Discrete-Time Signal processing
Z-transform

- Introduction
- z-Transform
- Properties of the Region of Convergence for the z-transform
- The inverse z-Transform
- z-Transform Properties

Introduction

- Fourier transform plays a key role in analyzing and representing discrete-time signals and systems, but does not converge for all signals.

- Continuous systems: Laplace transform is a generalization of the Fourier transform.

- Discrete systems : z-transform, generalization of DTFT, converges for a broader class of signals.

Introduction

Motivation of z-transform:

- The Fourier transform does not converge for all sequences and it is useful to have a generalization of the Fourier transform.

- In analytical problems the z-Transform notation is more convenient than the Fourier transform notation.

z-Transform

- z-Transform: two-sided, bilateral z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] Z^{-n} = \mathcal{Z}\{x[n]\}$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

- one-sided, unilateral z-transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

- If $z = e^{jw}$, z-transform is Fourier transform.

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

Relationship between z-transform and Fourier transform

- Express the complex variable z in polar form as

$$z = re^{jw}$$

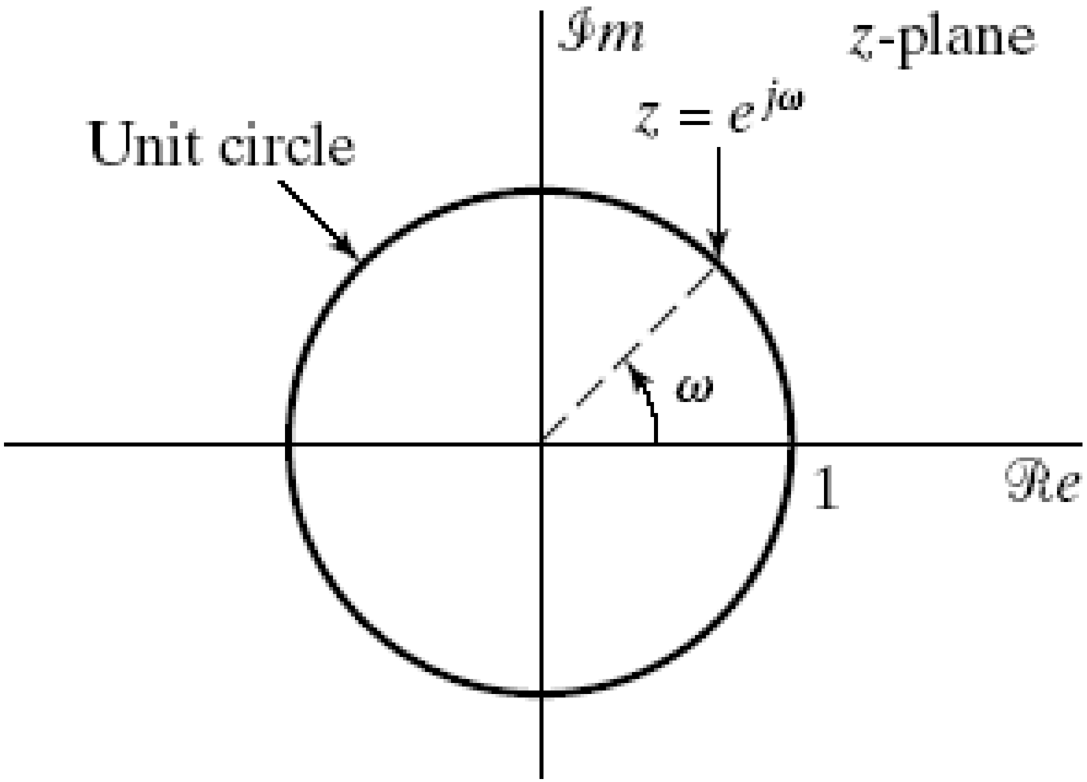
$$X\left(re^{jw}\right) = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right)e^{-jwn}$$

- The Fourier transform of the product of $x[n]$ and the exponential sequence r^{-n}

$$If\ \ r = 1,\ \ X(Z) \longrightarrow X\left(e^{jw}\right)$$

Complex z plane

$$-\pi \leq w < \pi \quad \Leftrightarrow \quad unit\ circle$$



Region of convergence (ROC)

For any given sequence, the set of values of z for which the z-transform converges is called the Region Of Convergence (ROC).

$$z = re^{jw}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad \Rightarrow \quad X\left(re^{jw}\right) = \sum_{n=-\infty}^{\infty} \left(x[n] r^{-n}\right) e^{-jwn}$$

Absolute Summability

$$\left| X\left(re^{jw}\right) \right| \le \sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty$$

$$\left| X(z) \right| \le \sum_{n=0}^{\infty} \left| x[n] \right| \left| z \right|^{-n} < \infty$$

The ROC consists of all values of z such that the inequality in the above holds

Region of convergence (ROC)

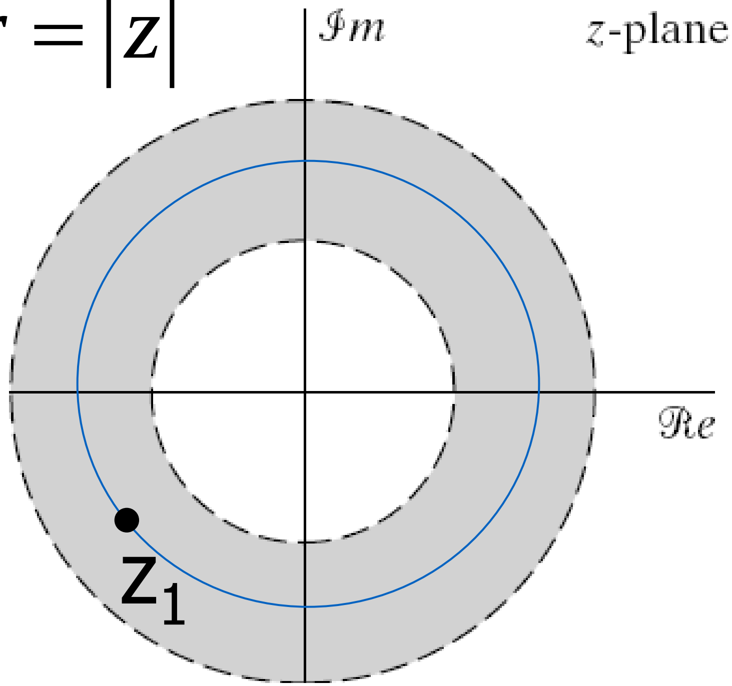
$$\left|X(z)\right| \le \sum_{n=0}^{\infty} \left|x[n]\right| \left|z\right|^{-n} < \infty$$

- Convergence of the z-transform for a given sequence depends only on

$$r = |z|$$

if some value of z, say, z $=z_1$, is in the ROC,

then all values of z on the circle defined by $|z|=|z_1|$ will also be in the ROC.



if ROC includes unit circle, then Fourier transform and all its derivatives with respect to *w* must be continuous functions of *w*.

- The z-transform is most useful when the infinite sum can be expressed in closed form, usually a ratio of polynomials in z (or z^{-1}).

$$X(z) = \frac{P(z)}{Q(z)}$$

- Zero: The value of z for which $X(z) = 0$

- Pole: The value of z for which $X(z) = \infty$

Example:

Ex.1 **Right-sided exponential sequence**

- Determine the z-transform, including the ROC in z-plane and a sketch of the pole-zero-plot, for sequence:

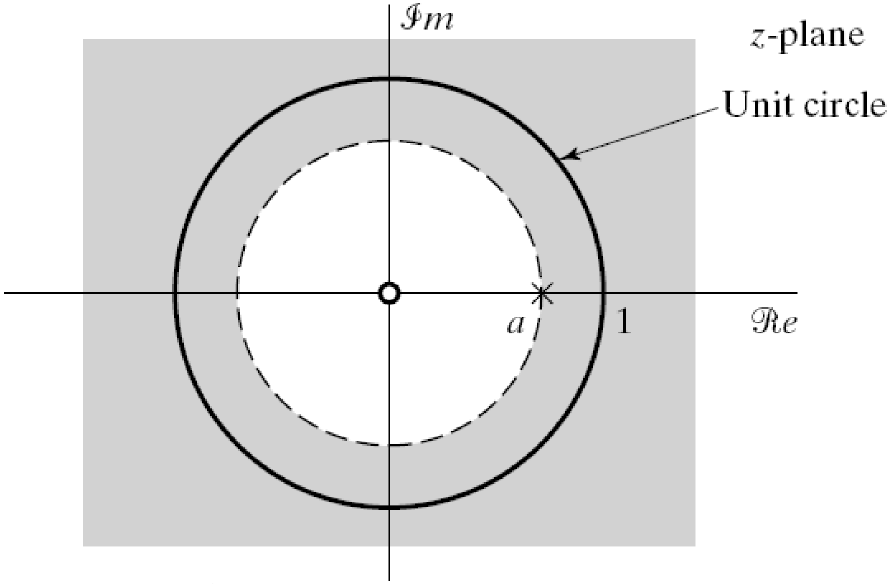
$$x[n] = a^n u[n]$$

Solution:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

ROC: $\left|az^{-1}\right| < 1 \; or \; |z| > |a|$

$$zero : z = 0 \qquad pole : z = a$$



$$x[n] = a^n u[n]$$

$$X(z) = \frac{z}{z-a}$$

$$for\ |z| > |a|$$

o : zeros

× : poles

Gray region: ROC

Ex.2 **Left-sided exponential sequence**

- Determine the z-transform, including the ROC, pole-zero-plot, for sequence:

$$x[n] = -a^n u[-n-1]$$

Solution:

