# Sums of Powers of Integers

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## Outline of the Talk

- Introduction
- Finding sum using telescoping method
- Finding sum using calculus
- Finding sum using Bernoulli numbers
- Finding sum using functional equations
- Sum of $k^{\text{th}}$ powers of odd integers
- Conclusion

## Introduction

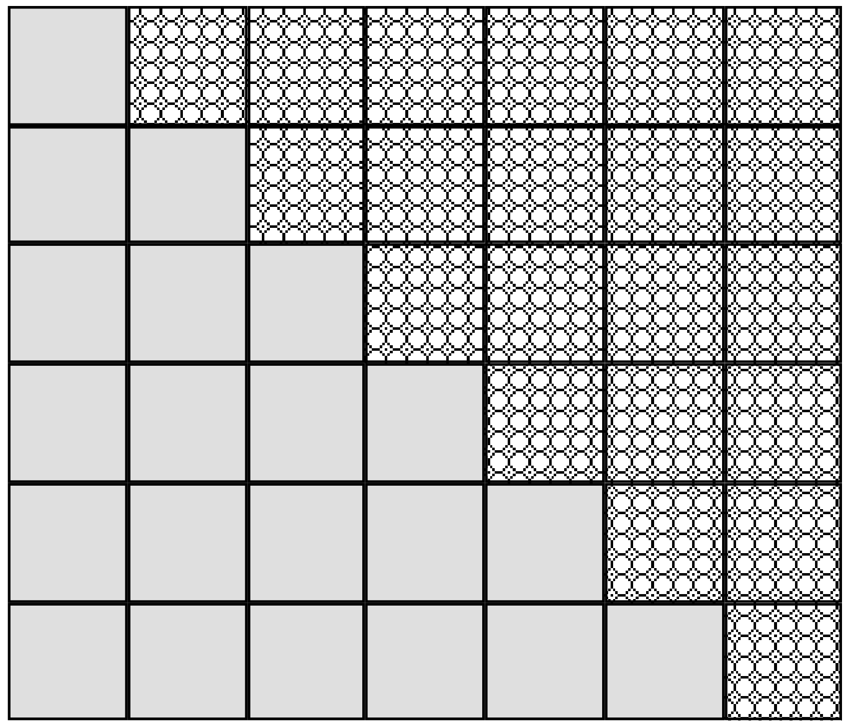
It is well-known fact that the sum of the first $n$ natural numbers is given by

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

If $f_1(n)$ denotes the sum $1 + 2 + 3 + \cdots + n$, then the $n$ by $n+1$ array of squares (see Figure 1) has $f_1(n)$ gray squares and $f_1(n)$ patterned squares, so $2f_1(n) = n(n+1)$. Therefore

$$f_1(n) = \frac{n(n+1)}{2}.$$

## Geometrical Illustration



$$2[1 + 2 + 3 + 4 + 5 + 6] = 6(6 + 1)$$

$$f_1(6) = \frac{6(6+1)}{2}$$

## An Anecdote

One day in 1787, when Gauss was only ten years old, his teacher had the students add up all the numbers from one to a hundred, with instructions that each should place his slate on a table as soon as he had completed the task. Almost immediately Gauss placed his slate on the table. The teacher looked at Gauss scornfully while the others worked diligently. When the instructor finally looked at the results, the slate of Gauss was the only one to have the correct answer, 5050, with no further calculation.

## How did young Gauss find the sum?

The ten-year-old boy evidently had computed mentally the sum $1+2+3+\cdots+99+100$ by grouping the numbers in pairs to total 101 as shown below.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 49 | 50 |
| + | + | + | + | + | + |
| 100 | 99 | 98 | 97 | 52 | 51 |
| 101 | 101 | 101 | 101 | 101 | 101 |

There are 50 pairs, each summing to 101, for a total sum of $50 \times 101$ or 5050

What is the value of the following sum for a given positive integer $k$?

$$1^k + 2^k + 3^k + \cdots + n^k.$$

Let us denote this sum by $f_k(n)$. In this talk we will explore several different ways of approaching this problem.

