Understanding the PRBS Signal as an Optimum Input Signal in the Wavelet-Correlation Method of System Identification using Multiresolution Analysis

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Abstract

This paper will focus on the relationship between the pseudo-random binary sequence (PRBS) signal and the dyadic lattice. A method for estimating the PRBS function by a series of Haar wavelets will be derived and explained. Relating the PRBS signal to the Haar wavelet is advantageous because the Haar wavelet forms an orthonormal wavelet basis. Multiresolution analysis will be used to write the PRBS signal as a finite series of Haar wavelets, and the dyadic period of the PRBS signal will be exploited to make this task easier. The PRBS signal and dyadic lattice will be shown to be complimentary choices for the wavelet-correlation (W-C) method.

1. Introduction

Many methods of system identification have been derived and are used to determine unknown system parameters. In any identification experiment, a suitable input signal must be chosen which is persistently exciting [1]. The wavelet-correlation (W-C) method is a method of system identification that has been derived as a generalization of the correlation method [2]. This method utilizes wavelets rather than correlation functions to determine the system transfer function estimate. In any wavelet experiment, a mother wavelet, scale parameter, and translation parameter must be selected.

The optimum input signal for the W-C method has been shown to be the pseudo-random binary sequence (PRBS) [2]. In this paper, it will be shown that a PRBS function can be generated as a sum of wavelet functions through multiresolution analysis. The periodic nature of the PRBS signal can be exploited to simplify the generation of the input signal.

2. Wavelet-Correlation Method

The transfer function for the W-C method can be written as:

$$\hat{H}(\omega) = \frac{\iint \frac{1}{\sqrt{a}} \int y(t) u\left(\frac{t-b}{a}\right) dt e^{-j\omega b} \, db\, da}{\iint \frac{1}{\sqrt{a}} \int u(t) u\left(\frac{t-b}{a}\right) dt e^{-j\omega b} \, db\, da} \qquad (1)$$

where *y(t)* and *u(t)* are the output and input signals, respectively. Also, the scale and translation parameters are *a* and *b*, respectively.

In this method of identification, the input signal is also the mother wavelet which means that the signal must be persistently exciting and must also meet the wavelet criteria**.** It has been shown that not all wavelets will be persistently exciting and a wavelet may not be persistently excited to all order [3]. For the W-C method, a pseudo-random binary sequence (PRBS) was chosen as the mother wavelet and was shown to produce more accurate identification results than other wavelets [3]. A scale parameter and translation parameter equal to two and one, respectively, were chosen.

3. PRBS Signal

A PRBS signal is a popular input signal for system identification because it is persistently exciting to the order of the period of the signal. A PRBS signal is a series of step functions generated by a series of shift registers with an exclusive or operator [4]. A maximum length PRBS signal has a correlation function that resembles a white noise correlation function. This property does not hold for non-maximum length sequences. The maximum possible period for a maximum length sequence is $\mu = 2^{\eta} - 1$ where η is the order of the PRBS. Thus the PRBS signal used in identification processes should be a maximum length PRBS signal. For example, consider a 3rd order PRBS signal shown in Figure 1. The maximum period is 7.

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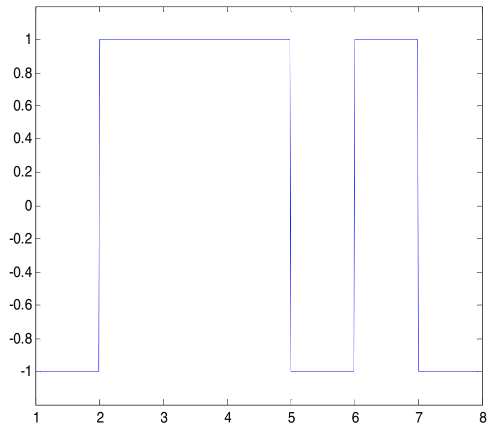

Figure 1. 3rd Order PRBS Signal

The PRBS is also a good choice as a mother wavelet. The PRBS signal is not a true wavelet in the sense of the classical wavelet forms because it does not decay to zero. However, the PRBS can be forced to zero at its beginning and end points with proper choice of signal length and can be said to decay very rapidly to zero. The PRBS will be compared to the Haar wavelet since the PRBS decays to zero in the same manner the Haar wavelet decays to zero. The Haar wavelet is defined by the following equation:

$$\psi(t) = \begin{cases} 1, & 0 \le t < 1/2 \\ -1, & 1/2 \le t < 1 \\ 0, & otherwise \end{cases} \tag{2}$$

and shown in Figure 2 below.