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

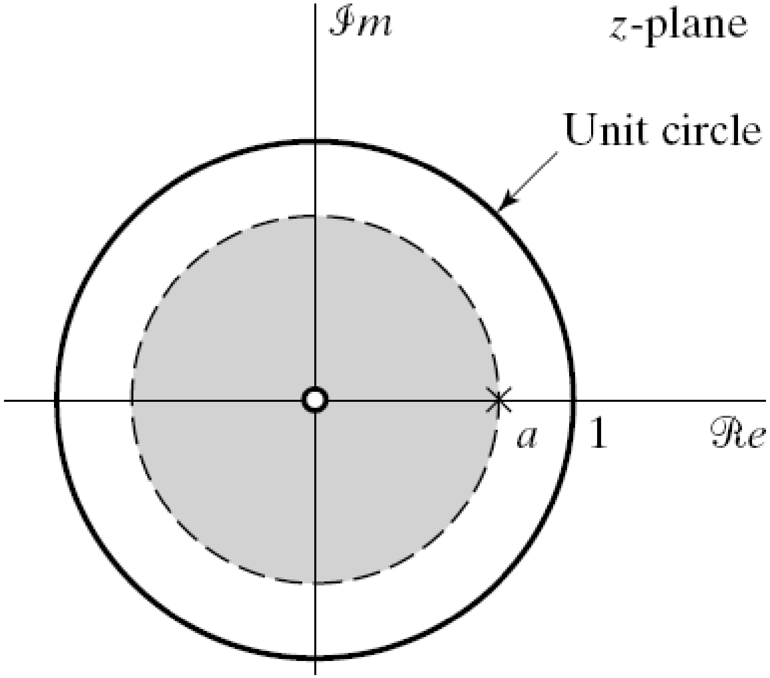
$$= -\sum_{n=1}^{\infty} a^{-n} z^n = -\sum_{n=1}^{\infty} \left(a^{-1} z\right)^n = -\frac{a^{-1} z}{1 - a^{-1} z} = \frac{z}{z-a}$$

ROC: $|z| < |a|, \quad zero : z = 0 \quad pole : z = a$

$$x[n] = -a^n u[-n-1]$$

$$X(z) = \frac{z}{z-a}$$

$$for\ |z| < |a|$$



Ex. 3 Sum of two exponential sequences

- Determine the z-transform, including the ROC, pole-zero-plot, for sequence:

Solution:

$$x[n]=\left(\frac{1}{2}\right)^n u[n]+\left(-\frac{1}{3}\right)^n u[n]$$

$$X(z)=\sum_{n=-\infty}^{\infty}\left\{\left(\frac{1}{2}\right)^n u[n]+\left(-\frac{1}{3}\right)^n u[n]\right\}z^{-n}$$

$$=\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^n u[n]z^{-n}+\sum_{n=-\infty}^{\infty}\left(-\frac{1}{3}\right)^n u[n]z^{-n}$$

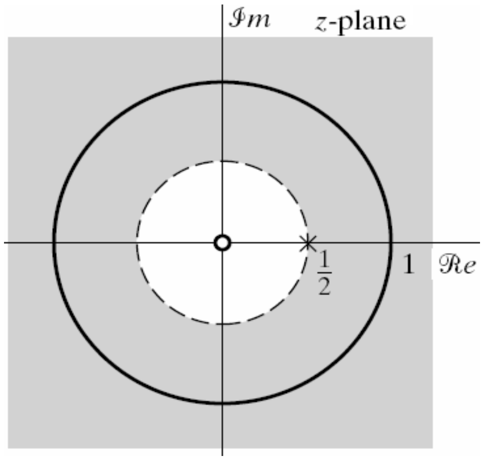
$$=\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n+\sum_{n=-\infty}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^n$$

Example 3
Sum of two exponential sequences

$$X(z)=\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^{n}+\sum_{n=-\infty}^{\infty}\left(-\frac{1}{3}z^{-1}\right)^{n}$$

$$=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{1}{1+\frac{1}{3}z^{-1}} \qquad =\frac{2z\left(z-\frac{1}{12}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}$$

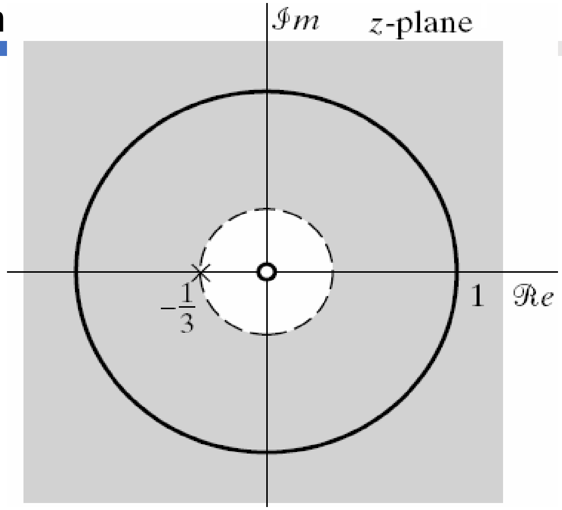
ROC: $|z|>\frac{1}{2}\ and\ |z|>\frac{1}{3}\Rightarrow ROC:|z|>\frac{1}{2}$



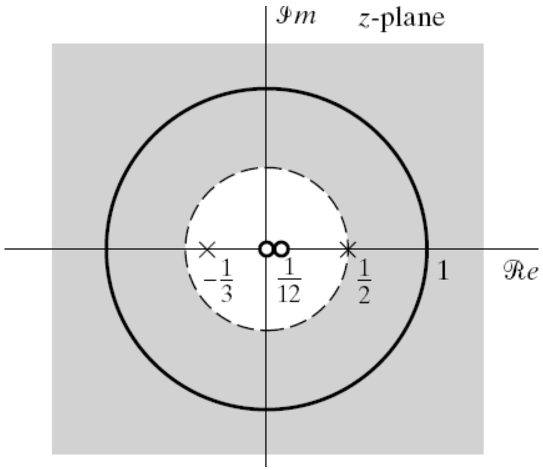
$$\frac{1}{1-\frac{1}{2}z^{-1}}$$

$+$

$$\frac{1}{1+\frac{1}{3}z}$$



$$\frac{2z\left(z-\frac{1}{12}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}$$



$$x[n]=\left(\frac{1}{2}\right)^{n}u[n]+\left(-\frac{1}{3}\right)^{n}u[n]$$

Example 4:Sum of two exponential

$$x[n]=\left(\frac{1}{2}\right)^n u[n]+\left(-\frac{1}{3}\right)^n u[n]$$

$$x[n]=a^n u[n]$$

Solution:

$$\left(\frac{1}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z|>\frac{1}{2}$$

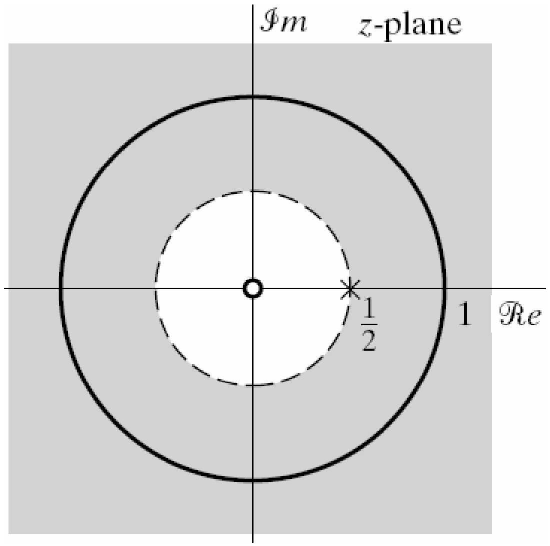
$$X(z)=\frac{z}{z-a}$$

$$\text{for } |z|>|a|$$

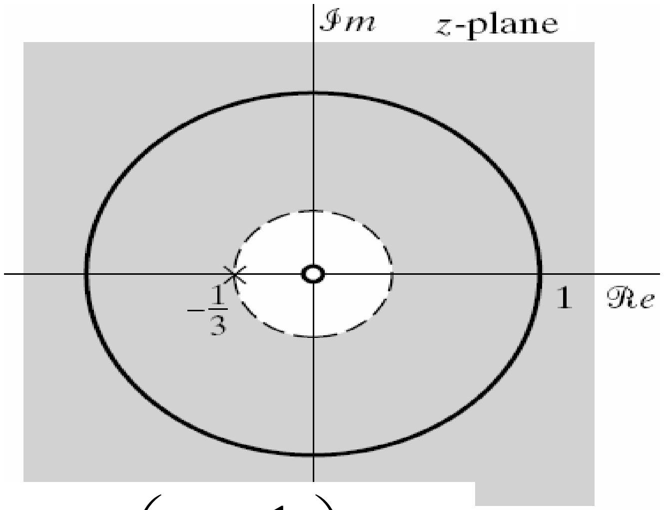
$$\left(-\frac{1}{3}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1+\frac{1}{3}z^{-1}}, \quad |z|>\frac{1}{3}$$

$$\left(\frac{1}{2}\right)^n u[n]+\left(-\frac{1}{3}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}+\frac{1}{1+\frac{1}{3}z^{-1}}, \quad |z|>\frac{1}{2}$$

ROC:



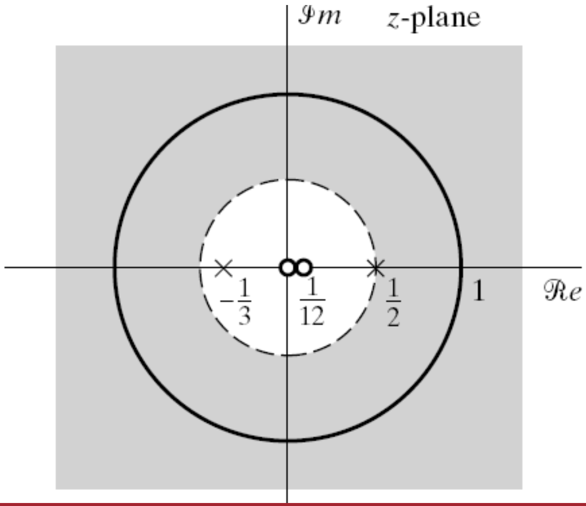
$$\frac{1}{1-\frac{1}{2}z^{-1}}$$



$$+$$

$$\frac{1}{1+\frac{1}{3}z^{-1}}$$

$$\frac{2z\left(z-\frac{1}{12}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}$$



$$x[n]=\left(\frac{1}{2}\right)^n u[n]+\left(-\frac{1}{3}\right)^n u[n]$$

Example 5:
Two-sided exponential sequence

$$x[n]=\left(-\frac{1}{3}\right)^n u[n]-\left(\frac{1}{2}\right)^n u[-n-1]$$

$$x[n]=-a^n u[-n-1]$$

$$X(z)=\frac{z}{z-a}$$

$$for\ |z|<|a|$$

Solution:

$$\left(-\frac{1}{3}\right)^n u[n]\overset{Z}{\leftrightarrow}\frac{1}{1+\frac{1}{3}z^{-1}},\quad |z|>\frac{1}{3}$$

