9.8 Curvilinear Motion

If a particle is moving along curved path then it is said to perform ***curvilinear motion.***

Let us discuss the terms - position, velocity and acceleration for curvilinear motion.

Position

Consider the motion of a particle along a curved path as shown. It is represented by a position vector $\overline{r}$ which is drawn from the origin '*O*' of the fixed reference axis to particle '*P*'. The line *OP* is called *position vector*. As the particle will move along the curved path, the value of $\overline{r}$ will go on changing.

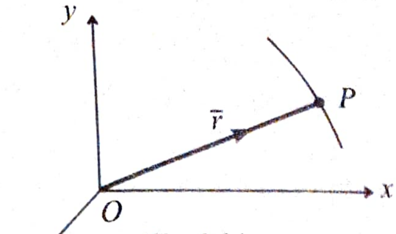

Fig. 9.34

Velocity

Consider after short interval of time Δt particle has occupied new position P' simultaneously the position vector $\overline{r}$ will change to $\overline{r}'$.

The vector joining P and P' is the change in position vector Δr during the time interval Δt.

$\therefore$ The average velocity $v = \dfrac{\Delta \overline{r}}{\Delta t}$

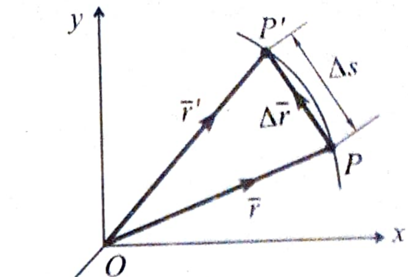

Fig. 9.35(a)

For very small interval of time $\Delta t \rightarrow 0$

Instantaneous velocity at P is $\lim\limits_{\Delta t \to 0} \dfrac{\Delta \overline{r}}{\Delta t}$

$$\therefore \; v = \frac{dr}{dt}$$

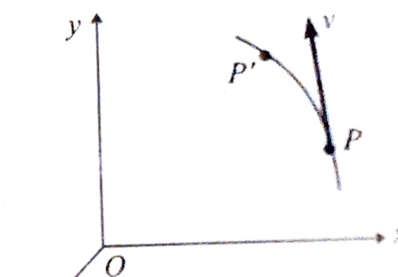

Fig. 9.35(b)

Here during a small interval of time Δt, the particle moves a distance Δs along the curve.

$\therefore$ The magnitude of velocity called speed is given by relation

$$\text{Speed} = |\overline{v}| = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

Note : In curvilinear motion velocity of particle is always tangent to the curved path at every instant.

Acceleration

As the direction of velocity is continuously changing instant to instant in curvilinear motion it is responsible to develop acceleration also at every instant.

Consider the velocity of the particle at P to be v and at position P' be v'.

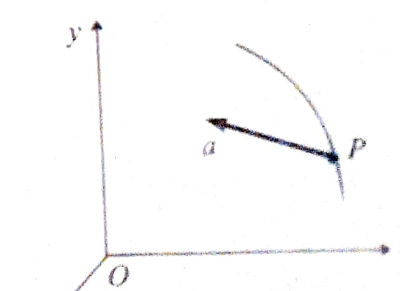

Fig. 9.36

$\therefore$ Average acceleration $a = \dfrac{\Delta \overline{v}}{\Delta t}$

For very small interval of time $\Delta t \rightarrow 0$

Instantaneous velocity at P is $\lim\limits_{\Delta t \to 0} \dfrac{\Delta \overline{v}}{\Delta t}$

$$\therefore \; a = \frac{dv}{dt}$$

Note : In rectilinear motion displacement, velocity and acceleration are always directed along the path of particle. Whereas in curvilinear motion it change its direction instant to instant. Therefore the analysis of curvilinear motion is done by considering different components system. There are two method for analysis in terms of different component system.

(1) Curvilinear motion by Rectangular Component System.

(2) Curvilinear motion by Tangential and Normal Component System.

Rectangular Component System

If a particle is moving along curved path its motion can be splitted into x, y and z direction as a independently performing rectilinear motions.

Thus for curvilinear motion we can have a relation as follows.

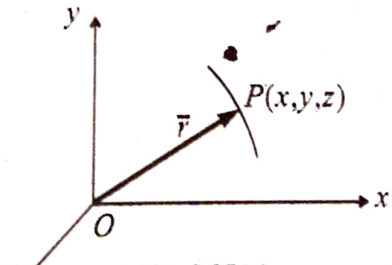

Fig. 9.37(a)

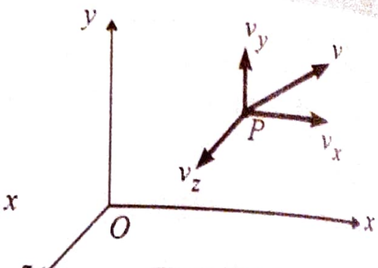

Fig. 9.37(b)

Vector Form	$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$	$\bar{v} = \dfrac{d\bar{r}}{dt} = v_x\bar{i} + v_y\bar{j} + v_z\bar{k}$	$a = \dfrac{dv}{dt} = a_x\bar{i} + a_y\bar{j} + a_z\bar{k}$
Magnitude	$r = \sqrt{x^2 + y^2 + z^2}$	$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$	$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$

Direction is given by the relation

$$\cos\alpha = \frac{x}{r} = \frac{v_x}{v} = \frac{a_x}{a}\ ;\ \cos\beta = \frac{y}{r} = \frac{v_y}{v} = \frac{a_y}{a}\ ;\ \cos\gamma = \frac{z}{r} = \frac{v_z}{v} = \frac{a_z}{a}\ ;$$

While dealing with coplanar motion we can consider particle is moving in xy plane. Its rectangular component system will be as follows.

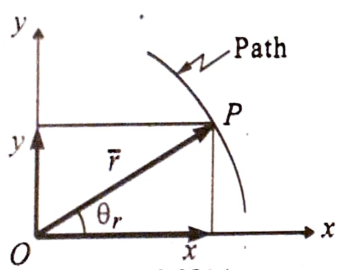

Fig. 9.38(a)

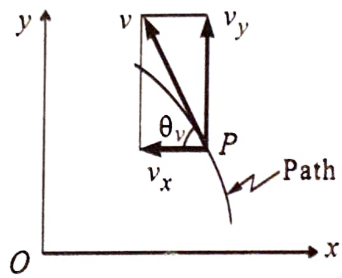

Fig. 9.38(b)

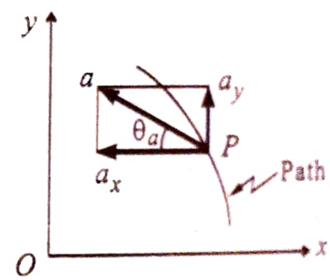

Fig. 9.38(c)

Vector Form	$\bar{r} = x\bar{i} + y\bar{j}$	$\bar{v} = v_x i + v_y j$	$a = a_x\bar{i} + a_y\bar{j}$
Magnitude	$r = \sqrt{x^2 + y^2}$	$v = \sqrt{v_x^2 + v_y^2}$	$a = \sqrt{a_x^2 + a_y^2}$
Direction	$\tan\theta_r = \dfrac{y}{x}$	$\tan\theta_v = \dfrac{v_y}{v_x}$	$\tan\theta_a = \dfrac{a_y}{a_x}$

Derivation of Rectangular Component of Velocity

As the direction of the velocity of a particle is curvilinear motion changes continuously, so it is convenient to deal with its components v_x and v_y.