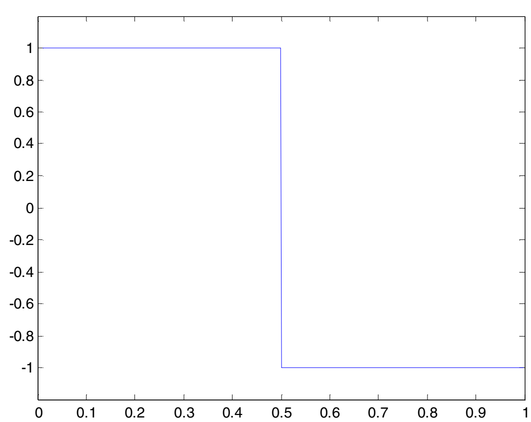

Figure 2. Haar Wavelet

4. Dyadic Lattice

The scale and translation parameters form the lattice. A dyadic lattice was selected for the W-C method [5]. This lattice is composed of a scale parameter that is equal to two and a translation parameter equal to one. A dyadic lattice is used in multiresolution analysis to generate orthonormal wavelet bases [6]. These bases have many desirable properties such as nonredundancy and energy preservation. A dyadic lattice is a popular choice in wavelet applications because it allows for ease in generating the wavelet functions. One reason for pairing the dyadic lattice with the PRBS input signal is that the factor of two is prevalent in both the PRBS signal and the dyadic scale.

5. Multiresolution Analysis

Any signal can be approximated as an infinite sum of Haar wavelets through multiresolution analysis [7]. Multiresolution analysis is a technique for estimating functions by generating successively finer approximations. Consider a function, *f*, which can be approximated by a compactly supported function on the interval $\left[-2^{J_1}, 2^{J_1}\right]$ where J_1 is a large integer. Also, the function, *f*, is piecewise constant on the interval $\left[-k2^{-J_0}, (k+1)2^{-J_0}\right]$ where J_0 is a large integer. By multiresolution analysis, this function can be written as:

$$f = \hat{f}^1 + \delta^1 \tag{3}$$

where $\hat{f}^1$ is the first order approximation of *f* and δ^1 is the new information available about the function. δ^1 can be calculated as will be discussed but $\hat{f}^1$ must be determined through multiresolution analysis by the following equation:

$$\hat{f}^1 = \hat{f}^2 + \delta^2 \; . \tag{4}$$

Again, δ^2 is new information available concerning the function estimate which can be easily calculated. $\hat{f}^2$ must be approximated through multiresolution analysis by

$$\hat{f}^2 = \hat{f}^3 + \delta^3 \; . \tag{5}$$

So that in general

$$\hat{f}^{r-1} = \hat{f}^r + \delta^r \tag{6}$$

where $\hat{f}^r$ is the r^{th} order approximation of *f*. As *r* increases, $\hat{f}^r$ will become coarser. To determine the approximation of *f*, the delta functions are calculated for $r=1, \ldots, J_0+J_1$. At this point, $\hat{f}^{J_0+J_1}$ is a step function and can easily be determined. Then the delta functions and $\hat{f}^{J_0+J_1}$ can be substituted back into Equations (3) and (6) to yield the approximation of *f* which can be written as

$$f = \hat{f}^{J_1+J_0} + \delta^1 + \ldots + \delta^{J_1+J_0} \quad . \tag{7}$$

Therefore, the multiresolution analysis approximation of a function is determined by a series of increasingly finer approximations.