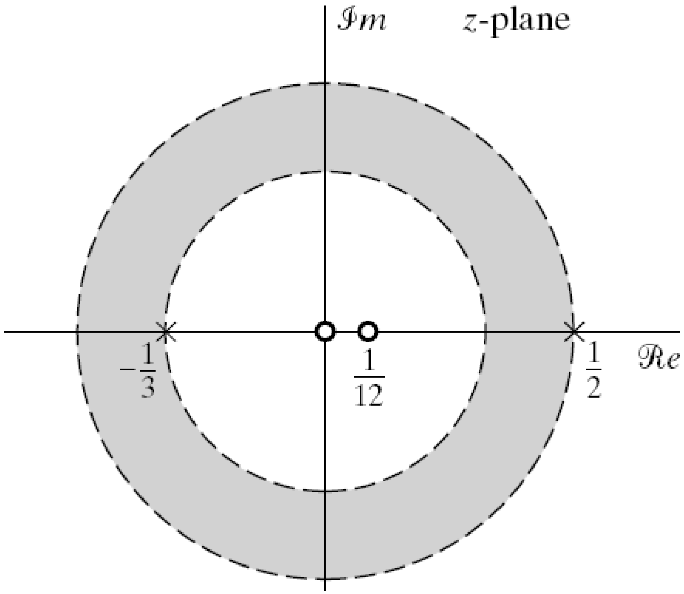
$$-\left(\frac{1}{2}\right)^n u[-n-1]\overset{Z}{\leftrightarrow}\frac{1}{1-\frac{1}{2}z^{-1}},\quad |z|<\frac{1}{2}$$

$$X(z)=\frac{1}{1+\frac{1}{3}z^{-1}}+\frac{1}{1-\frac{1}{2}z^{-1}}=\frac{2z\left(z-\frac{1}{12}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}\quad ROC:\frac{1}{3}<|z|<\frac{1}{2}$$

$$X(z)=\frac{1}{1+\frac{1}{3}z^{-1}}+\frac{1}{1-\frac{1}{2}z^{-1}}=\frac{2z\left(z-\frac{1}{12}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}\quad ROC:\frac{1}{3}<|z|<\frac{1}{2}$$

$$x[n]=\left(-\frac{1}{3}\right)^n u[n]-\left(\frac{1}{2}\right)^n u[-n-1]$$

ROC, pole-zero-plot



Finite-length sequence

$$X(z) = \sum_{n=N_1}^{N_2} x[n] Z^{-n}$$

Example :

$$x[n] = \delta[n] + \delta[n-5]$$

$$X(z) = 1 + z^{-5} \quad ROC : |z| > 0$$

Example 6: Finite-length sequence

- Determine the z-transform, including the ROC, pole-zero-plot, for sequence:

$$x[n]=\begin{cases} a^n, & 0\le n\le N-1 \\ 0, & otherwise \end{cases}$$

Solution:

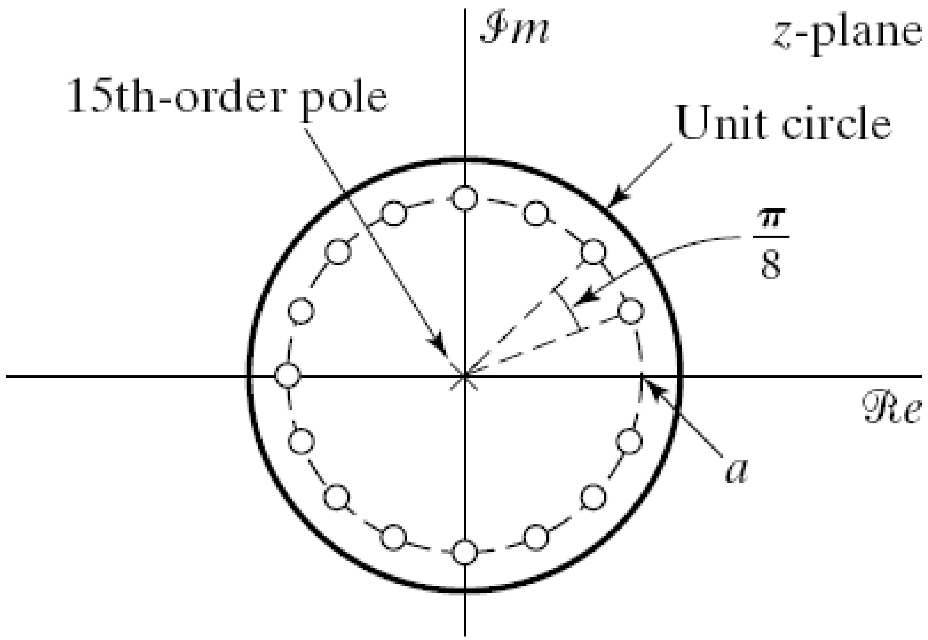
$$X(z)=\sum_{n=0}^{N-1} a^n z^{-n}=\sum_{n=0}^{N-1}\left(az^{-1}\right)^n$$

$$=\frac{1-\left(az^{-1}\right)^N}{1-az^{-1}}=\frac{1}{z^{N-1}}\frac{z^N-a^N}{z-a} \qquad ROC:|z|>0$$

N=16, *a* is real

$$X(z) = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a} \qquad ROC: |z| > 0$$

pole-zero-plot



z-transform pairs

$$\delta[n] \leftrightarrow 1, \quad ROC: all \ \ z$$

$$u[n] \leftrightarrow \frac{1}{1-z^{-1}}, \quad ROC: |z| > 1$$

$$-u[-n-1] \leftrightarrow \frac{1}{1-z^{-1}}, \quad ROC: |z| < 1$$

$$\delta[n-m] \leftrightarrow z^{-m},$$

$$ROC: all \ \ z \ except \ 0 \left(if \ \ m > 0\right) or \ \infty \left(if \ \ m < 0\right)$$

z-transform pairs

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}, \quad ROC: |z| > |a|$$

$$-a^n u[-n-1] \leftrightarrow \frac{1}{1-az^{-1}}, \quad ROC: |z| < |a|$$

$$na^n u[n] \leftrightarrow \frac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad ROC: |z| > |a|$$

$$-na^n u[-n-1] \leftrightarrow \frac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad ROC: |z| < |a|$$

z-transform pairs

$$[\cos w_0 n]u[n] \quad \leftrightarrow \quad \frac{1-[\cos w_0]z^{-1}}{1-2[\cos w_0]z^{-1}+z^{-2}}, \quad ROC: |z|>1$$

$$[\cos w_0 n]u[n] = \frac{1}{2}\left(e^{jw_0 n} + e^{-jw_0 n}\right)u[n]$$

$$\frac{1}{2}\left(\frac{1}{1-e^{jw_0}z^{-1}} + \frac{1}{1-e^{-jw_0}z^{-1}}\right)$$

$$[\sin w_0 n]u[n] \quad \leftrightarrow \quad \frac{[\sin w_0]z^{-1}}{1-2[\cos w_0]z^{-1}+z^{-2}}, \quad ROC: |z|>1$$

z-transform pairs

$$\left[r^n \cos w_0 n\right]u[n] \leftrightarrow \frac{1-[r\cos w_0]z^{-1}}{1-[2r\cos w_0]z^{-1}+r^2 z^{-2}}, \quad ROC: |z|>r$$

$$\left[r^n \cos w_0 n\right]u[n]=\frac{1}{2}\left(r^n e^{jw_0 n}+r^n e^{-jw_0 n}\right)u[n]$$

$$\frac{1}{2}\left(\frac{1}{1-re^{jw_0}z^{-1}}+\frac{1}{1-re^{-jw_0}z^{-1}}\right)$$

$$\left[r^n \sin w_0 n\right]u[n] \leftrightarrow \frac{[r\sin w_0]z^{-1}}{1-[2r\cos w_0]z^{-1}+r^2 z^{-2}}, \quad ROC: |z|>r$$

z-transform pairs

$$\begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & otherwise \end{cases} \leftrightarrow \frac{1-a^N z^{-N}}{1-az^{-1}},$$

$$ROC: |z| > 0$$

3.2 Properties of the ROC for the z-transform

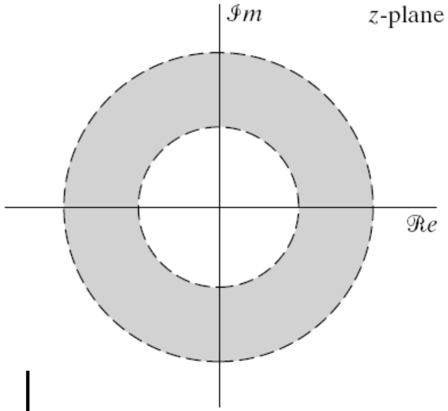
- Property 1: The ROC is a ring or disk in the z-plane centered at the origin.

$$0 \le r_R < |z| < r_L \le \infty$$

- For a given x[n], ROC is dependent only on $|z|$

$$r_R{}^n u[n] - r_L a^n u[-n-1]$$

$$\frac{1}{1 - r_R z^{-1}} + \frac{1}{1 - r_L z^{-1}}, \quad ROC: |r_R| < |z| < |r_L|$$



3.2 Properties of the ROC for the z-transform

- Property 2: The Fourier transform of $x[n]$ converges absolutely if the ROC of the z-transform of $x[n]$ includes the unit circle.