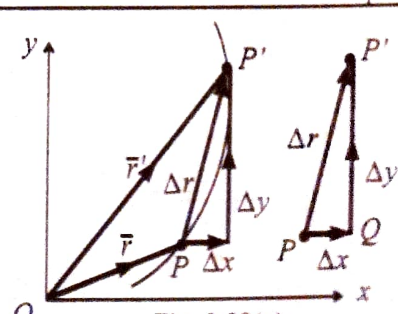

Fig. 9.39(a)

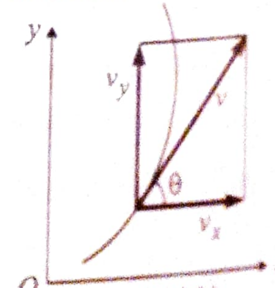

Fig. 9.39(b)

Resolve Δr into two rectangular components $\overline{PQ}$ (Δx) and $\overline{QP'}$ (Δy) parallel to the x and y axis as shown.

$$\Delta r = \overline{PQ} + \overline{QP'}$$

$$v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{PQ}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\overline{QP'}}{\Delta t}$$

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

$$v = \frac{dx}{dt} + \frac{dy}{dt}$$

$$v = v_x + v_y$$

$$\text{Magnitude} \quad v = \sqrt{v_x^2 + v_y^2}$$

$$\text{Direction} \quad \tan\theta = \frac{v_y}{v_x}$$

Derivation of Rectangular Component of Acceleration

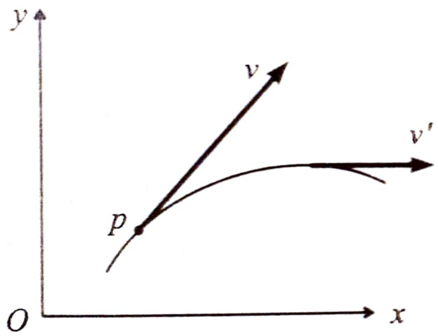

Fig. 9.40(a)

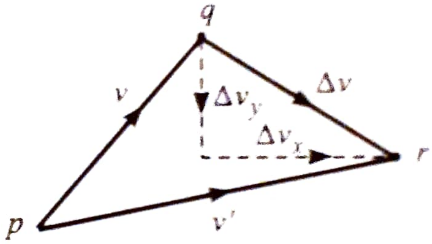

Fig. 9.40(b)

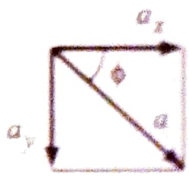

Fig. 9.40(c)

Vector $\overline{pq}$ and $\overline{pr}$ represents the velocities v and v' and the vector $\overline{qr}$ the change in the velocity Δv of the particle.

Resolving Δv into components Δv_x and Δv_y.

$$\Delta v = \overline{qr} = \overline{qs} + \overline{sr}$$

$$a = \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overline{qs}}{\Delta t} + \lim_{\Delta t \to 0} \frac{\overline{sr}}{\Delta t}$$

$$a = \lim_{\Delta t \to 0} \frac{\Delta v_y}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t}$$

$$a = \frac{dv_y}{dt} + \frac{dv_x}{dt}$$

$$a = a_x + a_y$$

$$\text{Magnitude of acceleration} \quad a = \sqrt{a_x^2 + a_y^2}$$

$$\text{Direction} \quad \tan\phi = \frac{a_y}{a_x}$$

Tangential and Normal Component System

Curvilinear motion can also be studies by considering tangential and normal components method.

The velocity vector v is always directed in the tangential direction as discussed in the previous contain. However, the net acceleration of the particle at a particular instant need not be along the tangential direction.

Therefore, sometime it is convenient to express the acceleration of the particle in tangential and normal component form. The components are named by *tangential acceleration* (a_t) and *normal acceleration* (a_n).

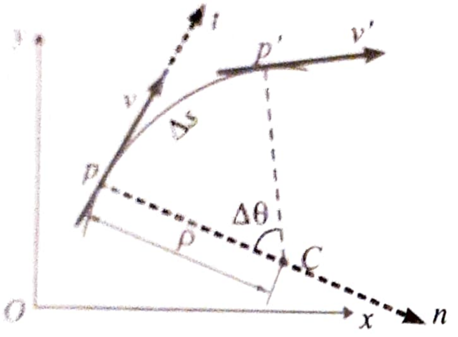

Fig. 9.41(a)

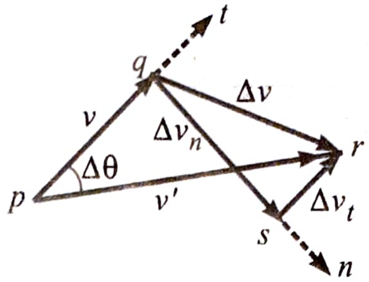

Fig. 9.41(b)

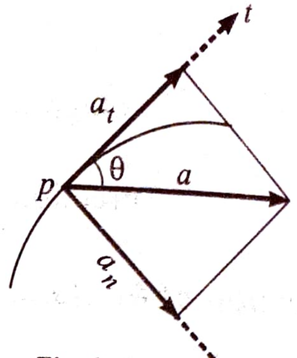

Fig. 9.41(c)

Let the particle has a velocity v at a time t and velocity v' at a later time $(t + \Delta t)$.

For getting the change in the velocity of particle draw $\overline{pq}$ and $\overline{pr}$ which represents v and v' respectively. The closing side $\overline{qr}$ of the triangle represents the change in the velocity Δv in time Δt.

Resolve Δv into two components Δv_t and Δv_n, along the tangent and normal to the path at p. Let the axis in these direction be denoted by t (tangential) and n (normal).

$$\Delta v = \overline{qr} = \overline{qs} + \overline{sr}$$

$$\Delta v = \Delta v_n + \Delta v_t$$

For acceleration, we have

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta v_t}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta v_n}{\Delta t}$$

$$a = \frac{dv_t}{dt} + \frac{dv_n}{dt}$$

$$a = a_t + a_n$$

Magnitude of acceleration $a = \sqrt{a_t^2 + a_n^2}$

Direction $\tan\theta = \dfrac{a_n}{a_t}$

When Δt is very small i.e. it tends to zero, the point p' coincides with the point p and direction of a_t and a_n coincides with the direction of tangent and normal to the path p.

Component of Tangential Acceleration (a_t)

It can be noted that $\overline{sr}$ represents the change in the magnitude of the velocity v.

$$\therefore\ a_t = \lim_{\Delta t \to 0}\left(\frac{v' - v}{\Delta t}\right) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

$$a_t = \frac{dv}{dt}$$

Thus, the component of tangential acceleration (a_t) *is equal to the rate of change of the speed of the particle*.

Component of Normal Acceleration (a_n)

It can be noted that $\overline{qs}$ represents rate of change of direction of the velocity

$$qs \approx v\,\Delta\theta \text{ for a small change in the angle } \theta$$

$$\therefore \Delta v_n \approx v \cdot \Delta\theta$$

$$\therefore a_n = \lim_{\Delta t \to 0} \left(\frac{v \cdot \Delta\theta}{\Delta t}\right)$$

If ρ is the radius of curvature of the curve then we have

$$\Delta s = \rho\, \Delta d\theta$$

$$\Delta\theta = \frac{\Delta s}{\rho}, \ \Delta s \text{ being the length of the arc } pp'$$

$$\therefore a_n = \lim_{\Delta t \to 0} \left(\frac{v \cdot \Delta\theta}{\Delta t}\right) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \cdot \frac{v}{\rho}$$

$$\therefore a_n = \frac{v}{\rho} \cdot \frac{ds}{dt} \qquad \text{But } \frac{ds}{dt} = v$$

$$\therefore a_n = \frac{v^2}{\rho}$$

Component of normal acceleration is always directed towards the centre of curvature of the path. It is also called as the *centripetal acceleration* (a_n).

