4.2 Centre of Gravity

It is a point where the whole weight of the body is assumed to act.

We use the term ***centre of gravity*** usually denoted by 'G' for all three dimensional rigid bodies, e.g. sphere, table, vehicle, dam, human etc.

Centroid

It is a point where the whole area of a plane lamina (figure) is assumed to act.

In other words *centroid is the geometrical centre of the figure.* We use the term *centroid* for two *dimension figures*, i.e. areas. e.g. rectangle, triangle, circle, semicircle, sector etc.

4.2.1 Centre of Gravity of a Flat Plate

Consider the flat plate having a uniform thickness (t) and lying in xy-plane as shown in *figure 4.3*. The plate can be divided into small elements having weights W_1, W_2, W_3, ...W_n. The coordinates of elements are denoted by (x_1, y_1), (x_2, y_2), ... (x_n, y_n).

The resultant of the elementary forces is the total weight W of the plate which is acting through point G having coordinates $(\bar{x}, \bar{y})$ and can be given as

$$W = W_1 + W_2 + ... + W_n$$

Taking moment about y-axis and applying Varignon's theorem, we have

$$W(\bar{x}) = W_1 x_1 + W_2 x_2 + ... + W_n x_n$$

$$\bar{x} = \frac{W_1 x_1 + W_2 x_2 + ... + W_n x_n}{W_1 + W_2 + ... + W_n} \quad (4.1)$$

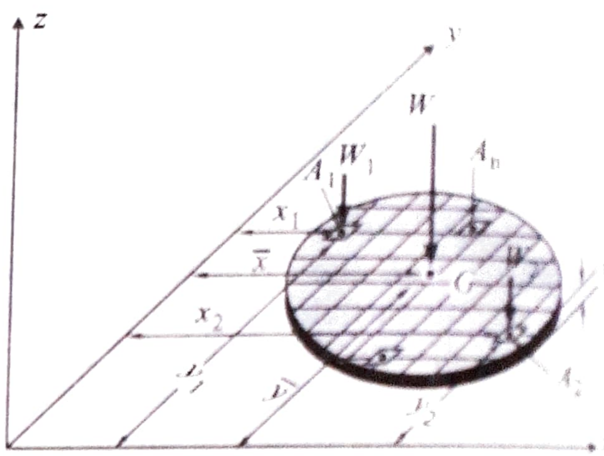


Fig. 4.3

Centroid of an Area

Let t be the uniform thickness, ρ be the mass density and A be the total surface area of the plate

We know, $W = m\,g = v\,\rho\,g = A\,t\,\rho\,g$. Similarly considering the small elements of the plate, we get $W_1 = A_1\,t\,\rho\,g$, $W_2 = A_2\,t\,\rho\,g$, ... , $W_n = A_n\,t\,\rho\,g$

Putting the above relation in *equation 4.1*, we get

$$\bar{x} = \frac{A_1 t \rho g x_1 + A_2 t \rho g x_2 + ... + A_n t \rho g x_n}{A_1 t \rho g + A_2 t \rho g + ... + A_n t \rho g}$$

$$\bar{x} = \frac{A_1 x_1 + A_2 x_2 + ... + A_n x_n}{A_1 + A_2 + ... + A_n} = \frac{\Sigma A_i x_i}{\Sigma A_i} \quad (4.2)$$

Similarly, taking moment about x-axis and applying Varignon's theorem, we get

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2 + ... + A_n y_n}{A_1 + A_2 + ... + A_n} = \frac{\Sigma A_i y_i}{\Sigma A_i} \quad (4.3)$$

Mathematically the above two expressions can also be represented as follows

$$\bar{x} = \frac{\int x\, dA}{\int dA} \; ; \quad \bar{y} = \frac{\int y\, dA}{\int dA} \quad (4.4)$$

Points to Remember

1. Centroid is the geometrical centre of a figure.
2. For a symmetric figure, centroid lies on the axis of symmetry.
3. If the figure is symmetric about x-axis then y coordinate of C.G. i.e. $\bar{y} = 0$.
4. If the figure is symmetric about y-axis then x coordinate of C.G. i.e. $\bar{x} = 0$.
5. If a figure has more than one axis of symmetry then the intersection of axis of symmetry is the centroid of the given figure.
6. Centroid may or may not lie on the given figure.
7. If axis of symmetry is inclined at 45° to horizontal line then $\bar{x} = \bar{y}$.

4.2.2 Centre of Gravity of a Thin Bent Bar / Wire

Consider a thin bent wire having a uniform cross section and lying in xy-plane as shown in *figure 4.4*. Thin wire can be divided into small elementary segments having weights W_1, W_2, W_3, ...W_n. The coordinates of elements are denoted by (x_1, y_1), (x_2, y_2), ... (x_n, y_n).

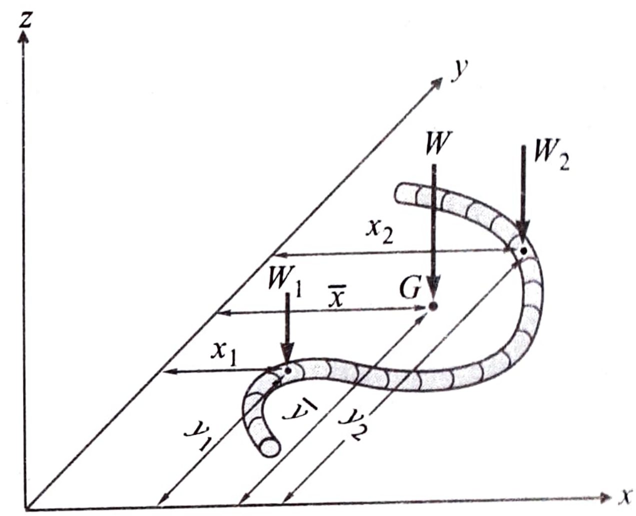


Fig. 4.4

The resultant of the elementary forces is the total weight W of the thin wire which is acting through point G having cordinates $(\bar{x}, \bar{y})$ and can be given as

$$W = W_1 + W_2 + ... + W_n$$

Taking moment about y-axis and applying Varignon's theorem, we have

$$W(\bar{x}) = W_1 x_1 + W_2 x_2 + ... + W_n x_n$$

$$\bar{x} = \frac{W_1 x_1 + W_2 x_2 + ... + W_n x_n}{W_1 + W_2 + ... + W_n} \qquad ...(4.5)$$

Let a be the cross section area, ρ be the mass density and L be the total length of the thin wire.

