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Closed-Loop Subspace Identification of a Quadrotor

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Abstract of the Thesis

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Professor Steve Gibson, Chair

Ne quo feugiat tractatos temporibus, eam te malorum sensibus. Impetus voluptua senserit id mel, lucilius adipiscing at duo. Suas noster sanctus cu pro, movet dicam intellegebat pri ad, esse utamur vulputate ut per. Admodum facilisi sea an, omittam molestiae pertinacia vim eu, et eam illud graeco. Enim persius duo ei, mea te posse congue putent.

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The thesis of Andrew G. Kee is approved.

Steve Gibson, Committee Chair

University of California, Los Angeles

2013

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Nomenclature

N	Number of block rows
s	Number of block columns
$\mathbb{R}$	Set of all real numbers
u_k	System input sequence
y_k	System output sequence
x_k	System state sequence
f	Future time horizon
p	Past time horizon

CHAPTER 1

Introduction

Unmanned Aerial Vehicles (UAVs) have seen explosive growth in the past thirty years, performing a multitude of military and civilian tasks including surveillance, reconnaissance, armed combat operations, search and rescue, forest fire management, and domestic policing [13, 15]. A class of modern UAVs which have recently grown in popularity are quadrotors - Vertical Take Off and Landing (VTOL) vehicles powered by four rotors arranged in a cross configuration. The main advantage of the quadrotor lies in its mechanical simplicity. Adjusting the speed of one or more of the vehicle's fixed-pitch rotors provides full attitude control, eliminating the need for the swash plate mechanism found on single rotor helicopters [1, 3]. In spite of its mechanical simplicity, the quadrotor exhibits somewhat complex dynamics that are best modeled as a Multi-Input Multi-Output (MIMO) system.

Advances in MEMS sensors and light-weight high-powered lithium polymer batteries have contributed to the recent popularity of quadrotors, making them an attractive choice for research applications in flight dynamics and control, as in [4, 7, 9, 10]. One problem of particular interest is the development of mathematical models representing system dynamics based on experimentally gathered data. System identification provides a mechanism to relate this input-output data to the underlying system dynamics. Traditionally, system identification techniques have focused on developing a system model which minimizes prediction error. Identification methods of this form are commonly known as Prediction Error Methods (PEMs). PEMs have seen widespread use in both theoretical and real-world applications, but experience difficulties with MIMO systems as noted in [11, 21]. Subspace identification methods have recently grown in popularity and offer an alternative approach to the identification problem. These methods have a foundation in linear algebra and overcome the issues found in PEMs when identifying MIMO systems [6]. It is the goal of this research project to apply subspace identification techniques to a quadrotor using experimentally gathered closed-loop input and output data.