The net acceleration of particle in vector form can be expressed as

$$a = a_t + a_n$$

where $a_t = \dfrac{dv}{dt}$ is responsible for changing the magnitude of speed and $a_n = \dfrac{v^2}{\rho}$ is responsible for changing the direction.

Magnitude of net acceleration $a = \sqrt{a_t^2 + a_n^2}$

and the direction $\tan\theta = \dfrac{a_n}{a_t}$

If particle is moving along curved path with uniform speed then component of tangential acceleration

$$a_t = \frac{dv}{dt} = 0$$

$$\therefore \text{ Net acceleration } a = a_n = \frac{v^2}{\rho}$$

If a_t is zero and a_n is constant then particle is moving along circular path with uniform speed.

If a_t is constant i.e. speed changes at uniform rate then to calculate distance covered by the particle along curved path or its speed at any instant or time interval of motion or a_t itself using the equation of rectilinear motion with constant acceleration.

$$v = u + a_t t$$

$$s = ut + \ a_t t^2$$

$$v^2 = u^2 + 2a_t s$$

here s is the distance covered along curved path.

u is initial speed and v is final speed and

a_t is the component of acceleration along tangential direction.

If equation of curve is given by $y = f(x)$ then at point $p(x, y)$ radius of curvature is calculated by following relation.

$$\rho = \left| \frac{\left[1 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\frac{d^2y}{dt^2}} \right|$$

If data is given in rectangular components form then radius of curvature is calculated by following relation.

$$\rho = \left| \frac{(v_x^2 + v_y^2)^{3/2}}{v_x a_y - v_y a_x} \right|$$

If particle is moving in space curve then radius of curvature is calculated by following relation.

$$|\bar{v} \times \bar{a}| = \frac{v^3}{\rho}$$

Components of velocity along x and y axis is given by

$$v_x = v \cos \theta \ , \ v_y = v \sin \theta$$

$$\tan \theta = \frac{dy}{dx}$$

and component of acceleration along x and y axis are given by

$$a_x = a \cos (\theta + \alpha)$$

$$a_y = a \sin (\theta + \alpha)$$

$$\tan \alpha = \frac{a_n}{a_t}$$

Relationship between rectangular components and tangential and normal components of acceleration.

$$a_x = a_n \sin \theta + a_t \cos \theta$$

$$a_y = a_n \cos \theta + a_t \sin \theta$$

$$a_n = a_x \sin \theta + a_y \cos \theta$$

$$a_t = a_x \cos \theta - a_y \sin \theta$$

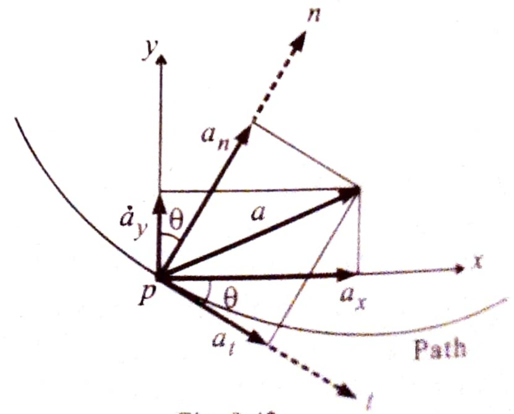

Fig. 9.42

9.9 Solved Problems Based on Curvilinear Motion

Problem 52

A bomb thrown from a plane flying at a height of 400 m moves along the path $\bar{r} = 50t\,\bar{i} + 4t^2\,\bar{j}$ where t is in second and distances are measured in m. The origin is taken as the point where the bomb is released and the y-axis is taken as pointing downwards. Find **(i)** the path of the bomb. **(ii)** the time taken to reach the ground. **(iii)** the horizontal distance traversed by the bomb.

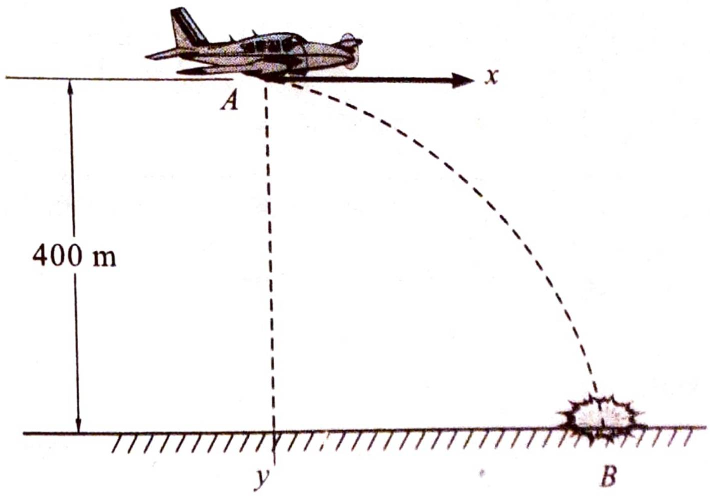

Fig. 9.43

Solution

Given : $\bar{r} = 50t\,\bar{i} + 4t^2\,\bar{j}$ $\quad x = 50t\ ; \ y = 4t^2$

$$\because\ y = 400 \text{ m} \qquad \therefore\ 400 = 4 \times t^2$$

$$\therefore\ t = 10 \text{ sec}$$

$$\because\ x = 50 \times t \qquad \therefore\ x = 500 \text{ m}$$

$$\because\ x = 50 \times t \qquad \therefore\ t = \frac{x}{50}$$

$$y = 4t^2 = 4 \times \left(\frac{x}{50}\right)^2$$

$$y = \frac{x^2}{625}$$ is the equation of path. ***Ans.***

Problem 53

The position vector of a particle is given by $\bar{r} = 2t^2\,\bar{i} + \left(\frac{4}{t^2}\right)\bar{j}$ m where t is in seconds.

Determine when $t = 1$ sec. **(i)** the magnitudes of normal and tangential components of acceleration of the particle and **(ii)** the radius of curvature of the path.

