K. J. Somaiya College of Engineering, Mumbai-77
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F.Y. Btech SEM-I

APPLIED MATHEMATICS-I

QUESTION BANK -1

TOPIC – COMPLEX NUMBERS

Type – 1: De-Moivre's Theorem

1. Simplify

(i) $\frac{(\cos 2\theta - i\sin 2\theta)^5(\cos 3\theta + i\sin 3\theta)^6}{(\cos 4\theta + i\sin 4\theta)^7(\cos\theta - i\sin\theta)^8}$ **(ii)** $\frac{(\cos 2\theta + i\sin 2\theta)^3(\cos 3\theta - i\sin 3\theta)^2}{(\cos 4\theta + i\sin 4\theta)^5(\cos 5\theta - i\sin 5\theta)^4}$

2. Prove that

(i) $\frac{(1+i)^8(1-i\sqrt{3})^3}{(1-i)^6(1+i\sqrt{3})^9} = \frac{i}{32}$ **(ii)** $\frac{(1+i\sqrt{3})^9(1-i)^4}{(\sqrt{3}+i)^{12}(1+i)^4} = -\frac{1}{8}$

3. Find the modulus and the principal value of the argument of $\frac{(1+i\sqrt{3})^{17}}{(\sqrt{3}-i)^{15}}$

4. Express in the form $a + ib$, $\frac{(1+i)^{10}}{(1+i\sqrt{3})^5}$

5. Express $(1+7i)(2-i)^{-2}$ in the form of $r(\cos\theta + i\sin\theta)$ and prove that the second power is a negative imaginary number and the fourth power is a negative real number.

6. If $x_n + iy_n = (1+i\sqrt{3})^n$, prove that $x_{n-1}y_n - x_n y_{n-1} = 4^{n-1}\sqrt{3}$.

7. Simplify

(i) $(1 + cos\,\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$ **(ii)** $\left(\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right)^n$

8. Prove that $\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$ Hence deduct that

$\left(1 + sin\frac{\pi}{5} + i\;cos\frac{\pi}{5}\right)^5 + i\left(1 + sin\frac{\pi}{5} - i\;cos\frac{\pi}{5}\right)^5 = 0.$

9. If $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ $and\ \bar{z}$ is the conjugate of z find the value of $(z)^{15} + (\bar{z})^{15}$.

10. Prove that, if n is a positive integer, then

(i) $(a+ib)^{m/n} + (a-ib)^{m/n} = 2\left(\sqrt{a^2+b^2}\right)^{m/n} cos\left(\frac{m}{n}tan^{-1}\frac{b}{a}\right)$

(ii) $\left(\sqrt{3}+i\right)^{120} + \left(\sqrt{3}-i\right)^{120} = 2^{121}$

11. If n is a positive integer, prove that $(1+i)^n + (1-i)^n = 2\ 2^{n/2}\cos n\ \pi/4$
Hence, deduce that $(1+i)^{10} + (1-i)^{10} = 0$

12. Prove that $\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$ is equal to -1 if $n = 3k \pm 1$ and 2 if $n = 3k$ where k is an integer.

13. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1} cos(n\pi/3)$.

(i) Deduce that $\alpha^{15} + \beta^{15} = -2^{16}$ **(ii)** Deduce that $\alpha^6 + \beta^6 = 128$

14. If α, β are the roots of the equation $z^2 sin^2\theta - z.\sin 2\theta + 1 = 0$, prove that $\alpha^n + \beta^n = 2\cos n\,\theta\, cosec^n\theta$

15. If $a = \cos 3\alpha + i\sin 3\alpha, b = \cos 3\beta + i\sin 3\beta, c = \cos 3\gamma + i\sin 3\gamma$, prove that $\sqrt[3]{\frac{ab}{c}} + \sqrt[3]{\frac{c}{ab}} = 2cos(\alpha + \beta - \gamma)$

16. If $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\emptyset$, $z + \frac{1}{z} = 2\cos\psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2\cos(\theta + \Phi + \psi)$ **(ii)** $\sqrt{xyz} + \frac{1}{\sqrt{xyz}} = 2\,cos\left(\frac{\theta+\Phi+\psi}{2}\right)$

(iii) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\Phi)$ **(iv)** $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\,cos\left(\frac{\theta}{m} - \frac{\emptyset}{n}\right)$

17. If $x + \frac{1}{x} = 2\cos\theta$ then prove that $\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{\cos n\theta}{\cos(n-1)\theta}$ and $\frac{x^{2n}-1}{x^{2n-1}-x} = \frac{\sin n\theta}{\sin(n-1)\theta}$

18. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$, prove that $\frac{(b+c)(c+a)(a+b)}{abc} = 8\,cos\frac{(\alpha-\beta)}{2}cos\frac{(\beta-\gamma)}{2}cos\frac{(\gamma-\alpha)}{2}$.