We know, $W = m\,g = v\,\rho\,g = L\,a\,\rho\,g$. Similarly considering the small elements of thin wire, we get $W_1 = l_1\,a\,\rho\,g$, $W_2 = l_2\,a\,\rho\,g$, ..., $W_n = l_n\,a\,\rho\,g$

Putting the above relation in *equation 4.5*, we get

$$\bar{x} = \frac{l_1 a \rho g x_1 + l_2 a \rho g x_2 + ... + l_n a \rho g x_n}{l_1 a \rho g + l_2 a \rho g + ... + l_n a \rho g}$$

$$\bar{x} = \frac{l_1 x_1 + l_2 x_2 + ... + l_n x_n}{l_1 + l_2 + ... + l_n} = \frac{\Sigma\, l_i x_i}{\Sigma\, l_i} \qquad ...(4.6)$$

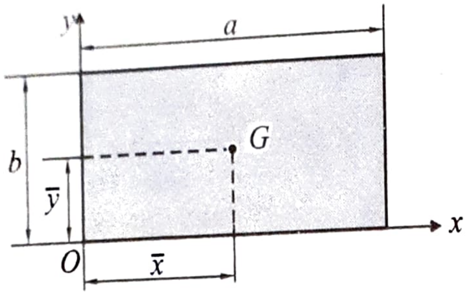
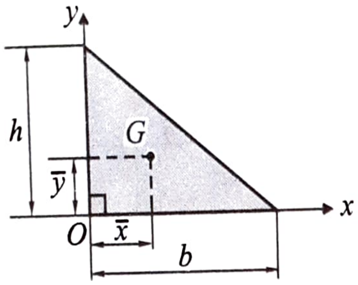
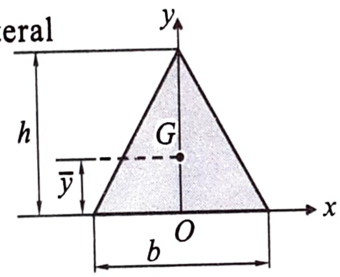
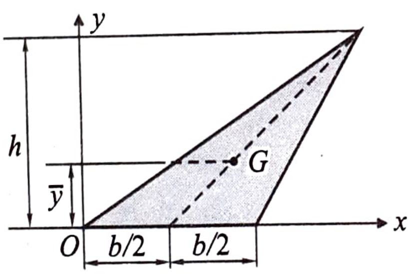
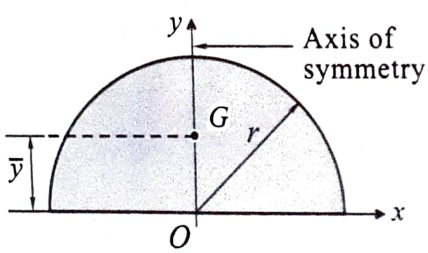
Similarly, taking moment about x-axis and applying Varignon's theorem, we get

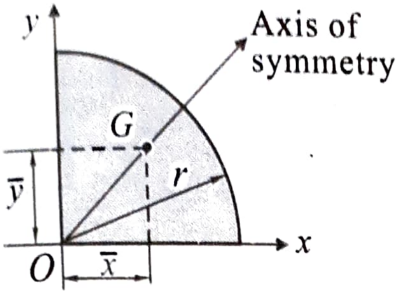
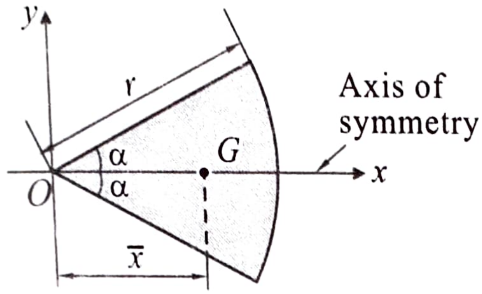
$$\bar{y} = \frac{l_1 y_1 + l_2 y_2 + ... + l_n y_n}{l_1 + l_2 + ... + l_n} = \frac{\Sigma\, l_i y_i}{\Sigma\, l_i} \qquad ...(4.7)$$

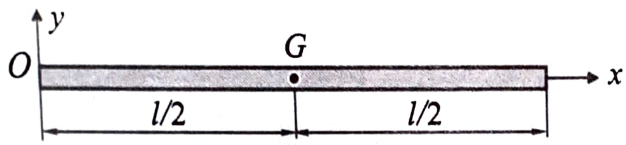
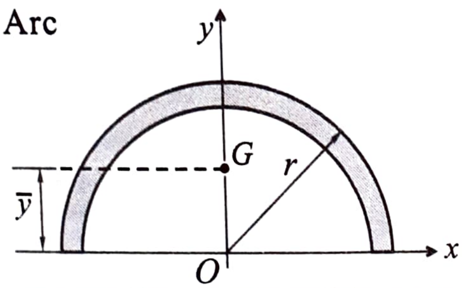
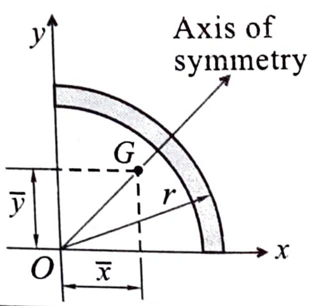
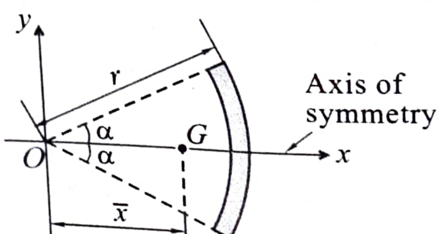
Mathematically the above two expressions can also be represented as follows

$$\bar{x} = \frac{\int x\,dl}{\int dl} \quad ; \quad \bar{y} = \frac{\int y\,dl}{\int dl} \qquad ...(4.8)$$

4.3 Centroid of Various Geometrical Shapes

Geometrical Shape		Area	$\overline{x}$	$\overline{y}$
1. Rectangle		ab	$\frac{a}{2}$	$\frac{b}{2}$
2. Right Angled Triangle		$\frac{bh}{2}$	$\frac{b}{3}$	$\frac{h}{3}$
3. Isosceles / Equilateral Triangle		$\frac{bh}{2}$	0	$\frac{h}{3}$
4. Triangle		$\frac{bh}{2}$	–	$\frac{h}{3}$
5. Semicircle		$\frac{\pi r^2}{2}$	0	$\frac{4r}{3\pi}$

Geometrical Shape	Area	$\bar{x}$	$\bar{y}$
6. Quarter Circle 	$\frac{\pi r^2}{4}$	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$
7. Circular Sector 	αr^2 (α in radians)	$\frac{2r \sin \alpha}{3\alpha}$ (α in radians)	0

