# HOMOGENEOUS FUNCTIONS

## FYBTECH SEM-I

## MODULE-5

# Homogeneous Functions

- **Def**- u = f(x, y, z) is called homogeneous function of degree n,
- If replacing $X = xt, Y = yt \ and \ Z = zt$ we get $f(X, Y, Z) = t^n f(x, y, z)$
- i.e. $f(xt, yt, zt) = t^n f(x, y, z)$
- Alternately, u = f(x, y) is homogeneous if it can be expressed as $u = x^n f\left(\frac{y}{x}\right)$ (two variable)
- And u = f(x, y, z) is homogeneous if it can be expressed as $u = x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$ (three variable)

# Euler's Theorem

- If u = f(x, y) is homogeneous function of deg n,

  then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

- For u = f(x, y, z) we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

## Corollary 1

- If u = f(x, y) is homogeneous function of deg n, then

- $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

- For u = f(x, y, z) we get,

- $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + 2yz\frac{\partial^2 u}{\partial y \partial z} + 2zx\frac{\partial^2 u}{\partial z \partial x} = n(n-1)u$

# EXAMPLE-1

- Verify Euler's theorem for $u = \sqrt{x} + \sqrt{y} + \sqrt{z}$
- **Part a)** $\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x}}, \ \frac{\partial u}{\partial y} = \frac{1}{2\sqrt{y}}, \ \frac{\partial u}{\partial z} = \frac{1}{2\sqrt{z}}$
- then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{\sqrt{x}}{2} + \frac{\sqrt{y}}{2} + \frac{\sqrt{z}}{2} = \frac{1}{2}u$
- **Part b)** consider $f(xt, yt, zt) = \sqrt{xt} + \sqrt{yt} + \sqrt{zt}$
- $= \sqrt{t} f(x, y, z)$
- Hence $(xt, yt, zt) = t^{\frac{1}{2}} f(x, y, z)$ ,
- u is homogeneous function of deg ½
- Then by Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = \frac{1}{2}u.$

  Hence Euler's theorem is verified.

# EXAMPLE-2

- If $u = sin^{-1}\left(\frac{x}{y}\right) + cos^{-1}\left(\frac{y}{z}\right) - log\left(\frac{z}{x}\right)$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$
- **Solution:** consider $f(xt, yt, zt)$
- $= sin^{-1}\left(\frac{xt}{yt}\right) + cos^{-1}\left(\frac{yt}{zt}\right) - log\left(\frac{zt}{xt}\right)$
- $= f(x, y, z) = t^0 f(x, y, z)$
- Hence u is homogeneous function of deg zero.
- By, Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 0$

# EXAMPLE-3

- If $u = \frac{\sqrt{x}+\sqrt{y}}{x+y}$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

- **Solution:** consider $f(xt, yt) = \frac{\sqrt{xt}+\sqrt{yt}}{xt+yt} = t^{-\frac{1}{2}} f(x, y)$

- Hence u is homogeneous function of deg -1/2.

- By, Euler's theorem,

- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = -\frac{1}{2}\left(\frac{\sqrt{x}+\sqrt{y}}{x+y}\right)$

# EXAMPLE-4

- If $u = \frac{x^3y+y^3x}{3x}$ then prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 6u$
- **Solution hint:** Check that u is homogeneous function of deg 3. Therefore By, Euler's theorem
- $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 3.2u = 6u$

# EXAMPLE-5



- If $u = \frac{x^2y^3z}{x^2+y^2+z^2} + sin^{-1}\left(\frac{xy+yz}{y^2+z^2}\right)$ then find value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$
- **Solution:** Here u is not homogeneous, consider u = v + w
- where $v = \frac{x^2y^3z}{x^2+y^2+z^2}$ $and\ \ w = sin^{-1}\left(\frac{xy+yz}{y^2+z^2}\right)$
- For v, consider $f(xt, yt, zt) = \frac{x^2t^2y^3t^3zt}{x^2t^2+y^2t^2+z^2t^2} = \frac{t^6}{t^2}f(x, y, z) = t^4 f(x, y, z)$
- v is homogeneous function of deg 4
- For w, consider $f(xt, yt, zt) = sin^{-1}\left(\frac{xyt^2+yzt^2}{y^2t^2+z^2t^2}\right) = t^0 f(x, y, z)$
- w is homogeneous function of deg zero