- The z-transform reduces to the Fourier transform when $|z| = 1$ ie. $z = e^{jw}$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \Longrightarrow X\left(e^{jw}\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

3.2 Properties of the ROC for the z-transform

Property 3: The ROC cannot contain any poles.

- $X(z)$ is infinite at a pole and therefore does not converge.

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = \frac{P(z)}{Q(z)}$$

3.2 Properties of the ROC for the z-transform

- Property 4: If $x[n]$ is a finite-duration sequence, i.e., a sequence that is zero except in a finite interval : $N_1 \le n \le N_2$

- then the ROC is the entire z-plane, except possible $z = 0$ or $z = \infty$

$$X(z) = \sum_{n=N_1}^{N_2} a^n z^{-n}$$

3.2 Properties of the ROC for the z-transform

- Property 5: If $x[n]$ is a right-sided sequence, i.e., a sequence that is zero for $n \le N_1 < \infty$, the ROC extends outward from the outermost finite pole in $X(z)$ to (may including) $z = \infty$

Proof: $$x[n] = \sum_{k=1}^{N} A_k (d_k)^n, n \ge N_1, \qquad \sum_{n=-\infty}^{\infty} |x[n]| r^{-n} < \infty$$

$$\sum_{n=N_1}^{\infty} \left| \sum_{k=1}^{N} A_k (d_k)^n \right| r^{-n} \le \sum_{k=1}^{N} |A_k| \left(\sum_{n=N_1}^{\infty} |d_k / r|^n \right) < \infty$$

$$if \quad r > |d_N| > |d_{N-1}| > \ldots > |d_1|, \quad i.e. \quad r > |d_N|$$

3.2 Properties of the ROC for the z-transform

- Property 6: If $x[n]$ is a left-sided sequence, i.e., a sequence that is zero for $n \geq N_2 > -\infty$, the ROC extends inward from the innermost nonzero pole in $X(z)$ to $z=0$.

Proof: $$x[n] = \sum_{k=1}^{N} A_k (d_k)^n, n < N_2, \qquad \sum_{n=-\infty}^{\infty} |x[n]| r^{-n} < \infty$$

$$\sum_{n=-\infty}^{n=N_2} \left| \sum_{k=1}^{N} A_k (d_k)^n \right| r^{-n} \leq \sum_{k=1}^{N} |A_k| \left(\sum_{n=-\infty}^{n=N_2} |d_k / r|^n \right) < \infty$$

$$r < |d_k|, \quad r < |d_1|, \ \ldots, \ r < |d_N|, \qquad r < |d_1|$$

3.2 Properties of the ROC for the z-transform

- Property 5: If $x[n]$ is a right-sided sequence, i.e., a sequence that is zero for $n \le N_1 < \infty$, the ROC extends outward from the outermost finite pole in $X(z)$ to (possibly including) $z = \infty$

Proof:

$$x[n] = \sum_{k=1}^{N} A_k (d_k)^n \qquad n \ge N_1$$

$$= \sum_{k=1}^{N} A_k (d_k)^n u[n], \qquad if \;\; N_1 = 0, ie. \;\; n \ge 0$$

Property 5: right-sided sequence

$$x[n] = \sum_{k=1}^{N} A_k (d_k)^n \quad = \sum_{k=1}^{N} A_k (d_k)^n u[n], \qquad n \geq 0$$

for $A_k (d_k)^n u[n]$ the z-transform:

$$\sum_{n=0}^{\infty} A_k (d_k)^n z^{-n} \quad = \frac{A_k}{1 - d_k z^{-1}}, \quad |z| = r > |d_k|$$

For other terms:

$$r > |d_1|, \; \ldots, \; r > |d_N|$$

$$if \;\; |d_N| > |d_{N-1}| > \ldots > |d_1|$$

ROC

$$r > |d_N|$$

3.2 Properties of the ROC for the z-transform

- Property 6: If $x[n]$ is a left-sided sequence, i.e., a sequence that is zero for $n \geq N_2 > -\infty$, the ROC extends inward from the innermost nonzero pole in $X(z)$ to $z = 0$

Proof:

$$x[n] = \sum_{k=1}^{N} A_k (d_k)^n \qquad n < N_2$$

$$= \sum_{k=1}^{N} A_k (d_k)^n u[-n-1], \qquad if \ N_2 = 0, ie. \ n < 0$$

Property 6: **left-sided sequence**

$$x[n]=\sum_{k=1}^{N} A_k (d_k)^n = \sum_{k=1}^{N} A_k (d_k)^n u[-n-1]$$

$$n<0$$

for $A_k'\left[-(d_k)^n u[-n-1]\right]$ the z-transform:

$$\sum_{n=-\infty}^{-1} -A_k'(d_k)^n z^{-n} = \frac{A_k'}{1-d_k z^{-1}}, \quad |z| = r < |d_k|$$

For other terms:

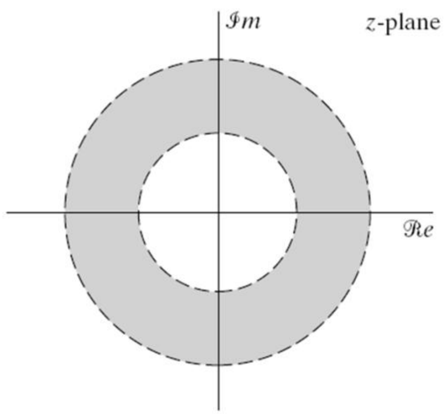
$$r<|d_1|, \ \ldots, \ r<|d_N| \longrightarrow r<|d_1|$$

ROC

3.2 Properties of the ROC for the z-transform

- Property 7: A two-sided sequence is an infinite-duration sequence that is neither right-sided nor left-sided.

If $x[n]$ is a two-sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and not containing any poles.



3.2 Properties of the ROC for the z-transform

Property 8: ROC must be a connected region.

- for finite-duration sequence

  ROC: $0 < |z| < \infty$ possible $z = 0$, $z = \infty$

- for right-sided sequence

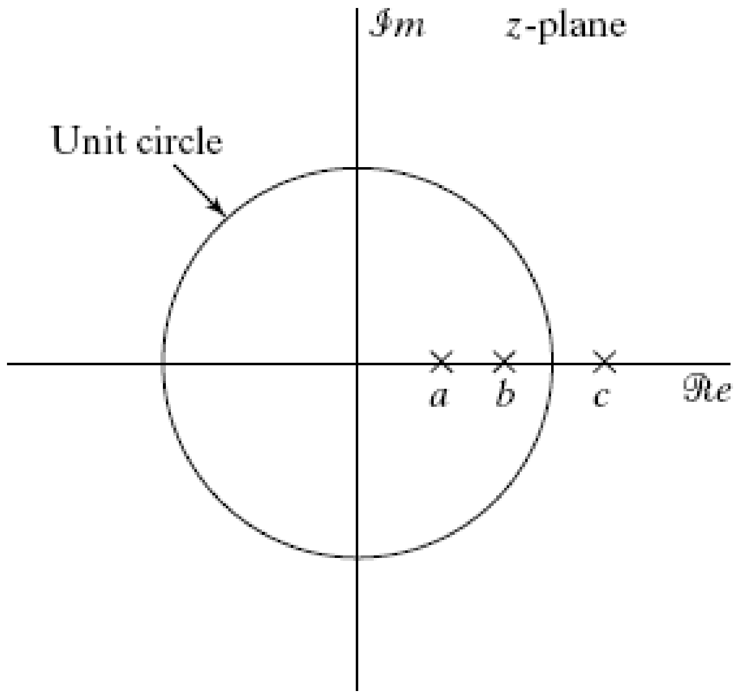
  ROC: $r_R < |z| < \infty$ possible $z = \infty$

- for left-sided sequence

  ROC: $0 < |z| < r_L$ possible $z = 0$

- for two-sided sequence $r_R < |z| < r_L$

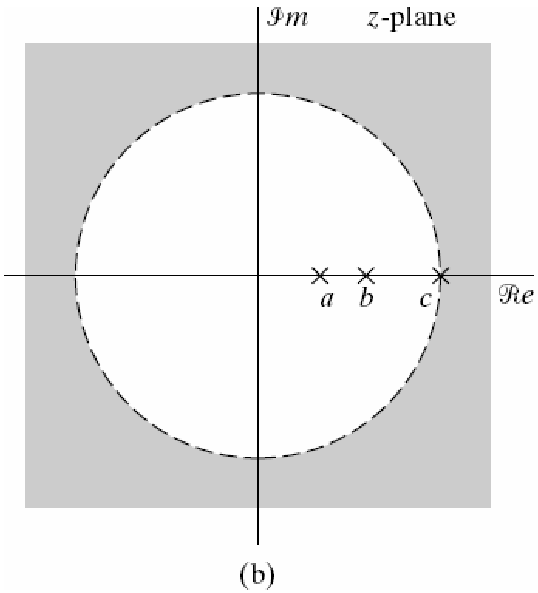
Example: Different possibilities of the ROC define different sequences



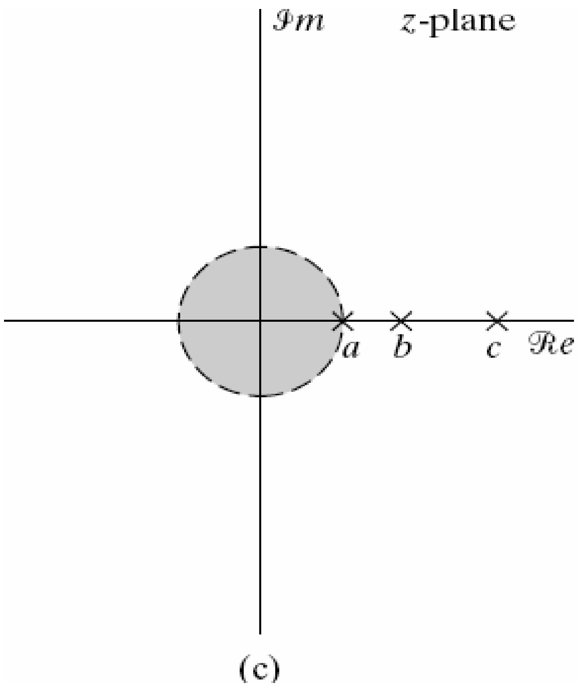
(a)

