Successive Differentiation.

Notations:-

If $y = f(x)$ is differentiable

Then we can write it's higher order derivatives using following notations

$$\frac{dy}{dx} = f'(x) = y_1$$

$$\frac{d^2y}{dx^2} = f''(x) = y_2$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = f^{(n)}(x) = y_n$$

① $y = e^{mx}$

$y_1 = e^{mx} \cdot m$

$y_2 = e^{mx} \cdot m^2$

$y_3 = e^{mx} m^2 \cdot m = e^{mx} m^3$

$=$ |

$y_n = e^{mx} m^n$

② $y = a^{mx}$

$y_1 = a^{mx} \log a \ (m)$

$y_2 = (m)(\log a) \ a^{mx} \ (\log a) \ m$

$= m^2 \ (\log a)^2 \ a^{mx}$

|

$y_n = m^n \ (\log a)^n \ a^{mx}.$

③ $y = (ax+b)^m$

$y_1 = m(ax+b)^{m-1} \ (a)$

$y_2 = m(m-1)(ax+b)^{m-2} \ (a)^2$

$y_3 = m(m-1)(m-2)(ax+b)^{m-3} \ (a)^3$

|

$y_n = m(m-1)(m-2) \text{———} (m-(n-1))(ax+b)^{m-n} a^n$

$n < m$

$y = x^4$ ✓

$y_1 = 4x^3$

$y_2 = 12x^2$

$y_3 = 24x$

$y_4 = 24$

$y_5 = 0$

for $n = m$

$y_n = m(m-1)(m-2) \text{——} (m-m+1)(ax+b)^{\cancel{m}-\cancel{m}} a^m$

$= m(m-1)(m-2) \text{——} 1 \ a^m$

$= m! \ a^m$

for $n > m$ $\quad y_n = 0.$

eg. $y = (ax+b)^5$

$y_1 = 5(ax+b)^4 \ a$

$y_2 = 5 \cdot 4 \cdot (ax+b)^3 \ a^2$

$y_3 = 5 \cdot 4 \cdot 3 \ (ax+b)^2 a^3$

$y_4 = 5 \cdot 4 \cdot 3 \cdot 2 \ (ax+b) a^4$

$y_5 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \ a^5 = 5! \ a^5$

$y_6 = 0$

(4) $y = (ax+b)^{-m}$

$y_1 = (-m)(ax+b)^{-m-1}(a)$

$y_2 = (-m)(-m-1)(ax+b)^{-m-2}a^2$

$y_3 = (-m)(-m-1)(-m-2)(ax+b)^{-m-3}a^3$

$|$

$y_n = (-m)(-m-1)(-m-2) \text{———} (-m-(n-1))(ax+b)^{-m-n}a^n$

$= (-1)^n \left[m(m+1)(m+2) \text{——} (m+n-1) \right] (ax+b)^{-(m+n)} a^n$

$= (-1)^n \dfrac{(1.2.3.\ldots.(m-1)\, m(m+1) \text{——} (m+n-1)}{1.2.3\ldots(m-1)} \dfrac{a^n}{(ax+b)^{m+n}}$

$= (-1)^n \dfrac{(m+n-1)!}{(m-1)!} \dfrac{a^n}{(ax+b)^{m+n}}$

eg. $y = \dfrac{1}{x^m}$ $\qquad y_n = (-1)^n \dfrac{(m+n-1)!}{(m-1)!} \dfrac{1}{x^{m+n}}$.

$y = x^m$

$y_n = m(m-1) \text{——} (m-n+1)x^{m-n} \qquad n < m$

$\left.\begin{array}{l} y_n = m(m-1) \text{——} (\cancel{m}-\cancel{m}+1) \\ = m! \end{array}\right\} \quad n = m$

$y_n = 0 \qquad n > m$

(5) $y = \frac{1}{(ax+b)} = (ax+b)^{-1}$

$y_1 = (-1)(ax+b)^{-2}(a)$

$y_2 = (-1)(-2)(ax+b)^{-3}(a)^2$

$y_3 = (-1)(-2)(-3)(ax+b)^{-4}(a)^3$

$|$

$y_n = (-1)(-2)\ldots\ldots(-n)(ax+b)^{-(n+1)}a^n$

$= (-1)^n(1\cdot 2 — n)(ax+b)^{-(n+1)}a^n$

$= \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

(6) $y = \log(ax+b)$

$y_1 = \frac{1}{(ax+b)}a = (ax+b)^{-1}a$

$y_2 = (-1)(ax+b)^{-2}a^2$

$y_3 = (-1)(-2)(ax+b)^{-3}a^3$

$|$

$y_n = (-1)(-2) — (-(n-1))(ax+b)^{-n}a^n$

$= \frac{(-1)^{n-1}(1.2.—(n-1))\, a^n}{(ax+b)^n}$

$= \frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$

(7) $y = \sin(ax+b)$

$y_1 = \cos(ax+b)\; a$

$= \sin\left(\frac{\pi}{2} + ax + b\right) a.$

$y_2 = \cos\left(\frac{\pi}{2} + ax + b\right) a^2$

$= \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + ax + b\right) a^2$

$= \sin\left(2\frac{\pi}{2} + ax + b\right) a^2$

$y_3 = \cos\left(2\frac{\pi}{2} + ax + b\right) a^3$

$= \sin\left(\frac{3\pi}{2} + ax + b\right) a^3$

$|$

$y_n = \sin\left(\frac{n\pi}{2} + ax + b\right) a^n$

(8) $y = \cos(ax+b)$

$\cos\left(\frac{\pi}{2} + \theta\right)$

$0 - \sin\theta$

$y_1 = -\sin(ax+b)\; a$

$= \cos\left(\frac{\pi}{2} + ax + b\right) a$

$y_2 = -\sin\left(\frac{\pi}{2} + ax + b\right) a^2 = \cos\left(\frac{\pi}{2} + \frac{\pi}{2} + ax + b\right) a^2$

$|$

$y_n = \cos\left(\frac{n\pi}{2} + ax + b\right) a^n$

① $y = \dfrac{x}{(x-1)(x-2)(x-3)}$

Note. We can use Partial fraction directly if $\deg(N^R) < \deg(D^R)$

$$\frac{f(x)}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

$$\frac{f(x)}{(x-a)^3(x-b)} = \frac{A}{(x-a)^3} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)} + \frac{D}{x-b}$$

$$y = \frac{x}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-3)}$$

$$= \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$x=1$ $\quad 1 = A(-1)(-2) \Rightarrow A = 1/2$