Solution : Method I

$$\bar{r} = 2t^2\,\bar{i} + \left(\frac{4}{t^2}\right)\bar{j} \text{ m}$$

$$\bar{v} = \frac{dr}{dt} = 4t\,\bar{i} - \frac{8}{t^3}\,\bar{j}$$

$$\therefore\ \bar{a} = \frac{dv}{dt} = 4\,\bar{i} + \frac{24}{t^4}\,\bar{j}$$

At $t = 1$ sec

$$\bar{v} = 4\,\bar{i} - 8\,\bar{j} \qquad \bar{e}_t = \frac{\bar{v}}{v} = \frac{4\,\bar{i} - 8\,\bar{j}}{\sqrt{(4)^2 + (-8)^2}}$$

$$\bar{a} = 4\,\bar{i} + 24\,\bar{j}$$

$$a = \sqrt{4^2 + 24^2} = 24.33 \text{ m/s}^2$$

$$a_t = \bar{a} \cdot \bar{e}_t = (4\,\bar{i} + 24\,\bar{j}) \cdot \left(\frac{4\,i - 8\,j}{\sqrt{(4)^2 + (-8)^2}}\right)$$

$$\therefore\ a_t = -19.67 \text{ m/s}^2 \quad \textbf{\textit{Ans.}}$$

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{(24.33)^2 - (-19.67)^2}$$

$$\therefore\ a_n = 14.319 \text{ m/s}^2 \quad \textbf{\textit{Ans.}}$$

But, $a_n = \dfrac{v^2}{\rho}$

$$\therefore\ \rho = \frac{v^2}{a_n} = \frac{80}{14.319} \qquad \left(\because\ v = \sqrt{4^2 + 8^2} = \sqrt{80}\right)$$

$$\rho = 5.586 \text{ m} \quad \textbf{\textit{Ans.}}$$

Method II

$$\bar{r} = 2t^2\,\bar{i} + \frac{4}{t^2}\,\bar{j}$$

Differentiating w.r.t. 't'

$$\bar{v} = 4t\,\bar{i} - \frac{8}{t^3}\,\bar{j}$$

Differentiating w.r.t. 't'

$$\therefore\ \bar{a} = 4\,\bar{i} + \frac{24}{t^4}\,\bar{j}$$

At t = 1 sec

$$\bar{v} = 4\,\bar{i} - 8\,\bar{j} \qquad \therefore\ \text{Magnitude } v = \sqrt{4^2 + 8^2} = 8.944 \text{ m/s}$$

$$\because\ \bar{v} = v_x\,\bar{i} + v_y\,\bar{j} \qquad \therefore\ v_x = 4\ ;\ v_y = -8$$

$$\bar{a} = 4\,\bar{i} + 24\,\bar{j}$$

$$\because\ \bar{a} = a_x\,\bar{i} + a_y\,\bar{j} \qquad \therefore\ \text{Magnitude } a = \sqrt{4^2 + 24^2} = 24.33 \text{ m/s}^2$$

$$\therefore\ a_x = 4\ ;\ a_y = 24$$

$$\rho = \left|\frac{(v_x^{\,2} + v_y^{\,2})^{3/2}}{v_x\,a_y - v_y\,a_x}\right| = \left|\frac{[4^2 + (-8)^2]^{3/2}}{4 \times 24 - (-8)(4)}\right| = 5.59 \text{ m}$$

$$a_n = \frac{v^2}{\rho} = \frac{8.444^2}{5.59}$$

$$\therefore\ a_n = 14.31 \text{ m/s}^2 \quad \textbf{\textit{Ans.}}$$

$$a^2 = a_n^{\,2} + a_t^{\,2}$$

$$\therefore\ a_t = \sqrt{a^2 - a_n^{\,2}} = \sqrt{(24.33)^2 - (14.31)^2}$$

$$\therefore\ a_t = 19.66 \text{ m/s}^2 \quad \textbf{\textit{Ans.}}$$

Problem 54

The Jet plane travels along a parabolic path. When it is at point A it has a speed of 200 m/s which is increasing at the rate of 0.8 m/s^2. Determine the magnitude and direction of acceleration of the plane when it is at A.

Solution

Given : $v = 200$ m/sec, $a_t = \dfrac{dv}{dt} = 0.8 \text{ m/s}^2$

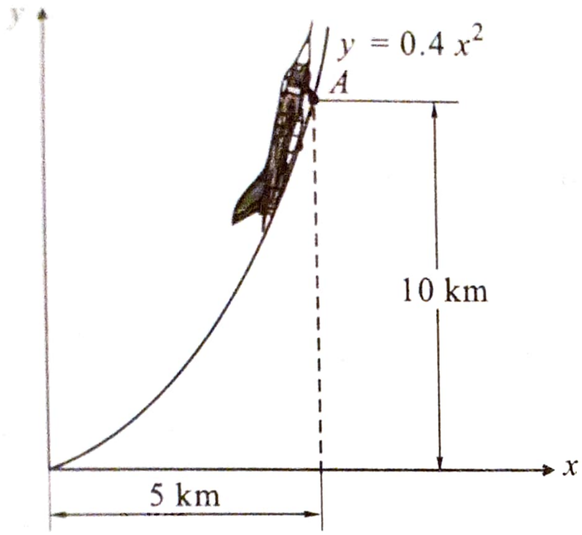

Fig. 9.44(a)

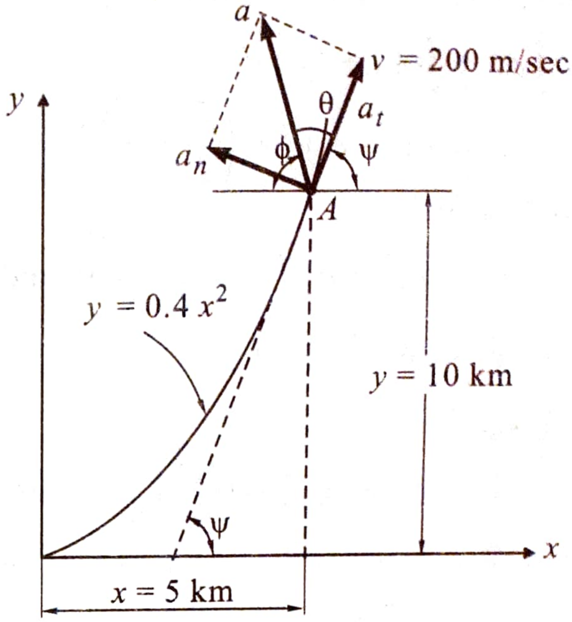

Fig. 9.44(b)

$$a_n = \frac{v^2}{\rho}$$

$$y = 0.4x^2$$

$$\frac{dy}{dx} = 0.8x \qquad \therefore \left(\frac{dy}{dx}\right)_{x=5} = 0.8 \times 5 = 4$$

$$\frac{d^2y}{dx^2} = 0.8 \qquad \therefore \left(\frac{d^2y}{dx^2}\right)_{x=5} = 0.8$$

$$\rho = \left| \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \right| = \left| \frac{\left[1 + (4)^2\right]^{3/2}}{0.8} \right|$$

$$\rho = 87.62 \text{ km}$$

$$a_n = \frac{v^2}{\rho} = \frac{(200)^2}{87.62}$$

$$a_n = 456.52 \text{ km/s}^2 = 0.4562 \text{ m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(0.8)^2 + (0.4562)^2}$$

$$a = 0.921 \text{ m/s}^2$$

$$\therefore a = 0.921 \text{ m/s}^2 \; (\phi) \quad \textbf{Ans.}$$

$$\tan \psi = \left(\frac{dy}{dx}\right)_{x=5} = 4$$

$$\therefore \psi = 75.96^\circ$$

$$\tan \theta = \frac{a_n}{a_t} = \frac{0.4562}{0.8}$$

$$\therefore \theta = 29.69^\circ$$

$$\phi = 180 - (\theta + \psi)$$

$$\phi = 180 - (29.69 + 75.96)$$

$$\therefore \phi = 74.35^\circ \quad \textbf{Ans.}$$

Problem 55

A rocket follows the path such that its acceleration is given by $\bar{a} = (4\,\bar{i} + t\,\bar{j})$ m/s^2. At $t = 0$ it starts from rest. At $t = 10$ sec; determine **(i)** speed of rocket **(ii)** radius of curvature of the path.

