Note : Here the x-axis is line of impact which can be treated as normal (n) and y-axis is the common tangent which can be treated as tangent (t).

13.5 Coefficient of Restitution (e)

When two bodies collides for a very small interval of time there will be phenomena of *Deformation* and *Restitution* (*Regain*) of shape.

By Impulse Momentum Principle for the process of deformation of the colliding body of mass m_1, we have

$$m_1 u_1 - \int F_D \, dt = m_1 u \qquad \text{..... (I)}$$

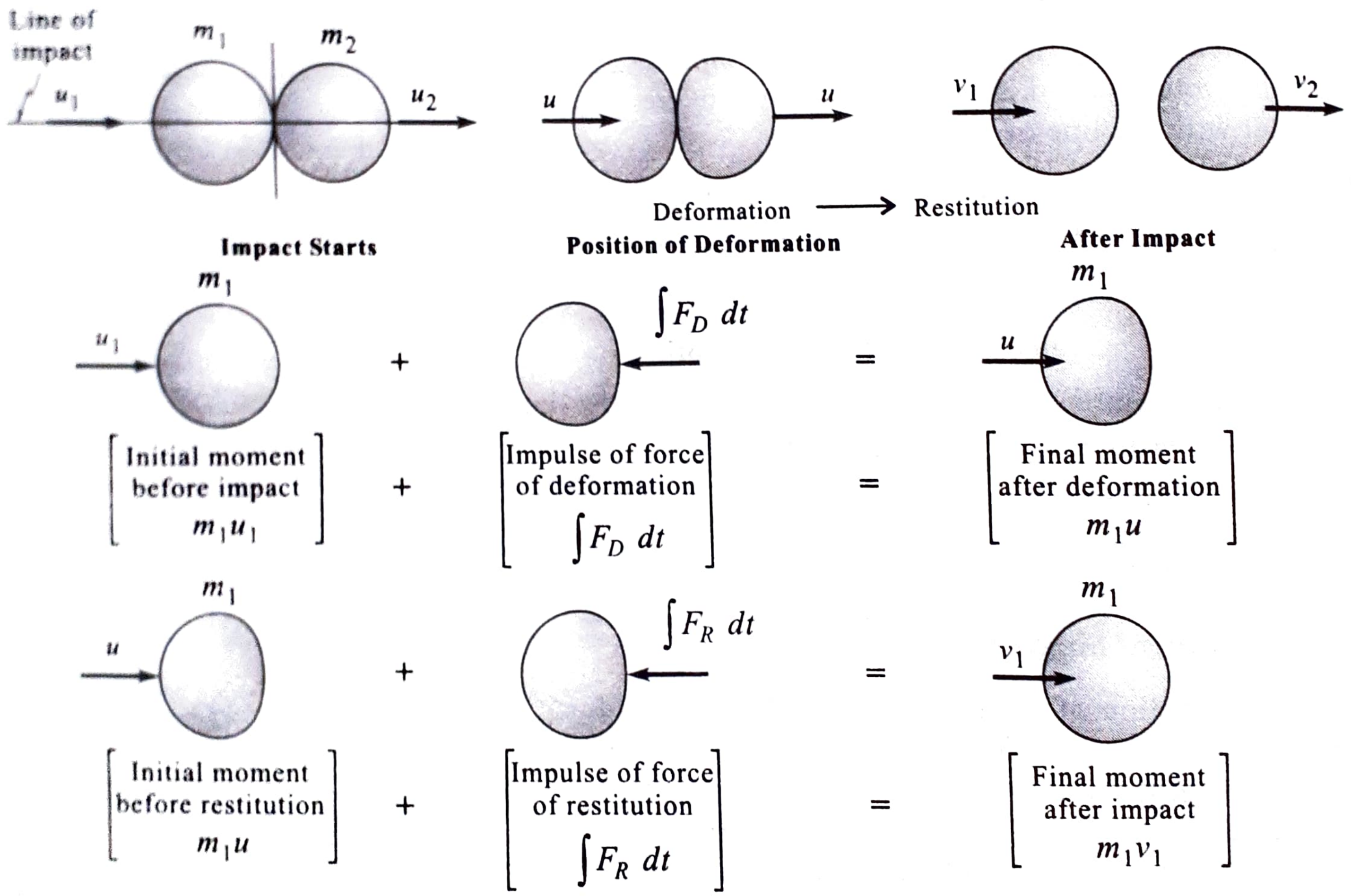


Fig: 13.7

Similarly, by Impulse Momentum Principle for the process of restitution of body of mass m_1, we have