# EXAMPLE-5



- By Euler's theorem,
- $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}=4v$ (1)
- and $x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=0$ (2)
- Adding (1) and (2)
- $x\frac{\partial (v+w)}{\partial x}+y\frac{\partial (v+w)}{\partial y}+z\frac{\partial (v+w)}{\partial z}=4v$
- Hence proved $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=4\frac{x^2y^3z}{x^2+y^2+z^2}$

# EXAMPLE-6

- If $u = \frac{x^2+xy}{y\sqrt{x}} + \frac{1}{x^7} sin^{-1}\left(\frac{y^2-xy}{x^2-y^2}\right)$ then find value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\ \ at\ x = 1, y = 2$
- **Solution:** Here u is not homogeneous, consider u = v + w
- where $v = \frac{x^2+xy}{y\sqrt{x}}$ $and\ \ w = \frac{1}{x^7} sin^{-1}\left(\frac{y^2-xy}{x^2-y^2}\right)$
- For v, consider $f(xt, yt) = \frac{x^2t^2+xyt^2}{y\sqrt{x}t^{\frac{3}{2}}} = \frac{t^2}{t^{\frac{3}{2}}} f(x, y) = t^{\frac{1}{2}} f(x, y)$
- v is homogeneous function of deg $\frac{1}{2}$
- For w, consider $f(xt, yt) = \frac{1}{x^7t^7} sin^{-1}\left(\frac{y^2t^2-xyt^2}{x^2t^2-y^2t^2}\right) = t^{-7} f(x, y)$
- w is homogeneous function of deg -7

# EXAMPLE-6



- By Euler's theorem,
- $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=\frac{1}{2}v$ (1) $\quad x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=-7w$ (2)
- $x^2\frac{\partial^2 v}{\partial x^2}+2xy\frac{\partial^2 v}{\partial x\partial y}+y^2\frac{\partial^2 v}{\partial y^2}=\frac{1}{2}\left(\frac{1}{2}-1\right)v=-\frac{1}{4}v$ (3)
- and $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-7(-7-1)w=56w$ (4)
- adding (1), (2), (3) and (4) in proper order
- LHS = $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$
- $\quad = 56w-7w-\frac{1}{4}v+\frac{1}{2}v=\frac{1}{4}v+49w$
- $at\ x=1\ and\ y=2,\ v=\frac{3}{2}\ and\ w=sin^{-1}(1)=\frac{\pi}{2}$
- put in above, LHS = $\frac{3}{8}+\frac{49}{2}\pi$

# Corollary of Euler’s Theorem

- If we have certain functions, say $u = \sin^{-1} \emptyset(x, y)$ *or* $\log \emptyset(x, y)$
- where u is not homogeneous but $\emptyset(x, y)$ is homogeneous function.
- In short in above examples $\sin u$ *or* $e^u$ will be homogeneous function. Then for this type of functions also we have corollary of Euler’s theorem.

## Corollary 1

- If f(u) is homogeneous function of deg n in two variable, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

- For three variable function we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{f(u)}{f'(u)}$

# Corollary of Euler's Theorem

- **Proof:** since t = f(u) is homogeneous function of deg n
- by Euler's theorem $x\frac{\partial t}{\partial x} + y\frac{\partial t}{\partial y} = nt$
- But $\frac{\partial t}{\partial x} = f'(u)\frac{\partial u}{\partial x}$ $\; and \;$ $\frac{\partial t}{\partial y} = f'(u)\frac{\partial u}{\partial y},$
- hence $xf'(u)\frac{\partial u}{\partial x} + yf'(u)\frac{\partial u}{\partial y} = nf(u).$
- Dividing by f'(u) we get the result. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$
- Corollary 2
- If f(u) is homogeneous function of deg n in two variable, then
- $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$ $\; where \; g(u) = n\frac{f(u)}{f'(u)}$

# EXAMPLE-7

- If $u = \frac{x^2y^2z^2}{x^2+y^2+z^2} + \cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$
- **Solution:** Here u is not homogeneous, consider u = v + w
- where $v = \frac{x^2y^2z^2}{x^2+y^2+z^2}$ $and\;\; w = \cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$
- For v, consider $f(xt, yt, zt) = \frac{x^2t^2y^2t^2z^2t^2}{x^2t^2+y^2t^2+z^2t^2} = \frac{t^6}{t^2}f(x,y) = t^4 f(x,y,z)$
- v is homogeneous function of deg 4
- By Euler's theorem, $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 4v$ --- (1)