Solution

Given : $\bar{a} = 4\,\bar{i} + t\,\bar{j}$; $\dfrac{d\bar{v}}{dt} = 4\,\bar{i} + t\,\bar{j}$

Integrating both the sides, we have

$$\int d\bar{v} = \int (4\,\bar{i} + t\,\bar{j})\,dt$$

$$\bar{v} = (4t + c_1)\,\bar{i} + \left(\frac{t^2}{2} + c_2\right)\bar{j}$$

At $t = 0,\ \bar{v} = 0$

$$0 = c_1\,\bar{i} + c_2\,\bar{j} \quad \therefore\ c_1 = 0 \text{ and } c_2 = 0$$

$$\therefore\ \bar{v} = (4t)\,\bar{i} + \left(\frac{t^2}{2}\right)\bar{j}$$

At $t = 10$ sec

$$\bar{v} = (4 \times 10)\,\bar{i} + \left(\frac{10^2}{2}\right)\bar{j} = 40\,\bar{i} + 50\,\bar{j}$$

$$\because\ \bar{v} = v_x\bar{i} + v_y\bar{j} \quad \therefore\ v_x = 40,\ v_y = 50$$

$$\therefore\ v = \sqrt{40^2 + 50^2} = 64.03 \text{ m/sec} \quad \therefore\ v = 64.03 \text{ m/sec} \quad \textbf{Ans.}$$

$$\bar{a} = 4\,\bar{i} + 10\,\bar{j} \qquad (\text{at } t = 10 \text{ sec})$$

$$\because\ \bar{a} = a_x\bar{i} + a_y\,\bar{j} \quad \therefore\ a_x = 4,\ a_y = 10$$

$$\rho = \left|\frac{\left[v_x^2 + v_y^2\right]^{3/2}}{v_x a_y - v_y a_x}\right| = \left|\frac{\left[(40)^2 + (50)^2\right]^{3/2}}{(40(10) - (50)(4)}\right| \quad \therefore\ \rho = 1312.6 \text{ m} \quad \textbf{Ans.}$$

Problem 56

A point moves along the path $y = \dfrac{1}{3}x^2$ with a constant speed of 8 m/sec. What are the x and y components of the velocity when $x = 3$? What is the acceleration of the point when $x = 3$?

Solution : Given : $v = 8$ m/s is constant;

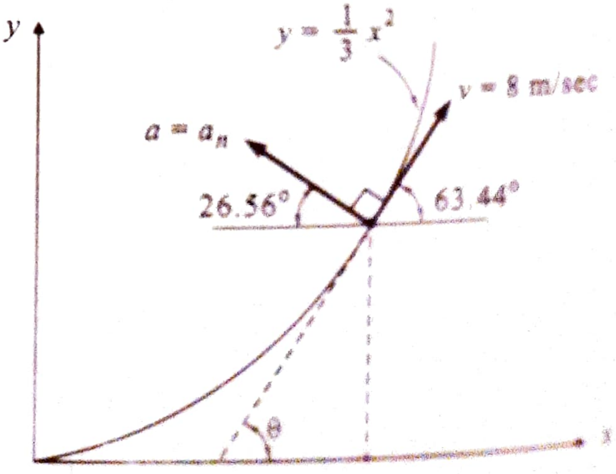

Fig. 9.45

$$a_t = 0 \qquad a_t = \frac{dv}{dt} = 0$$

$$\therefore\ a_n = a \qquad [\because a_t = 0]$$

$$\therefore\ a_n = \frac{v^2}{\rho}$$

$$y = \frac{1}{3}x^2$$

$$\frac{dy}{dx} = \frac{2}{3}x \qquad \left(\frac{dy}{dx}\right)_{x=3} = \frac{2}{3} \times 3 = 2$$

$$\frac{d^2y}{dx^2} = \frac{2}{3} \qquad \left(\frac{d^2y}{dx^2}\right)_{x=3} = \frac{2}{3} \quad \textbf{Ans.}$$

$$\therefore \rho = \left| \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \right| = \left| \frac{\left[1 + (2)^2\right]^{3/2}}{\frac{2}{3}} \right|$$

$$\rho = 16.77 \text{ m}$$

$$a_n = \frac{v^2}{\rho} = \frac{(8)^2}{16.77} = 3.82 \text{ m/s}^2$$

$$\tan\theta = \left(\frac{dy}{dx}\right)_{x=3} = 2$$

$$\theta = 63.44°$$

$$v_x = v\cos\theta = 8\cos 63.44 = 3.58 \text{ m/sec}$$ **Ans.**

$$v_y = v\sin\theta = 8\sin 63.44 = 7.15 \text{ m/sec}$$ **Ans.**

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{0 + (3.82)^2}$$

$$a = 3.82 \text{ m/s}^2 \left(26.56° \right)$$ **Ans.**

Problem 57

The movement of a particle is governed by $\bar{r} = 2t^2\,\bar{i} + 10t\,\bar{j} + t^3\bar{k}$, where r is in metre units and t is in seconds. Determine the normal and tangential components of acceleration and the radius of curvature of path traced at time $t = 2$ seconds.

Solution

Given : $\bar{r} = 2t^2\,\bar{i} + 10t\,\bar{j} + t^3\bar{k}$

$$\bar{v} = \frac{d\bar{r}}{dt} = 4t\,\bar{i} + 10\,\bar{j} + 3t^2\,\bar{k} \qquad \bar{a} = \frac{d\bar{v}}{dt} = 4\,\bar{i} + 6t\,\bar{k}$$

At $t = 2$ sec

$$\bar{v} = 8\,\bar{i} + 10\,\bar{j} + 12\,\bar{k} \qquad \bar{a} = 4\,\bar{i} + 12\,\bar{k}$$

We know unit vector along tangent direction is given by

$$\hat{e}_t = \frac{\bar{v}}{v} = \frac{8\,\bar{i} + 10\,\bar{j} + 12\,\bar{k}}{\sqrt{8^2 + 10^2 + 12^2}} = \frac{8\,\bar{i} + 10\,\bar{j} + 12\,\bar{k}}{\sqrt{308}}$$

$$a_t = \bar{a} \cdot \hat{e}_t \text{ (Dot product)}$$

$$a_t = (4\,\bar{i} + 12\,\bar{k}) \cdot \left(\frac{8\,\bar{i} + 10\,\bar{j} + 12\,\bar{k}}{\sqrt{308}}\right) = \frac{32 + 144}{\sqrt{308}} = \frac{176}{\sqrt{308}}$$

$$a_t = 10.02 \text{ m/sec}^2$$ **Ans.**

$$a = \sqrt{4^2 + 12^2} \qquad (\because\ \bar{a} = 4\,\bar{i} + 12\,\bar{k})$$

$$a = 12.649 \text{ m/s}^2$$

$$a^2 = a_t^2 + a_n^2$$

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{(12.649)^2 - (10.02)^2} \quad \therefore \; a_n = 7.72 \text{ m/s}^2 \quad \textbf{Ans.}$$

Now, $a_n = \dfrac{v^2}{\rho}$

$$\rho = \frac{v^2}{a_n} = \frac{\sqrt{8^2 + 10^2 + 12^2}}{7.72} \quad \therefore \; \rho = 39.89 \text{ m} \quad \textbf{Ans.}$$

Problem 58

A particle moves in the x-y plane with velocity components $v_x = 8t - 2$ and $v_y = 2$. If it passes through the point $(x, y) = (14, 4)$ at $t = 2$ seconds, determine the equation of the path traced by the particle. Find also the resultant acceleration at $t = 2$ seconds.