$$m_1 u - \int F_R \, dt = m_1 v_1 \qquad \text{..... (II)}$$

From equation (I) and (II), we have

$$\int F_D \, dt = m_1 u_1 - m_1 u \quad \text{and} \quad \int F_R \, dt = m_1 u - m_1 v_1$$

$$\therefore \quad \frac{\int F_R\,dt}{\int F_D\,dt} = \frac{m_1(u - v_1)}{m_1(u_1 - u)} = \frac{u - v_1}{u_1 - u} = e \qquad \text{..... (III)}$$

Similarly, by Impulse Momentum Principle for the process of deformation and restitution, we have

$$\frac{\int F_R\,dt}{\int F_D\,dt} = \frac{v_2 - u}{u - u_2} \qquad \text{..... (IV)}$$

From equation (III) and (IV), we get

$$\left[\frac{u - v_1 + v_2 - u}{u_1 - u + u - u_2}\right] = e$$

$$v_2 - v_1 = e(u_1 - u_2)$$

$$e = \frac{v_2 - v_1}{u_1 - u_2} = \frac{\text{Velocity of separation}}{\text{Velocity of approach}}$$

$$\therefore \; e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

Classification of Impact Based on Coefficient of Restitution

(a) Perfectly Elastic Impact

(1) Coefficient of restitution $e = 1$.

(2) Momentum is conserved along the line of impact

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

(3) K.E. is conserved. No loss of K.E.

$\therefore$ Total K.E. before impact = Total K.E. after impact

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

(b) Perfectly Plastic Impact

(1) Coefficient of restitution $e = 0$.

(2) After impact both the bodies collies and move together.

(3) Momentum is conserved

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

where v is the common velocity after impact.

(4) There is loss of K.E. during impact. Thus K.E. is not conserved

Loss of K.E. = Total K.E. before impact – Total K.E. after impact

$$\text{Loss of K.E.} = \left(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right) - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

(c) Semi Elastic Impact

Coefficient of restitution $(0 < e < 1)$

13.6 Solved Problems Based on Impact

Problem 1

Two particles of masses 10 kg and 20 kg are moving along a straight line towards each other at velocities of 4 m/s and 1 m/s respectively. If $e = 0.6$, determine the velocities of the particles immediately after their collision. Also find the loss of kinetic energy.

Solution

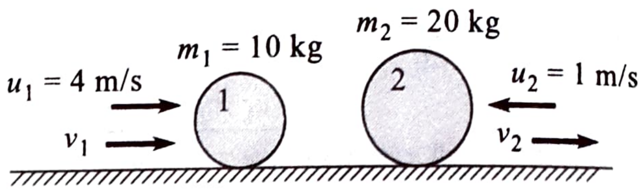
(i) By law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$10 \times 4 + 20 \times (-1) = 10v_1 + 20v_2$$

$$20 = 10v_1 + 20v_2$$

$$v_1 + 2v_2 = 2 \quad \text{..... (I)}$$



(v_1 and v_2 are the assumed directions toward right after impact)

Fig. 13.8

(ii) By coefficient of restitution, we have

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

$$0.6 = -\left[\frac{v_2 - v_1}{-1 - 4}\right]$$

$$v_2 - v_1 = 3 \quad \text{..... (II)}$$

Solving equation (I) and (II), we get

$v_2 = 1.667$ m/s ($\rightarrow$) ***Ans.***

$v_1 = -1.333$ m/s

$\therefore\ v_1 = 1.333$ m/s ($\leftarrow$) ***Ans.***

(iii) Loss of K.E. = Initial K.E. – Final K.E.