Finding formulas for $f_k(n)$ has interested mathematicians for more than 300 years since the time of Jacob Bernoulli.

## A Curious Fact

The sum of $k^{\text{th}}$ powers of first $n$ natural number $f_k(n)$ is a polynomial in $n$ of degree $k+1$. For each polynomial $f_k(n)$, the sum of its coefficients equals 1.

## Telescoping Method

One of the most powerful calculating tricks for evaluating the sum of a series is a technique called telescoping. The technique involves writing an expression for the difference between successive terms of a particular seris and then adding all the differences together. All terms excepts the first and the last vanish, and the sum magically appears!

To evaluate the sum

$$f_2(n) = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

let us use the telescoping method.

$$
\begin{array}{rcl}
n^3 - (n-1)^3 & = & 3\,n^2 - 3\,n + 1 \\
(n-1)^3 - (n-2)^3 & = & 3\,(n-1)^2 - 3\,(n-1) + 1 \\
(n-2)^3 - (n-3)^3 & = & 3\,(n-2)^2 - 3\,(n-2) + 1 \\
\vdots & = & \vdots \\
3^3 - 2^3 & = & 3\,(3^2) - 3\,(3) + 1 \\
2^3 - 1^3 & = & 3\,(2^2) - 3\,(2) + 1 \\
1^3 - 0^3 & = & 3\,(1^2) - 3\,(1) + 1 \\
\hline
n^3 - 0^3 & = & 3\,f_2(n) - 3\,f_1(n) + n
\end{array}
$$

Hence we have

$$f_2(n) = \frac{1}{3}\left(n^3 - 3\,f_1(n) + n\right)$$

Since $f_1(n) = \frac{n(n+1)}{2}$, we have from the last equality

$$f_2(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$= \frac{n(n+1)(2n+1)}{6}.$$

Using the telescoping method we can derive the formula for $f_3(n)$ , that is the the sum $1^3 + 2^3 + 3^3 + \cdots + n^3$. Since $n^4 - (n-1)^4 = 4\,n^3 - 6\,n^2 + 4\,n + 1$, similarly we have

$$n^4 - 0^4 = 4\,f_3(n) - 6\,f_2(n) + 4\,f_1(n) + 1.$$

As we know the formulas for $f_2(n)$ and $f_1(n)$, we can find a formula for the $f_3(n)$. With some calculations, we will have

$$f_3(n) = \left[\frac{n(n+1)}{2}\right]^2.$$

## Calculus Method

Note that the sum

$$f_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$$

of the $k^{\text{th}}$ powers of the first $n$ natural numbers is a polynomial of degree $k+1$. We can use calculus technique to find formula for $f_k(n)$ if we know the formula for $f_{k-1}(n)$.

Here is the set of rules for finding $f_k(n)$.

## Rules for Finding $f_k(n)$

- Write down $f_{k-1}(n)$ explicitly as a polynomial in $n$.
- Multiply it by $k$, that is find $k\, f_{k-1}(n)$.
- Then evaluate $\int_0^n k\, f_{k-1}(x)\, dx$.
- Finally add an appropriate linear term so that the sum of the co-efficients is 1.

**Example 1.** Find $f_2(n)$ if $f_1(n) = \frac{n(n+1)}{2}$.

**Answer:**

**Step 1.** Write $f_1(n)$ explicitly as a polynomial in $n$. Hence

$$f_1(n) = \frac{1}{2}\,n^2 + \frac{1}{2}\,n$$

**Step 2.** To find $f_2(n)$ multiply $f_1(n)$ by 2. Hence

$$2f_1(n) = n^2 + n.$$

**Step 3.** Evaluate $\int_0^n 2f_1(x)\,dx$. Hence

$$\int_0^n (x^2 + x)\,dx = \frac{n^3}{3} + \frac{n^2}{2}.$$

**Step 4.** Finally add an appropriate linear term so that the sum of the co-efficients is 1. That is add the linear term

$$\left(1 - \frac{1}{3} - \frac{1}{2}\right) n$$

to the result in Step 3. Hence $f_2(n)$ is given by

$$f_2(n) = \frac{n^3}{3} + \frac{n^2}{2} + \left(1 - \frac{1}{3} - \frac{1}{2}\right) n.$$