A system with three poles

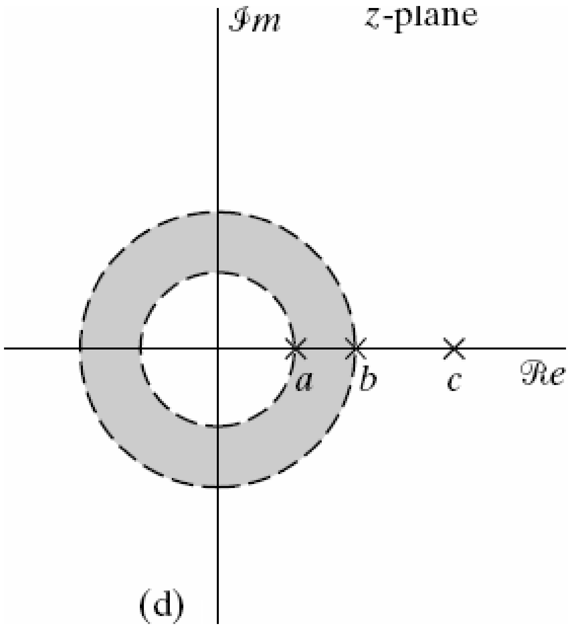
Different possibilities of the ROC.



(b) ROC to a right-sided sequence

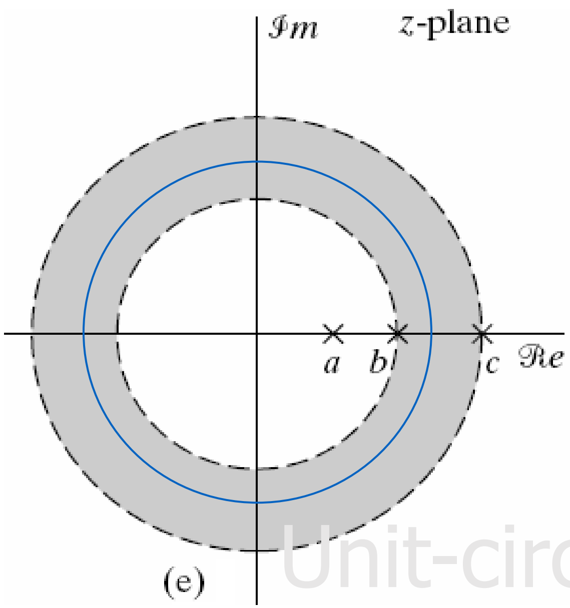


(c) ROC to a left-handed sequence



(d) ROC to a

two-sided sequence.



Unit-circle included

(e) ROC to another two-sided sequence

LTI system Stability, Causality, and ROC

A z-transform does not uniquely determine a sequence without specifying the ROC

- It's convenient to specify the ROC implicitly through time-domain property of a sequence

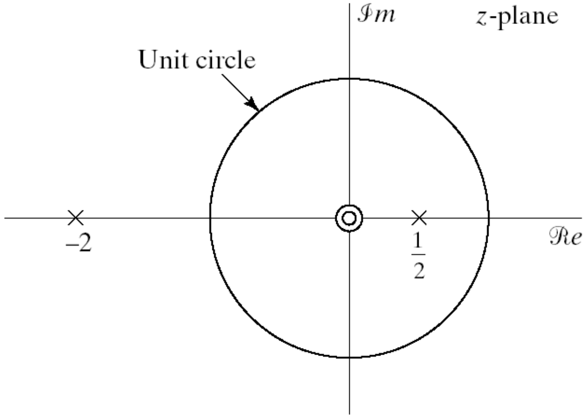
- Consider a LTI system with impulse response *h[n]*. The z-transform of *h[n]* is called the *system function H (z)* of the LTI system.

- stable system(*h[n]* is absolutely summable and therefore has a Fourier transform): ROC include unit-circle.
- causal system ($h[n]=0$,for $n<0$) : right sided

Ex. 3.7 Stability, Causality, and the ROC

Consider a LTI system with impulse response *h[n]*. The z-transform of *h[n]* i.e. the *system function H (z)* has the pole-zero plot shown in Figure. Determine the *ROC,* if the system is:

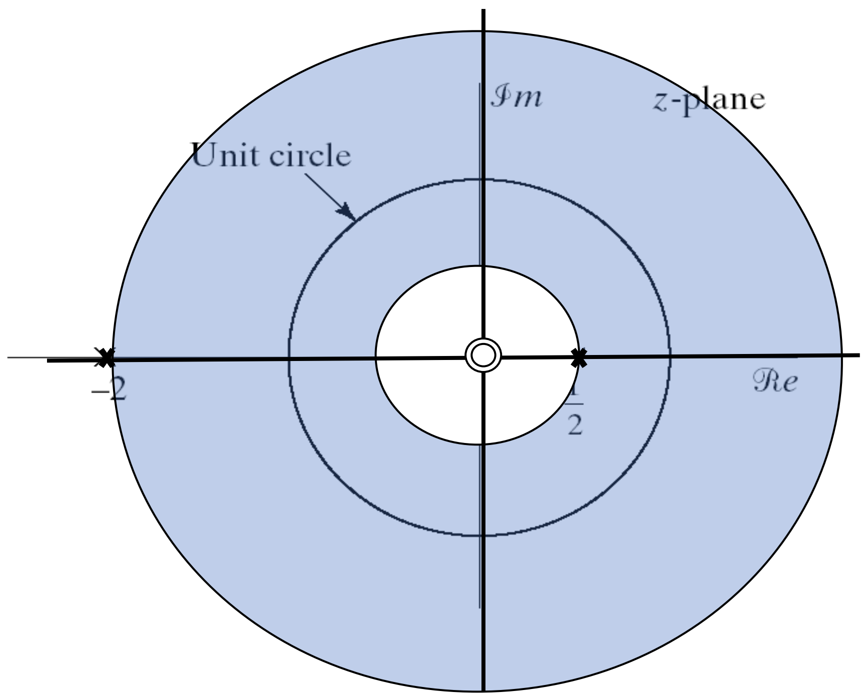
- (1) stable system: (ROC include unit-circle)
- (2) causal system: (right sided sequence)



Ex. 3.7 Stability, Causality, and the ROC

Solution: (1) stable system (ROC include unit-circle),

ROC: $\frac{1}{2} < |z| < 2$, the impulse response is two-sided, system is non-causal. **stable**.

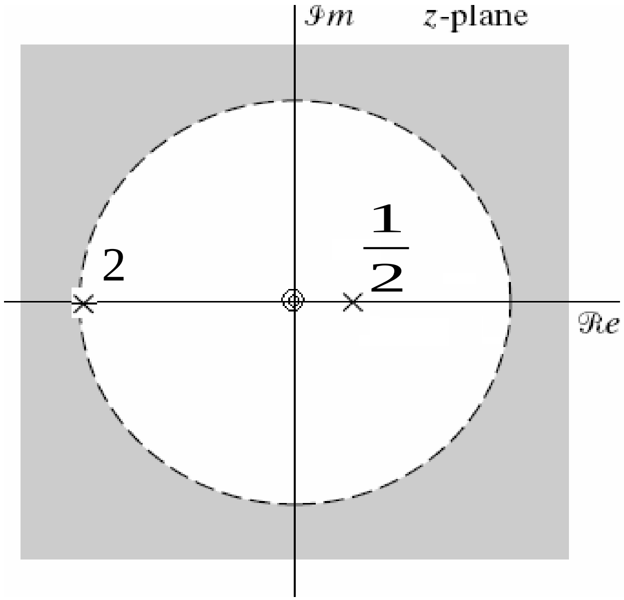


Ex. 3.7 Stability, Causality, and the ROC

- (2) causal system: (right sided sequence)

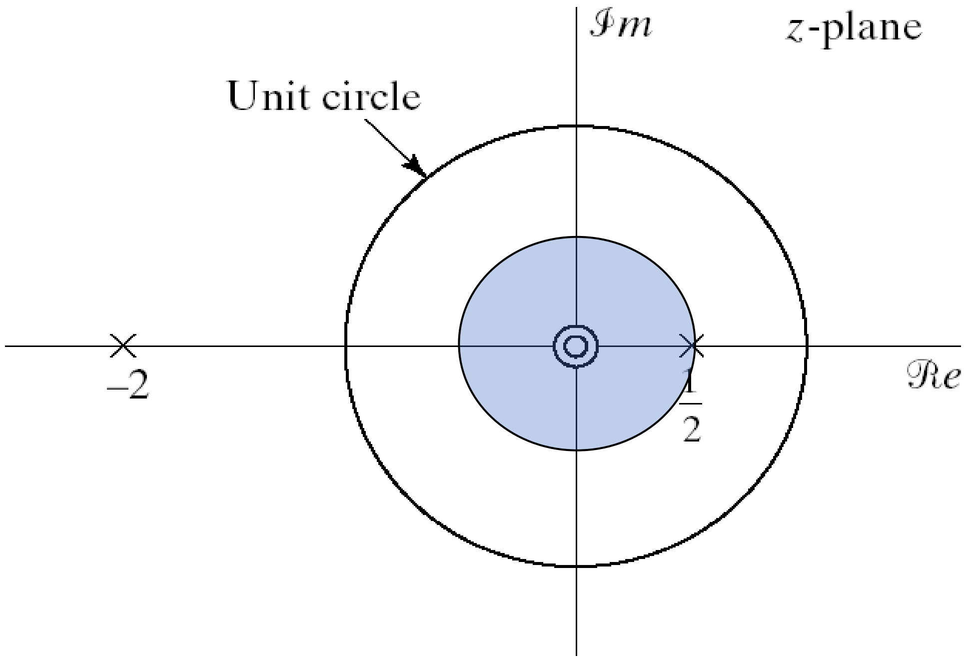
ROC: $|z| > 2$,the impulse response is right-sided. system is causal but unstable.