$x=2$ $\quad 2 = B(1)(-1) \Rightarrow B = -2$

$x=3$ $\quad 3 = C(2)(1) \Rightarrow C = 3/2$

$$y = \frac{x}{(x-1)(x-2)(x-3)} = \frac{1/2}{x-1} - \frac{2}{x-2} + \frac{3/2}{x-3}$$

$$\left[\begin{aligned} y &= \frac{1}{(ax+b)} = (ax+b)^{-1} \\ y_1 &= (-1)(ax+b)^{-2}\, a \\ y_2 &= (-1)(-2)(ax+b)^{-3}\, a^2 \\ &\vdots \\ y_n &= (-1)(-2) \cdots (-n)(ax+b)^{-(n+1)}\, a^n \\ &= \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}} \end{aligned}\right]$$

$$y_n = \frac{1}{2}\frac{(-1)^n n!}{(x-1)^{n+1}} - \frac{2(-1)^n n!}{(x-2)^{n+1}} + \frac{3}{2}\frac{(-1)^n n!}{(x-3)^{n+1}}$$

② $y = \frac{x^2}{(x+2)(2x+3)} = \frac{x^2}{2x^2+3x+4x+6} = \frac{x^2}{2x^2+7x+6}$

$$\begin{array}{r} \frac{1}{2} \\ 2x^2+7x+6 \overline{) \; x^2 \qquad\qquad} \\ x^2 + \frac{7}{2}x + 3 \\ - \quad - \quad - \\ \hline -\frac{7}{2}x - 3 \end{array}$$

$x^2 = (2x^2+7x+6)\frac{1}{2} + \left(-\frac{7}{2}x - 3\right)$

$\frac{x^2}{(2x^2+7x+6)} = \frac{1}{2} + \frac{\left(-\frac{7}{2}x-3\right)}{2x^2+7x+6}$

$= \frac{1}{2} + \frac{\left(-\frac{7}{2}x - 3\right)}{(x+2)(2x+3)}$

$\frac{-\frac{7}{2}x - 3}{(x+2)(2x+3)} = \frac{A}{(x+2)} + \frac{B}{(2x+3)}$

$-\frac{7}{2}x - 3 = A(2x+3) + B(x+2)$

$x = -2 \Rightarrow +7-3 = A(-4+3) \Rightarrow 4 = -A \Rightarrow A = -4$

$x = -3/2 \Rightarrow -\frac{7}{2}\left(\frac{-3}{2}\right) - 3 = B\left(\frac{-3}{2}+2\right)$

$\frac{21}{4} - 3 = B\left(\frac{1}{2}\right)$

$\Rightarrow \frac{9}{\cancel{4}2} = B\frac{1}{\cancel{2}} \Rightarrow B = 9/2$

$y = \frac{x^2}{(x+2)(2x+3)} = \frac{1}{2} - \frac{4}{(x+2)} + \frac{9/2}{2x+3} = \frac{1}{2} - 4\frac{1}{(x+2)} + \frac{9}{2}\frac{1}{(2x+3)}$

$y = \frac{1}{(ax+b)} \qquad y_n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$y_n = 0 - 4\frac{(-1)^n n!}{(x+2)^{n+1}} + \frac{9}{2}\frac{(-1)^n n!\, 2^n}{(2x+3)^{n+1}}$

$$y = \frac{1}{1+x+x^2+x^3} =$$

$$1+x+x^2+x^3=0$$
$$x=-1, \pm i$$

$$= \frac{1}{(x+1)(x+i)(x-i)}$$

$$= \frac{A}{x+1} + \frac{B}{x+i} + \frac{C}{x-i}$$

$$\frac{1}{(x+1)(x+i)(x-i)} = \frac{A(x+i)(x-i)+B(x+1)(x-i)+C(x+1)(x+i)}{(x+1)(x+i)(x-i)}$$

$x=-1$, $1 = A(-1+i)(-1-i) = A(1-(-1)) = 2A$

$A = 1/2$

$x=-i$ $\quad 1 = 0 + B(1-i)(-2i)$

$$B = -\frac{1}{2i(1-i)}$$

$x=i$ $\quad 1 = 0+0+C(i+1)(2i)$

$$C = \frac{1}{2i(1+i)}$$

$$y = \frac{1}{2}\frac{1}{x+1} - \frac{1}{2i(1-i)}\frac{1}{(x+i)} + \frac{1}{2i(1+i)}\frac{1}{(x-i)}$$

$$y = \frac{1}{(ax+b)} \Rightarrow y_n = \frac{(-1)^n n!(a)^n}{(ax+b)^{n+1}}$$

$$y_n = \frac{1}{2}\frac{(-1)^n n!}{(x+1)^{n+1}} - \frac{1}{2i(1-i)}\frac{(-1)^n n!}{(x+i)^{n+1}} + \frac{1}{2i(1+i)}\frac{(-1)^n n!}{(x-i)^{n+1}}$$

$$y = \frac{8x}{x^3-2x^2-4x+8}$$

$$x^3-2x^2-4x+8$$

$$x = -2, 2, 2$$

$$= \frac{8x}{(x-2)^2(x+2)}$$

$$\frac{8x}{(x-2)^2(x+2)} = \frac{A}{(x-2)^2} + \frac{B}{(x-2)} + \frac{C}{(x+2)}$$

$$\frac{8x}{\cancel{(x-2)^2(x+2)}} = \frac{A(x+2)+B(x-2)(x+2)+C(x-2)^2}{\cancel{(x-2)^2(x+2)}}$$

$$x=2 \Rightarrow 16 = 4A \Rightarrow A=4$$

$$x=-2 \Rightarrow -16 = C(16) \Rightarrow C=-1$$

$$x=0 \Rightarrow 0 = 2A-4B+4C$$

$$0 = 8-4B-4$$

$$4B=4 \Rightarrow B=1$$

$$y = \frac{4}{(x-2)^2} + \frac{1}{(x-2)} - \frac{1}{x+2}$$

If $y = \frac{1}{(ax+b)} \Rightarrow y_n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$$y = \frac{1}{(ax+b)^m} \Rightarrow y_n = \frac{(-1)^n (m+n-1)!\, a^n}{(m-1)!\,(ax+b)^{m+n}}$$

$$m=2$$

$$y_n = \frac{4(-1)^n(2+n-1)!}{(2-1)!\,(x-2)^{2+n}} + \frac{(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x+2)^{n+1}}$$

$$= \frac{4(-1)^n(n+1)!}{(x-2)^{n+2}} + \frac{(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x+2)^{n+1}}$$