Simplifying, we obtain

$$f_2(n) = \frac{n\,(n+1)\,(2n+1)}{6}.$$

**Example 2.** Find $f_3(n)$ if $f_2(n) = \frac{n(n+1)(2n+1)}{6}$.

**Answer:**

**Step 1.** Write $f_2(n)$ explicitly as a polynomial in $n$. Hence

$$f_2(n) = \frac{1}{3}\,n^3 + \frac{1}{2}\,n^2 + \frac{1}{6}\,n$$

**Step 2.** To find $f_3(n)$ multiply $f_2(n)$ by 3. Hence

$$3\,f_2(n) = n^3 + \frac{3}{2}\,n^2 + \frac{1}{2}\,n.$$

**Step 3.** Evaluate $\int_0^n 3f_2(x)\,dx$. Hence

$$\int_0^n \left(x^3 + \frac{3}{2}\,x^2 + \frac{1}{2}\,x\right)\,dx = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}.$$

**Step 4.** Finally add an appropriate linear term so that the sum of the co-efficients is 1. That is add the linear term

$$\left(1 - \frac{1}{4} - \frac{1}{2} - \frac{1}{4}\right) n$$

to the result in Step 3. Hence $f_2(n)$ is given by

$$f_3(n) = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} + \left(1 - \frac{1}{4} - \frac{1}{2} - \frac{1}{4}\right) n.$$

Simplifying, we obtain

$$f_3(n) = \left[\frac{n\,(n+1)}{2}\right]^2.$$

**Example 3.** Find $f_4(n)$ if $f_3(n) = \left[\frac{n\,(n+1)}{2}\right]^2$.

**Answer:**

**Step 1.** Write $f_3(n)$ explicitly as a polynomial in $n$. Hence

$$f_3(n) = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

**Step 2.** To find $f_3(n)$ multiply $f_3(n)$ by 3. Hence

$$4\,f_3(n) = n^4 + 2\,n^3 + n^2.$$

**Step 3.** Evaluate $\int_0^n 4\,f_2(x)\,dx$. Hence

$$\int_0^n \left(x^4 + 2\,x^3 + x^2\right)\,dx = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3}.$$

**Step 4.** Finally add an appropriate linear term so that the sum of the co-efficients is 1. That is add the linear term

$$\left(1-\frac{1}{5}-\frac{1}{2}-\frac{1}{3}\right) n$$

to the result in Step 3. Hence $f_2(n)$ is given by

$$f_4(n)=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}+\left(1-\frac{1}{5}-\frac{1}{2}-\frac{1}{3}\right) n.$$

Simplifying, we obtain

$$f_4(n)=\frac{n\,(n+1)\,(2n+1)\,(3n^2+3n-1)}{30}.$$

$$f_1(n) = \sum_{i=1}^{n} i = \frac{n\left(n+1\right)}{2}$$

$$f_2(n) = \sum_{i=1}^{n} i^2 = \frac{n\left(n+1\right)\left(2n+1\right)}{6}$$

$$f_3(n) = \sum_{i=1}^{n} i^3 = \frac{n^2\left(n+1\right)^2}{4}$$

$$f_4(n) = \sum_{i=1}^{n} i^4 = \frac{n\left(n+1\right)\left(2n+1\right)\left(3n^2+3n-1\right)}{30}$$

$$f_5(n) = \sum_{i=1}^{n} i^5 = \frac{n^2\left(n+1\right)^2\left(2n^2+2n-1\right)}{12}$$

$$f_6(n) = \sum_{i=1}^{n} i^6 = \frac{n\left(n+1\right)\left(2n+1\right)\left(3n^4+6n^3-3n+1\right)}{42}$$

## Sums of $k^{\text{th}}$ Powers by Bernoulli Numbers

One of the most unusual and effective methods uses the Bernoulli Numbers. Bernoulli numbers were discovered by Jacques Bernoulli (1654-1705), the older brother of Jean Bernoulli (1667-1748) who discovered the L'Hopital rule which can be found in every textbook. The Bernoulli family of Basel is one of the most celebrated family in the histrory of mathematics. Between Nicolaus Bernoulli (1623-1708) and Jean Gustav Bernoulli, the family produced twelve outstanding mathematicians and physicists.