1.1 Related Work

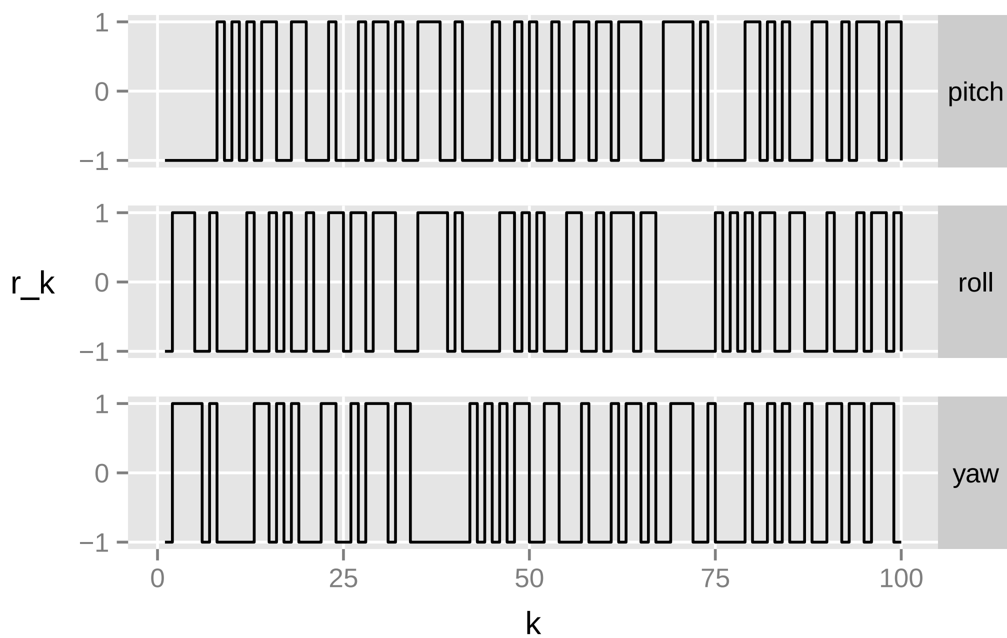

Figure 1.1: A figure caption

1.2 Motivation and Contributions

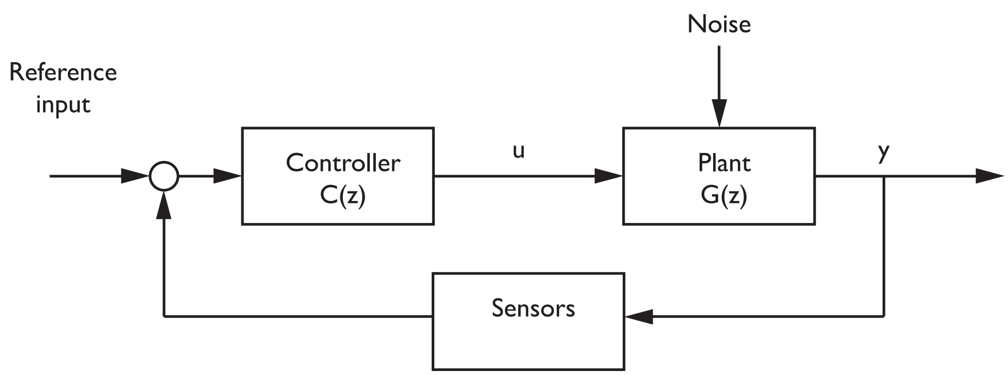

Figure 1.2: A figure caption

CHAPTER 2

Preliminaries

2.1 Linear Algebra Tools

Hankel Matrices

A Hankel matrix is a matrix $H \in \mathbb{R}^{m \times n}$ with constant skew-diagonals. In other words, the value of the $(i, j)^{\text{th}}$ entry of H depends only on the sum $i + j$.

$$H_{m,n} = \begin{bmatrix} h_1 & h_2 & \cdots & h_n \\ h_2 & h_3 & \cdots & h_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_m & h_{m+1} & \cdots & h_{m+n-1} \end{bmatrix}$$

If each entry in the matrix is also a matrix, it is called a block Hankel matrix.

2.1.1 Fundamental Matrix Subspaces

We require two of the fundamental matrix subspaces: the column space and the row space. The column space of a matrix $A \in \mathbb{R}^{m \times n}$ is the set of all linear combinations of the column vectors of A. The dimension of the column space is called the rank. The row space of a matrix $A \in \mathbb{R}^{m \times n}$ is the set of all linear combinations of the row vectors of A.

2.1.2 Projections

2.1.3 Singular Value Decomposition

Any matrix $A \in \mathbb{R}^{m \times n}$ can be decomposed by a singular value decomposition (SVD) given by

$$A = U \Sigma V^T$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices and $\Sigma \in \mathbb{R}^{m \times n}$ is diagonal matrix of the singular values of A ordered such that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$$

2.2 Linear Systems

2.2.1 Linear Time-Invariant Systems

2.2.2 Deterministic and Stochastic Systems

2.2.3 State Space Representations

We will consider a combined deterministic-stochastic LTI system written in innovation form as

$$x(k+1) = Ax(k) + Bu(k) + Ke(k) \tag{2.1a}$$

$$y(k) = Cx(k) + Du(k) + e(k) \tag{2.1b}$$

where $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^m$ is the system input, $y_k \in \mathbb{R}^l$ is the system output, and $e_k \in \mathbb{R}^l$ is the innovation. A, B, C, and D are the system matrices with appropriate dimensions and K is the Kalman filter gain. The system represented in (2.1) can also be represented in predictor form as

$$x(k+1) = A_K x(k) + B_K u(k) + Ky(k) \tag{2.2a}$$

$$y(k) = Cx(k) + Du(k) + e(k) \tag{2.2b}$$

where $A_K = A - KC$ and $B_K = B - KD$.