$y = \dfrac{x}{(x+1)^4}$ find y_n

$= \dfrac{(x+1)-1}{(x+1)^4}$

$= \dfrac{1}{(x+1)^3} - \dfrac{1}{(x+1)^4}$ $\rightarrow$ $m = 4$

$y = \dfrac{1}{(ax+b)^m}$ $\rightarrow$ $m = 3$

$y_n = \dfrac{(-1)^n (m+n-1)! (a)^n}{(m-1)! (ax+b)^{m+n}}$

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P.T. the value of nth differential coefficient of $\frac{x^3}{x^2-1}$ for $x=0$ is 0 if n is even & is $-n!$ if n is odd & greater than 1.

$$y = \frac{x^3}{x^2-1} = \frac{x^3-x+x}{x^2-1}$$
$$= \frac{x(x^2-1)+x}{(x^2-1)}$$
$$= x + \frac{x}{x^2-1}$$

or

$$x^2-1 \overline{)\, x^3 \,}\; x$$
$$\underline{-\cancel{x^3} - x}$$
$$x$$
$$x^3 = (x^2-1)x + x$$
$$\frac{x^3}{x^2-1} = x + \frac{x}{x^2-1}.$$

$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

$$x = A(x+1) + B(x-1)$$

$x=1$ $\quad 1 = 2A \Rightarrow A = 1/2$

$x=-1$ $\quad -1 = -2B \Rightarrow B = 1/2$

$$y = \underline{x} + \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x+1}\right]$$

If $y = \frac{1}{ax+b} \Rightarrow y_n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$$y_n = 0 + \frac{1}{2}\left[\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{(-1)^n n!}{(x+1)^{n+1}}\right]$$

for $x=0$

$$y_n = \frac{1}{2}\left[\frac{(-1)^n n!}{(-1)^{n+1}} + \frac{(-1)^n n!}{(1)^{n+1}}\right].$$
$$= \frac{n!}{2}\left[\frac{\cancel{(-1)^n}}{\cancel{(-1)^n}(-1)} + (-1)^n\right]$$
$$y_n = \frac{n!}{2}\left[-1 + (-1)^n\right]$$

If n is even $(-1)^n = 1$

$$\therefore y_n = \frac{n!}{2}[-1+1] = 0.$$

If n is odd $(-1)^n = -1$

$$\therefore y_n = \frac{n!}{2}[-1-1] = \frac{n!}{2}(-2) = -(n!).$$

$n > 1.$

If $y = x\log\left(\frac{x-1}{x+1}\right)$, prove that

$y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$.

$y = x(\log(x-1) - \log(x+1)) = x\log(x-1) - x\log(x+1)$

$y_1 = \frac{x}{x-1} + \log(x-1) - \left(\frac{x}{x+1} + \log(x+1)\right)$

$= \log(x-1) - \log(x+1) + \frac{x-1+1}{x-1} - \frac{x+1-1}{x+1}$

$= \log(x-1) - \log(x+1) + \cancel{x} + \frac{1}{x-1} - \left(\cancel{x} - \frac{1}{x+1}\right)$

$y_1 = \log(x-1) - \log(x+1) + \frac{1}{x-1} + \frac{1}{x+1}$

If $y = \log(ax+b)$ $\quad y_n = \frac{(-1)^{n-1}(n-1)!\,(a)^n}{(ax+b)^n}$ — (1)

If $y = \frac{1}{ax+b}$ $\quad y_n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$ — (2)

$Y = y_1$
$Y_1 = y_2$
$Y_2 = y_3$
$|$
$Y_{n-1} = y_n$
$Y_n = y_{n+1}$

Initially we have y_1
∴ If we take derivative of y_1 $(n-1)$ times then we get y_n

∴ Substitute in (1) & (2) n by $n-1$

If $y = \log(ax+b)$

$y_{n-1} = \frac{(-1)^{n-2}(n-2)!\,a^{n-1}}{(ax+b)^{n-1}}$

If $y = \frac{1}{ax+b}$ $\quad y_{n-1} = \frac{(-1)^{n-1}(n-1)!\,a^{n-1}}{(ax+b)^n}$.

$y_1 = \log(x-1) - \log(x+1) + \frac{1}{x-1} + \frac{1}{x+1}$

$(n-2)!(n-1)$

$y_n = \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}} + \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} + \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}$

$= (-1)^{n-2}(n-2)!\left[\frac{1}{(x-1)^{n-1}} - \frac{1}{(x+1)^{n-1}} + \frac{(-1)(n-1)}{(x-1)^n} + \frac{(-1)(n-1)}{(x+1)^n}\right]$

$= (-1)^{n-2}(n-2)!\left[\frac{1}{(x-1)^n(x-1)^{-1}} - \frac{1}{(x+1)^n(x+1)^{-1}} + \frac{(-n+1)}{(x-1)^n} + \frac{(-n+1)}{(x+1)^n}\right]$

$= (-1)^{n-2}(n-2)!\left[\frac{x-1}{(x-1)^n} + \frac{-n+1}{(x-1)^n} + \frac{(-n+1)}{(x+1)^n} - \frac{(x+1)}{(x+1)^n}\right]$

$= (-1)^{n-2}(n-2)!\left[\frac{x-\cancel{1}-n+\cancel{1}}{(x-1)^n} + \frac{-n+\cancel{1}-x-\cancel{1}}{(x+1)^n}\right]$