- A system is causal and stable if all the poles are **inside the unit circle**.



Ex. 3.7 Stability, Causality, and the ROC

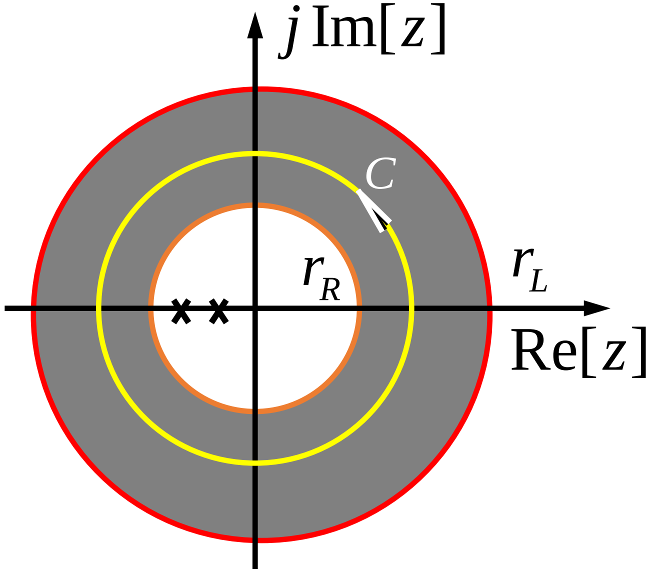
ROC: $|z| < \frac{1}{2}$, the impulse response is left-sided, system is non-causal, unstable since the ROC does not include unit circle.



3.3 The Inverse Z-Transform

Formal inverse z-transform is based on a Cauchy integral theorem.

$$x_n = \frac{1}{2\pi j}\oint_c X(z) z^{n-1} dz \qquad c \in ROC$$



3.3 The Inverse Z-Transform

- Less formal ways are sufficient and preferable in finding the inverse z-transform. :
  - Inspection method
  - Partial fraction expansion
  - Power series expansion

3.3 The inverse z-Transform

3.3.1 Inspection Method

$$a^n u[n] \overset{Z}{\leftrightarrow} \frac{1}{1-az^{-1}}, \quad |z|>|a|$$

$$X(z)=\frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z|>\frac{1}{2}$$

$$x[n]=\left(\frac{1}{2}\right)^n u[n]$$

3.3 The inverse z-Transform

3.3.1 Inspection Method

$$-a^n u[-n-1] \overset{Z}{\leftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| < |a|$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| < \frac{1}{2}$$

$$x[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$$

3.3 The inverse z-Transform

3.3.2 Partial Fraction Expansion

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}} = \frac{b_0}{a_0} \frac{\prod_{k=0}^{M} \left(1 - c_k z^{-1}\right)}{\prod_{k=0}^{N} \left(1 - d_k z^{-1}\right)}$$

$$= \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}} \qquad if\ M < N$$

$$where\ A_k = \left(1 - d_k z^{-1}\right) X(z)\Big|_{z=d_k}$$

Example 3.8: Second-Order z-Transform

$$X(z)=\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)},\quad |z|>\frac{1}{2}$$

$$X(z)=\frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

$$A_1=\left(1-\frac{1}{4}z^{-1}\right)X(z)\Big|_{z=\frac{1}{4}}=-1 \qquad A_2=\left(1-\frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}}=2$$

Example 3.8 : Second-Order z-Transform

$$X(z)=\frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{2}{\left(1-\frac{1}{2}z^{-1}\right)},\quad |z|>\frac{1}{2}$$

$$x[n]=2\left(\frac{1}{2}\right)^{n}u[n]-\left(\frac{1}{4}\right)^{n}u[n]$$

Inverse Z-Transform by Partial Fraction Expansion

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad if\ M \geq N$$

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

Br is obtained by long division

Inverse Z-Transform by Partial Fraction Expansion

- if M>N, and $X(z)$ has a pole of order s at $d=d_i$

$$X(z)=\sum_{r=0}^{M-N} B_r z^{-r}+\sum_{k=1,k\neq i}^{N}\frac{A_k}{1-d_k z^{-1}}+\sum_{m=1}^{s}\frac{C_m}{\left(1-d_i z^{-1}\right)^m}$$

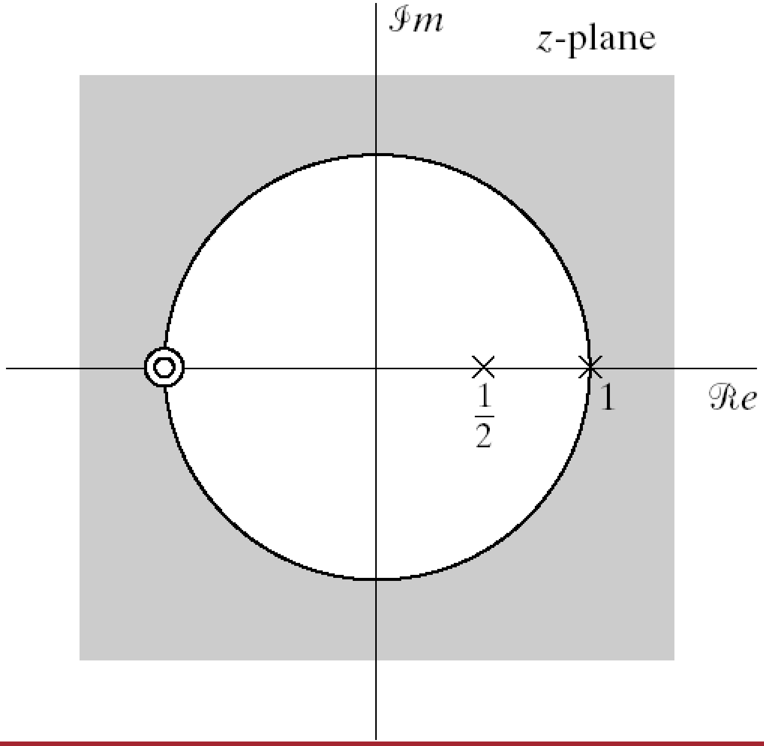
Br is obtained by long division

$$A_k=\left(1-d_k z^{-1}\right)X(z)\Big|_{z=d_k}$$

$$C_m=\frac{1}{(s-m)!(-d_i)^{s-m}}\left\{\frac{d^{s-m}}{dw^{s-m}}\left[(1-d_i w)^s X\left(w^{-1}\right)\right]\right\}_{w=d_i^{-1}}$$

Example 3.9:
Inverse by Partial Fractions

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}=\frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)},\quad |z|>1$$



$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}=B_0+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}$$

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & z^{-2}+2z^{-1}+1 \\ & \underline{z^{-2}-3z^{-1}+2} \\ & \quad 5z^{-1}-1 \end{array}$$

$$X(z)=2+\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)}$$

$$X(z) = 2 + \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} = 2 + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-z^{-1}}$$

$$A_1 = \left[\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} \times \left(1-\frac{1}{2}z^{-1}\right)\right]_{z=\frac{1}{2}} = -9$$

$$A_2 = \left[\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} \times \left(1-z^{-1}\right)\right]_{z=1} = 8$$

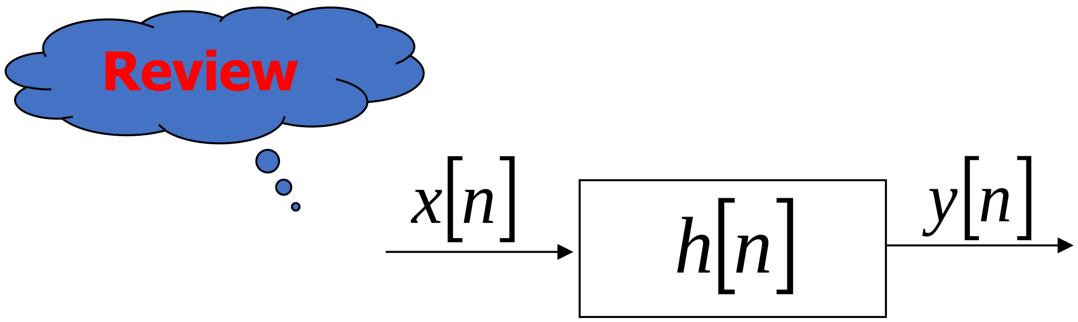
$$X(z) = 2 - \frac{9}{\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{8}{\left(1 - z^{-1}\right)}, \quad |z| > 1$$

$$2 \xleftrightarrow{Z} 2\delta[n] \qquad \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)} \xleftrightarrow{z} \left(\frac{1}{2}\right)^n u[n]$$

$$\frac{1}{\left(1 - z^{-1}\right)} \xleftrightarrow{z} u[n]$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n]$$

LTI system Stability, Causality, and ROC



- For a LTI system with impulse response h[n], if it is causal, what do we know about h[n]? Is h[n] one-sided or two-sided sequence? Left-sided or right-sided?
- Then what do we know about the ROC of the *system function H (z)?*
- If the poles of H (z) are all in the unit circle, is the system stable?