20. If $\cos\alpha + \cos\beta + \cos\gamma = 0\ and \sin\alpha + \sin\beta + \sin\gamma = 0$, Prove that

(i) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$, $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$.

(ii) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$

(iii) $cos(\alpha+\beta) + cos(\beta+\gamma) + cos(\gamma+\alpha) = 0$.

(iv) $\sin(\alpha+\beta) + sin(\beta+\gamma) + sin(\gamma+\alpha) = 0$.

(v) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\,cos(\alpha+\beta+\gamma)$

(vi) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

21. If $a\cos\alpha + b\cos\beta + c\cos\gamma = a\sin\alpha + b\sin\beta + c\sin\gamma = 0$, Prove that

$a^3\cos 3\alpha + b^3\cos 3\beta + c^3\cos 3\gamma = 3abc\cos(\alpha+\beta+\gamma)$ and

$a^3\sin 3\alpha + b^3\sin 3\beta + c^3\sin 3\gamma = 3\,abc\,sin(\alpha+\beta+\gamma)$

22. If $x_r = cos\left(\frac{2}{3}\right)^r\pi + i\,sin\left(\frac{2}{3}\right)^r\pi$, prove that

(i) $x_1x_2x_3 \ldots \infty = 1$, **(ii)** $x_0x_1x_2 \ldots \infty = -1$

23. If $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots\ldots\ldots\ldots (\cos n\,\theta + i\sin n\,\theta) = i$, then show that the general value of $\theta = \left[2r + \frac{1}{n(n+1)}\right]\pi$

Type -2: Roots of Complex numbers

1. Find the cube roots of unity. If ω is a complex cube root of unity prove that

(i) $1 + \omega + \omega^2 = 0$ **(ii)** $\frac{1}{1+2\omega} + \frac{1}{2+\omega} - \frac{1}{1+\omega} = 0$

2. Prove that the n nth roots of unity are in geometric progression.

3. Show that the sum of the n nth roots of unity is zero.

4. Prove that the product of n nth roots of unity is $(-1)^{n-1}$

5. Find all the values of the following :

(i) $(-1)^{1/5}$ **(ii)** $(-i)^{1/3}$ **(ix)** $\left(1-i\sqrt{3}\right)^{1/4}$

6. Find the continued product of all the values of $\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{3/4}$

7. Find all the value of $(1+i)^{2/3}$ and find the continued product of these values.

8. Solve the equations

(i) $x^9+8x^6+x^3+8=0$ **(ii)** $x^4-x^3+x^2-x+1=0$

(iii) $(x+1)^8+x^8=0$

9. If $(x+1)^6=x^6, show\ that\ x=-\frac{1}{2}-i\ cot\frac{\theta}{2}\ where\ \theta=\frac{2k\pi}{6}, k=0,1,2,3,4,5.$

10. Show that the roots of $(x+1)^7=(x-1)^7$are given by $\pm i\ cot\frac{r\pi}{7}, r=1,2,3.$

11. If $\alpha,\alpha^2,\alpha^3,\dots\alpha^6$ are the roots of $x^7-1=0$, find them and prove that

$(1-\alpha)(1-\alpha^2)\dots\dots\dots\dots(1-\alpha^6)=7.$

12. Prove that $x^5-1=(x-1)\left(x^2+2x\ cos\frac{\pi}{5}+1\right)\left(x^2+2x\ cos\frac{3\pi}{5}+1\right)=0.$

13. Solve the equation $z^n=(z+1)^n$ and show that the real part of all the roots is $-1/2$.

14. If $a=e^{i\,2\pi/7}$ and $b=a+a^2+a^4$, $c=a^3+a^5+a^6$**.** then prove that b & c are roots of quadratic equation $x^2+x+2=0.$