6. Multiresolution Analysis with the Haar Wavelet

Consider multiresolution of a function *f* defined in Equation (6). Then the delta functions can be written as

$$\delta^r = \sum_{k=-2^{J_1+J_0-1}+r}^{2^{J_1+J_0-1}-1-r} \alpha_{2k}^r \psi\left(2^{J_0-r}x-k\right), \tag{8}$$

where $\psi(x)$ is the Haar wavelet [5]. The alpha coefficients are defined as

$$\begin{aligned} \alpha_{2k}^r &= \beta_{2k}^{r-1} - \beta_k^r = \frac{1}{2}\left(\beta_{2k}^{r-1} - \beta_{2k+1}^{r-1}\right) \\ \alpha_{2k+1}^r &= \beta_{2k+1}^{r-1} - \beta_k^r = \frac{1}{2}\left(\beta_{2k+1}^{r-1} - \beta_{2k}^{r-1}\right) = -\alpha_{2k}^r \end{aligned} \tag{9}$$

where the beta coefficients represent the approximation of the function, *f*, at various stages and are defined as

$$\beta_k^r = \frac{1}{2}\left(\beta_{2k}^{r-1} + \beta_{2k+1}^{r-1}\right) \quad . \tag{10}$$

The coarsest approximation of *f* occurs at $r = J_0 + J_1$ and can be written as

$$\begin{aligned} \hat{f}^{J_=+J_1} = {} & \beta_{-1}^{J_0+J_1}\left[u\left(x+2^{J_1}\right)-u(x)\right]+ \\ & \beta_0^{J_0+J_1}\left[u(x)-u\left(x-2^{J_1}\right)\right] \end{aligned} \tag{11}$$

where *u(x)* is a step function. Equations (8) – (11) can then be substituted back into Equation (3) to generate an estimate of a PRBS function by a series of Haar wavelets which can be written as:

$$\begin{aligned} f = {} & \beta_{-1}^{J_0+J_1}\left[u\left(x+2^{J_1}\right)-u(x)\right]+ \\ & \beta_{-1}^{J_0+J_1}\left[u(x)-u\left(x+2^{J_1}\right)\right] \\ & + \sum_{r=1}^{J_1+J_0} \sum_{k=2^{J_1+J-1}+r}^{2^{J_1+J-1}} \alpha_{2k}^r \psi\left(2^{J_0-r}x-k\right) \end{aligned} \tag{12}$$

where $u(x)$ is a step function, $\psi(x)$ is a Haar wavelet, the α and β coefficients are defined in Equations (9) and (9), and J_0 and J_1 are constants previously defined.

For example, consider the maximum length PRBS signal of order 3 shown in Figure 1. The signal is compactly supported on the interval [-8, 8] and thus $J_1=3$. Since the PRBS is only guaranteed to be piecewise constant on intervals of length 1, $J_0=0$. First assign values of *f* to β_k^0 for $k=-8, 8$. Then Equations (9) and (10) can be used to compute the alpha and beta coefficients as shown in Tables 1 and 2 below. From Equation (8), the delta functions can be written as

$$\begin{aligned} \delta^1 &= \sum_{k=-4}^{3} \alpha_{2k}^r \psi(x/2-k) \\ \delta^2 &= \sum_{k=-2}^{1} \alpha_{2k}^r \psi(x/4-k). \\ \delta^3 &= \sum_{k=-1}^{0} \alpha_{2k}^r \psi(x/8-k) \end{aligned} \tag{13}$$

$\hat{f}^3$ is the coarsest approximation of the function *f* and from Equation (11) becomes

$$\begin{aligned} \hat{f}^3 = {} & \beta_{-1}^3\left[u(x+8)-u(x)\right]+ \\ & \beta_0^3\left[u(x)-u(x-8)\right]. \end{aligned} \tag{14}$$

Then by substituting these equations in to Equation (12), the third order PRBS signal can be written as

$$\begin{aligned} f = {} & \psi(x/2+4)-\psi(x/2+3)-\psi(x/2+1) \\ & +\psi(x/2-2)+\psi(x/2-3)+(1/2)\psi(x/4+1) \\ & -\psi(x/4)-(1/4)\psi(x/8+1) \\ & +(1/4)(u(x+8)-u(x)). \end{aligned} \tag{15}$$

Therefore the PRBS can be written as a finite sum of Haar wavelets rather than an infinite sum. All other orders of the PRBS can also be written as a finite sum of Haar wavelets in a similar manner. Thus an exact representation of the PRBS signal can be written in terms of Haar wavelets and not merely an approximation.