Jacob Bernoulli introduced the famous Bernoulli numbers in connection with finding sums of powers of integers in his book *Ars Conjectandi* which was published in 1713 after his death. Jacob was very enthusiastic about the technique that he had developed and in comparing his calculations with those of Bullialdus (1605-1694) he wrote *with the help of this table it took me less than half of a quarter of an hour to find that the tenth powers of the first 1000 numbers being added together will yield the sum*

$$91,409,924,241,424,243,424,241,924,242,500.$$

## Definition of Bernoulli Numbers

The sequence of numbers $B_0, B_1, B_2, \ldots,$ are called Bernoulli numbers if $B_0 = 1$ and, for $m > 0$,

$$B_m = -\frac{1}{m+1} \sum_{k=0}^{m-1} \binom{m+1}{k} B_k$$

where

$$\binom{m+1}{k} = \frac{(m+1)!}{k!\,(m+1-k)!}.$$

Using this definition one can find the first 16 Bernoulli Numbers to be

$$
\begin{array}{llll}
B_0 = 1 & B_1 = -\frac{1}{2} & B_2 = \frac{1}{6} & B_3 = 0 \\[2ex]
B_4 = -\frac{1}{30} & B_5 = 0 & B_6 = \frac{1}{42} & B_7 = 0 \\[2ex]
B_8 = -\frac{1}{30} & B_9 = 0 & B_{10} = \frac{3}{66} & B_{11} = 0 \\[2ex]
B_{12} = -\frac{7}{6} & B_{13} = 0 & B_{14} = \frac{691}{2730} & B_{15} = 0
\end{array}
$$

Although Bernoulli numbers seem quite complicated and even patternless, they do have many fabulous mathematical properties. Here are some of these remarkable properties:

**B1.** For all positive integers $m$, $B_{2m+1} = 0$, $B_{2m}$ alternates in sign for successive values of $m$.

**B2.** If $m$ is a positive integer, then

$$B_{2m} = \frac{(-1)^{m+1}\,(2m)!\,\zeta(2m)}{2^{2m-1}\,\pi^{2m}},$$

where $\zeta(m) = \frac{1}{1^m} + \frac{1}{2^m} + \cdots + \frac{1}{k^m} + \cdots$ is the zeta function.

**B3.** Bernoulli number $B_k$ satisfies

$$B_k = \lim_{x \to 0} \left[ \frac{d^k}{dx^k} \left( \frac{x}{e^x - 1} \right) \right].$$

**B4.** Bernoulli numbers satisfy the following equality

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Bernoulli numbers appear in the computations of the sums of powers on consecutive integers, zeta function $\zeta(2m)$, and in the expansion of many functions such as $\tan(x)$, $\tanh(x)$, $1/\sin(x)$.

## Formula for $f_k(n)$ in terms of Bernoulli Numbers

It turns out from the works of Jacques Bernoulli (1654-1705) that the sum of $k^{\text{th}}$ powers of first $n$ natural numbers is given by

$$f_k(n) = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j \; (n+1)^{k+1-j}$$

Although, the general formula is usually attributed to James Bernoulli, A. W. F. Edwards (1986) traces the knowledge of the formula to Johann Faulhaber who published these formulas (for up to $k = 17$) in 1631.

## Another Formula for $f_k(n)$

For each $k > 0$, the sum $f_k(n)$ satisfies the following first order differential equation

$$\frac{d}{dn} f_k(n) = k\, f_{k-1}(n) + (-1)^k\, B_k. \tag{DE}$$

Integrating equation (DE) on both sides we have

$$f_k(n) = \int_0^n k\, f_{k-1}(x)\, dx + (-1)^k\ n\ B_k \tag{IE}$$

which was the main tool for computing the sum $f_k(n)$ using calculus technique earlier. This formula helps us to compute $f_k(n)$ using Bernoulli numbers $B_k$ and $f_{k-1}(n)$.