Solution

Given : $v_x = 8t - 2$; $\quad v_y = 2$

$$\frac{dx}{dt} = 8t - 2 \; ; \quad \frac{dy}{dt} = 2$$

Integrating we get

$\therefore \; x = 4t^2 - 2t + c_1$

At $t = 2$ sec, $x = 14$ m

$14 = 4(2)^2 - 2(2) + c_1$

$c_1 = 2$

$\therefore \; x = 4t^2 - 2t + 2$

$y = 2t + c_2$

At $t = 2$ sec, $y = 4$ m

$4 = 2(2) + c_2$

$c_2 = 0$

$y = 2t$

$$x = (2t)^2 - 2t + 2$$

$$x = y^2 - y + 2 \qquad (\because y = 2t)$$

$x = y^2 - y + 2$ is the equation of path. **Ans.**

(Any equation of path does not have time)

$$\therefore \; \bar{v} = (8t - 2)\,\bar{i} + 2\,\bar{j}$$

$$\bar{a} = \frac{d\bar{v}}{dt} = 8\,\bar{i} \text{ m/s}^2$$

$$\therefore \; \bar{a} = 8\,\bar{i} \text{ m/s}^2 \quad \textbf{Ans.}$$

Problem 59

A particle moves in the x-y plane with acceleration components $a_x = -3$ m/s^2 and $a_y = -16$ m/s^2. If its initial velocity is $u = 50$ m/s directed at 30° to the x-axis. Compute the radius of curvature of the path at $t = 2$ sec.

Solution

Given : $u = 50$ m/s ; $a_x = -3$ m/s^2 ; $a_y = -16$ m/s^2

$u_x = 50 \cos 30 = 43.3$ m/s

$u_y = 50 \sin 30 = 25$ m/s

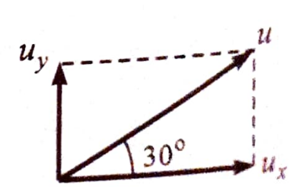

Fig. 9.46

At $t = 2$ sec

$$v = u + at$$

$$v_x = u_x + a_x t \; ; \qquad v_y = u_y + a_y t$$

$$v_x = 43.3 + (-3)(2) \; ; \qquad v_y = 25 + (-16)(2)$$

$$v_x = 37.3 \text{ m/s} \qquad v_y = -7 \text{ m/s.}$$

$$\rho = \left| \frac{(v_x^2 + v_y^2)^{3/2}}{v_x a_y - v_y a_x} \right|$$

$$\therefore \rho = \left| \frac{[(37.3)^2 + (-7)^2]^{3/2}}{37.3 \times (-16) - (-7)(-3)} \right|$$

$$\therefore \rho = 88.48 \text{ m} \quad \textbf{Ans.}$$

Problem 60

A particle moves along a circular track with constant tangential acceleration of 0.28 m/s^2. It starts at rest from a point A shown in the figure. Find the velocity and acceleration components of the particle along x and y directions when it reaches point B.

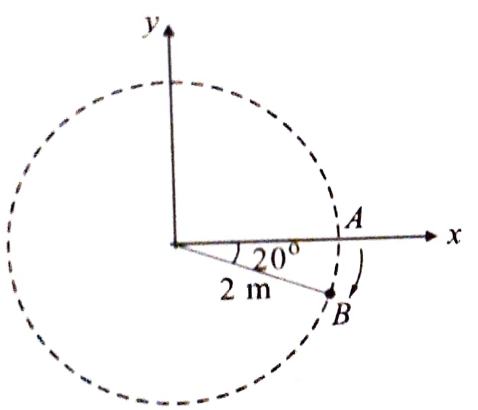

Fig. 9.47(a)

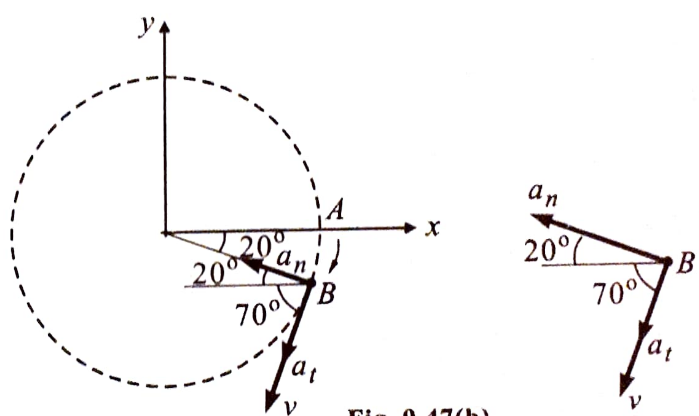

Fig. 9.47(b)

Solution

Given : $a_t = 0.28 \text{ m/s}^2 \; ; \; u = 0, \quad v = ?$

$$s = r\theta = 2 \times \left(20 \times \frac{\pi}{180}\right) \qquad \left[\because \; 1^\circ = \left(\frac{\pi}{180}\right)^c \quad 20^\circ = \left(\frac{\pi}{180} \times 20\right)^c\right]$$

$$s = 0.698 \text{ m}$$

$$\therefore v^2 = u^2 + 2as$$

$$v^2 = 0 + 2(0.28)(0.698)$$

$$v = 0.625 \text{ m/sec} \quad \textbf{Ans.}$$

$$\therefore v_x = -v \cos 70 = -0.625 \times \cos 70 = -0.213 \text{ m/sec}$$

$$v_y = -v \sin 70 = -0.625 \times \sin 70 = -0.587 \text{ m/sec}$$

$$\therefore a_t = 0.28 \text{ m/s}^2$$

$$a_n = \frac{v^2}{\rho} = \frac{(0.625)^2}{2} = 0.1953 \text{ m/s}^2$$

$$\therefore\ a_x = -a_n \cos 20 - a_t \cos 70 = -0.1953 \cos 20 - 0.28 \cos 70$$

$$a_x = -0.2793 \text{ m/s}^2 \quad \textbf{Ans.}$$

$$a_y = +a_n \sin 20 - a_t \sin 70 = 0.1953 \sin 20 - 0.28 \sin 70$$

$$a_y = -0.1963 \text{ m/s}^2 \quad \textbf{Ans.}$$

Problem 61

A particle moves along a hyperbolic path $\frac{x^2}{16} - y^2 = 28$. If the x-component of velocity is $v_x = 4$ m/sec and remains constant, determine the magnitudes of particles velocity and acceleration when it is at point (32, 6) m.