Table 1. Beta Coefficients for 3rd Order PRBS Signal

β_m^j	Stage 0 (j=0)	Stage 1 (j=1)	Stage 2 (j=2)	Stage 3 (j=3)
m=-8	1			
m=-7	-1			
m=-6	-1			
m=-5	1			
m=-4	1	0		
m=-3	1	0		
m=-2	-1	1	0	
m=-1	1	0	0.5	0.25
m=0	-1	-1	0	0
m=1	-1	1	0	
m=2	1	0		
m=3	1	0		
m=4	1			
m=5	-1			
m=6	1			
m=7	-1			

Table 2. Alpha Coefficients for 3rd Order PRBS Signal

α_m^j	Stage 0 (j=0)	Stage 1 (j=1)	Stage 2 (j=2)	Stage 3 (j=3)
m=-8	1			
m=-6	-1			
m=-4	0	0		
m=-2	-1	0.5	0.25	
m=-0	0	-1	0	0.125
m=2	0	0		
m=4	1			
m=6	1			

7. Simplification of Multiresolution Analysis Approximation

When writing a signal as a sum of Haar wavelets, the computation of the delta functions becomes rather tedious because every alpha and beta coefficient must be calculated. Due to the periodic nature of the PRBS signal, the computation of the delta functions can be simplified. Define a new vector as

$$\underline{\gamma}^r = \left[\alpha^r_{-2^{J_1+J_0-r+1}} \ \dots \ \alpha^r_0 \quad \alpha^r_2 \quad \dots \quad \alpha^r_{2^{J_1+J_0-r+1}-2}\right]. \tag{16}$$

This vector contains the alpha coefficients required to compute δ^r. Notice in Equation (9) that only α_l^r for l an even number need to be computed to generate the delta functions and thus only the even alpha coefficients are included in the new gamma vector. Referring to the example of the 3rd order PRBS where $J_1=3$, $J_0=0$, the gamma vectors can be written as

$$\begin{aligned} \underline{\gamma}^1 &= \left[\alpha^1_{-8} \quad \alpha^1_{-6} \quad \dots \quad \alpha^1_6\right] \\ &= \left[1 \quad -1 \quad 0 \quad -1 \quad 0 \quad 0 \quad 1 \quad 1\right] \\ \underline{\gamma}^2 &= \left[\alpha^2_{-4} \quad \alpha^2_{-2} \quad \alpha^2_0 \quad \alpha^2_2\right] \\ &= \left[0 \quad 1/2 \quad -1 \quad 0\right] \\ \underline{\gamma}^3 &= \left[\alpha^3_{-2} \quad \alpha^3_0\right] = \left[-1/4 \quad 0\right]. \end{aligned} \tag{17}$$

Notice that each vector contains 2^{J_1+1-r} elements. For this example, J_1 equals the order of the PRBS signal. For larger values of J_1, more periods of the PRBS would be contained within the interval $\left[-2^{J_1}, 2^{J_1}\right]$. For $J_1 = \mu$, two periods of the PRBS would be contained within the interval. For $J_1 = \mu + 1$, four periods are contained in the interval. As the value of J_1 increases, the number of periods of the PRBS signal contained in the interval $\left[-2^{J_1}, 2^{J_1}\right]$ will increase in a similar manner.