$$\text{Loss of K.E.} = \left(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right) - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

$$\text{Loss of K.E.} = \left[\frac{1}{2} \times 10 \times 4^2 + \frac{1}{2} \times 20 \times (-1)^2\right] - \left[\frac{1}{2} \times 10 \times (-1.333)^2 + \frac{1}{2} \times 20 \times 1.667^2\right]$$

$$\text{Loss of K.E.} = 90 - 36.67$$

$$= 53.33 \text{ J}$$

$$\%\ \text{loss in K.E.} = \frac{\text{Loss in K.E.}}{\text{Initial K.E.}} \times 100$$

$$= \frac{53.33}{90} \times 100$$

$\therefore$ % loss in K.E. = 59.27 % ***Ans.***

Problem 2

Three perfectly elastic balls A, B and C masses 2 kg, 4 kg and 8 kg move along a line with velocities 4 m/s, 1 m/s and 0.75 m/s respectively. If the ball A strikes ball B which in turn strikes C, determine the velocities of the three balls after impact.

Solution

Consider Collision Between Two Balls A and B

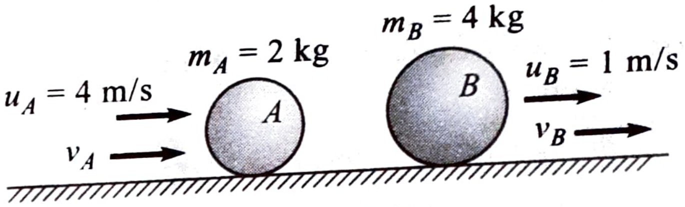


Fig. 13.9(a)

(i) By law of conservation of momentum, we have

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B$$

$$2 \times 4 + 4 \times 1 = 2v_A + 4v_B$$

$$v_A + 2v_B = 6 \qquad \text{..... (I)}$$

(ii) By coefficient of restitution, we have

$$e = -\left[\frac{v_B - v_A}{u_B - u_A}\right]$$

$$1 = -\left[\frac{v_B - v_A}{0.75 - 3}\right]$$

$$v_B - v_A = 3 \qquad \text{..... (II)}$$

Solving equations (I) and (II), we get

$v_B = 3$ m/s (→) and $v_A = 0$ (Ball A comes to rest) ***Ans.***

Consider Collision Between Two Balls B and C

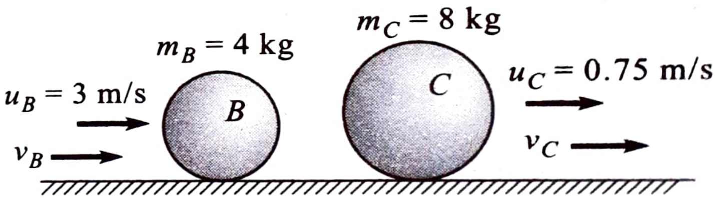


Fig. 13.9(b)

(i) By law of conservation of momentum, we have

$$m_B u_B + m_C u_C = m_B v_B + m_C v_C$$

$$4 \times 3 + 8 \times 0.75 = 4v_B + 8v_C$$

$$18 = 4v_B + 8v_C$$

$$2v_B + 4v_C = 9 \qquad \text{..... (I)}$$

(ii) By coefficient of restitution, we have

$$e = -\left[\frac{v_C - v_B}{u_C - u_B}\right]$$

$$1 = -\left[\frac{v_C - v_B}{0.75 - 3}\right]$$

$$v_C - v_B = 2.25 \qquad \text{..... (II)}$$

Solving equations (I) and (II), we get

$v_C = 2.25$ m/s (→) and $v_B = 0$ (Ball B comes to rest) ***Ans.***

Problem 4

Two balls having masses 20 kg and 30 kg are moving towards each other with velocities 10 m/s and 5 m/s respectively. If after impact the ball having mass 30 kg is moving with velocity 6 m/s to the right then determine the coefficient of restitution between the two balls.

Solution

(i) By law of conservation of momentum, we have

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$20 \times 10 + 30 \times (-5) = 20 v_1 + 30 \times 6$$