# EXAMPLE-7



- check that w is not homogeneous, Let $f(w) = \cos w = \frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$
- consider $h(xt, yt, zt) = \frac{xt+yt+zt}{\sqrt{xt}+\sqrt{yt}+\sqrt{zt}} = t^{\frac{1}{2}} h(x, y, z)$
- hence f(w) = cos w is homogeneous function of deg ½
- By corollary of Euler's theorem,
- $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = n\frac{f(w)}{f'(w)} = \frac{1}{2}\frac{\cos w}{-\sin w} = -\frac{1}{2}\cot w$ ---(2)
- Adding (1) and (2),
- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4v - \frac{1}{2}\cot w$
- $= 4\left(\frac{x^2y^2z^2}{x^2+y^2+z^2}\right) - \frac{1}{2}\cot\left(\cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)\right)$

# EXAMPLE-8

- If $u = cosec^{-1}\sqrt{\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}}$ then prove that
- $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13 + \tan^2 u)$

- **Solution:** since u is not homogeneous, consider f(u) = cosec u = $\sqrt{\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}}$
- $h(xt, yt) = \sqrt{\frac{\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)t^{\frac{1}{2}}}{\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)t^{\frac{1}{3}}}} = t^{\frac{1}{12}}h(x, y)$
- Thus f(u) is homogeneous function of degree $\frac{1}{12}$
- Then by corollary, $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

# EXAMPLE-8

- $where\ g(u) = n\dfrac{f(u)}{f'(u)} = \dfrac{1}{12}\dfrac{cosec\ u}{(-cosec\ u\cot u)} = -\dfrac{1}{12}\tan u$
- And $[g'(u) - 1] = -\dfrac{1}{12}\sec^2 u - 1$
- $= -\dfrac{1}{12}(1 + \tan^2 u) - 1 = -\dfrac{13}{12} - \dfrac{\tan^2 u}{12}$
- $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$
- $= -\dfrac{1}{12}\tan u\left[-\dfrac{1}{12}(13 + \tan^2 u)\right]$
- $= \dfrac{1}{144}\tan u\,[13 + \tan^2 u]$

# EXAMPLE-9

- If $u = \tan^{-1}(x^2 + 2y^2)$ then prove that
- (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$
- (ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$
- **Solution:** since u is not homogeneous,
- consider f(u) = tan u = $x^2 + 2y^2$
- $h(xt, yt) = x^2t^2 + 2y^2t^2 = t^2 h(x, y)$
- Thus f(u) is homogeneous function of degree 2
- Then by corollary,
- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 2\frac{tanu}{\sec^2 u} = 2\sin u\cos u = \sin 2u$
- $g(u) = sin2u$

# EXAMPLE-9

- $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g^{'}(u)-1]$
- And $[g^{'}(u)-1]=[cos2u(2)-1]$
- Hence $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$
- $=g(u)[g^{'}(u)-1]=\sin 2u\,[2cos2u-1]$
- $=2\sin 2u\cos 2u-sin2u$
- $=\sin 4u-sin2u$

# EXAMPLE-10

- If $x = e^u \tan v\,,\ y = e^u \sec v$ then find $\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$
  - **Solution:** First we have to express u and v as functions of x and y
  - Consider $y^2 - x^2 = e^{2u}\sec^2 v - e^{2u}\tan^2 v = e^{2u}$
  - thus $u = \frac{1}{2}\log(y^2 - x^2)$
  - And divide to get $\frac{x}{y} = \frac{e^u \tan v}{e^u \sec v} = \sin v$ thus $v = \sin^{-1}\left(\frac{x}{y}\right)$
  - Now we check for v,
  - v is homogeneous function of deg zero.
  - By Euler's theorem, $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 0$
  - Then required product $\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 0$

# EXAMPLE-11

- If $u = log\left(\frac{x^3+y^3}{x^2+y^2}\right)$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$
- **Solution:** u is not homogeneous but $e^u$ is homogeneous function of deg 1 (prove!!)
- Hence by corollary of Euler's theorem,
- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 1\frac{e^u}{e^u} = 1$

# EXAMPLE-12

## For practice

- If $u = e^{\frac{x}{y}} + log(x^3 + y^3 - x^2y + xy^2)$
- $then\ find$
- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$
- Can you guess above expression for any deg homogeneous function inside log function