Infact using the earlier formula (IE) one can find a formula for the Bernoulli numbers $B_k$ in terms of $f_k(n)$ as follows.

For $n = 1$, the equation (IE) yields

$$1 = \int_0^1 k\, f_{k-1}(x)\, dx + (-1)^k\, B_k.$$

Hence we have

$$B_k = (-1)^k \left[ 1 - \int_0^1 k\, f_{k-1}(x)\, dx \right]. \tag{BE}$$

Combining equations (IE) and (BE) we have the following recursive algorithm

$$f_k(n) = \int_0^n k\, f_{k-1}(x)\, dx + (-1)^k \; n \; B_k$$

$$B_k = (-1)^k \left[ 1 - \int_0^1 k\, f_{k-1}(x)\, dx \right].$$

with initial conditions $f_0(n) = n$ and $B_0 = 1$. With this recursive algorithm we easily derive formulas for the sums of powers $1^k + 2^k + 3^k + \cdots + n^k$.

## Functional Equation Approach to Finding $f_k(n)$

The sum of $k^{\text{th}}$ powers of first $n$ natural numbers can also be found by use of a well known functional equation called Cauchy functional equation. Functional equations are equations where the unknowns are functions. The Cauchy functional equation is the following:

$$f(x+y) = f(x) + f(y) \qquad \forall\, x, y \in \mathbb{R},$$

where $f : \mathbb{R} \to \mathbb{R}$ is the unknown function to be determined.

The continuous solution of the Cauchy functional is given by

$$f(x) = c\,x \tag{S}$$

where $c$ is any arbitrary real constant.

If continuity is not imposed then the Cauchy functional equation has strange solution in the sense that the graph of the solution is dense in the plane.

However, on the set of rationals the solution of the Cauchy functional equation is of the form (S) without any continuity or other regularity condition.

## Sum of First $n$ Natural Numbers

From the definition of $f_1(n)$, we get

$$\begin{aligned} f_1(m+n) &= 1+2+3+\cdots+m+m+1+\cdots+m+n \\ &= f_1(m)+(m+1)+(m+2)+\cdots+(m+n) \\ &= f_1(m)+1+2+\cdots+n+mn \\ &= f_1(m)+f_1(n)+mn, \quad \text{for } m,n\in\mathbb{N}. \end{aligned}$$

If we define $g:\mathbb{N}\to\mathbb{N}$ by

$$g_1(n) = f_1(n) - \frac{1}{2}\,n^2 \qquad \text{for } n\in\mathbb{N},$$

then we see that

$$g_1(m+n) = g_1(m)+g_1(n), \quad \text{for } m,n\in\mathbb{N}.$$

The solution of the above Cauchy functional equation on $\mathbb{N}$ is given by

$$g_1(n) = cn$$

where $c$ is a constant. Further we have that

$$f_1(n) = cn + \frac{n^2}{2}.$$

Since $f_1(1) = 1$, we get $1 = c + \frac{1}{2}$ which yields $c = 1 - \frac{1}{2} = \frac{1}{2}$. Therefore

$$f_1(n) = \frac{n(n+1)}{2}.$$

## Sum of Squares of First $n$ Natural Numbers

Since

$$
\begin{aligned}
& f_2(m+n) \\
& = 1^2 + 2^2 + \cdots + m^2 + (m+1)^2 + \cdots + (m+n)^2 \\
& = f_2(m) + [1^2 + 2^2 + \cdots + n^2] + 2m[1 + 2 + \cdots + n] + m^2 n \\
& = f_2(m) + f_2(n) + mn^2 + m^2 n + mn,
\end{aligned}
$$

by defining $g_2 : \mathbb{N} \to \mathbb{N}$ as

$$g_2(n) = f_2(n) - \frac{n^2}{2} - \frac{n^3}{3}, \quad \text{for } n \in \mathbb{N}$$

we have

$$g_2(m+n) = g_2(m) + g_2(n) \qquad \forall\, m, n \in \mathbb{N}.$$

Thus $g_2(n) = c\,n$, where $c$ is a constant.