15. Prove that $\sqrt{1-sce(\theta/2)}=\left(1+e^{i\theta}\right)^{-1/2}-\left(1+e^{-i\theta}\right)^{-1/2}$

16. If $1+2i$ is a root of the equation $x^4-3x^3+8x^2-7x+5=0$, find all the other roots.

17. Find the roots common to $x^{12}-1=0$ and $x^4-x^2+1=0$

Type-3 : Hyperbolic Functions

1. If $\tanh x=2/3$, find the value of x and then $cosh\ 2x$.

2. Solve the equation for real values of x, $17\cosh x+18\sinh x=1.$

3. If $6\sinh x+2\ cosh\ x+7=0$, find $tanh\ x$.

4. If $cosh^{-1}a+cosh^{-1}b=cosh^{-1}x$, then prove that $a\sqrt{b^2-1}+b\sqrt{a^2-1}=\sqrt{x^2-1}.$

5. If $cosh^6x=a\cosh 6x+b\cosh 4x+c\cosh 2x+d$, Prove that $25a-5b+3c-4d=0$

6. Prove that $cosh^7x=\frac{1}{64}[\cosh 7x+7\cosh 5x+21\cosh 3x+35\cosh x]$

7. If $\cos\alpha\cosh\beta=x/2$, $\sin\alpha\sinh\beta=y/2$, show that

(i) $\sec(\alpha-i\,\beta)+\sec(\alpha+i\,\beta)=\frac{4x}{x^2+y^2}$ **(ii)** $\sec(\alpha-i\,\beta)-sec(\alpha+i\,\beta)=\frac{-4iy}{x^2+y^2}$

8. Prove that $cosech\ x+\coth x=\coth\frac{x}{2}$

9. Prove that $(\cosh x+sinh\ x)^n=\cosh nx+\sinh nx$

10. Prove that $\left(\frac{\cosh x+\sinh x}{\cosh x-\sinh x}\right)^n = \cosh 2nx + \sinh 2nx$

11. If $\log\tan x = y$, prove that $\cosh ny = \frac{1}{2}[tan^n x + cot^n x]$ and

$\sinh(n+1)y + \sinh(n-1)\,y = 2\sinh ny\ \ cosec\ 2x$

12. Prove that $\frac{1}{1-\frac{1}{1-\frac{1}{1+sinh^2 x}}} = -sinh^2 x$

13. If $\cosh u = \sec\theta$, $prove\ that$

(i) $\sinh u = \tan\theta$ **(ii)** $\tanh u = \sin\theta$ **(iii)** $u = \log\left[tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$

Type -4: Separation into real and Imaginary parts

1. Separate into real and imaginary parts.

(i) $\cosh(x+iy)$ **(ii)** $\cos(x+iy)$ **(iii)** $\coth(x+iy)$

(iv) $\operatorname{sech}(x+iy)$ **(v)** $\coth i(x+iy)$ **(vi)** $\tan(x+iy)$

(vii) $\cot(x+iy)$

2. Separate into real and imaginary parts $tan^{-1}(\alpha + i\,\beta)$

3. Separate into real and imaginary parts $sin^{-1}\left(e^{i\,\theta}\right)$

4. If A $+i\,B = C\tan(x+iy)$, prove that $tan2x = \frac{2CA}{C^2-A^2-B^2}$

5. If cos $(\theta + i\,\Phi) = r(\cos\alpha + i\sin\alpha)$, prove that $r^2 = \frac{1}{2}[\cosh 2\,\Phi + \cos 2\,\theta]$ &

$\tan\alpha = -\tan\theta\tanh\Phi$

6. If $\cos(\alpha + i\,\beta) = x + i\,y$, Prove that $\frac{x^2}{cosh^2\beta} + \frac{y^2}{sinh^2\beta} = 1,\quad \frac{x^2}{cos^2\alpha} - \frac{y^2}{sin^2\alpha} = 1$

7. If $\sinh(a+i\,b) = x + i\,y$, prove that $x^2 cosech^2 a + y^2 sech^2 a = 1$

$and\ \ y^2 cosec^2 b - x^2 sec^2 b = 1$

8. If sin $(x+iy) = \cos\alpha + i\sin\alpha$, Prove that

(i) $\cosh 2y - \cos 2x = 2$ **(ii)** $y = \frac{1}{2}\ log\frac{\cos(x-\alpha)}{\cos(x+\alpha)}$