Bent Wires / Bars	Length	$\bar{x}$	$\bar{y}$
1. Straight Line 	l	$\frac{l}{2}$	0
2. Semicircular Arc 	πr	0	$\frac{2r}{\pi}$
3. Quarter Circular Arc 	$\frac{\pi r}{2}$	$\frac{2r}{\pi}$	$\frac{2r}{\pi}$
4. Arc of Circle 	$2\alpha r^2$ (α in radians)	$\frac{r \sin \alpha}{\alpha}$ (α in radians)	0

Procedure to Locate Centroid of Composite Figure

1. Place the given figure into the first quadrant touching the outer edge of figure to the axis selected, if the axes are not given.
2. Divide the given composite figure into the standard geometrical shapes such as rectangle, triangle, circle, semicircle, quarter circle, sector of circle etc.
3. Mark the centroids G_1, G_2, ... etc. on the composite figure and find their co-ordinates w.r.t. the given axes, i.e. x_1, x_2, ... etc. and y_1, y_2, ... etc.
4. Find the co-ordinates of centroid using the following equations

$$\bar{x} = \frac{\Sigma A_i x_i}{\Sigma A_i} = \frac{A_1 x_1 + A_2 x_2 + ... + A_n x_n}{A_1 + A_2 + ... + A_n}$$

$$\bar{y} = \frac{\Sigma A_i y_i}{\Sigma A_i} = \frac{A_1 y_1 + A_2 y_2 + ... + A_n y_n}{A_1 + A_2 + ... + A_n}$$

4.4 Solved Problems

Problem 1

Find the centroid of shaded area in *figure 4.14(a)*.

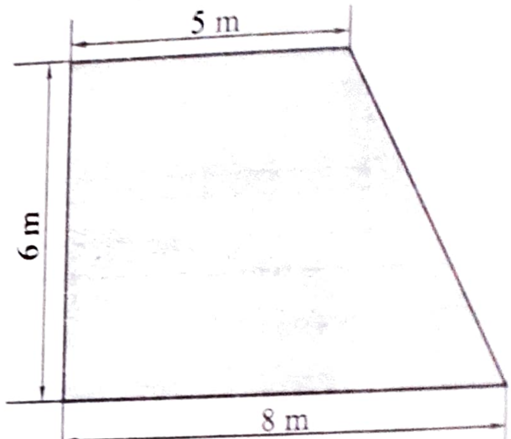


Fig. 4.14(a)

Solution

1. The area can be viewed as a rectangle and a triangle combined together. The areas and centroidal coordinates for each of these shapes can be determined referring *figure 4.14(b)*.

2. Consider rectangle $OABD$

$$A_1 = 5 \times 6 = 30 \text{ m}^2$$

$$x_1 = \frac{5}{2} = 2.5 \text{ m}$$

$$y_1 = \frac{6}{2} = 3 \text{ m}$$

3. Consider triangle DBC

$$A_2 = \frac{1}{2} \times 6 \times 3 = 9 \text{ m}^2$$

$$x_2 = 5 + \frac{3}{3} = 6 \text{ m}$$

$$y_2 = \frac{6}{3} = 2 \text{ m}$$

4. Coordinates of centroid shaded area

$$\bar{x} = \frac{30 \times 2.5 + 9 \times 6}{30 + 9} = 3.308 \text{ m}$$

$$\bar{y} = \frac{30 \times 3 + 9 \times 2}{30 + 9} = 2.769 \text{ m}$$

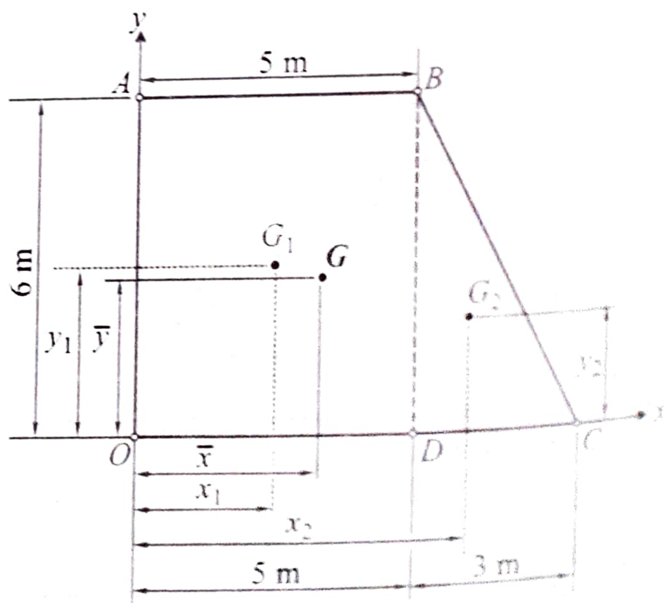


Fig. 4.14(b)

$\therefore$ Coordinates of centroid w.r.t. origin O are G (3.308, 2.769) m. **Ans.**

Problem 2

Find the centroid of given shaded area in *figure 4.15(a)*.

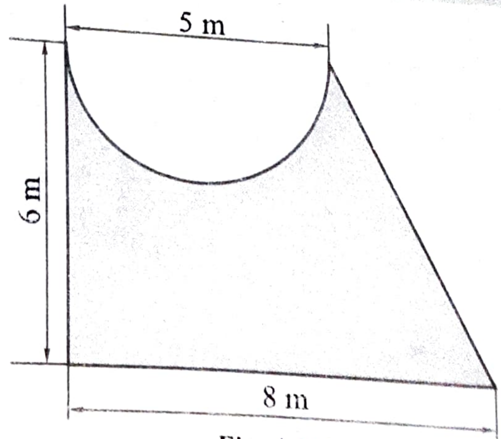


Fig. 4.15(a)

Solution

1. The area can be viewed as a rectangle from which a semicircle is removed and a triangle combined together. The areas and centroidal coordinates for each of these shapes can be determined referring *figure 4.15(b)*.