$$f_2(n) = cn + \frac{n^2}{2} + \frac{n^3}{3}.$$

Using the condition $f_2(1) = 1$, we get

$$1 = c + \frac{1}{2} + \frac{1}{3}.$$

Hence $c = \frac{1}{6}$. Therefore

$$f_2(n) = \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3} = \frac{n + 3n^2 + 2n^3}{6} = \frac{n(n+1)(2n+1)}{6}.$$

## Sum of $k^{\text{th}}$ Powers of First $n$ Natural Numbers

For arbitrary $k$, we use the Binomial Theorem to get a functional equation (that is, recurrence relation). From the functional equation, we can determine the sum of $k^{\text{th}}$ powers of first $n$ natural numbers

$$
\begin{aligned}
&f_k(n+m) \\
&= 1^k + 2^k + \cdots + n^k + (n+1)^k + \cdots + (n+m)^k \\
&= f_k(n) + \sum_{i=0}^{k} \binom{k}{i} n^i 1^{k-i} + \cdots + \sum_{i=0}^{k} \binom{k}{i} n^i m^{k-i} \\
&= f_k(n) + \sum_{i=0}^{k} \binom{k}{i} n^i [1^{k-i} + \cdots + m^{k-i}] \\
&= f_k(n) + \sum_{i=0}^{k} \binom{k}{i} n^i f_{k-i}(m) \\
&= f_k(n) + f_k(m) + \sum_{i=1}^{k} \binom{k}{i} n^i f_{k-i}(m), \quad \text{for } m, n, k \in \mathbb{N}.
\end{aligned}
$$

Hence we have

$$f_k(m+n) - f_k(m) - f_k(n) = \sum_{i=1}^{k} \binom{k}{i} n^i f_{k-i}(m) \quad \text{for } m, n \in \mathbb{N}.$$

There are several ways of solving this above equation. We will discuss some. Note that $f_k(1) = 1$ for all $k \in \mathbb{N}$, $f_0(m) = m$.

## Evaluation at $n = 1$

This is probably the most direct and simplest method of all. Letting $n = 1$, we have

$$f_k(m+1) - f_k(m) - f_k(1) = \sum_{i=1}^{k} \binom{k}{i} f_{k-i}(m),$$

that is,

$$(m+1)^k - 1 = \sum_{i=1}^{k} \binom{k}{i} f_{k-i}(m), \quad \text{for } m \in \mathbb{N},$$

a simple recurrence relation.

Set $k = 2$ to get

$$m^2 + 2m = 2f_1(m) + f_0(m) = 2f_1(m) + m$$

or

$$f_1(m) = \frac{m(m+1)}{2}.$$

Set $k = 3$ to obtain

$$\begin{aligned} m^3 + 3m^2 + 3m &= 3f_2(m) + 3f_1(m) + f_0(m) \\ &= 3f_2(m) + \frac{3m(m+1)}{2} + m \end{aligned}$$

or

$$f_2(m) = \frac{m(m+1)(2m+1)}{6}.$$

## General case

The left side of (*) is symmetric with respect to $m$ and $n$, therefore we obtain

$$\sum_{i=1}^{k} \binom{k}{i} n^i f_{k-i}(m) = \sum_{i=1}^{k} \binom{k}{i} m^i f_{k-i}(n), \quad \text{for } m, n \in \mathbb{N}.$$

Substituting $m = 1$ and using the fact that $f_k(1) = 1$ we have

$$\sum_{i=1}^{k} \binom{k}{i} n^i f_{k-i}(1) = \sum_{i=1}^{k} \binom{k}{i} f_{k-i}(n),$$

that is,

$$\sum_{i=1}^{k} \binom{k}{i} f_{k-i}(n) = (1+n)^k - 1.$$

From the last equality, we get

$$k f_{k-1}(n) = (1+n)^k - 1 - \sum_{i=2}^{k} \binom{k}{i} f_{k-i}(n), \quad \text{for } n \in \mathbb{N},$$

that is, a recurrence relation

$$f_{k-1}(n) = \frac{(1+n)^k - 1 - \sum_{i=2}^{k} \binom{k}{i} f_{k-i}(n)}{k}, \quad \text{for } n \in \mathbb{N}$$