$= (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$

$$y = \cos x \ \underline{\cos 2x \ \cos 3x}$$

$$y = \cos x\left(\frac{\cos 5x + \cos(-x)}{2}\right)$$

$$= \frac{1}{2}\left(\cos x \cos 5x + \cos x \cos x\right)$$

$$= \frac{1}{2}\left(\frac{\cos 6x + \cos(-4x)}{2} + \frac{\cos 2x + 1}{2}\right)$$

$$= \frac{1}{4}\left(\cos 6x + \cos 4x + \cos 2x + 1\right)$$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$y = \cos(ax+b) \qquad y_n = \cos\left(\frac{n\pi}{2} + ax + b\right)(a)^n$

$$y_n = \frac{1}{4}\left[6^n \cos\left(\frac{n\pi}{2} + 6x\right) + 4^n \cos\left(\frac{n\pi}{2} + 4x\right) + 2^n \cos\left(\frac{n\pi}{2} + 2x\right) + 0\right]$$

$$y = \cos^4 x$$

$$= (\cos^2 x)^2$$

$$= \left(\frac{1+\cos 2x}{2}\right)^2$$

$$y = \cos(ax+b)$$

$$y_n = \cos\left(\frac{n\pi}{2}+ax+b\right)a^n$$

$$1+\cos 2x = 2\cos^2 x$$

$$1-\cos 2x = 2\sin^2 x$$

$$= \frac{1}{4}\left(1+2\cos 2x+\cos^2 2x\right)$$

$$= \frac{1}{4}\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right).$$

$$= \frac{1}{8}\left(2+4\cos 2x+1+\cos 4x\right)$$

$$= \frac{1}{8}\left(3+4\cos 2x+\cos 4x\right)$$

$$y_n = \frac{1}{8}\left(0+4\cos\left(\frac{n\pi}{2}+2x\right)2^n+\cos\left(\frac{n\pi}{2}+4x\right)4^n\right).$$

$$y = \sin^2 x \cos^3 x$$

$$\sin 2x = 2\sin x \cos x$$

$$2\sin^2 x = 1 - \cos 2x$$

$$= \sin^2 x \cos^2 x \cos x$$

$$= (\sin x \cos x)^2 \cos x$$

$$= \left(\frac{\sin 2x}{2}\right)^2 \cos x .$$

$$= \frac{1}{4}(\sin^2 2x) \quad \cos x$$

$$= \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right) \cos x$$

$$= \frac{1}{8}\left(\cos x - \underline{\cos 4x \cos x}\right)$$

$$= \frac{1}{8}\left(\cos x - \left(\frac{\cos 5x + \cos 3x}{2}\right)\right)$$

$$= \frac{1}{16}\left(2\cos x - \cos 5x - \cos 3x\right)$$

$$y_n = \frac{1}{16}\left(2\cos\left(\frac{n\pi}{2}+x\right) - \cos\left(\frac{n\pi}{2}+5x\right)5^n - \cos\left(\frac{n\pi}{2}+3x\right)3^n\right).$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

$$y = \sin^5 x$$
$$= (\sin x)^5$$
$$= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^5$$
$$= \frac{1}{(2i)^5}\Big((e^{ix})^5 - 5(e^{ix})^4(e^{-ix}) + 10(e^{ix})^3(e^{-ix})^2 - 10(e^{ix})^2(e^{-ix})^3$$
$$+ 5(e^{ix})(e^{-ix})^4 - (e^{-ix})^5\Big).$$

$$\begin{matrix} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{matrix}$$

$$y = \frac{1}{2^5 i}\left(e^{i5x} - 5e^{i3x} + 10e^{ix} - 10e^{-ix} + 5e^{-i3x} - e^{-i5x}\right)$$
$$= \frac{1}{2^5 i}\left(e^{i5x} - e^{-i5x} - 5\left(e^{i3x} - e^{-i3x}\right) + 10\left(e^{ix} - e^{-ix}\right)\right).$$
$$= \frac{1}{2^5 i}\left(2i\sin 5x - 5(2i\sin 3x) + 10(2i\sin x)\right)$$
$$= \frac{1}{2^5 i}\, 2i\left(\sin 5x - 5\sin 3x + 10\sin x\right)$$

$$\frac{e^{i\theta}-e^{-i\theta}}{2i} = \sin\theta$$

$$y = \frac{1}{2^4}\left(\sin 5x - 5\sin 3x + 10\sin x\right)$$

$$y_n = \frac{1}{2^4}\Big(\sin\left(\frac{n\pi}{2}+5x\right)5^n - 5\sin\left(\frac{n\pi}{2}+3x\right)3^n + 10\sin\left(\frac{n\pi}{2}+x\right)$$

If y = sinrx+cosrx then prove that
$y_n = r^n[1 + (-1)^n sin2rx]^{1/2}$

$$y_n = \sin\left(\frac{n\pi}{2}+rx\right) r^n + \cos\left(\frac{n\pi}{2}+rx\right) r^n$$

$$= r^n\left[\sin\left(\frac{n\pi}{2}+rx\right) + \cos\left(\frac{n\pi}{2}+rx\right)\right]$$

$$= r^n\left[\left(\sin\left(\frac{n\pi}{2}+rx\right) + \cos\left(\frac{n\pi}{2}+rx\right)\right)^2\right]^{1/2}$$

$$= r^n\left[\sin^2\left(\frac{n\pi}{2}+rx\right) + \cos^2\left(\frac{n\pi}{2}+rx\right) + 2\sin\left(\frac{n\pi}{2}+rx\right)\cos\left(\frac{n\pi}{2}+rx\right)\right]^{1/2}$$

$$= r^n\left[1 + \sin 2\left(\frac{n\pi}{2}+rx\right)\right]^{1/2} = r^n\left[1 + \sin(n\pi+2rx)\right]^{1/2}$$

$$= r^n\left[1 + \sin n\pi \cos 2rx + \cos n\pi \sin 2rx\right]^{1/2}$$

$$= r^n\left[1 + 0 + (-1)^n \sin 2rx\right]^{1/2}$$

$$y = \cos(ax+b)$$

$$y_n = \cos\left(\frac{n\pi}{2}+ax+b\right) a^n$$

De Moivre's Thm.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

① $y = \dfrac{1}{x^2+a^2}$

P.T. $y_n = \dfrac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \ \sin(n+1)\theta$

where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$.