Solution

From given data $v_x = 4$ m/sec is constant $\quad \therefore\ a_x = 0$ or $\frac{d^2x}{dt^2} = 0$

$$\frac{x^2}{16} - y^2 = 28$$

Differentiating with respect to t, we get

$$\frac{2x}{16}\frac{dx}{dt} - 2y \cdot \frac{dy}{dt} = 0$$

$$\frac{x}{8}\frac{dx}{dt} - 2y \cdot \frac{dy}{dt} = 0 \qquad \text{..... (I)}$$

$$\frac{32}{8} v_x - 2 \times 6 \cdot v_y = 0$$

$$4 \times 4 - 12\, v_y = 0$$

$$v_y = 1.33 \text{ m/sec} \quad \textbf{Ans.}$$

$$\therefore\ v = \sqrt{{v_x}^2 + {v_y}^2} = \sqrt{4^2 + 1.33^2} = 4.22 \text{ m/sec} \quad \textbf{Ans.}$$

We have equation (I),

$$\frac{x}{8}\frac{dx}{dt} - 2y \cdot \frac{dy}{dt} = 0$$

Again differentiating with respect to t, we get,

$$\frac{1}{8} \cdot \left[x \cdot \frac{d^2x}{dt^2} + \frac{dx}{dt} \cdot \frac{dx}{dt}\right] - 2\left[y \cdot \frac{d^2y}{dt^2} + \frac{dy}{dt} \cdot \frac{dy}{dt}\right] = 0$$

$$\frac{1}{8} \cdot \left[32 \cdot \frac{d^2x}{dt^2} + (4)^2\right] - 2\left[6 \cdot \frac{d^2y}{dt^2} + (1.33)^2\right] = 0$$

$$\therefore\ \frac{1}{8} \cdot [32 \times 0 + 16] - 2\,[6 \cdot a_y + (1.33)^2] = 0$$

$$\therefore\ a_y = -0.129 \text{ m/s}^2$$

$$\therefore\ a = \sqrt{{a_x}^2 + {a_y}^2} = \sqrt{0 + (-0.129)^2}$$

$$a = 0.129 \text{ m/sec}^2 \quad \textbf{Ans.}$$

$$\left[\begin{array}{l} \because\ P(32, 6) \text{ m} \\ \therefore\ x = 32,\ y = 6 \\ \frac{dy}{dx} = v_x = 4 \text{ m/s} \\ \frac{dy}{dt} = v_y = 1.33 \text{ m/sec} \\ \frac{d^2x}{dt^2} = 0 \end{array}\right]$$

Problem 62

Motion of a particle is defined by a relation $v = 4t^2 - 3t - 1$ where v is in m/s and t is in seconds. If the displacement $x = -4$ m at $t = 0$, determine the displacement and acceleration when $t = 3$ sec. Find also the time when the velocity becomes zero and distance travelled by the particle during that time.

Solution

Given : $v = 4t^2 - 3t - 1$ ..... (I)

Differentiating with respect to t, we have

$$a = 8t - 3 \quad \text{..... (II)}$$

Integrating equation (I) with respect to t, we have

$$x = \frac{4t^3}{3} - \frac{3t^2}{2} - t + c$$

At $t = 0, x = -4$

$$\therefore \; c = -4$$

$$\therefore \; x = \frac{4t^3}{3} - \frac{3t^2}{2} - t - 4 \quad \text{..... (III)}$$

At $t = 3$ sec. from equation (II) and (III)

$a = 21 \text{ m/s}^2$ and $x = 15.5$ m ***Ans.***

When the velocity will be zero, i.e. $v = 0$

From equation (I)

$$0 = 4t^2 - 3t - 1$$

$$\therefore \; t = 1 \text{ sec}$$

Displacement from equation (III)

$$x = -5.167 \text{ m}$$

$\therefore$ Distance traveled $= 5.167 - 4$

$d = 1.167$ m ***Ans.***

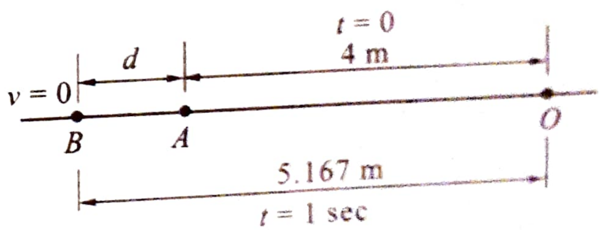

Fig. 9.48

Problem 63

A car travels along a vertical curve on a road, the equation of the curve being $x^2 = 200y$ (x-horizontal and y-vertical distances in m). The speed of the car is constant and equal to 72 km/hr.

(i) Find its acceleration when the car is at the deepest point on the curve.

(ii) What is the radius of curvature of the curve at this point ?

Solution

$$x^2 = 200y$$

$$\therefore \; y = \frac{x^2}{200}$$

$$v = 72 \text{ km/hr} = 72 \times \frac{5}{18} \text{ m/sec}$$

$$v = 20 \text{ m/sec (constant)}$$

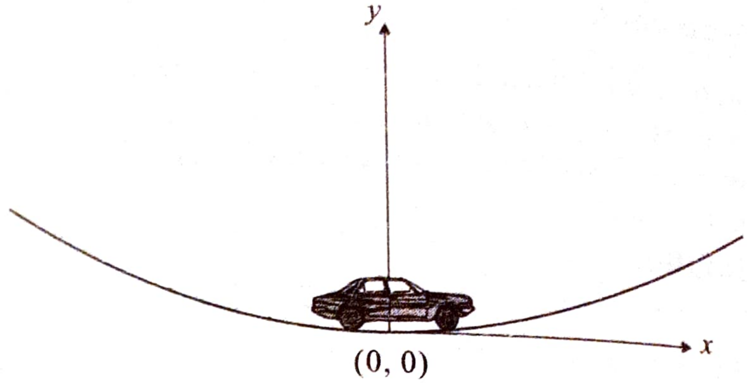

Fig. 9.49

Now, $a_t = 0$

$$y = \frac{x^2}{200}$$

$$\frac{dy}{dx} = \frac{1}{200}(2x)$$

$$\left(\frac{dy}{dx}\right)_{at\ x=0} = 0$$

$$\left(\frac{d^2y}{dx^2}\right)_{at\ x=0} = \frac{1}{100}$$

$$\therefore \rho = \left|\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}\right| = \left|\frac{\left[1+(0)^2\right]^{3/2}}{\frac{1}{100}}\right|$$

$\rho = 100$ m **Ans.**

$$a_n = \frac{v^2}{\rho} = \frac{(20)^2}{100}$$

$\therefore a_n = 4$ m/s^2 **Ans.**

Problem 64

A point moves along the path $y = \frac{x^2}{3}$ with a constant speed of 8 m/s. What are the x and y component of the velocity when $x = 3$? What is the acceleration of the point when $x = 3$?

Solution

Given : $y = \frac{x^2}{3}$

At $x = 3$

$$\frac{dy}{dx} = \frac{2x}{3} = 2 \ ; \quad \frac{d^2y}{dx} = \frac{2}{3}. \quad \text{Now, } \tan\theta = \frac{dy}{dx} \quad \therefore \theta = 63.43$$

(i) θ is the inclination of velocity with horizontal when $x = 3$

$\therefore$ x component of velocity and y component of velocity

$v_x = v\cos\theta = 8\cos 63.43$

$v_x = 3.58$ m/s **Ans.**

$v_y = v\sin\theta = 8\sin 63.43$

$v_y = 7.16$ m/s **Ans.**

(ii) Velocity is constant

$$\therefore a_t = \frac{dy}{dx} = 0$$

$$\rho = \left|\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}\right| = \left|\frac{\left[1+(2)^2\right]^{3/2}}{\frac{2}{3}}\right| \quad \therefore \rho = 16.77 \text{ m}$$ **Ans.**

$$a_n = \frac{v^2}{\rho} = \frac{8^2}{16.77} \quad \therefore a_n = 3.82 \text{ m/s}$$

$$a = \sqrt{a_n^2 + a_t^2} = \sqrt{(3.82)^2 + (0)^2}$$

$$a = 3.82 \text{ m/s}^2 \quad \textbf{Ans.}$$

Problem 65

A particle travels along the path defined by the parabola $y = 0.5x^2$. If the x component of velocity is $v_x = 5t$ m/sec, determine the particles distance from the origin 0 and the magnitude of acceleration when $t = 1$ sec, when $t = 0$, $x = 0$, $v = 0$.