LTI system Stability, Causality, and ROC

- For H (z) with the poles as shown in figure ,

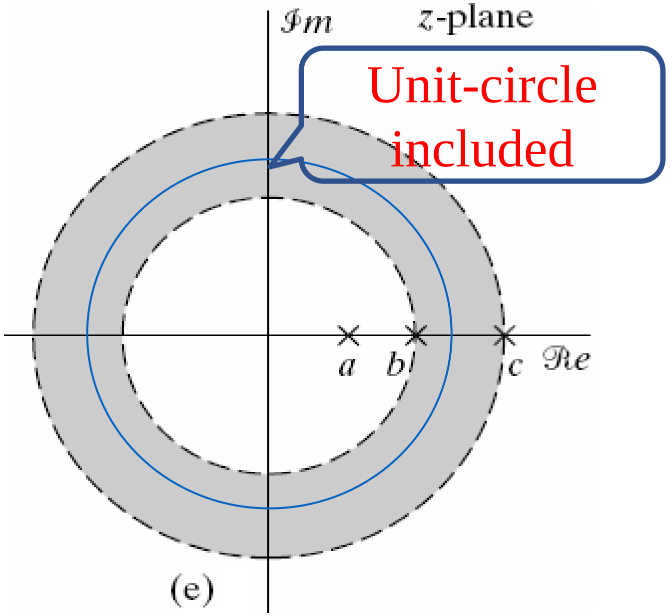
Review

$$H(z) = \frac{1}{(1 - az^{-1})(1 - bz^{-1})(1 - cz^{-1})}$$

can we uniquely determine *h[n]* ?

- If ROC of H(z) is as shown in figure, can we uniquely determine *h[n]* ?

- is the system stable ?

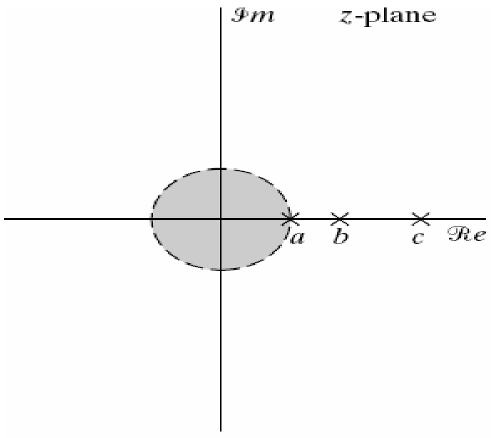


LTI system Stability, Causality, and ROC

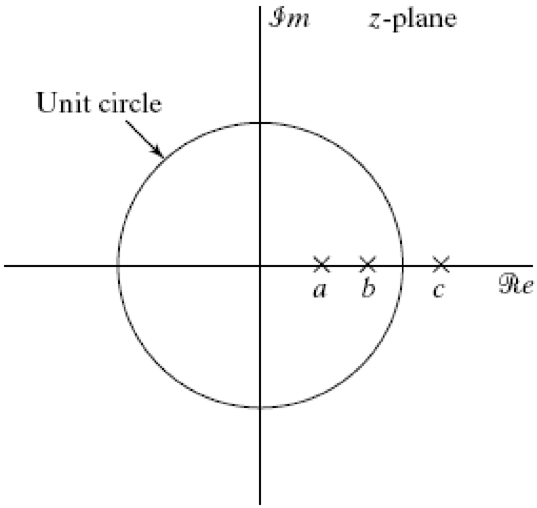
- For H (z) with the poles as shown in figure ,

Review

$$H(z)=\frac{1}{\left(1-az^{-1}\right)\left(1-bz^{-1}\right)\left(1-cz^{-1}\right)}$$



(c)



(a)

- If the system is causal (h[n]=0,for n<0,right-sided), What's the ROC like?

- If ROC is as shown in figure, is *h[n]* one-sided or two-sided? Is the system causal or stable?

3.3 The Inverse Z-Transform

Review

- Inspection method
- Partial fraction expansion
- Power series expansion

$$\delta[n] \leftrightarrow 1, \quad ROC: all \ \ z$$

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}, \quad ROC: |z| > |a|$$

$$-a^n u[-n-1] \leftrightarrow \frac{1}{1-az^{-1}}, \quad ROC: |z| < |a|$$

Review

Partial Fraction Expansion

- if M>N, and $X(z)$ has a pole of order s at d=d_i

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1,k\neq i}^{N} \frac{A_k}{1-d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1-d_i z^{-1}\right)^m}$$

Br is obtained by long division

$$A_k = \left(1-d_k z^{-1}\right)X(z)\Big|_{z=d_k}$$

$$C_m = \frac{1}{(s-m)!(-d_i)^{s-m}}\left\{\frac{d^{s-m}}{dw^{s-m}}\left[\left(1-d_i w\right)^s X\left(w^{-1}\right)\right]\right\}_{w=d_i^{-1}}$$

3.3 The inverse z-Transform

3.3.3 Power Series Expansion

$$X(z) = \sum_{n=-\infty}^{n=\infty} x[n] z^{-n}$$

$$= \cdots + x[-2]z^{2} + x[-1]z^{1} + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

Example 3.10:
Finite-Length Sequence

$$X(z) = z^2\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})(1-z^{-1}) = z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}$$

$$x[n] = \begin{cases} 1, & n=-2 \\ -\frac{1}{2}, & n=-1 \\ -1, & n=0 \\ \frac{1}{2}, & n=1 \\ 0, & otherwise \end{cases}$$

$$x[n] = \delta[n+2] - \frac{1}{2}\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1]$$

Ex. 3.11: Inverse Transform by power series expansion

$$X(z) = \log\left(1 + az^{-1}\right), \quad |z| > |a|$$

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \qquad |x| < 1$$

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}, \quad \left|az^{-1}\right| < 1$$

$$x(n) = \begin{cases} (-1)^{n+1} \dfrac{a^n}{n}, & n \geq 1 \\ 0, & n < 1 \end{cases} = (-1)^{n+1} \frac{a^n}{n} u[n-1]$$

Example 3.12: Power Series Expansion by Long Division

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a| \qquad \overset{Z}{\longleftrightarrow} a^n u[n]$$

$$\begin{array}{r l} & 1 + az^{-1} + a^2 z^{-2} \\ \hline 1 - az^{-1} \big) & 1 \\ & 1 - az^{-1} \\ \hline & az^{-1} \\ & az^{-1} - a^2 z^{-2} \\ \hline & a^2 z^{-2} \cdots \end{array}$$

$$\frac{1}{1 - az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + \cdots$$

$$x[n] = a^n u[n]$$

Example 3.13: Power Series Expansion for a Left-sided Sequence

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| < |a| \quad \overset{z}{\longleftrightarrow} -a^n u[-n-1]$$

$$\begin{array}{r|l} & -a^{-1}z - a^{-2}z^2 \\ \hline -az^{-1}+1 & 1 \\ & 1 - a^{-1}z \\ \hline & a^{-1}z \\ & a^{-1}z - a^{-2}z^2 \\ \hline & a^{-2}z^2 \cdots \end{array}$$

$$\frac{1}{1 - az^{-1}} = -a^{-1}z - a^{-2}z^2 + \cdots$$

$$x[n] = -a^n u[-n-1]$$

3.4 z-Transform Properties

$$x[n] \xleftrightarrow{Z} X(z), \qquad ROC = R_x$$

3.4.1 Linearity

$$x_1[n] \xleftrightarrow{Z} X_1(z), \qquad ROC = R_{x_1}$$

$$x_2[n] \xleftrightarrow{Z} X_2(z), \qquad ROC = R_{x_2}$$

$$ax_1[n] + bx_2[n] \xleftrightarrow{Z} aX_1(z) + bX_2(z)$$

ROC *contains* (*may* more than) $R_{x_1} \cap R_{x_2}$

Example of Linearity

$$x[n] = a^n u[n] - a^n u[n-N]$$

$$a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, \quad ROC: |z| > a$$

$$a^n u[n-N] \xleftrightarrow{Z} \frac{a^N z^{-N}}{1-az^{-1}}, \quad ROC: |z| > a$$

$$\frac{1}{1-az^{-1}} - \frac{a^N z^{-N}}{1-az^{-1}} \xleftrightarrow{Z} a^n u[n] - a^n u[n-N]$$

$$= \left(1 + az^{-1} + a^2 z^{-2} + \cdots + a^{N-1} z^{-(N-1)}\right) \qquad |z| > 0$$

$$= \frac{\left(1-az^{-1}\right)\left(1 + az^{-1} + a^2 z^{-2} + \cdots + a^{N-1} z^{-(N-1)}\right)}{1-az^{-1}}$$

3.4.2 Time Shifting

$$x[n-n_0] \xleftrightarrow{Z} z^{-n_0} X(z)$$

- n_0 is an integer
  - n_0 is positive, $x[n]$ is shifted right
  - n_0 is negative, $x[n]$ is shifted left

$$ROC = R_x \begin{pmatrix} \text{except for the possible addition} \\ \text{or deletion of } z=0 \text{ or } z=\infty \end{pmatrix}$$

Time Shifting: Proof

$$if\ \ y[n] = x[n - n_0]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n - n_0] z^{-n} \qquad Let\ m = n - n_0$$

$$Y(z) = \sum_{m=-\infty}^{\infty} x[m] z^{-(m+n_0)}$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m] z^{-m} = z^{-n_0} X(z)$$