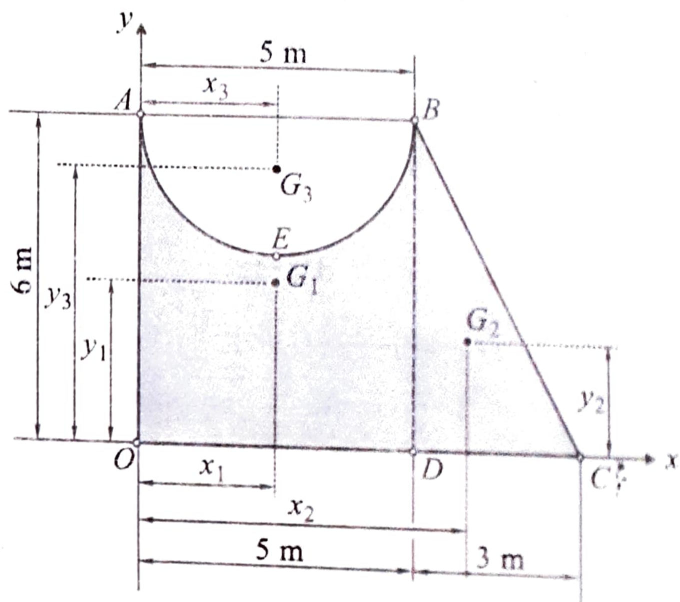


Fig. 4.15(b)

2. Consider rectangle *OABD*

$$A_1 = 5 \times 6 = 30 \text{ m}^2$$

$$x_1 = \frac{5}{2} = 2.5 \text{ m}$$

$$y_1 = \frac{6}{2} = 3 \text{ m}$$

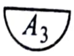
3. Consider triangle *DBC*

$$A_2 = \frac{1}{2} \times 6 \times 3 = 9 \text{ m}^2$$

$$x_2 = 5 + \frac{3}{3} = 6 \text{ m}$$

$$y_2 = \frac{6}{3} = 2 \text{ m}$$

4. Consider semicircle *AEB*



$$r = 2.5 \text{ m}$$

$$-A_3 = -\frac{\pi \times 2.5^2}{2} = -9.817 \text{ m}^2$$

$$x_3 = 2.5 \text{ m}$$

$$y_3 = 6 - \frac{4 \times 2.5}{3\pi} = 4.939 \text{ m}$$

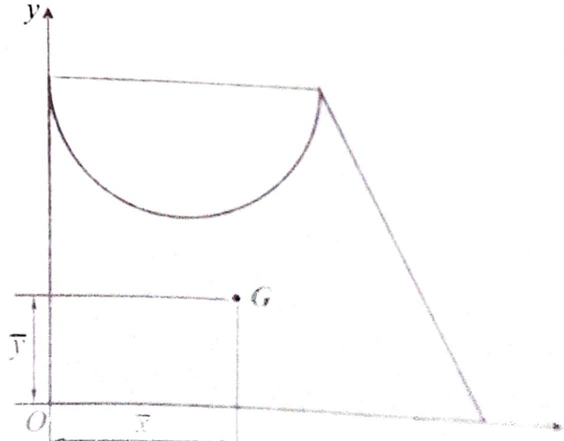
5. Centroid of the given shaded area is given as

$$\bar{x} = \frac{30 \times 2.5 + 9 \times 6 - 9.817 \times 2.5}{30 + 9 - 9.817}$$

$$\bar{x} = 3.579 \text{ m}$$

$$\bar{y} = \frac{30 \times 3 + 9 \times 2 - 9.817 \times 4.939}{30 + 9 - 9.817}$$

$$\bar{y} = 2.039 \text{ m}$$



∴ Coordinates of centroid w.r.t. origin *O* are *G* (3.579, 2.039) m. **Ans.**

Problem 3

Find the centroid of shaded area shown in *figure 4.16(a)*.

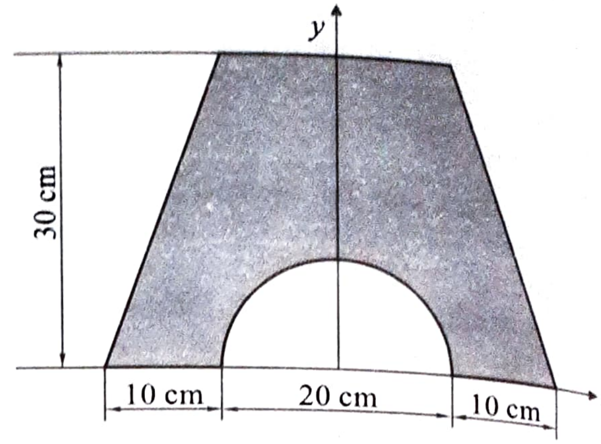


Fig. 4.16(a)

Solution

1. The given figure is symmetric about y-axis

$\therefore \quad \bar{x} = 0.$

The composite area can be viewed as a triangle $\oplus$ rectangle $\oplus$ triangle $\ominus$ semicircle. The areas and centroidal coordinates for each of these shapes can be determined referring *figure 4.16(b)*.

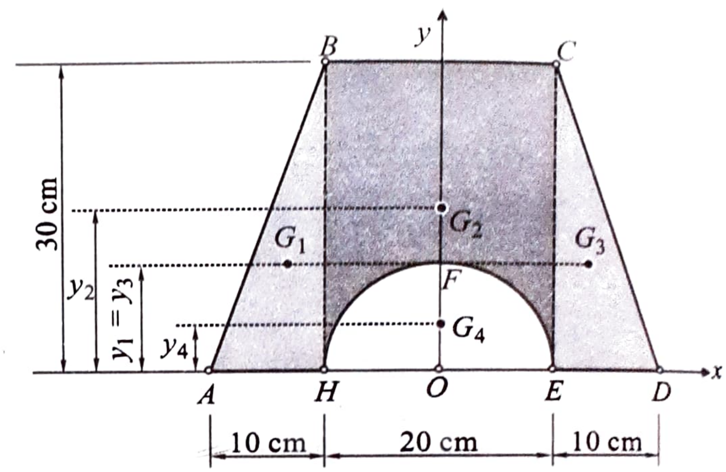


Fig. 4.16(b)

2. Consider triangle ABH

$$A_1 = \frac{1}{2} \times 10 \times 30 = 150 \text{ cm}^2$$

$$y_1 = \frac{30}{3} = 10 \text{ cm}$$

3. Consider rectangle $HBCE$

$$A_2 = 20 \times 30 = 600 \text{ cm}^2$$

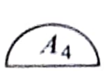
$$y_2 = \frac{30}{2} = 15 \text{ cm}$$

4. Consider triangle ECD

$$A_3 = \frac{1}{2} \times 10 \times 30 = 150 \text{ cm}^2$$

$$y_3 = \frac{30}{3} = 10 \text{ cm}$$

5. Consider semicircle HFE

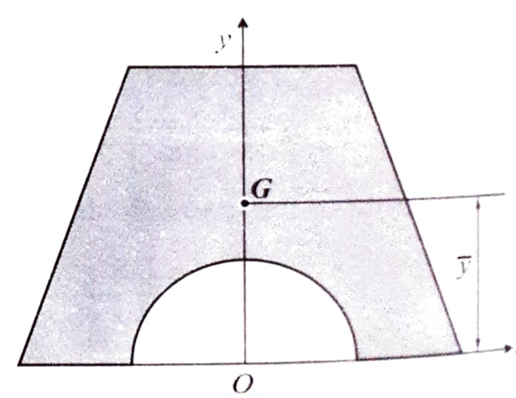


$$r = 10 \text{ cm}$$

$$-A_4 = -\frac{\pi \times 10^2}{2} = -157.08 \text{ cm}^2$$

$$y_4 = \frac{4 \times 10}{3\pi} = 4.24 \text{ cm}$$

6. Centroid of the given shaded area is given as



$$\bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3 + (-A_4) y_4}{A_1 + A_2 + A_3 + (-A_4)}$$

