Taylor's series Expansion. / Taylor's Thm.

(1) Expansion of $f(x+h)$ in powers of h is

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \text{———}$$

(2) Expansion of $f(x+h)$ in powers of x $x \leftrightarrow h$

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \text{———}$$

(3) Expansion of $f(x)$ in powers of $(x-a)$ is

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \text{———}$$

① Apply taylor's thm to find approx. value of $f\left(\frac{11}{10}\right)$ where $f(x) = x^3 + 3x^2 + 15x - 10$

→ $f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \text{———}$ ——— ①

$f\left(\frac{11}{10}\right) = f(1\cdot1)$

$= f(1 + 0\cdot1)$ → nearest int. to 1·1

$\therefore x = 1, \quad h = 0\cdot1$

from ①

$f(1 + 0\cdot1) = f(1) + (0\cdot1) f'(1) + \frac{(0\cdot1)^2}{2!} f''(1) + \text{———}$ ——— ①

$f(x) = x^3 + 3x^2 + 15x - 10$ $\quad f(1) = 9$

$f'(x) = 3x^2 + 6x + 15$ $\quad f'(1) = 24$

$f''(x) = 6x + 6$ $\quad f''(1) = 12$

$f'''(x) = 6$ $\quad f'''(1) = 6$

$f^{iv}(x) = 0$

$f(1\cdot1) = 9 + 0\cdot1\,(24) + \frac{(0\cdot1)^2}{\cancel{2}_1}(\cancel{12}^{6}) + \frac{(0\cdot1)^3}{\cancel{6}}(\cancel{6})$

$= 11\cdot46$ ✓

(2) . Expand e^x in powers of $(x-1)$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \text{———}$$

$$a = 1$$

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \text{———} \quad \text{—— (1)}$$

$f(x) = e^x$ $\qquad$ $f(1) = e$

$f'(x) = e^x$ $\qquad$ $f'(1) = e$

$f''(x) = e^x$ $\qquad$ $f''(1) = e$

from (1)

$$\therefore e^x = e + (x-1)e + \frac{(x-1)^2}{2!}e + \text{———}$$

$$= e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \text{———}\right]$$

Expand $\tan^{-1}x$ in powers of $\left(x-\frac{\pi}{4}\right)$

$\rightarrow$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \text{———}$$

$$a = \frac{\pi}{4}$$

$$f(x) = \tan^{-1}x = f\left(\frac{\pi}{4}\right) + \left(x-\frac{\pi}{4}\right)f'(\pi/4) + \frac{(x-\pi/4)^2}{2!}f''(\pi/4) + \text{———}. \quad \text{—} (1).$$

$f(x) = \tan^{-1}x$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = \frac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$

$f(\pi/4) = \tan^{-1}(\pi/4) = 1$

$$f'(\pi/4) = \frac{1}{1+\frac{\pi^2}{16}} = \frac{16}{16+\pi^2}$$

$$f''(\pi/4) = \frac{-2\pi/4}{\left(1+\frac{\pi^2}{16}\right)^2}$$

$$= \frac{-\pi}{2\left(\frac{16+\pi^2}{16}\right)^2}$$

$$= \frac{-\pi(16)^2}{2(16+\pi^2)^2}$$

$$\tan^{-1}x = 1 + (x-\pi/4)\frac{16}{16+\pi^2} + \frac{(x-\pi/4)^2}{\cancel{2}}\left(\frac{-\pi\times16\times\cancel{16}^{\cancel{8}\,4}}{\cancel{2}(16+\pi^2)^2}\right) + \text{———}$$

$$= 1 + \frac{16}{16+\pi^2}(x-\pi/4) - \frac{64\pi}{(16+\pi^2)^2}(x-\pi/4)^2 + \text{———}$$

Expand $\log x$ in powers of $(x-1)$

Hence show that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \text{———}$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) \text{———}$$

$$a = 1$$

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \text{———}$$

$f(x) = \log x \quad \Rightarrow f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x} \quad \Rightarrow f'(1) = 1$