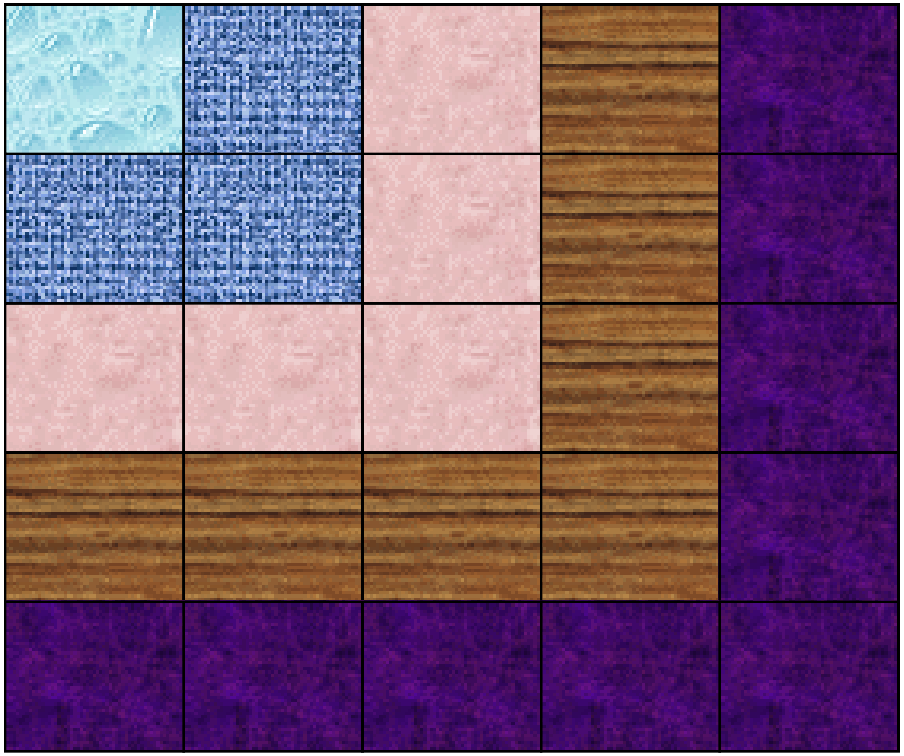
and $k = 1, 2, \ldots$. Using $f_0(n) = n$, we can determine the rest.

## Sum of first $n$ odd integers

When Kolmogorov was young, he discovered the formula for the sum of first $n$ odd integers. He showed that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

## Geometrical Illustration



$$1 + 3 + 5 + 7 + 9 = 5^2$$

$$
\begin{aligned}
1+3+5+\cdots+(2n-1) &= \sum_{i=1}^{2n} i - 2\sum_{i=1}^{n} i \\
&= f_1(2n) - 2\, f_1(n) \\
&= \frac{2\, n\, (2n+1)}{2} - 2\, \frac{n\, (n+1)}{2} \\
&= n\, (2n+1-n-1) \\
&= n^2.
\end{aligned}
$$

## Sum of powers odd integers

$$1^k + 3^k + 5^k + \cdots + (2n-1)^k$$

$$= \sum_{i=1}^{2n} i^k - 2^k \sum_{i=1}^{n} i^k$$

$$= f_k(2n) - 2^k f_k(n).$$

## An Example

$$
\begin{aligned}
1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 &= \sum_{i=1}^{2n} i^2 - 2^2 \sum_{i=1}^{n} i^2 \\
&= f_2(2n) - 2^2\, f_2(n) \\
&= \frac{1}{3}\, n\, \left(4n^2 - 1\right).
\end{aligned}
$$

$$
\begin{aligned}
1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 &= \sum_{i=1}^{2n} i^3 - 2^3 \sum_{i=1}^{n} i^3 \\
&= f_3(2n) - 2^2\, f_3(n) \\
&= n^2\, \left(2n^2 - 1\right).
\end{aligned}
$$

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