$$y = \frac{1}{(x-ai)(x+ai)} = \frac{A}{x-ai} + \frac{B}{x+ai} \qquad x^2+a^2=0$$

$$x^2 = -a^2$$

$$x = \pm ai$$

$$1 = A(x+ai) + B(x-ai)$$

$x = ai \quad 1 = 2ai A \Rightarrow A = 1/2ai$

$x = -ai \quad 1 = -2ai B \Rightarrow B = -1/2ai$

$$y = \frac{1}{2ai}\left[\frac{1}{x-ai} - \frac{1}{x+ai}\right]$$

$$y = \frac{1}{ax+b} \qquad y_n = \frac{(-1)^n (n)!\, a^n}{(ax+b)^{n+1}}$$

$$y_n = \frac{1}{2ai}\left[\frac{(-1)^n n!}{(x-ai)^{n+1}} - \frac{(-1)^n n!}{(x+ai)^{n+1}}\right]$$

put $x = r\cos\theta$, $a = r\sin\theta$ $\qquad x^2+a^2 = r^2 \qquad \tan\theta = \frac{a}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{a}{x}\right)$

$$= \frac{1}{2ai}(-1)^n n! \left[\frac{1}{(r\cos\theta - ir\sin\theta)^{n+1}} - \frac{1}{(r\cos\theta + ir\sin\theta)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2ai\, r^{n+1}}\left[(\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}\right]$$

By De Moivre's thm.

$$= \frac{(-1)^n n!}{2ai\, r^{n+1}}\left[\cos(-(n+1)\theta) - i\sin(-(n+1)\theta) - \left(\cos(-(n+1)\theta) + i\sin(-(n+1)\theta)\right)\right]$$

$$= \frac{(-1)^n n!}{2ai\, r^{n+1}}\left[\cancel{\cos(n+1)\theta} + i\sin(n+1)\theta - \left(\cancel{\cos(n+1)\theta} - i\sin(n+1)\theta\right)\right]$$

$$= \frac{(-1)^n n!}{\cancel{2}a\cancel{i}\, r^{n+1}} \cancel{2i}\sin(n+1)\theta$$

as $a = r\sin\theta \Rightarrow r = \dfrac{a}{\sin\theta}$

$$y_n = \frac{(-1)^n n!}{a\left(\frac{a}{\sin\theta}\right)^{n+1}} \sin(n+1)\theta = \frac{(-1)^n n!}{a^{n+2}} \sin^{n+1}\theta \ \sin(n+1)\theta$$

$y = \tan^{-1} x$

P. T. $\quad y_n = (-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta$

where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$.

$\rightarrow$ $y = \tan^{-1} x$

$$y_1 = \frac{1}{1+x^2} = \frac{1}{(x-i)(x+i)}$$

$x^2+1=0$

$x = \pm i.$

$$= \frac{A}{x-i} + \frac{B}{x+i}$$

$1 = A(x+i) + B(x-i)$

$x = i \qquad A = 1/2i$

$x = -i \qquad B = -1/2i$

$$y_1 = \frac{1}{2i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

$$y = \frac{1}{ax+b} \Rightarrow y_n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$$

$y = y_1$

$y_1 = y_2$

$$y_{n-1} = \frac{(-1)^{n-1}(n-1)!\,a^{n-1}}{(ax+b)^n}$$

$y_{n-1} = y_n$

$y_n = y_{n+1}$

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]$$

$x = r\cos\theta \qquad x^2+1 = r^2$

$1 = r\sin\theta \qquad \tan\theta = \frac{1}{x}$

$\theta = \tan^{-1}(1/x)$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^n} - \frac{1}{(r\cos\theta + ir\sin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i\,r^n}\left[(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin\theta)^{-n}\right]$$

By DeMoivre's thm

$$= \frac{(-1)^{n-1}(n-1)!}{2i\,r^n}\left[\cancel{\cos n\theta} + i\sin n\theta - (\cancel{\cos n\theta} - i\sin n\theta)\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{\cancel{2i}\,r^n}\,\cancel{2i}\sin n\theta$$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$1 = r\sin\theta \qquad r = \frac{1}{\sin\theta}$

$$= \frac{(-1)^{n-1}(n-1)!}{1/\sin^n\theta}\sin n\theta = (-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta$$

or

$$\left[y_n = \frac{(-1)^{n-1}(n-1)!}{(x^2+1)^{n/2}}\sin n\theta\right]$$

$$y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$$

P. T. $y_n = (-1)^{n-1}(n-1)!\, \sin^n\theta \sin n\theta$ | $\theta = \tan^{-1}\left(\frac{1}{x}\right)$.

$$\tan^{-1}(a) + \tan^{-1}(b) = \tan^{-1}\left(\frac{a+b}{1-ab}\right)$$

$$\tan^{-1}(1) + \tan^{-1}(x) = \tan^{-1}\left(\frac{1+x}{1-x}\right) = y$$

$$\therefore y = \frac{\pi}{4} + \tan^{-1}x \Rightarrow y_1 = 0 + \frac{1}{x^2+1}$$

$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ / $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ | $y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$

P.T. $y_n = 2(-1)^{n-1}(n-1)! \cdot \sin^n\theta \sin n\theta$

where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

$x = \tan\alpha$

$$y = \sin^{-1}\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right) = \sin^{-1}(\sin 2\alpha)$$

$$= 2\alpha$$

$$= 2\tan^{-1} x$$

$$y_1 = \frac{2}{1+x^2}$$

$\cos^{-1}(-\cos$

$\cos(\pi + \theta))$

If $y = \dfrac{1}{x^2+x+1}$ prove that

$$y_n = \frac{2(-1)^n}{\sqrt{3}} \frac{n!}{r^{n+1}} \sin(n+1)\theta$$

where $\theta = \cot^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)$ &

$r = \sqrt{x^2+x+1}$

$$y = \frac{1}{x^2+x+1}$$

x^2+x+1

$\Rightarrow x = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

$$= \frac{1}{\left(x-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)\left(x-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)}$$

$X = x + \dfrac{1}{2}$

$$= \frac{1}{\left(x+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\left(x+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{\left(X-i\frac{\sqrt{3}}{2}\right)\left(X+i\frac{\sqrt{3}}{2}\right)} = \frac{A}{X-i\frac{\sqrt{3}}{2}} + \frac{B}{X+i\frac{\sqrt{3}}{2}}$$

Leibnitz's Thm

If $y = uv$.