$$= \frac{150 \times 10 + 600 \times 15 + 150 \times 10 - 157.08 \times 4.24}{150 + 600 + 150 - 157.08}$$

$$\bar{y} = 15.26 \text{ cm}$$

$\therefore$ Coordinates of centroid w.r.t. origin O are $\mathbf{G}$ (0, 15.26) cm. ***Ans.***

Problem 4

Find the centroid of shaded area about x and y axis in *figure 4.17(a)*.

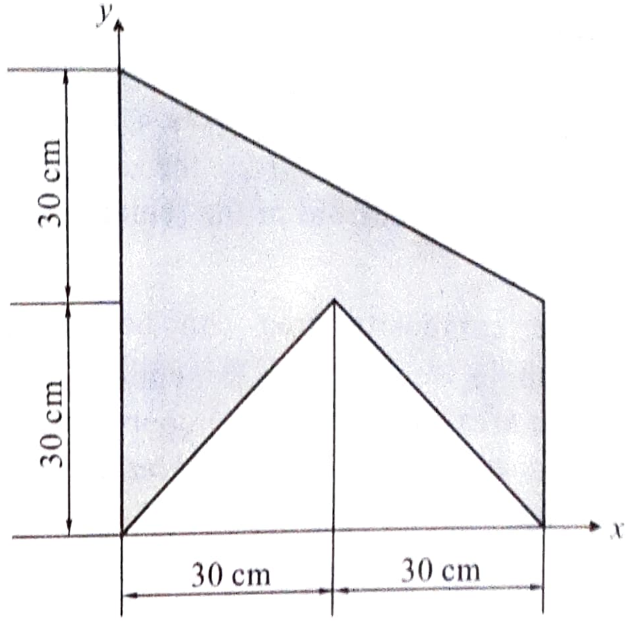


Fig. 4.17(a)

Solution

1. The area can be viewed as three triangles combined together. The areas and centroidal coordinates for each of these shapes can be determined referring *figure 4.17(b)*.

   Distances of x_i and y_i are from origin O.

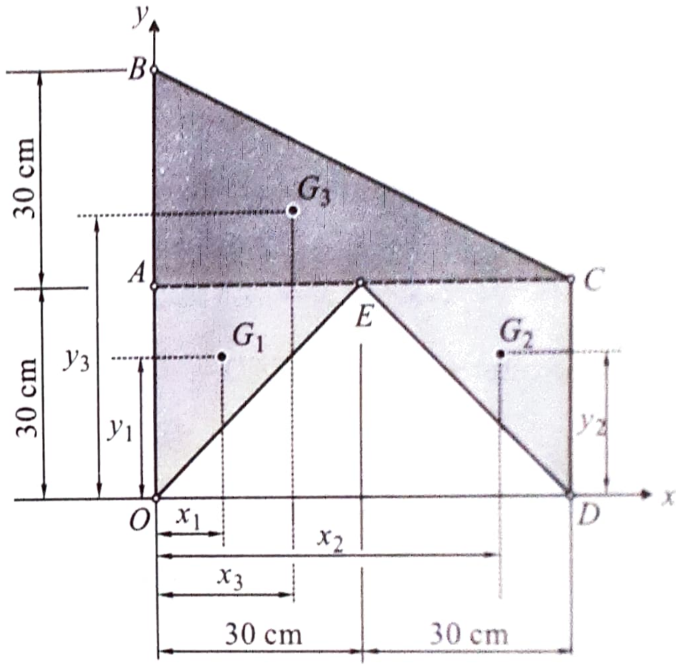


Fig. 4.17(b)

2. Consider triangle OAE

$$A_1 = \frac{1}{2} \times 30 \times 30 = 450 \text{ cm}^2$$

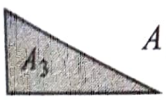
$$x_1 = 10 \text{ cm} \; ; \; y_1 = 20 \text{ cm}$$

3. Consider triangle ECD

$$A_2 = \frac{1}{2} \times 30 \times 30 = 450 \text{ cm}^2$$

$$x_2 = 50 \text{ cm} \; ; \; y_2 = 20 \text{ cm}$$

4. Consider triangle ABC



$$A_3 = \frac{1}{2} \times 60 \times 30 = 900 \text{ cm}^2$$

$$x_3 = 20 \text{ cm} \; ; \; y_3 = 30 + 10 = 40 \text{ cm}$$

5. Centroid of the given shaded area is given as

$$\bar{x} = \frac{A_1 x_1 + A_2 x_2 + A_3 x_3}{A_1 + A_2 + A_3}$$

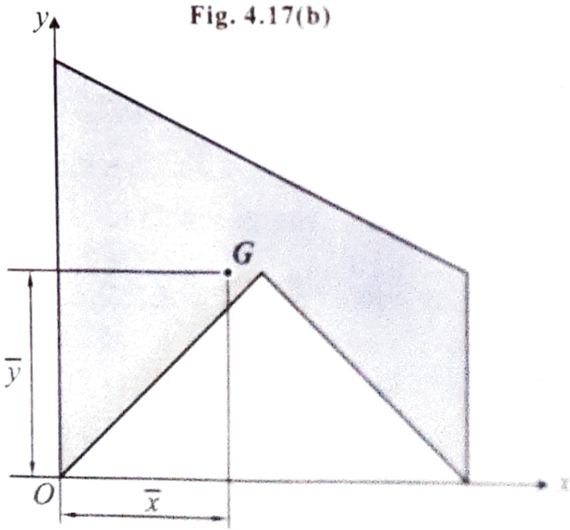
$$= \frac{450 \times 10 + 450 \times 50 + 900 \times 20}{450 + 450 + 900}$$

$$\bar{x} = 25 \text{ cm}$$

$$\bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3}$$

$$= \frac{450 \times 20 + 450 \times 20 + 900 \times 40}{450 + 450 + 900}$$

$$\bar{y} = 30 \text{ cm}$$



$\therefore$ Coordinates of centroid w.r.t. origin O are **G** (25, 30) cm. ***Ans.***