$f''(x) = -\frac{1}{x^2} = -x^{-2} \quad \Rightarrow f''(1) = -1$

$f'''(x) = 2x^{-3} \quad \Rightarrow f'''(1) = 2$

$f^{iv}(x) = -6x^{-4} \quad \Rightarrow f^{iv}(1) = -6$

$$\therefore \log x = 0 + (x-1)(1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6}(2) + \frac{(x-1)^4}{4!}(-6) + \text{———}$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4 \cancel{6}}{\cancel{24}\ 4} \text{———}$$

$$\frac{\cancel{6}}{\cancel{24}\ 4}$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} \text{———}$$

To find $\log 2$ put $x = 2$

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \text{———}$$

Using Taylor's Thm arrange in powers of x

$$7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$$

here,

$$f(x+2) = 7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$$

By Taylor's thm

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \text{———}$$

here $h=2$

$$f(x+2) = f(2) + x f'(2) + \frac{x^2}{2!} f''(2) + \text{———}$$

$f(x) = 7 + x + 3x^3 + x^4 - x^5$	$f(2) = 17$
$f'(x) = 1 + 9x^2 + 4x^3 - 5x^4$	$f'(2) = -11$
$f''(x) = 18x + 12x^2 - 20x^3$	$f''(2) = -76$
$f'''(x) = 18 + 24x - 60x^2$	$f'''(2) = -174$
$f^{iv}(x) = 24 - 120x$	$f^{iv}(2) = -216$
$f^{v}(x) = -120$	$f^{v}(2) = -120$

$$f(x+2) = 17 + x(-11) + \frac{x^2}{\cancelto{1}{2!}}(\cancelto{38}{-76}) + \frac{x^3}{3!}(-174) + \frac{x^4}{4!}(-216) + \frac{x^5}{5!}(-120)$$

$$= 17 - 11x - 38x^2 - 29x^3 - 9x^4 - x^5 \quad //.$$

Expand $\tan^{-1}(x+h)$ in powers of h & hence find $\tan^{-1}(1.003)$ upto 5 places of decimals given $\pi = 3.141593$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \text{———} \quad \text{—}①$$

$$f(x+h) = \tan^{-1}(x+h)$$

$$f(x) = \tan^{-1}x$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = -\frac{1}{(1+x^2)^2}(2x) \quad \Big| \quad \frac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2}$$

$$= \frac{-2x}{(1+x^2)^2}$$

$$f'''(x) = \frac{(1+x^2)^2(-2) + (2x)\,2(1+x^2)(2x)}{(1+x^2)^4}$$

$$= \frac{2(1+x^2)\left(-(1+x^2) + 4x^2\right)}{(1+x^2)^4}$$

$$= 2\left[\frac{-1+3x^2}{(1+x^2)^3}\right]$$

$$\tan^{-1}(x+h) = \tan^{-1}x + h\left(\frac{1}{1+x^2}\right) + \frac{h^2}{\cancel{2!}}\left(\frac{-\cancel{2}x}{(1+x^2)^2}\right) + \frac{h^3}{3!}\,2\left(\frac{3x^2-1}{(1+x^2)^3}\right) + \text{———}$$

$$= \tan^{-1}x + \frac{h}{1+x^2} - \frac{x}{(1+x^2)^2}h^2 + \frac{h^3}{3}\left(\frac{3x^2-1}{(1+x^2)^3}\right) + \text{———}$$

$$\tan^{-1}(1.003) = \tan^{-1}(1 + 0.003)$$

$$x = 1 \quad h = 0.003$$

$$= \underset{\Downarrow\ \frac{\pi}{4}}{\tan^{-1}(1)} + \frac{0.003}{2} - \frac{1}{4}(0.003)^2 + \frac{(0.003)^3}{3}\frac{(2)}{2^3}$$

$$= \frac{3.141593}{4} + \frac{0.003}{2} - \frac{(0.003)^2}{4} + \frac{(0.003)^3}{12}$$

$$= 0.786896 \approx 0.78690$$

Maclaurin' Series.

