot," *Networking, Sensing and Control, 2008. ICNSC 2008. IEEE International Conference on* , vol., no., pp.938-942, 6-8 April 2008

[5] González-Vázquez, S.; Moreno-Valenzuela, J.; , "A New Nonlinear PI/PID Controller for Quadrotor Posture Regulation," *Electronics, Robotics and Automotive Mechanics Conference (CERMA), 2010* , vol., no., pp.642-647, Sept. 28 2010-Oct. 1 2010

[6] Salih, A.L.; Moghavvemi, M.; Mohamed, H.A.F.; Gaeid, K.S.; , "Modelling and PID controller design for a quadrotor unmanned air vehicle," *Automation Quality and Testing Robotics (AQTR), 2010 IEEE International Conference on* , vol.1, no., pp.1-5, 28-30 May 2010.