Problem 5(a)

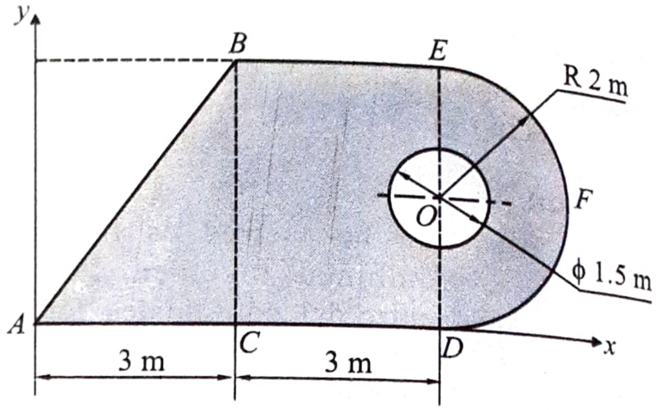


Fig. 4.18(a)

Three plates ABC and $BCDE$ and DEF are welded together as shown in the *figure 4.18(a)*. Circle of diameter 1.5 m is cut from the composite plate. Determine the centroid of the remaining area.

Solution

1. The composite area can be viewed as a triangle ⊕ rectangle ⊕ semicircle ⊖ circle. The areas and centroidal coordinates for each of these shapes can be determined referring *figure 4.18(b)*.

2. Consider triangle ABC

$$A_1 = \frac{1}{2} \times 3 \times 4 = 6 \text{ m}^2$$

$$x_1 = \frac{2}{3} \times 3 = 2 \text{ m}$$

$$y_1 = \frac{1}{3} \times 4 = 1.33 \text{ m}$$

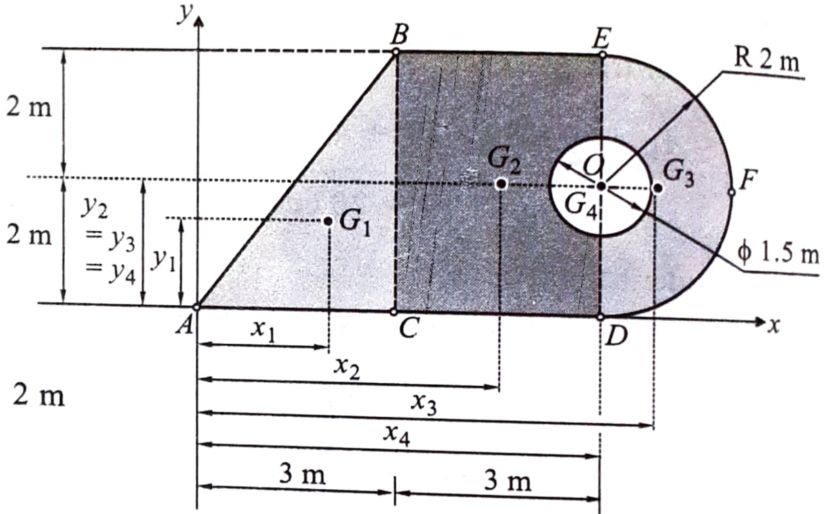


Fig. 4.18(b)

3. Consider rectangle $BCDE$

$$A_2 = 3 \times 4 = 12 \text{ m}^2$$

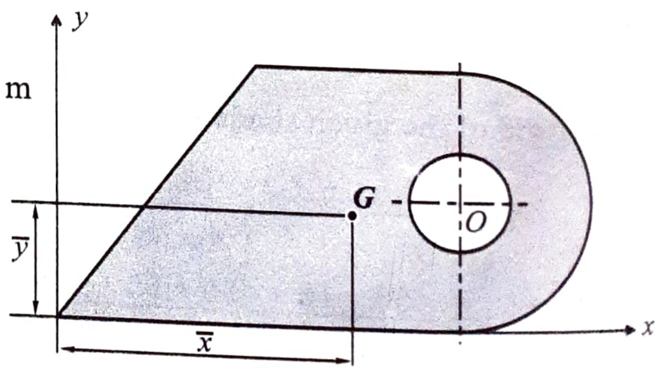
$$x_2 = 3 + 1.5 = 4.5 \text{ m} \; ; \; y_2 = 2 \text{ m}$$

4. Consider semicircle EFD

$$r = 2 \text{ m}$$

$$A_3 = \frac{\pi \times 2^2}{2} = 6.283 \text{ m}^2$$

$$x_3 = 3 + 3 + \frac{4 \times 2}{3\pi} = 6.848 \text{ m} \; ; \; y_3 = 2 \text{ m}$$



5. Consider circle with centre O and radius r

$$r = 0.75 \text{ m}$$

$$-A_4 = -(\pi \times 0.75^2) = -1.767 \text{ m}^2$$

$$x_4 = 6 \text{ m} \; ; \; y_4 = 2 \text{ m}$$

6. Centroid of the given shaded area is given as

$$\bar{x} = \frac{6 \times 2 + 12 \times 4.5 + 6.283 \times 6.848 - 1.767 \times 6}{6 + 12 + 6.283 - 1.767}$$

$$\bar{x} = 4.37 \text{ m}$$

$$\bar{y} = \frac{6 \times 1.33 + 12 \times 2 + 6.283 \times 2 - 1.767 \times 2}{6 + 12 + 6.283 - 1.767}$$

$$\bar{y} = 1.82 \text{ m}$$

∴ Coordinates of centroid w.r.t. origin A are $\boldsymbol{G}$ (4.37, 1.82) m. ***Ans.***

4.5 Solved Problems on Bent Bar / Wire

Problem 25(a)

Locate the centroid of the 10 mm diameter bar ; bent in xy - plane as shown in figure 4.38(a).

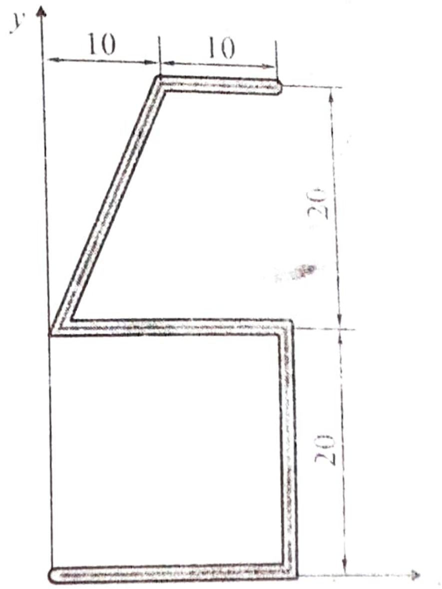


Fig. 4.38(a)

[All dimensions are in cm]

Solution

Divide the bent bar into various parts.