(iii) $\sin\alpha = \pm\, cos^2 x = \pm\, sinh^2 y$

9. If $\cosh(\theta + i\,\Phi) = e^{i\,\alpha}$, prove that $sin^2\alpha = sin^4\Phi = sinh^4\theta$

10. If $\cos(u + i\,v) = x + i\,y$ Prove that, $(1+x)^2 + y^2 = (\cosh v + \cos u)^2$ and

$(1-x)^2 + y^2 = (\cosh v - \cos u\,)^2$

11. If $tan(\alpha + i\,\beta) = x + i\,y$, prove that $x^2 + y^2 + 2x\cot 2\alpha = 1,\ \ x^2 + y^2 - 2y\coth 2\beta + 1 = 0$

12. If tan $\left(\frac{\pi}{3} + i\,\alpha\right) = x + i\,y$, prove that, $x^2 + y^2 - \frac{2x}{\sqrt{3}} - 1 = 0$

13. If $cot(\alpha + i\,\beta) = x + i\,y$, prove that $x^2 + y^2 - 2x\cot 2\alpha = 1,\ \ x^2 + y^2 + 2y\coth 2\beta + 1 = 0$

14. If $tanh\left(\alpha + \frac{i\,\pi}{6}\right) = x + i\,y$, prove that, $x^2 + y^2 + \frac{2y}{\sqrt{3}} = 1$

15. If $\coth(\alpha + i\,\pi/8) = x + i\,y$, prove that $x^2 + y^2 + 2y = 1$

16. If $\sinh(x+i\,y) = e^{i\,\pi/3}$, prove that

(i) $3cos^2y - sin^2y = 4sin^2y\,cos^2y$ **(ii)** $3sinh^2x + cosh^2x = 4sinh^2xcosh^2x$

17. If $x+i\,y = 2\cosh\left(\alpha + \frac{i\,\pi}{3}\right)$, prove that $3x^2 - y^2 = 3$

18. If $\cot(u+i\,v) = cosec(x+i\,y)$, prove that $\coth y \sinh 2v = \cot x \sin 2u$

19. Show that $tan\left(\frac{u+i\,v}{2}\right) = \frac{\sin u + i\sinh v}{\cos u + \cosh v}$

20. If $\sin^{-1}(\alpha + i\,\beta) = x + i\,y$, show that $\sin^2 x$ and $\cos h^2 y$ are the roots of the equation $\lambda^2 - (\alpha^2 + \beta^2 + 1)\lambda + \alpha^2 = 0$

Type – 5: Inverse hyperbolic functions

1. Prove that **(i)** $tanh\left(log\sqrt{3}\right) = 1/2$ **(ii)** $\tanh\left(log\sqrt{5}\right) = 2/3.$

2. Prove that **(i)** $cosech^{-1}x = log\left[\frac{1+\sqrt{1+x^2}}{x}\right]$ **(ii)** $tanh^{-1}x = cosh^{-1}\frac{1}{\sqrt{1-x^2}}$

(iii) $coth^{-1}x = \frac{1}{2}log\left(\frac{x+1}{x-1}\right)$

3. Prove that **(i)** $tanh^{-1}\cos\theta = cosh^{-1}cosec\,\theta$ **(ii)** $sinh^{-1}tan\theta = log(\sec\theta + \tan\theta)$

4. Separate into real and imaginary parts.

(i) $sin^{-1}(3i/4)$ **(ii)** $cosh^{-1}(i\,x)$ **(iii)** $cos^{-1}\left(\frac{16\,i}{63}\right)$

5. Prove that $cosh^{-1}(3i/4) = \log 2 + i\,\pi/2$

6. Prove that $cos^{-1}(\sec\theta) = i\,log(\sec\theta + \tan\theta)$

7. Prove that $cos^{-1}i\,x = \frac{\pi}{2} - i\,log\left(x + \sqrt{x^2+1}\right)$

8. If $\tan z = \frac{i}{2}(1-i)$, prove that $z = \frac{1}{2}tan^{-1}2 + \frac{i}{4}\log 5.$

9. If $sinh^{-1}(x+i\,y) + sinh^{-1}(x-i\,y) = sinh^{-1}a$, prove that $2(x^2+y^2)\sqrt{a^2+1} = a^2 - 2x^2 + 2y^2$

10. Find all the roots of the equation $\cos z = 2.$

11. If $cos\left(\frac{\pi}{4} + ia\right).co\,h\left(b + \frac{i\pi}{4}\right) = 1$ where a,b arereal, prove that $2b = log\left(2+\sqrt{3}\right)$