Example 3.14: Shifted Exponential Sequence

$$X(z)=\frac{1}{z-\frac{1}{4}}, \quad |z|>\frac{1}{4} \quad \xleftrightarrow{Z} \left(\frac{1}{4}\right)^n u[n]$$

$$X(z)=\frac{z^{-1}}{1-\frac{1}{4}z^{-1}}=-4+\frac{4}{1-\frac{1}{4}z^{-1}}$$

$$x[n]=-4\delta[n]+4\left(\frac{1}{4}\right)^n u[n]$$

$$X(z)=z^{-1}\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$$

$$x[n]=\left(\frac{1}{4}\right)^{n-1} u[n-1]$$

3.4.3 Multiplication by an Exponential sequence

$$z_0^n x[n] \xleftrightarrow{Z} X(z/z_0), \quad ROC = |z_0| R_x$$

$$if \;\; R_x \; is \; r_R < |z| < r_L, \; |z_0| R_x \; is \;\; |z_0| r_R < |z| < |z_0| r_L$$

$$if \;\; z_0 = e^{jw_0} \qquad\qquad z = e^{jw}$$

$$e^{jw_0 n} x[n] \xleftrightarrow{F} X\left(e^{j(w-w_0)}\right): AM$$

Example 3.15:
Exponential Multiplication

$$u[n] \overset{Z}{\longleftrightarrow} \frac{1}{1-z^{-1}}, \quad |z|>1$$

$$x[n]=r^n \cos(w_0 n)u[n]$$

$$=\frac{1}{2}\left(re^{jw_0}\right)^n u[n]+\frac{1}{2}\left(re^{-jw_0}\right)^n u[n]$$

$$\frac{1}{2}\left(re^{jw_0}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{\frac{1}{2}}{1-re^{jw_0}z^{-1}}, \quad |z|>r$$

$$x[n]=\frac{1}{2}\left(re^{jw_0}\right)^n u[n]+\frac{1}{2}\left(re^{-jw_0}\right)^n u[n]$$

$$\frac{1}{2}\left(re^{-jw_0}\right)^n u[n]\xleftrightarrow{Z}\frac{\frac{1}{2}}{1-re^{-jw_0}z^{-1}},\quad |z|>r$$

$$X(z)=\frac{\frac{1}{2}}{1-re^{jw_0}z^{-1}}+\frac{\frac{1}{2}}{1-re^{-jw_0}z^{-1}}$$

$$=\frac{\left(1-r\cos w_0 z^{-1}\right)}{1-2r\cos w_0 z^{-1}+r^2z^{-2}},\quad |z|>r$$

3.4.4 Differentiation of X(z)

$$nx[n] \xleftrightarrow{Z} -z\frac{dX(z)}{dz}, \quad ROC = R_x$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$-z\frac{dX(z)}{dz} = -z\sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = Z\{nx[n]\}$$

Example 3.16:
Inverse of Non-Rational z-Transform

$$X(z) = \log\left(1 + az^{-1}\right), \quad |z| > |a|$$

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}}$$

$$nx[n] \xleftrightarrow{Z} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1-(-a)z^{-1}}, \quad |z| > |a|$$

$$nx[n] = a(-a)^{n-1}u[n-1]$$

Look

$$(-1)^{n-1}\frac{a^n}{n}u[n-1] \xleftrightarrow{Z} \log\left(1 + az^{-1}\right), \quad |z| > |a|$$

Example 3.17: Second-Order Pole

$$x[n] = na^n u[n] = n\left(a^n u[n]\right)$$

$$a^n u[n] \xleftrightarrow{\;Z\;} \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$X(z) = -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right), \quad |z| > |a|$$

$$= \frac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad |z| > |a|$$

$$na^n u[n] \xleftrightarrow{\;Z\;} \frac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad |z| > |a|$$

3.4.5 Conjugation of a complex Sequence

$$x^*[n] \overset{Z}{\longleftrightarrow} X^*(z^*), \quad ROC = R_x$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$\sum_{n=-\infty}^{\infty} x^*[n] z^{-n} = \left(\left(\sum_{n=-\infty}^{\infty} x^*[n] z^{-n} \right)^* \right)^*$$

$$= \left(\sum_{n=-\infty}^{\infty} x[n] (z^*)^{-n} \right)^* = X^*(z^*)$$

$$X(z^*)$$

3.4. 6 Time Reversal

$$x^*[-n] \xleftrightarrow{Z} X^*\left(1/z^*\right), \quad ROC = 1/R_x$$

$$R_x : r_R < |z| < r_L \Rightarrow 1/R_x : 1/r_L < |z| < 1/r_R$$

$$if \;\; x[n] \; is \; real \; or \; we \; do \; not \; conjugate \; x[-n]$$

$$x[-n] \xleftrightarrow{Z} X\left(1/z\right), \quad ROC = 1/R_x$$

Example 3.18:
Time-Reverse Exponential Sequence

$$x[n] = a^{-n}u[-n] \qquad a^n u[n] \xleftrightarrow{Z} \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

$$x[-n] \xleftrightarrow{Z} X\left(1/z\right), \quad ROC = 1/R_x$$

$$X(z) = \frac{1}{1-az} \qquad = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, \quad |z| < |a^{-1}|$$

3.4. 7 Convolution of Sequences

$$x_1[n] * x_2[n] \xleftrightarrow{Z} X_1(z)X_2(z), \quad ROC\ contains\ R_{x_1} \cap R_{x_2}$$

$$y[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right\} z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n} \quad = \sum_{k=-\infty}^{\infty} x_1[k] \left\{ \sum_{m=-\infty}^{\infty} x_2[m]z^{-m} \right\} z^{-k}$$

$$= X_2(z) \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \quad = X_1(z)X_2(z)$$

Ex. 3.19: Evaluating a Convolution Using the z-transform

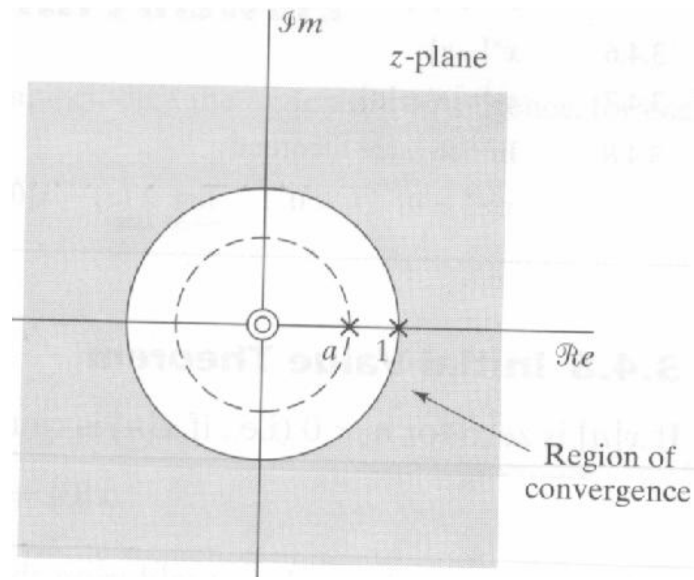
Solution: $y[n] = x_1[n] * x_2[n]$

$$x_1[n] = a^n u[n] \xleftrightarrow{Z} X_1(z) = \frac{1}{1 - az^{-1}}, |z| > |a|$$

$$x_2[n] = u[n] \xleftrightarrow{Z} X_2(z) = \frac{1}{1 - z^{-1}}, |z| > 1$$

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})} \quad \text{if } |a| < 1$$

$|z| > 1$



Example 3.19: Evaluating a Convolution Using the z-transform

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})}, \quad |z| > 1$$

$$= \frac{1}{1-a}\left(\frac{1}{(1 - z^{-1})} - \frac{a}{(1 - az^{-1})}\right), \quad |z| > 1$$

$$y[n] = \frac{1}{1-a}\left(u[n] - a^{n+1}u[n]\right)$$

3.4. 8 Initial Value Theorem

$$if \;\; x[n]=0 \;\; for \; n<0$$

$$x[0]=\lim_{z\to\infty} X(z)$$

$$X(z)=\sum_{n=0}^{\infty} x[n]z^{-n}=x[0]+\sum_{n=1}^{\infty}\frac{x[n]}{z^n}$$

$$\lim_{z\to\infty} X(z)=x[0]+\lim_{z\to\infty}\sum_{n=1}^{\infty}\frac{x[n]}{z^n}=x[0]$$

Region of convergence (ROC)

$$\cos(w_0 n) = \frac{1}{2}(e^{jw_0 n} + e^{-jw_0 n}) \qquad -\infty<n<\infty,$$

$$\Updownarrow$$

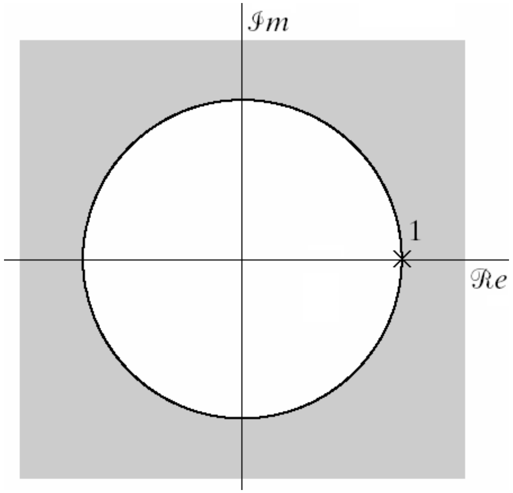
$$\sum_{k=-\infty}^{\infty}\left[\pi\delta(w - w_0 + 2\pi k) + \pi\,\delta(w + w_0 + 2\pi k)\right]$$

$$h_{lp}[n] = \frac{\sin w_c n}{\pi n} \quad -\infty<n<\infty \qquad H_{lp}(e^{jw}) = \begin{cases} 1, & |w| < w_c \\ 0, & w_c < |w| \le \pi \end{cases}$$

$$\sum_{n=-\infty}^{\infty} \frac{\sin w_c n}{\pi n} e^{-jwn}$$

- does not converge uniformly to the discontinuous function $H_{lp}(e^{jw})$.

Example



- There's no Z-Transform for $x[n]=1, \ -\infty<n<\infty$

For $\quad x[n]=u[n]$

$$X\left(e^{jw}\right)=\sum_{n=0}^{\infty}e^{-jwn} \ =\frac{1}{1-e^{-jw}}+\sum_{k=-\infty}^{\infty}\pi\,\delta\left(w+2\pi\, k\right)$$

does not absolutely converge

$$X\left(z\right)=X\left(re^{jw}\right)=\sum_{n=0}^{\infty}r^{-n}e^{-jwn}=\frac{1}{1-r^{-1}e^{-jw}}$$

$$\textit{absolutely converge if } |r|>1\Rightarrow ROC:|z|>1$$

Thank You! Keep Learning