1. OA

   $l_1 = 20$ cm ; $x_1 = 10$ cm ; $y_1 = 0$

2. AB

   $l_2 = 20$ cm ; $x_2 = 20$ cm ; $y_2 = 10$ cm

3. BC

   $l_3 = 20$ cm ; $x_3 = 10$ cm ; $y_3 = 20$ cm

4. CD

   $l_4 = \sqrt{20^2 + 10^2} = 22.36$ cm

   $x_4 = 5$ cm ; $y_4 = 20 + 10 = 30$ cm

5. DE

   $l_5 = 10$ cm ; $x_5 = 15$ cm ; $y_5 = 40$ cm

6. Centroid of given bent bar can be calculated as

$$\bar{x} = \frac{l_1 x_1 + l_2 x_2 + l_3 x_3 + l_4 x_4 + l_5 x_5}{l_1 + l_2 + l_3 + l_4 + l_5}$$

$$\bar{x} = \frac{20 \times 10 + 20 \times 20 + 20 \times 10 + 22.36 \times 5 + 10 \times 15}{20 + 20 + 20 + 22.36 + 10}$$

$$\bar{x} = 11.50 \text{ cm}$$

$$\bar{y} = \frac{l_1 y_1 + l_2 y_2 + l_3 y_3 + l_4 y_4 + l_5 y_5}{l_1 + l_2 + l_3 + l_4 + l_5}$$

$$\bar{y} = \frac{20 \times 0 + 20 \times 10 + 20 \times 20 + 22.36 \times 30 + 10 \times 40}{20 + 20 + 20 + 22.36 + 10}$$

$$\bar{y} = 18.09 \text{ cm}$$

$\therefore$ Coordinates of centroid w.r.t. origin O are **G** (11.50, 18.09) cm. ***Ans.***

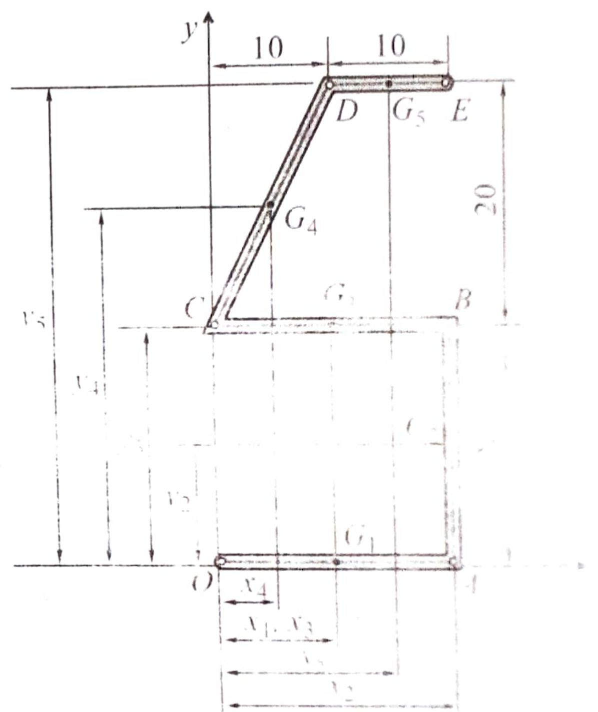
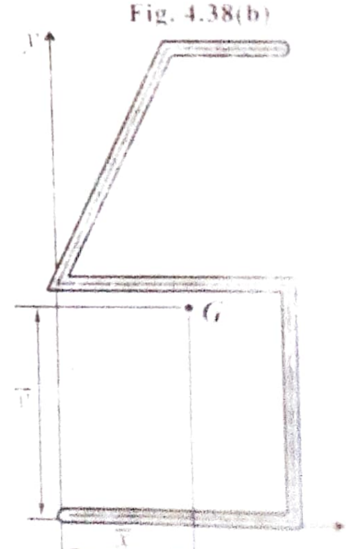


Fig. 4.38(b)



Problem 26

A uniform wire is bent into a Sharpe as shown in *figure 4.39(a)*. Calculate position of C.G. of the wire.

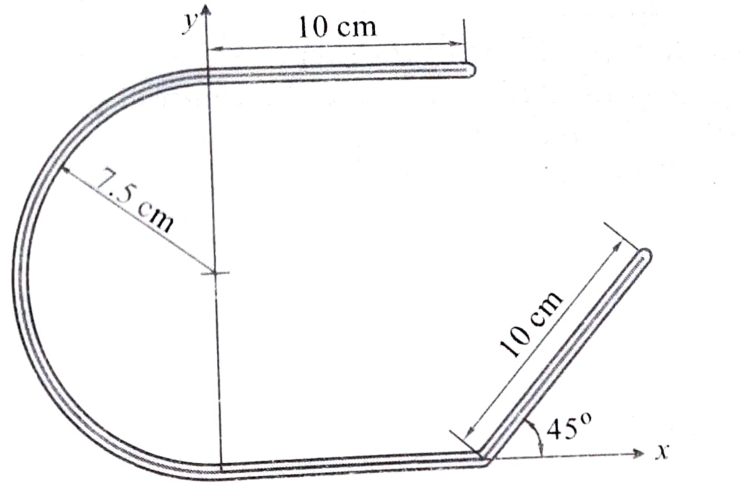


Fig. 4.39(a)

Solution

Divide the bent wire into various parts.

1. *AB*

$l_1 = 10$ cm ; $x_1 = 5$ cm ; $y_1 = 15$ cm

2. *BO*

$l_2 = \pi \times 7.5 = 23.56$ cm

$x_2 = \dfrac{-2 \times 7.5}{\pi} = -4.77$ cm

$y_2 = 7.5$ cm

3. *OC*

$l_3 = 10$ cm ; $x_3 = 5$ cm ; $y_3 = 0$

4. *CD*

$l_4 = 10$ cm

$x_4 = 10 + 5\cos 45 = 13.54$ cm ;

$y_4 = 5\sin 45 = 3.54$ cm

Centroidal coordinates for each of the parts are shown in *figure 4.39(b)*.