12. If $\tan(x + i\,y) = i$ and x, y are real, prove that x is indeterminate and y is infinite.

13. If $tan\left(\frac{\pi}{4} + i\,v\right) = re^{i\,\theta}$, show that,

(i) $r = 1.$ **(ii)** $tan\theta = \sinh 2v.$ **(iii)** $\tanh v = tan\frac{\theta}{2}$

Type - 6: Logarithm of Complex Numbers

1. Prove that $\tan\left[i\log\frac{a-ib}{a+ib}\right]=\frac{2ab}{a^2-b^2}$.

2. Prove that $\sin\left[i\log\left(\frac{a-ib}{a+ib}\right)\right]=\frac{2ab}{a^2+b^2}$.

3. Prove that $\cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right]=\frac{a^2-b^2}{a^2+b^2}$.

4. Prove that $\log\left(e^{i\alpha}-e^{i\beta}\right)=log\left[2\sin\left(\frac{\alpha-\beta}{2}\right)\right]+i\left(\frac{\pi+\alpha+\beta}{2}\right)$.

5. Prove that $\log\left(e^{i\alpha}+e^{i\beta}\right)=log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\right]+i\left(\frac{\alpha+\beta}{2}\right)$.

6. Prove that $\log\left(\frac{1}{1-e^{i\theta}}\right)=\log\left(\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right)+i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$.

7. Separate real and imaginary parts of

   (i) $\left(\sqrt{i}\right)^{\sqrt{i}}$ (ii) $\log_{(1-i)}(1+i)$ (iii) $(1+i)^{i}$ (iv) $i^{log(1+i)}$

8. Prove that real part of principal value of $(1+i)^{\log i}$ *is* $e^{-\frac{\pi^2}{8}}\cos\left(\frac{\pi}{4}\log 2\right)$.

9. Considering only principal values, separate real and imaginary parts of $i^{\log(1+i)}$.

10. By Considering only the principal value express $\left(1+i\sqrt{3}\right)^{\left(1+i\sqrt{3}\right)}$ in the form of $a+ib$.

11. If $\tan[log(x+iy)]=a+ib, prove\ that\ tan[log(x^2+y^2)]=\frac{2a}{1-a^2-b^2}, a^2+b^2\neq 1$.

12. Show that for real values of a and b, $e^{2ai\,co^{-1}b}.\left[\frac{bi-1}{bi+1}\right]^{-a}=1$.

13. Considering only the principal value, if $\left(1+i\tan\alpha\right)^{1+i\tan\beta}$ is real,

    prove that its value is $\left(\sec\alpha\right)^{\sec^2\beta}$.

14. Prove that general value of $(1+itan\alpha)^{-i}$ $is\ e^{2m\pi+\alpha}[cos(logcos\alpha)+isin(logcos\alpha)]$.

15. If $i^{\alpha+i\beta}=\alpha+i\beta, prove\ that\ \alpha^2+\beta^2=e^{-(4n+1)\pi\beta}, where\ n\ is\ positive\ integer.$

16. Prove that $i\log\left(\frac{x-i}{x+i}\right)=\pi-2\tan^{-1}x.$

17. If $i^{i^{i\ldots\ldots\infty}}=A+iB, considering\ principal\ value,\ \ prove\ that\ \tan\left(\frac{\pi A}{2}\right)=\frac{B}{A}\ and\ A^2+B^2=e^{-\pi B}.$

18. If $\sqrt{i}^{\sqrt{i}^{\sqrt{i}\ldots\ldots\infty}}=\alpha+i\beta,\ prove\ that\ 1)\ \alpha^2+\beta^2=e^{\frac{-\pi\beta}{2}}\quad 2)\ \tan^{-1}\left(\frac{\beta}{\alpha}\right)=\left(\frac{\pi\alpha}{4}\right).$

19. Prove that $\log\left[\frac{\sin(x+iy)}{\sin(x-iy)}\right]=2i\tan^{-1}(\cot x\tanh y).$

20. Prove that $\log\tan\left(\frac{\pi}{4}+i\frac{x}{2}\right)=i\tan^{-1}(\sinh x).$

21. If $\log\left[\log(x+iy)\right]=p+iq$, then prove that $y=x\tan\left[\tan q\ \log\sqrt{x^2+y^2}\right].$

22. Prove that $i^{i^i}=\cos\theta+i\sin\theta\ ,\ where\ \ \theta=\left(2n+\frac{1}{2}\right)\pi e^{-\left(2m+\frac{1}{2}\right)\pi}.$