$$v_1 = -6.5 \text{ m/s}$$

$$\therefore\ v_1 = 6.5 \text{ m/s } (\leftarrow) \quad \textbf{Ans.}$$

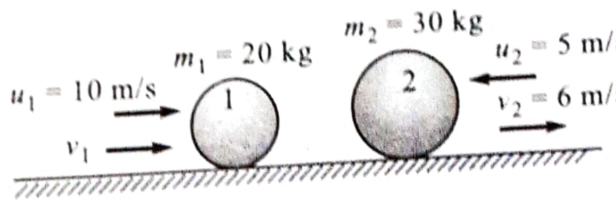


Fig. 13.11

(ii) For coefficient of restitution, we have the relation

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

$$e = -\left[\frac{6-(-6.5)}{-5-10}\right]$$

$$e = 0.8333 \quad \textbf{Ans.}$$

Problem 5

A series of '*n*' identical balls is shown on a smooth horizontal surface [*figure 13.12(a)*]. If number '1' ball moves horizontally with a velocity '*u*' and collides with ball number '2' which in turn collides with ball '3' and so on, and if the coefficient of restitution for each impact is '*e*', show that the velocity of the n^{th} ball is given by

$$v_n = \frac{(1+e)^{n-1} u}{2^{n-1}}$$

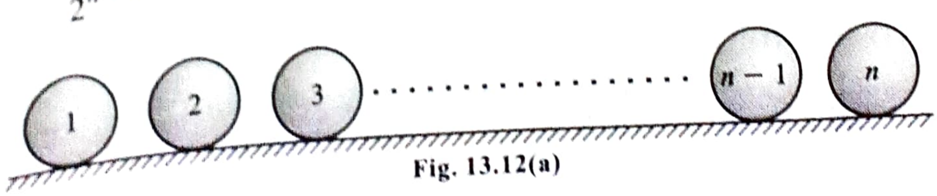


Fig. 13.12(a)

Solution

(i) Consider the collision between ball 1 and 2
By law of conservation of momentum, we have

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$u_1 + u_2 = v_1 + v_2$$

$$u + 0 = v_1 + v_2$$

$$\therefore\ v_1 + v_2 = u \qquad \ldots\ldots \text{(I)}$$

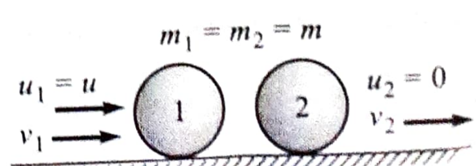


Fig. 13.12(b)

Problem 3

Two small discs of masses 25 gm and 5 gm are kept on horizontal surface. Second disc is strike by the first disc and produces direct central impact. After impact each disc slides and comes to rest. First slides 95 mm to the right and second slides 480 mm to the right before coming to rest. Determine the value of coefficient of restitution. Assume common coefficient of friction for both discs.

Solution

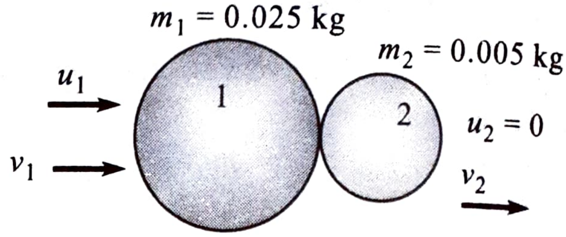


Fig. 13.10

(i) By law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$0.025 \times u_1 + 0.005 \times 0 = 0.025 \times v_1 + 0.005 \times v_2$$

$$5v_1 + v_2 = 5u_1 \qquad \text{..... (I)}$$

Kinetic energy of each disc after impact is lost in doing work against friction.

(ii) By work energy principle, we have

Work done = Change in K.E.

$$-\mu(0.025 \times 9.81)95 = 0 - \frac{1}{2} \times 0.025 \times v_1^2 \qquad \text{..... (II)}$$

$$-\mu(0.005 \times 9.81)480 = 0 - \frac{1}{2} \times 0.005 \times v_2^2 \qquad \text{..... (III)}$$

Taking ratio of equations (II) and (III), we get

$$\frac{95}{480} = -\frac{0.025 \times v_1^2}{0.005 \times v_2^2}$$

$$\therefore \frac{v_1}{v_2} = 0.445$$

$$v_1 = 0.445v_2$$

Substituting in equation (I), we get

$$5 \times 0.445v_2 + v_2 = 5u_1$$

$$v_2 = 1.55u_1 \text{ and } v_1 = 0.69u_1$$

(iii) Coefficient of restitution gives

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

$$= -\left[\frac{1.55u_1 - 0.69u_1}{0 - u_1}\right]$$

$e = 0.86$ ***Ans.***

Problem 4

Two balls having masses 20 kg and 30 kg are moving towards each other with velocities 10 m/s and 5 m/s respectively. If after impact the ball having mass 30 kg is moving with velocity 6 m/s to the right then determine the coefficient of restitution between the two balls.