The systems represented by (2.1) and (2.2) are equivalent from an input/output point of view, but because A_K is guaranteed stable even if the original process matrix A is unstable, the predictor form proves advantageous when considering unstable open-loop systems. We will use the state space model in innovation form to derive the general subspace algorithm for identifying combined deterministic-stochastic LTI systems but will rely on the prediction form of the model when considering identification of closed-loop systems.

2.3 Assumptions

[**Assumption 1**]**:** $A_K = A - KC$ is stable (i.e. its eigenvalues lie within the unit circle)

[**Assumption 2**]**:** The system is represented in its minimal form

CHAPTER 3

Subspace Identification Methods

Subspace identification methods (SIM) provide an approach to identifing LTI systems in their state space form using input-output data. SIMs provide an attractive alternative to Prediction Error Methods (PEM) because of their ability to identify MIMO systems and their non-iterative solution nature, making them suitable for working with large data sets. In general, the subspace identification problem is: given a set of input and output data, estimate the system matrices (A, B, C, D) up to within a similarity transform.

Extensive work in both the theory and application of SIMs in the last 20 years has resulted in the development of a number of popular algorithms, including the canonical variate analysis (CVA) method proposed by Larimore [8], the multi-variable output-error state space (MOESP) method proposed by Verhaegen [19], and the numerical algorithms for subspace state space system identification (N4SID) proposed by Van Overschee and De Moor [16]. A unifying theorem proposed by Van Overschee and De Moor [17] links these algorithms and provides a generalized approach to the subspace identification problem.

As described in Van Overschee and De Moor's unifying theorem, all SIMs follow the same general two step procedure. First, estimate the subspace spanned by the columns of the extended observability matrix (Γ_s) from input-output data u_k, y_k. Because the dimension of Γ_s determines the order n of the estimated system, we reduce the order of the estimated subspace before proceeding. Second, the system matrices are determined, either directly from the extended observability matrix or from the realized state sequence X_s.

Until recently, SIMs were unable to identify systems operating in the presence of feedback control (i.e. closed-loop). In the open-loop case, input data is uncorrelated with past noise. When a system is operating in closed-loop, the presence of feedback control causes the input to be correlated with past noise as the controller attempts to eliminate system disturbances [11]. The result is a system that is biased when identified using traditional SIMs. Several new approaches to identifying closed-loop systems by decoupling inputs from past noise (thus removing any bias) have been proposed, most notably the innovation estimation method (IEM) proposed by Qin and Ljung [12] and the whitening filter approach (WFA) proposed by Chiuso and Picci [2].

3.1 Open-Loop Subspace Identification

When a system is operating in open-loop (i.e. no feedback), the input data is assumed to be independent of past noise. In this case, the traditional SIMs (MOESP, N4SID, CVA) can be used without modification.

3.1.1 Extended State Space Model

Recalling the combined deterministic-stochastic LTI system is given in its innovation form as

$$x(k+1) = Ax(k) + Bu(k) + Ke(k) \tag{3.1a}$$

$$y(k) = Cx(k) + Du(k) + e(k) \tag{3.1b}$$

Application of the subspace algorithms require we represent the input, output, and noise sequences in Hankel form with $2k$ block rows and N columns. For the input matrix,

$$U_s = \left[\begin{array}{cccc} u(0) & u(1) & \cdots & u(N-1) \\ u(1) & u(2) & \cdots & u(N) \\ \vdots & \vdots & \ddots & \vdots \\ u(k-1) & u(k) & \cdots & u(k+N-2) \\ \hline u(k) & u(k+1) & \cdots & u(k+N-1) \\ u(k+1) & u(k+2) & \cdots & u(k+N) \\ \vdots & \vdots & \ddots & \vdots \\ u(2k-1) & u(2k) & \cdots & u(2k+N-2) \end{array}\right] = \left[\begin{array}{c} U_p \\ \hline U_f \end{array}\right] \tag{3.2}$$

where p and f denote past and future horizons, respectively. The input matrix is partitioned into these two overlapping "past" and "future" blocks to later construct an instrumental variable matrix used to eliminate the influence of noise. We delay the explanation of this procedure until it is needed in Section 3.1.2. We construct similar matrices Y_s and E_s for the output and noise data.