$$y_n = (uv)_n$$
$$= {}^nC_0 u_n v_0 + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \text{———} + {}^nC_n u_0 v_n$$

$$(uv)_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \frac{n(n-1)(n-2)}{3!} u_{n-3} v_3 + \text{—}$$
$$+ u v_n.$$

Note: choose v as that funⁿ whose derivative is vanishing.

$$\left[(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \text{———} + {}^nC_n a^0 b^n\right]$$

(1) $y = x^2 e^{mx} = e^{mx} x^2$

$u = e^{mx}$ $\qquad\qquad$ $v = x^2$

$u_1 = e^{mx}\, m$ $\qquad\qquad$ $v_1 = 2x$

$u_2 = m^2 e^{mx}$ $\qquad\qquad$ $v_2 = 2$

$\vdots$ $\qquad\qquad$ $v_3 = 0$

$u_n = m^n e^{mx}$ $\qquad\qquad$ $\vdots$

$v_r = 0$

$$(uv)_n = u_n v + n\, u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \text{——} + u v_n$$

$$= m^n e^{mx} x^2 + n\, m^{n-1} e^{mx} (2x) + \frac{n(n-1)}{\cancel{2}} m^{n-2} e^{mx} (\cancel{2}) + 0$$

$y = x^3 \sin 2x.$

$= \sin 2x \quad x^3$

$u = \sin 2x$

$u_n = \sin\left(\frac{n\pi}{2} + 2x\right) 2^n$

$v = x^3$

$v_1 = 3x^2$

$v_2 = 6x$

$v_3 = 6$

$v_4 = 0$

$\downarrow$

$v_n = 0$

$$y_n = (uv)_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \frac{n(n-1)(n-2)}{3!} u_{n-3} v_3$$

$$\text{———} + u v_n$$

$$y_n = \sin\left(\frac{n\pi}{2} + 2x\right) 2^n x^3 + n \sin\left[\frac{(n-1)\pi}{2} + 2x\right] 2^{n-1} 3x^2$$

$$+ \frac{n(n-1)}{\cancel{2}} \sin\left[\frac{(n-2)\pi}{2} + 2x\right] 2^{n-2} \overset{3}{\cancel{6}}x$$

$$+ \frac{n(n-1)(n-2)}{\cancel{6}} \sin\left[\frac{(n-3)\pi}{2} + 2x\right] 2^{n-3} \cancel{6}$$

$y = x\log(x+1)$

$$y_n = \frac{(-1)^{n-2}(n-2)!\,(x+n)}{(x+1)^n}$$

(D.)

$$y_1 = \frac{x}{x+1} + \log(x+1)$$

$$y_1 = \frac{x+1-1}{x+1} + \log(x+1) = 1 - \frac{1}{x+1} + \log(x+1)$$

$$y = \frac{1}{ax+b} \qquad y_n = \frac{(-1)^n (n)!\, a^n}{(ax+b)^{n+1}}$$

$$y = \log(ax+b) \qquad y_n = \frac{(-1)^{n-1}(n-1)!\, a^{n-1}}{(ax+b)^n} \Big\} y_{n-1}$$

Taking derivative of y_1 $(n-1)$ times then we get y_n

$y = x\log(x+1) = \log(x+1)\,x$

$u = \log(x+1) \qquad v = x$

$$u_n = \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} \qquad v_1 = 1, \quad v_2 = 0, \quad \ldots, \quad v_n = 0$$

$$(uv)_n = u_n v + n\,u_{n-1}v_1 + \frac{n(n-1)}{2!}u_{n-2}v_2 + \cdots + uv_n$$

$$= \frac{(-1)^{n-1}(n-1)!}{(x+1)^n}x + \frac{n(-1)^{n-2}(n-2)!}{(x+1)^{n-1}}$$

$$= (-1)^{n-2}\left[\frac{(-1)(n-2)!\,(n-1)}{(x+1)^n}x + \frac{n(n-2)!}{(x+1)^n\cdot(x+1)^{-1}}\right]$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x+1)^n}\left[-(n-1)x + n(x+1)\right]$$

$$= \frac{(-1)^{n-2}(n-2)!}{(x+1)^n}\left[\cancel{-nx} + x + \cancel{nx} + n\right]$$

If $y = x^n \log x$ P.T. $y_{n+1} = \frac{n!}{x}$.

$y_1 = x^n \frac{1}{x} + \log x \; n x^{n-1}$

$= x^n \frac{1}{x} + n \log x \frac{x^n}{x}$

$x y_1 = x^n + n y$

By taking n^{th} ordrivative. weget

$(x y_1)_n = (x^n)_n + (n y)_n$

$(y_1 x)_n = (x^n)_n + n y_n$ —— 1

$(y_1 x)_n$ $\quad u = y_1 \quad v = x$

$u_1 = y_2 \quad v_1 = 1$

$u_2 = y_3 \quad v_2 = 0$

$(uv)_n = u_n v + n u_{n-1} v_1$ $\quad u_n = y_{n+1} \quad v_n = 0$

$(y_1 x)_n = y_{n+1} x + n y_n (1)$

$y = x^n$

$y_1 = n x^{n-1}$

$y_2 = n(n-1) x^{n-2}$.

$y_n = n(n-1) \text{——} (n-(n-1)) x^{n-n}$

$= n(n-1) \text{——} (\cancel{n} - \cancel{n} + 1) x^0$

$= n!$

from 1

$y_{n+1} x + \cancel{n y_n} = n! + \cancel{n y_n} \Rightarrow y_{n+1} = \frac{n!}{x}$

$y = \cos^{-1}x$ prove that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0.$

$\rightarrow$ $y_1 = \dfrac{-1}{\sqrt{1-x^2}}$ $\qquad y_2 \quad y_1 \quad y$

$\sqrt{1-x^2}\, y_1 = -1$

Taking diff. wrt x.