Solution

(i) By law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$20 \times 10 + 30 \times (-5) = 20v_1 + 30 \times 6$$

$$v_1 = -6.5 \text{ m/s}$$

$$\therefore \; v_1 = 6.5 \text{ m/s } (\leftarrow) \quad \textbf{\textit{Ans.}}$$

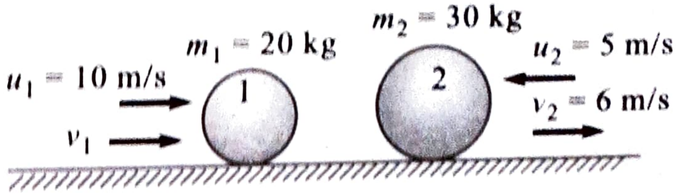


Fig. 13.11

(ii) For coefficient of restitution, we have the relation

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

$$e = -\left[\frac{6 - (-6.5)}{-5 - 10}\right]$$

$$e = 0.8333 \quad \textbf{\textit{Ans.}}$$

Problem 5

A series of 'n' identical balls is shown on a smooth horizontal surface [*figure 13.12(a)*]. If number '1' ball moves horizontally with a velocity 'u' and collides with ball number '2' which in turn collides with ball '3' and so on, and if the coefficient of restitution for each impact is 'e', show that the velocity of the n^{th} ball is given by

$$v_n = \frac{(1 + e)^{n-1}\, u}{2^{n-1}}$$

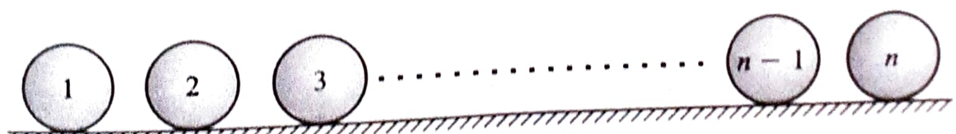


Fig. 13.12(a)

Solution

(i) Consider the collision between ball 1 and 2

By law of conservation of momentum, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$u_1 + u_2 = v_1 + v_2$$

$$u + 0 = v_1 + v_2$$

$$\therefore \; v_1 + v_2 = u \qquad \text{.....(I)}$$

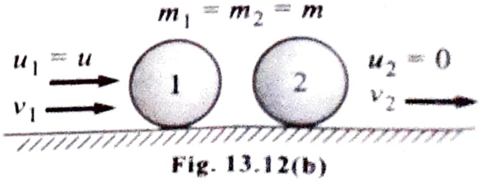


Fig. 13.12(b)

By coefficient of restitution, we have

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

$$e = -\left[\frac{v_2 - v_1}{0 - u}\right]$$

$$v_2 - v_1 = eu \qquad \text{..... (II)}$$

Solving equations (I) and (II), we get

$$v_2 = \frac{u(1+e)}{2}$$

(ii) Consider the collision between ball 2 and 3

We have

$$u_2 = \frac{u(1+e)}{2} \quad \text{and } u_3 = 0$$

By law of conservation of momentum, we have

$$m_2u_2 + m_3u_3 = m_2v_2 + m_3v_3$$

$$u_2 + u_3 = v_2 + v_3$$

$$\frac{u(1+e)}{2} + 0 = v_2 + v_3$$

$$v_2 + v_3 = \frac{u(1+e)}{2} \qquad \text{..... (III)}$$

Coefficient of restitution gives

$$e = -\left[\frac{v_3 - v_2}{u_3 - u_2}\right]$$

$$e = -\left[\frac{v_3 - v_2}{0 - \frac{u(1+e)}{2}}\right]$$

$$v_3 - v_2 = \frac{eu(1+e)}{2} \qquad \text{..... (IV)}$$

Solving equations (I) and (II), we get

$$2v_3 = \frac{u(1+e)}{2} + \frac{eu(1+e)}{2} = \frac{u(1+e)(1+e)}{2} = \frac{u(1+e)^2}{2}$$