Based on the state space model in (3.1) and considering the future horizon, an extended state space model can be formulated as

$$Y_f = \Gamma_f X + H_f U_f + G_f E_f \tag{3.3}$$

The extended observability matrix is

$$\Gamma_f = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{f-1} \end{bmatrix} \tag{3.4}$$

and H_f and G_f are Toeplitz matrices of the Markov parameters of the deterministic and stochastic subsystems, respectively

$$H_f = \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ CAB & CB & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{f-2}B & CA^{f-3}B & CA^{f-4}B & \cdots & D \end{bmatrix} \tag{3.5a}$$

$$G_f = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ CK & I & 0 & \cdots & 0 \\ CAK & CK & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{f-2}K & CA^{f-3}K & CA^{f-4}K & \cdots & I \end{bmatrix} \tag{3.5b}$$

We will leverage this structure of (3.3) to identify the unknown system matrices from known input-output data. In particular, we will estimate the column space of the extended observability matrix. Knowledge of this subspace is sufficient to then estimate the unknown system matrices.

3.1.2 Estimation of the Extended Observability Matrix

Determination of the system matrices relies on an estimate of the column space of Γ_f. In order to estimate the column space of Γ_f in (3.3), we must eliminate the influence of the input and noise terms. The general procedure, as outlined in [11, 20] is as follows: First, we eliminate the influence of the input U_f by post-multiplying (3.3) by $\Pi^{\perp}_{U_f}$ where $\Pi^{\perp}_{U_f}$ is an orthogonal projection onto the column space of U_f given by

$$\Pi^{\perp}_{U_f} = I - U_f^T(U_f U_f^T)^{-1}U_f$$

By definition, $U_f \Pi^{\perp}_{U_f} = 0$ so (3.3) becomes

$$Y_f \Pi^{\perp}_{U_f} = \Gamma_f X_k \Pi^{\perp}_{U_f} + G_f E_f \Pi^{\perp}_{U_f} \tag{3.6}$$

Under open-loop conditions, E_f is uncorrelated with U_f. That is,

$$E_f \Pi_{U_f}^{\perp} = E_f(I - U_f^T(U_f U_f^T)^{-1} U_f) = E_f$$

so

$$Y_f \Pi_{U_f}^{\perp} = \Gamma_f X_k \Pi_{U_f}^{\perp} + G_f E_f \tag{3.7}$$

Next we eliminate the influence of the noise E_f. In order to remove the influence of the noise on the extended observability matrix, we must introduce an instrumental variable matrix as described in [20]. We seek a matrix $Z \in \mathbb{R}^{2k \times N}$ which exhibits the following properties:

$$\lim_{N\to\infty} \frac{1}{N} E_f Z^T = 0 \tag{3.8a}$$

$$\operatorname{rank}\left(\lim_{N\to\infty} \frac{1}{N} X_k \Pi_{U_f}^{\perp} Z^T\right) = n \tag{3.8b}$$

Satisfying condition (3.8a) ensures that we can eliminate E_f by multiplying (3.7) on the right by Z^T and take the limit for $N \to \infty$:

$$\lim_{N\to\infty} \frac{1}{N} Y_f \Pi_{U_f}^{\perp} Z^T = \lim_{N\to\infty} \frac{1}{N} \Gamma_f X_k \Pi_{U_f}^{\perp} Z^T \tag{3.9}$$

Satisfying condition (3.8b) ensures multiplication by Z^T does not change the rank of the remaining term on the right hand side of (3.9) so we have

$$\operatorname{range}\left(\lim_{N\to\infty} \frac{1}{N} Y_f \Pi_{U_f}^{\perp} Z^T\right) = \operatorname{range}(\Gamma_f) \tag{3.10}$$