For the PRBS signal, the elements of the gamma vectors can be shown to be periodic [5]. Alpha coefficients which are periodic will allow the computation of the delta functions to become easier. Recall that for the PRBS $J_0=0$ and the PRBS has a period μ. Then the gamma vector for $r=1$ becomes

$$\underline{\gamma}^1 = \left[\alpha^1_{-2^{J_1}} \dots \alpha^1_{-2^{J_1}+2} \quad \dots \quad \alpha^1_0 \quad \alpha^1_2 \quad \dots \quad \alpha^1_{2^{J_1}-2}\right]. \tag{18}$$

However, the following equations can be shown to be true

$$\begin{aligned} \alpha^1_\mu &= \frac{1}{2}\left(\beta^0_\mu - \beta^0_{\mu+1}\right) = \frac{1}{2}\left(\beta^0_0 - \beta^0_1\right) = \alpha^1_0 \\ \alpha^1_{\mu+2} &= \frac{1}{2}\left(\beta^0_{\mu+2} - \beta^0_{\mu+3}\right) = \frac{1}{2}\left(\beta^0_2 - \beta^0_3\right) = \alpha^1_2 \\ &\vdots \\ \alpha^1_{2\mu-2} &= \frac{1}{2}\left(\beta^0_{2\mu-2} - \beta^0_{2\mu-1}\right) = \frac{1}{2}\left(\beta^0_{\mu-2} - \beta^0_{\mu-1}\right) = \alpha^1_{\mu-2} \end{aligned} \tag{19}$$

because the beta coefficients are periodic as shown below

$$
\begin{aligned}
&f(0)=\beta_0^0=\beta_\mu^0=\beta_{2\mu}^0=\dots\\
&f(1)=\beta_1^0=\beta_{\mu+1}^0=\beta_{2\mu+1}^0=\dots\\
&\vdots\\
&f(\mu-1)=\beta_{\mu-1}^0=\beta_{2\mu-1}^0=\dots \quad .
\end{aligned} \tag{20}
$$

Notice the beta coefficients are actual values of the f function to be approximated. Therefore Equation (17) can be written as

$$
\underline{\gamma}^1=\left[\alpha_0^1\dots\alpha_{\mu-1}^1 \quad \alpha_0^1 \quad \dots \quad \alpha_{\mu-1}^1 \quad \alpha_0^1 \quad \dots \quad \alpha_{\mu-1}^1\right]. \tag{21}
$$

The periodic nature of the gamma vector can be generalized. The alpha coefficients can be written as

$$
\begin{aligned}
\alpha_{k+\mu}^r&=\frac{1}{2^r}\left[\sum_{j=0}^{2^{r-1}-1}\beta_{2^{r-1}k+j}^0-\sum_{j=2^{r-1}}^{2^{r-1}-1}\beta_{2^{r-1}(k+\mu)+j}^0\right]\\
&=\frac{1}{2^r}\left[\sum_{j=0}^{2^{r-1}-1}\beta_{2^{r-1}k+j}^0-\sum_{j=2^{r-1}}^{2^r-1}\beta_{2^{r-1}(+j}^0\right]\\
&=\alpha_k^r \ .
\end{aligned} \tag{22}
$$

The gamma vectors have coefficients that are periodic with a period that equals the period of the PRBS signal. Only the first μ alpha coefficients need to be computed to generate the gamma vector and thus the delta functions. The periodic nature of the PRBS has reduced the complexity of determining the Haar wavelet representations.

The gamma vectors are periodic with a period n that is the number of shift registers used to generate the PRBS signal. Consider the first gamma vector and the $(n+1)^{th}$ gamma vector which can be written as

$$
\begin{aligned}
&\underline{\gamma}^1=\left[\alpha_0^1\dots\alpha_{\mu-1}^1 \quad \alpha_0^1 \quad \dots \qquad \alpha_{\mu-1}^1\right]\\
&\underline{\gamma}^{\eta+1}=\left[\alpha_0^{\eta+1}\dots\alpha_{\mu-1}^{\eta+1} \quad \alpha_0^{\eta+1} \quad \dots \qquad \alpha_{\mu-1}^{\eta+1}\right].
\end{aligned} \tag{23}
$$

The k^{th} element of the $(n+1)^{th}$ gamma vector can be written as:

$$
\begin{aligned}
\alpha_k^{\eta+1}&=\frac{1}{2^{\eta+1}}\left(\beta_k^0-\beta_{k+1}^0\right)=\frac{1}{2^\eta}\left[\frac{1}{2}\left(\beta_k^0-\beta_{k+1}^0\right)\right]\\
&=\frac{1}{2^\eta}\alpha_k^1 \ .
\end{aligned} \tag{24}
$$