$$v_3 = \frac{u(1+e)^2}{2^2}$$

(iii) Similarly considering the collision between ball 3 and 4

$$v_4 = \frac{u(1+e)^3}{2^3} \quad \text{and so on .....}$$

(iv) $\therefore$ velocity of n^{th} ball when it gets hit by $(n-1)^{\text{th}}$ ball

$$v_n = \frac{u(1+e)^{n-1}}{2^{n-1}} \qquad \textbf{\textit{Ans.}}$$

Problem 6

A 50 gm ball is dropped from a height of 600 mm on a small plate. It rebounds to a height of 400 mm when the plate directly rests on the ground and to a height of 250 mm when a foam rubber mat is placed between the plate and the ground. Determine **(i)** the coefficient of restitution between the plate and the ground **(ii)** mass of the plate.

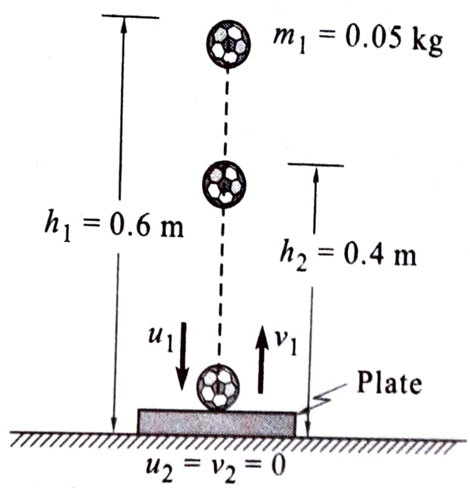


Fig. 13.13(a)

Solution

(i) The plate is kept directly on the ground

$$u_1 = \sqrt{2gh_1}\ (\downarrow)\ \text{(velocity before impact)}$$

$$u_1 = \sqrt{2 \times 9.81 \times 0.6}$$

$$u_1 = 3.43 \text{ m/s } (\downarrow)$$

$$v_1 = \sqrt{2gh_2}\ (\uparrow)\ \text{(velocity after impact)}$$

$$v_1 = \sqrt{2 \times 9.81 \times 0.4}$$

$$v_1 = 2.8 \text{ m/s } (\uparrow)$$

Coefficient of restitution

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right] = -\left[\frac{0 - 2.8}{0 - (-3.43)}\right]$$

$e = 0.816$ ***Ans.***

(ii) The plate is kept on the foam rubber mat

$$u_1 = \sqrt{2 \times 9.81 \times 0.6}$$

$$u_1 = 3.43 \text{ m/s } (\downarrow)$$

$$v_1 = \sqrt{2 \times 9.81 \times 0.25}$$

$$v_1 = 2.215 \text{ m/s } (\uparrow)$$

Coefficient of restitution gives

$$e = -\left[\frac{v_2 - v_1}{u_2 - u_1}\right]$$

$$0.816 = -\left[\frac{-v_2 - 2.215}{0 - (-3.43)}\right]$$

$$\therefore\ v_2 = 0.584 \text{ m/s } (\downarrow)$$

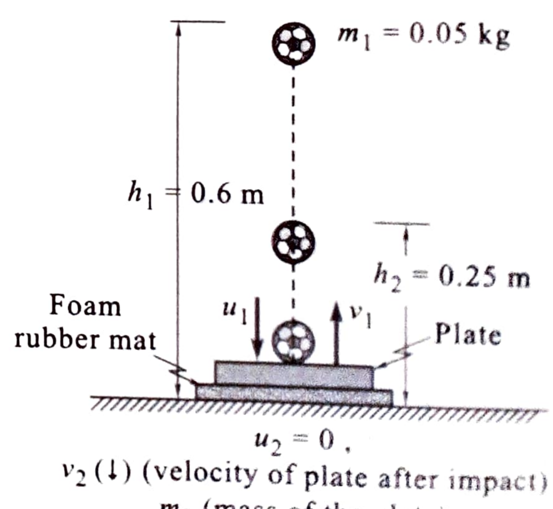


Fig. 13.13(b)

(iii) By law of conservation of momentum, we have

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$0.05 \times (-3.43) + m_2 \times 0 = 0.05 \times 2.215 + m_2 \times (-0.584)$$

$\therefore\ m_2 = 0.483$ kg (mass of the plate) ***Ans.***