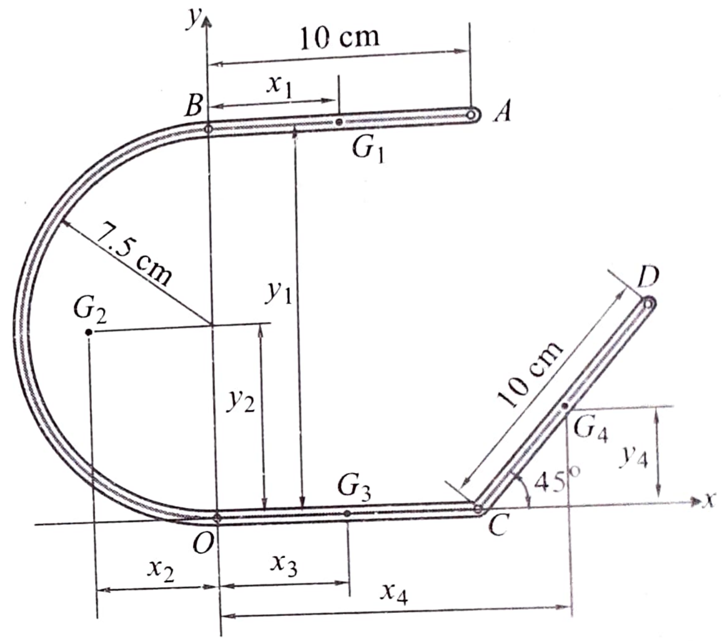


Fig. 4.39(b)

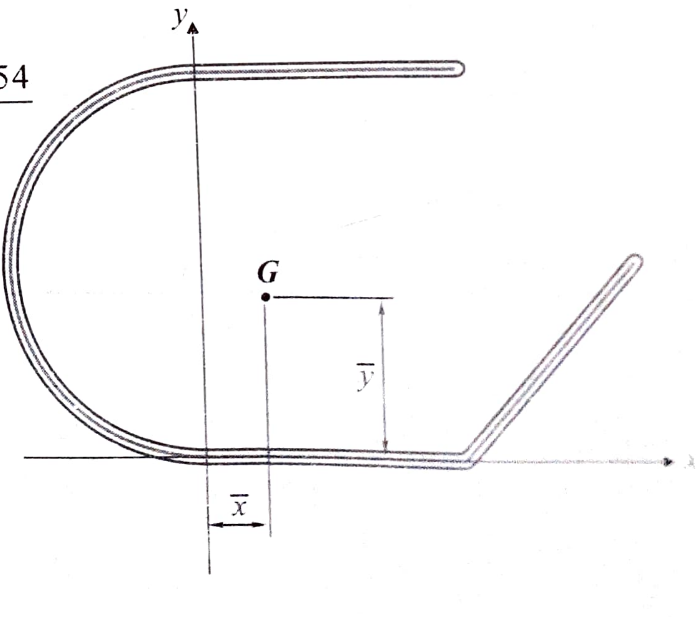
6. Centroid of given bent wire can be calculated as

$$\bar{x} = \frac{10 \times 5 + 23.56 \times (-4.77) + 10 \times 5 + 10 \times 13.54}{10 + 23.56 + 10 + 10}$$

$\bar{x} = 2.297$ cm

$$\bar{y} = \frac{10 \times 15 + 23.56 \times 7.5 + 10 \times 0 + 10 \times 3.54}{10 + 23.56 + 10 + 10}$$

$\bar{y} = 6.76$ cm



$\therefore$ Coordinates of centroid w.r.t. origin *O* are ***G*** (2.297, 6.76) cm. **Ans.**

Problem 28

A thin homogeneous wire of uniform section is built into a shape as shown in figure 4.41(a). Determine the position of C.G. of the wire. [$\theta = 30°$]

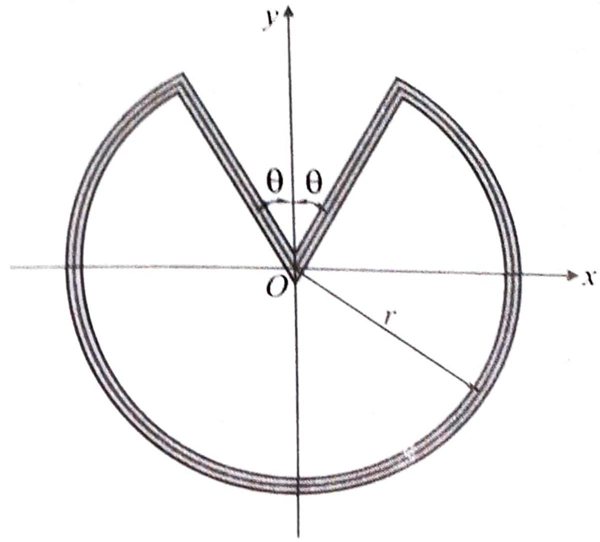


Fig. 4.41(a)

Solution

The given bent thin wire is symmetric about y-axis.

$\bar{x} = 0$

1. ABC

$$l_1 = 2\,r\left(\frac{150}{180}\right)\pi = 5.24\,r \qquad (\because l_1 = 2\,r\,\alpha)$$

$$y_1 = -\frac{r\sin 150}{\left(\frac{150}{180}\pi\right)} = -0.191\,r \qquad \left(\because y_1 = \frac{r\sin\alpha}{\alpha}\right)$$

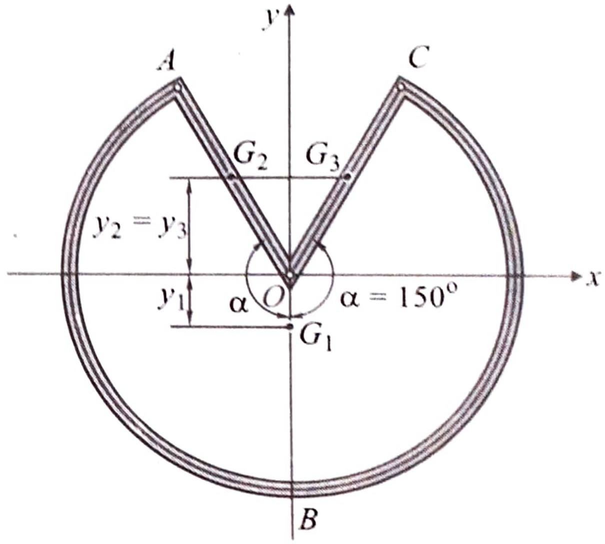


Fig. 4.41(b)

2. AO

$$l_2 = r$$

$$y_2 = \frac{r}{2}\cos 30 = 0.433\,r$$

3. CO

$$l_3 = r$$

$$y_3 = \frac{r}{2}\cos 30 = 0.433\,r$$

4. Centroid of given bent thin wire can be calculated as

$$\bar{y} = \frac{5.24\,r \times (-0.191\,r) + 2(r \times 0.433\,r)}{5.24\,r + 2\,r}$$

$$\bar{y} = -0.0186\,r$$

$\therefore$ Coordinates of centroid w.r.t. origin O are $G\,(0, -0.0186\,r)$. **Ans.**