Solution

(i)

$$v_x = 5t$$

$$\therefore \frac{dx}{dt} = 5t$$

$$\int dx = \int 5t$$

$$x = 5\frac{t^2}{2} + c$$

When $t = 0$, $x = 0$

$$0 = 0 + c$$

$$c = 0$$

$$\therefore x = \frac{5}{2}t^2$$

When $t = 1$ sec

$$x = \frac{5}{2}(1)^2 = \frac{5}{2}$$

$$x = 2.5 \text{ m} \quad \textbf{Ans.}$$

$$\therefore y = 0.5x^2 = 0.5(2.5)^2$$

$$y = 3.125 \text{ m} \quad \textbf{Ans.}$$

Particles distance from origin = (2.5, 3.125) m

(ii)

$$y = 0.5x^2$$

$$\frac{dy}{dt} = 0.5(2x)\frac{dx}{dt}$$

$$\frac{dy}{dt} = x \times \frac{dx}{dt}$$

$$\frac{dy}{dt} = x(5t) = \left(\frac{5}{2}t^2\right)(5t)$$

$$v_y = \frac{dy}{dt} = \frac{25}{2}t^3$$

$$a_y = \frac{d^2y}{dt^2} = \frac{25}{2} \times (3t^2)$$

$$a_y = \frac{75}{2}t^2$$

When $t = 1$ sec

$$a_y = \frac{75}{2}(1)^2$$

$$a_y = 37.5 \text{ m/sec}^2 \quad \textbf{Ans.}$$

$$v_x = \frac{dx}{dt} = 5t$$

$$a_x = \frac{d^2x}{dt^2} = 5$$

$$a_x = 5 \text{ m/sec}^2 \quad \textbf{Ans.}$$

$$\therefore a = \sqrt{a_x^2 + a_y^2} = \sqrt{(5)^2 + (37.5)^2}$$

$$a = 37.83 \text{ m/sec}^2 \quad \textbf{Ans.}$$

$$a_n = \frac{v^2}{\rho} = \frac{8^2}{16.77} \quad \therefore a_n = 3.82 \text{ m/s}$$

$$a = \sqrt{a_n^2 + a_t^2} = \sqrt{(3.82)^2 + (0)^2}$$

$$a = 3.82 \text{ m/s}^2 \quad \textbf{Ans.}$$

Problem 65

A particle travels along the path defined by the parabola $y = 0.5x^2$. If the x component of velocity is $v_x = 5t$ m/sec, determine the particles distance from the origin 0 and the magnitude of acceleration when $t = 1$ sec, when $t = 0$, $x = 0$, $v = 0$.

Solution

(i)

$$v_x = 5t$$

$$\therefore \frac{dx}{dt} = 5t$$

$$\int dx = \int 5t$$

$$x = 5\frac{t^2}{2} + c$$

When $t = 0$, $x = 0$

$$0 = 0 + c$$

$$c = 0$$

$$\therefore x = \frac{5}{2}t^2$$

When $t = 1$ sec

$$x = \frac{5}{2}(1)^2 = \frac{5}{2}$$

$$x = 2.5 \text{ m} \quad \textbf{Ans.}$$

$$\therefore y = 0.5\,x^2 = 0.5\,(2.5)^2$$

$$y = 3.125 \text{ m} \quad \textbf{Ans.}$$

Particles distance from origin = (2.5, 3.125) m

(ii)

$$y = 0.5x^2$$

$$\frac{dy}{dt} = 0.5\,(2x)\frac{dx}{dt}$$

$$\frac{dy}{dt} = x \times \frac{dx}{dt}$$

$$\frac{dy}{dt} = x\,(5t) = \left(\frac{5}{2}t^2\right)(5t)$$

$$v_y = \frac{dy}{dt} = \frac{25}{2}t^3$$

$$a_y = \frac{d^2y}{dt^2} = \frac{25}{2} \times (3t^2)$$

$$a_y = \frac{75}{2}t^2$$

When $t = 1$ sec

$$a_y = \frac{75}{2}(1)^2$$

$$a_y = 37.5 \text{ m/sec}^2 \quad \textbf{Ans.}$$

$$v_x = \frac{dx}{dt} = 5t$$

$$a_x = \frac{d^2x}{dt^2} = 5$$

$$a_x = 5 \text{ m/sec}^2 \quad \textbf{Ans.}$$

$$\therefore a = \sqrt{a_x^2 + a_y^2} = \sqrt{(5)^2 + (37.5)^2}$$

$$a = 37.83 \text{ m/sec}^2 \quad \textbf{Ans.}$$

Problem 66

A particle at the position (4, 6, 3) m at start, is accelerated at $\bar{a} = 4t\,\bar{i} - 10t^2\,\bar{j}$ m/s^2. Determine magnitude of the acceleration, velocity and the displacement after 2 seconds.

Solution

$$\bar{a} = 4t\,\bar{i} - 10t^2\bar{j} + 0\,\bar{k}$$

Integrating

$$\bar{v} = \left(\frac{4t^2}{2} + c_1\right)\bar{i} + \left(-\frac{10t^3}{3} + c_2\right)\bar{j} + (0 + c_3)\,\bar{k}$$

At $t = 0,\ v_x = 0,\ v_y = 0,\ v_z = 0$

$\therefore\ c_1 = 0,\ c_2 = 0,\ c_3 = 0$

$$\bar{v} = 2t^2\bar{i} + \left(\frac{-10}{3}t^3\right)\bar{j} + 0\,\bar{k}$$

Integrating

$$\bar{r} = \left(\frac{2t^3}{3} + c_4\right)\bar{i} + \left(\frac{-10t^4}{4\times 3} + c_5\right)\bar{j} + c_6\,\bar{k}$$

At $t = 0,\ x = 4,\ y = 6,\ z = 3$

$\therefore\ c_4 = 4,\ c_5 = 6,\ c_6 = 3$

$$\therefore\ \bar{r} = \left(\frac{2t^3}{3} + 4\right)\bar{i} + \left(\frac{-10t^4}{4\times 3} + 6\right)\bar{j} + 3\,\bar{k}$$

At $t = 2$ sec

$$\bar{a} = 8\,\bar{i} - 40\,\bar{j} + 0\,\bar{k}$$

$$\text{Magnitude } |\bar{a}| = \sqrt{8^2 + 40^2}$$

$$a = 40.79 \text{ m/s}^2 \qquad \textbf{Ans.}$$

$$\bar{v} = 8\,\bar{i} - \frac{80}{3}\,\bar{j} + 0\,\bar{k}$$

$$\text{Magnitude } |\bar{v}| = \sqrt{8^2 + \left(\frac{80}{3}\right)^2}$$

$$v = 27.84 \text{ m/s} \qquad \textbf{Ans.}$$

$$\bar{r} = \left(\frac{16}{3} + 4\right)\bar{i} + \left(\frac{-160}{12} + 6\right)\bar{j} + 3\,\bar{k}$$

$$\bar{r} = \left(\frac{28}{3}\right)\bar{i} - \left(\frac{88}{12}\right)\bar{j} + 3\,\bar{k}$$

$$\text{Magnitude } |\bar{r}| = \sqrt{\left(\frac{28}{3}\right)^2 + \left(\frac{-88}{12}\right)^2 + (3)^2}$$

$$r = 12.24 \text{ m} \qquad \textbf{Ans.}$$