Therefore the $(n+1)^{th}$ gamma vector can be written as

$$
\begin{aligned}
\underline{\gamma}^{\eta+1}&=\left[\alpha_0^{\eta+1}\dots\alpha_{\mu-1}^{\eta+1} \quad \alpha_0^{\eta+1} \quad \dots \qquad \alpha_{\mu-1}^{\eta+1}\right]\\
&=\left[\frac{1}{2^\eta}\alpha_0^1\dots\frac{1}{2^\eta}\alpha_{\mu-1}^1 \quad \frac{1}{2^\eta}\alpha_0^1 \quad \dots \qquad \frac{1}{2^\eta}\alpha_{\mu-1}^1\right]\\
&=\frac{1}{2^\eta}\left[\alpha_0^1 \quad \dots \quad \alpha_{\mu-1}^1 \quad \alpha_0^1 \quad \dots \quad \alpha_{\mu-1}^1\right]\\
&=\frac{1}{2^\eta}\underline{\gamma}^1 \ .
\end{aligned} \tag{25}
$$

The alpha coefficients are periodic with period of the PRBS signal, μ. Also the gamma vectors have been shown to be a scalar multiple of other gamma vectors. The first η gamma vectors are unique where η is the order of the PRBS. Then by applying the previous derivation, the gamma vectors are determined by

$$
\underline{\gamma^{\eta+k}}=\frac{1}{2^\eta}\underline{\gamma^k} \quad . \tag{26}
$$

These periodic relationships are useful in reducing the amount of computation required for determining the Haar estimates of the PRBS signal.

The estimation of a PRBS function by a series of Haar wavelets is now quite simple. Recall Equation (3). Assign the data points of the PRBS to the beta coefficients as

$$
\beta_k^0=f(k). \tag{27}
$$

The $\{\beta_k^r\}$ coefficients for $r=1, \dots J_0+J_1$ are calculated as

$$
\beta_k^r=\frac{1}{2}\left(\beta_{2k}^{r-1}+\beta_{2k+1}^{r-1}\right) \quad . \tag{28}
$$

Now for each $r=1, \dots,n$ determine $\{\alpha_k^r\}_{k=0,\dots,\mu-1}$ by

$$
\begin{aligned}
\alpha_{2k}^r&=\beta_{2k}^{r-1}-\beta_k^r=\frac{1}{2}\left(\beta_{2k}^{r-1}-\beta_{2k+1}^{r-1}\right)\\
\alpha_{2k+1}^r&=\beta_{2k+1}^{r-1}-\beta_k^r=\frac{1}{2}\left(\beta_{2k+1}^{r-1}-\beta_{2k}^{r-1}\right)=-\alpha_{2k}^r \ .
\end{aligned} \tag{29}
$$

The remaining alpha coefficients can be determined by the periodic relationships as

$$
\underline{\gamma}^k=\left[\alpha_0^k\dots\alpha_{\mu-1}^k \quad \alpha_0^k \quad \dots \qquad \alpha_{\mu-1}^k\right] \tag{30}
$$

and

$$
\underline{\gamma^{\eta+k}}=\frac{1}{2^\eta}\underline{\gamma^k} \quad . \tag{31}
$$

After determining the alpha and beta coefficients, the results can be substituted into Equation (3) to compute the estimate of the PRBS signal. Relating the PRBS signal to the Haar wavelet may be advantageous because the Haar wavelet forms an orthonormal wavelet basis. Multiresolution analysis can be used to write the PRBS signal as a finite series of Haar wavelets, and the dyadic period of the PRBS is exploited to make this task easier.

8. Conclusion

Any signal can be written as a finite series of Haar wavelets using multiresolution analysis. This process requires the computation of a large number of coefficients. This process has been simplified by taking advantage of the periodic nature of the PRBS signal. It has been shown that the coefficients of the approximation can be simply found by determining a minimum set of values which are periodic.

9. References

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