$\sqrt{1-x^2}\,.y_2 + y_1 \dfrac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x) = 0$

$(1-x^2)\, y_2 - x y_1 = 0$

By taking n^{th} order derivative

$((1-x^2)y_2)_n - (xy_1)_n = 0$

$(\underset{u}{y_2}\,\underset{v}{(1-x^2)})_n - (\underset{u}{y_1}\,\underset{v}{x})_n = 0$

$(uv)_n = u_n v + n u_{n-1} v_1 + \dfrac{n(n-1)}{2!} u_{n-2} v_2 + \text{---} + u v_n$

$\Rightarrow \left[y_{n+2}(1-x^2) + n\, y_{n+1}(-2x) + \dfrac{n(n-1)}{\cancel{2!}} y_n(-\cancel{2}) \right]$

$- (y_{n+1}x + n\, y_n(1)) = 0$

$\Rightarrow (1-x^2)y_{n+2} - \underbrace{2xn\,y_{n+1}} - n^2 y_n + \cancel{n y_n}$

$- \underbrace{x y_{n+1}} - \cancel{y_n n} = 0$

$\Rightarrow (1-x^2)\,y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n = 0$

HW

$y = (x^2-1)^n$ prove that

$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$

If $y = (x + \sqrt{a^2+x^2})^m$. prove that

$a^2 y_{n+2} + (n^2 - m^2) y_n = 0$ at $x = 0$

$y_1 = m(x + \sqrt{a^2+x^2})^{m-1} \cdot \left(1 + \frac{1}{\cancel{2}\sqrt{a^2+x^2}} \cancel{2}x\right)$ $y_2 \to y$

$y_1 = m(x + \sqrt{a^2+x^2})^{m-1} \left[\frac{\sqrt{a^2+x^2} + x}{\sqrt{a^2+x^2}}\right]$

$y_1 = \frac{m(x + \sqrt{a^2+x^2})^m}{\sqrt{a^2+x^2}}$

$\sqrt{a^2+x^2}\, y_1 = my$ —— (1)

Diff. wrt. x

$(\sqrt{a^2+x^2}) y_2 + y_1 \frac{1}{\cancel{2}\sqrt{a^2+x^2}}(\cancel{2}x) = m y_1$

$(a^2+x^2) y_2 + y_1 x = m y_1 \sqrt{a^2+x^2}$

$(a^2+x^2) y_2 + y_1 x = m^2 y$ — from (1)

By taking n^{th} order derivative

$((a^2+x^2) y_2)_n + (y_1 x)_n = m^2 y_n$

By Leibnitz's thm

$(uv)_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \dots + u v_n$

$(\underset{u}{y_2} \underset{v}{(a^2+x^2)})_n + (\underset{u}{y_1} \underset{v}{x})_n = m^2 y_n$

$y_{n+2}(a^2+x^2) + n y_{n+1}(2x) + \frac{n(n-1)}{\cancel{2!}} y_n \cancel{(2)}$

$+ y_{n+1}(x) + n y_n (1) = m^2 y_n$

$(a^2+x^2) y_{n+2} + 2xn y_{n+1} + n^2 y_n - \cancel{n y_n} + x y_{n+1} + \cancel{n y_n}$
$= m^2 y_n$

$(a^2+x^2) y_{n+2} + (2n+1) x y_{n+1} + (n^2 - m^2) y_n = 0$

at $x = 0$

$a^2 y_{n+2} + 0 + (n^2 - m^2) y_n = 0$

If $y = \sin(m \sin^{-1}x)$ / $m \sin^{-1}x = \sin^{-1}y$, P.T.

$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} + (m^2-n^2)y_n = 0$

Hence deduce that $y_n(0) = 0$ if n is even

$= ((n-2)^2 - m^2) \cdots (3^2-m^2)(1^2-m^2)m$ if n is odd.

→ S.

$y_1 = \cos(m \sin^{-1}x) \dfrac{m}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, y_1 = m\cos(m\sin^{-1}x)$ —— (1)

Diff. wrt x.

$\sqrt{1-x^2}\, y_2 + y_1 \dfrac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x) = -m \sin(m\sin^{-1}x)\dfrac{m}{\sqrt{1-x^2}}$

$(1-x^2)y_2 - x y_1 = -m^2 \sin(m \sin^{-1}x)$

$(1-x^2)y_2 - xy_1 = -m^2 y$

$[\underset{u}{y_2}\,\underset{v}{(1-x^2)}]_n - (y_1 x)_n = (-m^2 y)_n$

By Leibnitz's thm.

$(uv)_n = u_n v + n u_{n-1} v_1 + \dfrac{n(n-1)}{2!} u_{n-2} v_2 + \text{——} + u v_n$

$y_{n+2}(1-x^2) + n\, y_{n+1}(-2x) + \dfrac{n(n-1)}{2!} y_n(-2)$

$-\left(y_{n+1}(x) + n y_n(1)\right) = -m^2 y_n$

$y_{n+2}(1-x^2) - (2n+1)x\, y_{n+1} + (-n^2 + \cancel{n} - \cancel{n} + m^2) y_n = 0$

$(1-x^2)y_{n+2} - (2n+1)x\, y_{n+1} + (m^2-n^2)y_n = 0$

at $x=0$

$y_{n+2}(0) - 0 + (m^2-n^2)y_n(0) = 0$

$y_{n+2}(0) = -(m^2-n^2)y_n(0)$

$y_{n+2}(0) = (n^2-m^2)y_n(0)$ —— (*)

$y = \sin(m\sin^{-1}x) \Rightarrow y(0) = 0$

$y_1 = \cos(m\sin^{-1}x)\dfrac{m}{\sqrt{1-x^2}} \Rightarrow y_1(0) = \dfrac{1\,m}{1} = m$

in (*) put $n=0$

$y_2(0) = (0-m^2)y_0(0) = (-m^2)y(0)$
$= 0$

In (*) put $n=1$

$y_3(0) = (1^2-m^2)\,y_1(0) = (1^2-m^2)m$

in (*) put $n=2$

$y_4(0) = (2^2-m^2)\,y_2(0) = 0$

in (*) put $n=3$

$y_5(0) = (3^2-m^2)\,y_3(0) = (3^2-m^2)(1^2-m^2)m$

$\therefore$ $y_n(0) = 0$ for n even

$= ((n-2)^2-m^2) \cdots (3^2-m^2)(1^2-m^2)m$ for n odd.