From (3.10) we see that an SVD of the matrix $Y_f \Pi_{U_f}^{\perp} Z^T$ will provide an estimate of the column space of Γ_f. All that remains is to identify a suitable instrumental variable matrix Z. As described in [14, 20], instrumental variable matrices are typically constructed from input-output data. Recalling that we partitioned our input and output data into "past" and "future" sets in Section 3.1.1, we will use the "future" input-output data to identify the system and the "past" input-output data as the instrumental variable matrix Z where

$$Z_p = \begin{bmatrix} U_p \\ Y_p \end{bmatrix}$$

Recalling that E_f is uncorrelated with U_f for open-loop systems, and enforcing the assumption that E_f is white-noise, we have from [20] that

$$\lim_{N\to\infty} \frac{1}{N} E_f Z_p^T = 0$$

which satisfies condition (3.8a). Jannson showed in [5] that if the input sequence is persistently exciting, the rank condition (3.8b) is satisfied. We will enforce the persistence of excitation criteria during experiment design to ensure Z_p is a valid instrumental variable matrix.

In the presence of noise, the matrix $Y_f \Pi_{U_f}^{\perp} Z^T$ is full rank while the true system order is smaller. We choose the order of the identified system by partitioning the SVD matrices as follows:

$$Y_f \Pi_{U_f}^{\perp} Z^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} V_1^T & V_2^T \end{bmatrix}$$

where the number of singular values n in S_1 is equal to the system order and the remaining submatrices are scaled appropriately. The reduced rank estimate of the extended observability matrix is then

$$\hat{\Gamma}_f = U_1 {S_1}^{1/2} \tag{3.11}$$

Numerically Efficient Estimation of Γ_f by LQ Factorization

In the case where N is large, the construction of the matrix $Y_f \Pi_{U_f}^{\perp} Z^T$ and the calculation of its SVD is computationally intensive. Verhaegen has shown in [18] that from the following LQ factorization

$$\begin{bmatrix} U_f \\ U_p \\ Y_p \\ Y_f \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} \tag{3.12}$$

we have

$$\text{range}\left(\lim_{N\to\infty}\frac{1}{\sqrt{N}}\begin{bmatrix}R_{42} & R_{43}\end{bmatrix}\right) = \text{range}\,(\Gamma_f) \tag{3.13}$$

This shows an equivalency between (3.10) and (3.13), therefore we can estimate the column space of $\hat{\Gamma}_f$ by computing the LQ factorization (3.12), taking the SVD of the matrix $\begin{bmatrix}R_{42} & R_{43}\end{bmatrix}$, and reducing the system order as described above. The remainder of this document assumes $\hat{\Gamma}_f$ is estimated via LQ factorization.

3.1.3 Determination of the System Matrices

We will exploit the structure of the extended observability matrix to recover the A and C matrices. The matrix C can be read directly from the first block row of $\hat{\Gamma}_f$. In order to recover A, we define the following two matrices

$$\hat{\overline{\Gamma}}_f = \begin{bmatrix} C \\ \vdots \\ CA^{f-2} \end{bmatrix}, \qquad \hat{\underline{\Gamma}}_f = \begin{bmatrix} CA \\ \vdots \\ CA^{f-1} \end{bmatrix} \tag{3.14}$$

Inspecting the structure of $\hat{\overline{\Gamma}}_f$ and $\hat{\underline{\Gamma}}_f$, we see that $\hat{\overline{\Gamma}}_f$ is equal to $\hat{\Gamma}_f$ without the last block row and $\hat{\underline{\Gamma}}_f$ is equal to $\hat{\Gamma}_f$ without the first block row. The structure of (3.14) implies

$$\hat{\overline{\Gamma}}_f A = \hat{\underline{\Gamma}}_f \tag{3.15}$$

which is linear in A and can be solved by least squares.

All that remains is to recover the B and D matrices. Verhaegen has shown in [18] that given the LQ factorization in (3.12), we have

$$\begin{bmatrix}R_{31} & R_{42}\end{bmatrix} = H_f \begin{bmatrix}R_{11} & R_{22}\end{bmatrix}$$

3.2 Closed-Loop Subspace Identification

3.2.1 Identifying Systems Operating Under Feedback Control

3.2.2 Innovation Estimation Method

3.2.3 Whitening Filter

CHAPTER 4

Experiments

4.1 Experiment Design

CHAPTER 5

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6.1 Future Work

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