Taylor's series in powers of x

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \text{———}$$

put $h = 0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \text{———}$$

This series is called as Maclaurin Series.

1) $f(x) = e^x$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \text{———} \quad \text{—} (1)$$

$f(x) = e^x$ $\quad$ $f(0) = 1$

$f'(x) = e^x$ $\quad$ $f'(0) = 1$

$f''(x) = e^x$ $\quad$ $f''(0) = 1$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{———}$$

similarly by replacing x by $(-x)$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \text{———}$$

$$a^x = e^{\log a^x} = e^{(x \log a)}$$

$$= 1 + (x \log a) + \frac{(x \log a)^2}{2!} + \text{———}$$

$f(x) = \cos x.$ $\qquad\qquad f(0) = 1$

$f'(x) = -\sin x$ $\qquad\qquad f'(0) = 0$

$f''(x) = -\cos x$ $\qquad\qquad f''(0) = -1$

$f'''(x) = \sin x$ $\qquad\qquad f'''(0) = 0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} - \cdots$$

HW $\qquad$ **f(x) = sinx**

① $\sinh x$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) \text{———}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$f(x) = \sinh x \Rightarrow f(0) = \sinh(0) = 0$

$f'(x) = \cosh x \Rightarrow f'(0) = \cosh(0) = 1$

$f''(x) = \sinh x \Rightarrow f''(0) = 0$

$f'''(x) = \cosh x \Rightarrow f'''(0) = 1$

$$f(x) = \sinh x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + 0 + \frac{x^5}{5!} \text{———}$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \text{———}$$

HW

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \text{———}$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} + \text{———}$$

$$= \text{———}$$

$\log(1+x)$

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \_\_\_\_$

$f(x) = \log(1+x).$ $\qquad f(0) = \log 1 = 0$

$f'(x) = \frac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2} = -(1+x)^{-2}$ $\qquad f''(0) = -1$

$f'''(x) = 2(1+x)^{-3}$ $\qquad f'''(0) = 2$

$f^{iv}(x) = -6(1+x)^{-4}$ $\qquad f^{iv}(0) = -6$

$f(x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \_\_\_\_$

24

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \_\_\_\_$

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \_\_\_\_$

$\tanh^{-1}x$

$f(x) = \tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

$= \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]$

$= \frac{1}{2}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \_\_\_\_ - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \_\_\_\_\right)\right]$

$= \frac{1}{2}\left[x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} + \frac{x^5}{5} \_\_\_\_ + x + \cancel{\frac{x^2}{2}} + \frac{x^3}{3} + \cancel{\frac{x^4}{4}} + \frac{x^5}{5} \_\_\_\_\right]$

$= \frac{1}{2}\left[2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} \_\_\_\_\right]$

$= x + \frac{x^3}{3} + \frac{x^5}{5} + \_\_\_\_$ ✓

H.W

$\sinh^{-1}x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} \_\_\_\_$

$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \_\_\_\_$

$\cos^{-1}x = \frac{\pi}{2} - \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \_\_\_\_\right)$

$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \_\_\_\_$

$f(x) = (1+x)^m$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \text{———} \quad \text{——} (1)$$

$f(x) = (1+x)^m \qquad f(0) = 1$

$f'(x) = m(1+x)^{m-1} \qquad f'(0) = m$

$f''(x) = m(m-1)(1+x)^{m-2} \qquad f''(0) = m(m-1)$

$f'''(x) = m(m-1)(m-2)(1+x)^{m-3} \qquad f'''(0) = m(m-1)(m-2)$

from (1)

$$(1+x)^m = 1 + xm + \frac{x^2}{2!} m(m-1) + \frac{x^3}{3!} m(m-1)(m-2) + \text{———}$$

If m is +ve int. then we'll get only finite terms in above series.

If $m = -1$

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + \frac{x^2}{\cancel{2!}}\cancel{(-1)(-2)} + \frac{x^3}{3!}(-1)(-2)(-3) + \text{———}$$

$$= 1 - x + x^2 - x^3 + \text{————}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \text{———}$$