$$y = e^{m\cos^{-1}x}$$

P.T. $(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - (n^2+m^2)y_n = 0$

$$\rightarrow \quad y_1 = e^{m\cos^{-1}x}\,\frac{m(-1)}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\;y_1 = -my \quad \text{——} \; (1)$$

$$\sqrt{1-x^2}\;y_2 + y_1\,\frac{1(-\cancel{2}x)}{\cancel{2}\sqrt{1-x^2}} = -my_1$$

$$(1-x^2)\,y_2 - (x y_1) = -m\sqrt{1-x^2}\;y_1$$

from (1)

$$(1-x^2)y_2 - (x y_1) = -m(-my)$$

$$(1-x^2)y_2 - (x y_1) = m^2 y$$

By Leibnitz's thm

$$(uv)_n = u_n v + n\,u_{n-1}v_1 + \frac{n(n-1)}{2!}u_{n-2}v_2 + \text{---} + u v_n$$

$$[\underset{u}{y_2}\,\underset{v}{(1-x^2)}]_n - (\underset{u}{y_1}\,\underset{v}{x})_n = (m^2 y)_n$$

$$\left[\, y_{n+2}(1-x^2) + n\,y_{n+1}(-2x) + \frac{n(n-1)}{\cancel{2}!}\,y_n(-\cancel{2}) \right]$$

$$-\left(y_{n+1}\,x + n\,y_n(1) \right) = m^2 y_n$$

$$(1-x^2)y_{n+2} - 2xn\,y_{n+1} - n^2 y_n + \cancel{n}y_n - x\,y_{n+1} - \cancel{n}y_n = m^2 y_n$$

$$(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n - m^2 y_n = 0.$$

$$(1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - (n^2+m^2)\,y_n = 0$$

If $\sin^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^{n}$

P.T. $x^2 y_{n+2} + (2n+1)\, x\, y_{n+1} + 2n^2 y_n = 0$

$$\rightarrow \quad \frac{y}{b} = \sin\left(\log\left(\frac{x}{n}\right)^{n}\right)$$

$$y = b \sin\left(n \log\left(\frac{x}{n}\right)\right)$$

$$y_1 = b\cos\left(n\log\left(\frac{x}{n}\right)\right) \times \cancel{n}\ \frac{1}{\left(x/n\right)}\ \frac{1}{\cancel{n}}(1)$$

$$y_1 = b\cos\left(n\log\left(\frac{x}{n}\right)\right)\frac{n}{x}$$

$$xy_1 = bn\cos\left(n\log\left(\frac{x}{n}\right)\right)$$

Diff wrt. x

$$xy_2 + y_1 = -bn\sin\left(n\log\left(\frac{x}{n}\right)\right) \times \cancel{n}\ \frac{1}{\left(x/n\right)}\ \frac{1}{\cancel{n}}(1)$$

$$xy_2 + y_1 = -bn \times \frac{n}{x}\sin\left(n\log\left(\frac{x}{n}\right)\right)$$

$$x^2y_2 + xy_1 = -n^2\left(b\sin\left(n\log\left(\frac{x}{n}\right)\right)\right)$$

$$x^2y_2 + xy_1 = -n^2 y$$

$$\left[y_2 x^2\right]_n + (y_1 x)_n = -n^2 y_n$$

Use Leibnitz's thm.

$$y = \sin\left[\log\left(x^2+2x+1\right)\right] \quad \text{p.T.}$$

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)\,y_{n+1} + (n^2+4)\,y_n = 0$$

$\rightarrow$

$$y = \sin\left[\log(x+1)^2\right]$$

$$= \sin\left(2\log(x+1)\right)$$

$y = \sec^{-1}x$ p. T.

$$x(x^2-1)y_{n+2} + [(2+3n)x^2-(n+1)]y_{n+1} + n(3n+1)xy_n + n^2(n-1)y_{n-1} = 0$$

$\longrightarrow$

$$y_1 = \frac{1}{x\sqrt{x^2-1}}$$

$$(x\sqrt{x^2-1})\,y_1 = 1$$

$$(x\sqrt{x^2-1})\,y_2 + y_1\left(x\cdot\frac{1}{\cancel{2}\sqrt{x^2-1}}(\cancel{2}x) + \sqrt{x^2-1}\right) = 0$$

$$x(x^2-1)y_2 + y_1(x^2 + (x^2-1)) = 0.$$

$$(x^3-x)\,y_2 + (2x^2-1)y_1 = 0.$$

$$[\underset{u}{y_2}\,\underset{v}{(x^3-x)}]_n + [\underset{u}{y_1}\,\underset{v}{(2x^2-1)}] = 0$$

By Leibnitz's thm

$$(uv)_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!}u_{n-2}v_2 + \frac{n(n-1)(n-2)}{3!}u_{n-3}v_3 + \text{———} + uv_n$$

$$y_{n+2}(x^3-x) + n\,y_{n+1}(3x^2-1) + \frac{n(n-1)}{\cancel{2}}y_n(\cancel{6}^{3}x) + \frac{n(n-1)(n-2)}{\cancel{6}}y_{n-1}(\cancel{6})$$

$$+\left[y_{n+1}(2x^2-1) + n y_n(4x) + \frac{n(n-1)}{\cancel{2!}}y_{n-1}(\cancel{4})\right] = 0$$

$$(x^3-x)y_{n+2} + [n(3x^2-1) + (2x^2-1)]y_{n+1}$$

$$+ (3(n^2-n) + 4n)x\,y_n + n(n-1)y_{n-1}(n\cancel{-2}\ \cancel{+2})$$

If $x = \sin\theta$ $y = \sin 2\theta$,

p.T.

$(1-x^2) y_{n+2} - (2n+1)x\, y_{n+1} - (n^2-4) y_n = 0$

$y = \sin 2\theta$

$= 2\sin\theta \cos\theta$

$y = 2x\sqrt{1-\sin^2\theta}$

$= 2x\sqrt{1-x^2}$

If $x = e^t$ & $y = \cos mt$ p.T.

$x^2 y_{n+2} + (2n+1)x\, y_{n+1} + (m^2+n^2) y_n = 0.$

$y = \cos mt$ as $x = e^t \Rightarrow t = \log x$.

$= \cos m(\log x) = \cos(m\log x)$.

If $x = \cos\theta$ $\theta = \frac{1}{m}\log y$

$(1-x^2) y_{n+2} - (2n+1)x\, y_{n+1} - (n^2+m^2) y_n = 0$

$x = \cos\left(\frac{1}{m}\log y\right)$

$\frac{1}{m}\log y = \cos^{-1}x \Rightarrow \log y = m\cos^{-1}x$

$\Rightarrow y = e^{m\cos^{-1}x}$